MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let $f(x)=\frac{x-1}{x+1}, x\in R-\{0,-1,1\}$. If $f^{n+1}(x) = f(f^n(x))$ for all $n \in \mathbf{N}$, then $f^6(6) + f^7(7)$ is equal to :

(A) $\frac{7}{6}$ (B) $-\frac{3}{2}$

(C) $\frac{7}{12}$ (D) $-\frac{11}{12}$

Answer (B)

Sol. $f(x)=\frac{x-1}{x+1} \Rightarrow f(f(x))=\frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1}=-\frac{1}{x}$

$$\Rightarrow f^3(x)=-\frac{x+1}{x-1} \Rightarrow f^4(x)=-\frac{\frac{x-1}{x+1}+1}{\frac{x-1}{x+1}-1}=x$$

So, $f^6(6)+f^7(7)=f^2(6)+f^3(7)$

$$=-\frac{1}{6}-\frac{7+1}{7-1}=-\frac{9}{6}=-\frac{3}{2}$$

2. Let $A=\left\{z\in\mathbf{C}:\left|\frac{z+1}{z-1}\right|<1\right\}$

and $B=\left\{z\in\mathbf{C}:\arg\left(\frac{z-1}{z+1}\right)=\frac{2\pi}{3}\right\}$.

Then $A\cap B$ is :

(A) A portion of a circle centred at $\left(0,-\frac{1}{\sqrt{3}}\right)$ that lies in the second and third quadrants only

(B) A portion of a circle centred at $\left(0,-\frac{1}{\sqrt{3}}\right)$ that lies in the second quadrant only

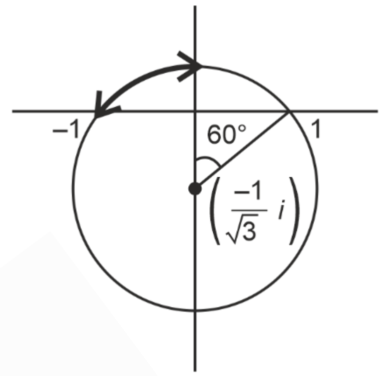
(C) An empty set

(D) A portion of a circle of radius $\frac{2}{\sqrt{3}}$ that lies in the third quadrant only

Answer (B)

Sol. $\left|\frac{z+1}{z-1}\right|<1 \Rightarrow |z+1|<|z-1| \Rightarrow \mathrm{Re}(z)<0$

and $\arg\left(\frac{z-1}{z+1}\right)=\frac{2\pi}{3}$ is a part of circle as shown.



3. Let A be a 3 × 3 invertible matrix. If |adj (24A)| = |adj (3 adj (2A))|, then $|A|^2$ is equal to :

(A) 6^6 (B) 2^{12}

(C) 2^6 (D) 1

Answer (C)

Sol. $|\mathrm{adj}(24A)|=|\mathrm{adj}(3\,\mathrm{adj}(2A))|$

$$\Rightarrow |24A|^2=|3\ \mathrm{adj}(2A)|^2$$

$$\Rightarrow \left(24^3\right)^2\cdot|A|^2=\left(3^3\right)^2|\mathrm{adj}(2A)|^2$$

$$\Rightarrow 24^6\cdot|A|^2=3^6\,|2A|^4$$

$$\Rightarrow 24^6\,|A|^2=3^6\cdot(2^3)^4\,|A|^4$$

$$\Rightarrow |A|^2=\frac{24^6}{3^6\cdot 2^{12}}=\frac{2^{18}\cdot 3^6}{3^6\cdot 2^{12}}=2^6$$

4. The ordered pair (a, b), for which the system of linear equations

$3x - 2y + z = b$

$5x - 8y + 9z = 3$

$2x + y + az = -1$

has no solution, is :

(A) $\left(3,\frac{1}{3}\right)$ (B) $\left(-3,\frac{1}{3}\right)$

(C) $\left(-3,-\frac{1}{3}\right)$ (D) $\left(3,-\frac{1}{3}\right)$

Answer (C)

Sol. $\begin{vmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{vmatrix} = 0 \Rightarrow -14a - 42 = 0 \Rightarrow a = -3$

Now 3(equation (1)) – (equation (2)) – 2(equation (3)) is

$3(3x - 2y + z - b) - (5x - 8y + 9z - 3)$

$- 2(2x + y + az + 1) = 0$

$\Rightarrow -3b + 3 - 2 = 0 \Rightarrow b = \dfrac{1}{3}$

So for no solution $a = -3$ and $b \neq \dfrac{1}{3}$

5. The remainder when $(2021)^{2023}$ is divided by 7 is :

(A) 1 (B) 2

(C) 5 (D) 6

Answer (C)

Sol. $2021 \equiv -2 \pmod 7$

$\Rightarrow (2021)^{2023} \equiv -(2)^{2023} \pmod 7$

$\equiv -2(8)^{674} \pmod 7$

$\equiv -2(1)^{674} \pmod 7$

$\equiv -2 \pmod 7$

$\equiv 5 \pmod 7$

So when $(2021)^{2023}$ is divided by 7, remainder is 5.

6. $\lim\limits_{x \to \frac{1}{\sqrt{2}}} \dfrac{\sin\left(\cos^{-1} x\right) - x}{1 - \tan\left(\cos^{-1} x\right)}$ is equal to :

(A) $\sqrt{2}$ (B) $-\sqrt{2}$

(C) $\dfrac{1}{\sqrt{2}}$ (D) $-\dfrac{1}{\sqrt{2}}$

Answer (D)

Sol. $\lim\limits_{x \to \frac{1}{\sqrt{2}}} \dfrac{\sin\left(\cos^{-1} x\right) - x}{1 - \tan\left(\cos^{-1} x\right)}$ let $\cos^{-1} x = \dfrac{\pi}{4} + \theta$

$$= \lim_{\theta \to 0} \frac{\sin\left(\frac{\pi}{4} + \theta\right) - \cos\left(\frac{\pi}{4} + \theta\right)}{1 - \tan\left(\frac{\pi}{4} + \theta\right)}$$

$$= \lim_{\theta \to 0} \frac{\sqrt{2}\sin\left(\frac{\pi}{4} + \theta - \frac{\pi}{4}\right)}{1 - \frac{1 + \tan\theta}{1 - \tan\theta}}$$

$$= \lim_{\theta \to 0} \frac{\sqrt{2}\sin\theta}{-2\tan\theta}(1 - \tan\theta) = -\frac{1}{\sqrt{2}}$$

7. $f, g : \mathbf{R} \to \mathbf{R}$ be two real valued functions defined as $f(x) = \begin{cases} -|x+3|, & x < 0 \\ e^x, & x \geq 0 \end{cases}$ and

$g(x) = \begin{cases} x^2 + k_1 x, & x < 0 \\ 4x + k_2, & x \geq 0 \end{cases}$, where k_1 and k_2 are real constants. If (gof) is differentiable at $x = 0$, then $(gof)(-4) + (gof)(4)$ is equal to :

(A) $4(e^4 + 1)$ (B) $2(2e^4 + 1)$

(C) $4e^4$ (D) $2(2e^4 - 1)$

Answer (D)

Sol. $\because$ gof is differentiable at $x = 0$

So R.H.D = L.H.D

$$\frac{d}{dx}\left(4e^x + k_2\right) = \frac{d}{dx}\left(\left(-|x+3|\right)^2 - k_1|x+3|\right)$$

$\Rightarrow 4 = 6 - k_1 \Rightarrow k_1 = 2$

Also $g(f(0^+)) = g(f(0^-))$

$\Rightarrow 4 + k_2 = 9 - 3k_1 \Rightarrow k_2 = -1$

Now $g(f(-4)) + g(f(4))$

$= g(-1) + g(e^4) = (1 - k_1) + (4e^4 + k_2)$

$= 4e^4 - 2$

$= 2(2e^4 - 1)$

8. The sum of the absolute minimum and the absolute maximum values of the function $f(x) = |3x - x^2 + 2| - x$ in the interval $[-1, 2]$ is :

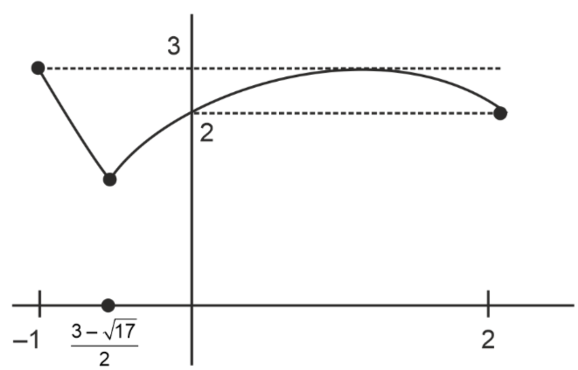
(A) $\dfrac{\sqrt{17}+3}{2}$ (B) $\dfrac{\sqrt{17}+5}{2}$

(C) 5 (D) $\dfrac{9-\sqrt{17}}{2}$

Answer (A)

Sol. $f(x) = |x^2 - 3x - 2| - x \ \forall x \in [-1, 2]$

$$\Rightarrow f(x) = \begin{cases} x^2 - 4x - 2 & \text{if } -1 \leq x < \frac{3-\sqrt{17}}{2} \\ -x^2 + 2x + 2 & \text{if } \frac{3-\sqrt{17}}{2} \leq x \leq 2 \end{cases}$$



$f(x)_{max} = 3$

$$f(x)_{\min} = f\left(\frac{3-\sqrt{17}}{2}\right)$$

$$= \frac{\sqrt{17}-3}{2}$$

9. Let S be the set of all the natural numbers, for which the line $\frac{x}{a}+\frac{y}{b}=2$ is a tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at the point (a, b), $ab \neq 0$. Then :

(A) $S = \phi$ (B) $n(S) = 1$

(C) $S = \{2k : k \in \mathbf{N}\}$ (D) $S = \mathbf{N}$

Answer (D)

Sol. $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$

$$\Rightarrow \frac{n}{a}\left(\frac{x}{a}\right)^{n-1}+\frac{n}{b}\left(\frac{y}{b}\right)^{n-1}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{b}{a}\left(\frac{bx}{ay}\right)^{n-1}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(a,b)}=-\frac{b}{a}$$

So line always touches the given curve.

10. The area bounded by the curve $y=\left|x^2-9\right|$ and the line $y = 3$ is

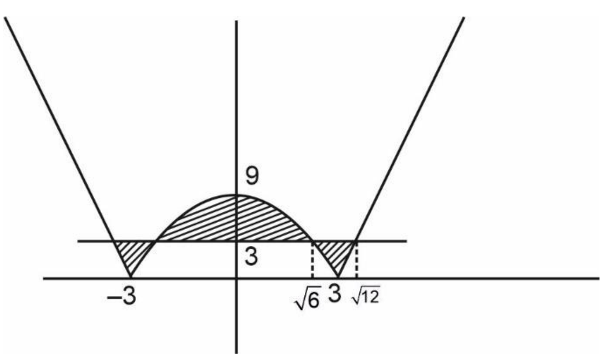
(A) $4\left(2\sqrt{3}+\sqrt{6}-4\right)$ (B) $4\left(4\sqrt{3}+\sqrt{6}-4\right)$

(C) $8\left(4\sqrt{3}+3\sqrt{6}-9\right)$ (D) $8\left(4\sqrt{3}+\sqrt{6}-9\right)$

Answer (*)

Sol. $y = 3$ and $y = |x^2 - 9|$

Intersect in first quadrant at $x=\sqrt{6}$ and $x=\sqrt{12}$



Required area

$$=2\left[\frac{2}{3}\left(6\times\sqrt{6}\right)+\int_{\sqrt{6}}^{3}\left(3-\left(9-x^2\right)\right)dx+\int_{3}^{\sqrt{12}}\left(3-\left(x^2-9\right)\right)dx\right]$$

$$=2\left[4\sqrt{6}+\left(\frac{x^3}{3}-6x\right)\Bigg|_{\sqrt{6}}^{3}+\left(12x-\frac{x^3}{3}\right)\Bigg|_{3}^{\sqrt{12}}\right]$$

$$=2\left[4\sqrt{6}+\left(4\sqrt{6}-9\right)+\left(8\sqrt{12}-27\right)\right]$$

$$=2\left[8\sqrt{6}+16\sqrt{3}-36\right]=8\left[2\sqrt{6}+4\sqrt{3}-9\right]$$

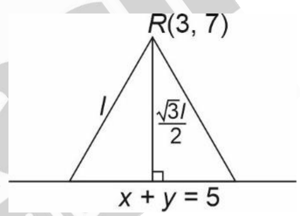
11. Let R be the point (3, 7) and let P and Q be two points on the line $x + y = 5$ such that PQR is an equilateral triangle, Then the area of ΔPQR is :

(A) $\frac{25}{4\sqrt{3}}$ (B) $\frac{25\sqrt{3}}{2}$

(C) $\frac{25}{\sqrt{3}}$ (D) $\frac{25}{2\sqrt{3}}$

Answer (D)

Sol.



Altitude of equilateral triangle,

$$\frac{\sqrt{3}l}{2}=\frac{5}{\sqrt{2}}$$

$$l=\frac{5\sqrt{2}}{\sqrt{3}}$$

Area of triangle $=\frac{\sqrt{3}}{4}l^2=\frac{\sqrt{3}}{4}.\frac{50}{3}=\frac{25}{2\sqrt{3}}$

12. Let C be a circle passing through the points $A(2, –1)$ and B (3, 4). The line segment AB is not a diameter of C. If r is the radius of C and its centre lies on the circle $(x-5)^2+(y-1)^2=\frac{13}{2}$, then r^2 is equal to :

(A) 32 (B) $\frac{65}{2}$

(C) $\frac{61}{2}$ (D) 30

Answer (B)

Sol. Equation of perpendicular bisector of AB is

$$y-\frac{3}{2}=-\frac{1}{5}\left(x-\frac{5}{2}\right)\Rightarrow x+5y=10$$

Solving it with equation of given circle,

$$(x-5)^2+\left(\frac{10-x}{5}-1\right)^2=\frac{13}{2}$$

$$\Rightarrow\ (x-5)^2\left(1+\frac{1}{25}\right)=\frac{13}{2}$$

$$\Rightarrow\ x-5=\pm\frac{5}{2}\Rightarrow x=\frac{5}{2}\text{ or }\frac{15}{2}$$

But $x\neq\frac{5}{2}$ because AB is not the diameter.

So, centre will be $\left(\frac{15}{2},\frac{1}{2}\right)$

Now $r^2=\left(\frac{15}{2}-2\right)^2+\left(\frac{1}{2}+1\right)^2$

$$=\frac{65}{2}$$

13. Let the normal at the point P on the parabola $y^2 = 6x$ pass through the point (5, –8). If the tangent at P to the parabola intersects its directrix at the point Q, then the ordinate of the point Q is :

(A) –3 (B) $-\frac{9}{4}$

(C) $-\frac{5}{2}$ (D) –2

Answer (B)

Sol. Let $P(at^2, 2at)$ where $a=\frac{3}{2}$

$T : yt = x + at^2$ So point Q is $\left(-a, at-\frac{a}{t}\right)$

$N : y = -tx + 2at + at^3$ passes through (5, –8)

$$-8=-5t+3t+\frac{3}{2}t^3$$

$\Rightarrow$ $3t^3 - 4t + 16 = 0$

$\Rightarrow$ $(t + 2)(3t^2 - 6t + 8) = 0$

$\Rightarrow$ $t = -2$

So ordinate of point Q is $-\frac{9}{4}$.

14. If the two lines $l_1:\frac{x-2}{3}=\frac{y+1}{-2}, z=2$ and $l_2:\frac{x-1}{1}=\frac{2y+3}{\alpha}=\frac{z+5}{2}$ are perpendicular, then an angle between the lines l_2 and $l_3:\frac{1-x}{3}=\frac{2y-1}{-4}=\frac{z}{4}$ is :

(A) $\cos^{-1}\left(\frac{29}{4}\right)$ (B) $\sec^{-1}\left(\frac{29}{4}\right)$

(C) $\cos^{-1}\left(\frac{2}{29}\right)$ (D) $\cos^{-1}\left(\frac{2}{\sqrt{29}}\right)$

Answer (B)

Sol. $\because$ L_1 and L_2 are perpendicular, so

$$3\times1+(-2)\left(\frac{\alpha}{2}\right)+0\times2=0$$

$\Rightarrow$ $\alpha = 3$

Now angle between l_2 and l_3,

$$\cos\theta=\frac{1(-3)+\frac{\alpha}{2}(-2)+2(4)}{\sqrt{1+\frac{\alpha^2}{4}+4}\sqrt{9+4+16}}$$

$$\Rightarrow\ \cos\theta=\frac{2}{\frac{29}{2}}\Rightarrow\theta=\cos^{-1}\left(\frac{4}{29}\right)=\sec^{-1}\left(\frac{29}{4}\right)$$

15. Let the plane $2x + 3y + z + 20 = 0$ be rotated through a right angle about its line of intersection with the plane $x - 3y + 5z = 8$. If the mirror image of the point $\left(2,-\frac{1}{2},2\right)$ in the rotated plane is $B(a, b, c)$, then :

(A) $\frac{a}{8}=\frac{b}{5}=\frac{c}{-4}$ (B) $\frac{a}{4}=\frac{b}{5}=\frac{c}{-2}$

(C) $\frac{a}{8}=\frac{b}{-5}=\frac{c}{4}$ (D) $\frac{a}{4}=\frac{b}{5}=\frac{c}{2}$

Answer (A)

Sol. Consider the equation of plane,

$P : (2x + 3y + z + 20) + \lambda(x - 3y + 5z - 8) = 0$

$P : (2 + \lambda)x + (3 - 3\lambda)y + (1 + 5\lambda)z + (20 - 8\lambda) = 0$

$\because$ Plane P is perpendicular to $2x + 3y + z + 20 = 0$

So, $4 + 2\lambda + 9 - 9\lambda + 1 + 5\lambda = 0$

$\Rightarrow$ $\lambda = 7$

$P : 9x - 18y + 36z - 36 = 0$

Or $P : x - 2y + 4z = 4$

If image of $\left(2, -\frac{1}{2}, 2\right)$ in plane P is (a, b, c) then

$$\frac{a-2}{1} = \frac{b+\frac{1}{2}}{-2} = \frac{c-2}{4}$$

and $\left(\frac{a+2}{2}\right) - 2\left(\frac{b-\frac{1}{2}}{2}\right) + 4\left(\frac{c+2}{2}\right) = 4$

Clearly $a = \frac{4}{3}, b = \frac{5}{6}$ and $c = -\frac{2}{3}$

So, $a : b : c = 8 : 5 : -4$

16. If $\vec{a}\cdot\vec{b} = 1, \vec{b}\cdot\vec{c} = 2$ and $\vec{c}\cdot\vec{a} = 3$, then the value of $\left[\vec{a}\times\left(\vec{b}\times\vec{c}\right), \vec{b}\times\left(\vec{c}\times\vec{a}\right), \vec{c}\times\left(\vec{b}\times\vec{a}\right)\right]$ is :

(A) 0 (B) $-6\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$

(C) $12\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$ (D) $-12\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)$

Answer (A)

Sol. $\because$ $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = 3\vec{b} - \vec{c} = \vec{u}$

$\vec{b}\times\left(\vec{c}\times\vec{a}\right) = \vec{c} - 2\vec{a} = \vec{v}$

$\vec{c}\times(\vec{b}\times\vec{a}) = 3\vec{b} - 2\vec{a} = \vec{w}$

$\therefore$ $\vec{u} + \vec{v} = \vec{w}$

So vectors $\vec{u}, \vec{v}$ and $\vec{w}$ are coplanar, hence their Scalar triple product will be zero.

17. Let a biased coin be tossed 5 times. If the probability of getting 4 heads is equal to the probability of getting 5 heads, then the probability of getting atmost two heads is:

(A) $\frac{275}{6^5}$ (B) $\frac{36}{5^4}$

(C) $\frac{181}{5^5}$ (D) $\frac{46}{6^4}$

Answer (D)

Sol. Let probability of getting head = p

So, ${}^5C_4 p^4(1-p) = {}^5C_5 p^5$

$\Rightarrow$ $p = 5(1-p) \Rightarrow p = \frac{5}{6}$

Probability of getting atmost two heads =

${}^5C_0(1-p)^5 + {}^5C_1 p(1-p)^4 + {}^5C_2 p^2(1-p)^3$

$= \frac{1+25+250}{6^5}$

$= \frac{276}{6^5} = \frac{46}{6^4}$

18. The mean of the numbers a, b, 8, 5, 10 is 6 and their variance is 6.8. If M is the mean deviation of the numbers about the mean, then 25 M is equal to:

(A) 60 (B) 55

(C) 50 (D) 45

Answer (A)

Sol. $\because$ $\bar{x} = 6 = \frac{a+b+8+5+10}{5} \Rightarrow a+b = 7 \ldots\text{(i)}$

And $\sigma^2 = \frac{a^2+b^2+8^2+5^2+10^2}{5} - 6^2 = 6.8$

$\Rightarrow$ $a^2 + b^2 = 25$ ...(ii)

From (i) and (ii) $(a, b) = (3, 4)$ or $(4, 3)$

Now mean deviation about mean

$M = \frac{1}{5}(3+2+2+1+4) = \frac{12}{5}$

$\Rightarrow$ $25M = 60$

19. Let $f(x) = 2\cos^{-1}x + 4\cot^{-1}x - 3x^2 - 2x + 10, \chi \in [-1, 1]$, If $[a, b]$ is the range of the function, f then $4a - b$ is equal to :

(A) 11

(B) $11 - \pi$

(C) $11 + \pi$

(D) $15 - \pi$

Answer (B)

Sol. $f(x) = 2\cos^{-1}x + 4\cot^{-1}x - 3x^2 - 2x + 10 \; \forall x \in [-1, 1]$

$\Rightarrow$ $f'(x) = -\frac{2}{\sqrt{1-x^2}} - \frac{4}{1+x^2} - 6x - 2 < 0 \; \forall x \in [-1, 1]$

So $f(x)$ is decreasing function and range of $f(x)$ is $[f(1), f(-1)]$, which is $[\pi+5, 5\pi+9]$

Now $4a - b = 4(\pi + 5) - (5\pi + 9)$

$= 11 - \pi$

20. Let $\Delta, \nabla \in \{\wedge, \vee\}$ be such that $p \nabla q \Rightarrow ((p \Delta q) \nabla r)$ is a tautology. Then $(p \nabla q) \Delta r$ is logically equivalent to :

(A) $(p \Delta r) \vee q$ (B) $(p \Delta r) \wedge q$

(C) $(p \wedge r) \Delta q$ (D) $(p \nabla r) \wedge q$

Answer (A)

Sol. Case-I

If ∇ is same as $\wedge$

Then $(p \wedge q) \Rightarrow ((p \Delta q) \wedge r)$ is equivalent to $\sim(p \wedge q) \vee ((p \Delta q) \wedge r)$ is equivalent to $(\sim(p \wedge q) \vee (p \Delta q)) \wedge (\sim(p \wedge q) \vee r)$

Which cannot be a tautology

For both Δ (i.e. $\vee$ or $\wedge$)

Case-II

If ∇ is same as $\vee$

Then $(p \vee q) \Rightarrow ((p \Delta q) \vee r)$ is equivalent to

$\sim(p \vee q) \vee (p \Delta q) \vee r$ which can be a tautology if Δ is also same as $\vee$.

Hence both Δ and ∇ are same as $\vee$.

Now $(p \nabla q) \Delta r$ is equivalent to $(p \vee q \vee r)$.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. The sum of the cubes of all the roots of the equation $x^4 - 3x^3 - 2x^2 + 3x + 1 = 0$ is _______.

Answer (36)

Sol. $x^4 - 3x^3 - x^2 - x^2 + 3x + 1 = 0$

$(x^2 - 1)(x^2 - 3x - 1) = 0$

Let the root of $x^2 - 3x - 1 = 0$ be α and β and other two roots of given equation are 1 and –1

So sum of cubes of roots $= 1^3 + (-1)^3 + \alpha^3 + \beta^3$

$= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= (3)^3 - 3(-1)(3)$

$= 36$

2. There are ten boys $B_1, B_2, \ldots, B_{10}$ and five girls $G_1, G_2, \ldots, G_5$ in a class. Then the number of ways of forming a group consisting of three boys and three girls, if both B_1 and B_2 together should not be the members of a group, is ________.

Answer (1120)

Sol. Required number of ways = Total ways of selection – ways in which B_1 and B_2 are present together.

$= {}^{10}C_3 \cdot {}^5C_3 - {}^8C_1 \cdot {}^5C_3 = 10(120 - 8)$

$= 1120$

3. Let the common tangents to the curves $4(x^2 + y^2) = 9$ and $y^2 = 4x$ intersect at the point Q. Let an ellipse, centered at the origin O, has lengths of semi-minor and semi-major axes equal to OQ and 6, respectively. If e and l respectively denote the eccentricity and the length of the latus rectum of this ellipse, then $\frac{l}{e^2}$ is equal to ________.

Answer (4)

Sol. Let $y = mx + c$ is the common tangent

So $c = \frac{1}{m} = \pm\frac{3}{2}\sqrt{1+m^2} \Rightarrow m^2 = \frac{1}{3}$

So equation of common tangents will be

$y = \pm\frac{1}{\sqrt{3}}x \pm \sqrt{3}$, which intersects at $Q(-3, 0)$

Major axis and minor axis of ellipse are 12 and 6. So eccentricity

$e^2 = 1 - \frac{1}{4} = \frac{3}{4}$ and length of latus rectum

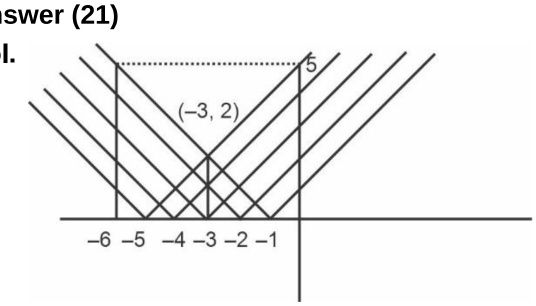
$= \frac{2b^2}{a} = 3$

Hence $\frac{\ell}{e^2} = \frac{3}{3/4} = 4$

4. Let $f(x) = \max\{|x+1|, |x+2|, \ldots\ldots, |x+5|\}$. Then $\int_{-6}^{0} f(x)dx$ is equal to _______.

Answer (21)

Sol.



$$\int_{-6}^{0} f(x)dx = 2\left[\frac{1}{2}(2+5)3\right] = 21$$

5. Let the solution curve $y = y(x)$ of the differential equation $(4 + x^2)dy - 2x(x^2 + 3y + 4)dx = 0$ pass through the origin. Then $y(2)$ is equal to ________.

Answer (12)

Sol. $(4 + x^2)\, dy - 2x(x^2 + 3y + 4)dx = 0$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{6x}{x^2+4}\right)y + 2x$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{6x}{x^2+4}\right)y = 2x$$

$$\text{I.F.} = e^{-3\ln\left(x^2+4\right)} = \frac{1}{\left(x^2+4\right)^3}$$

$$\text{So } \frac{y}{\left(x^2+4\right)^3} = \int \frac{2x}{\left(x^2+4\right)^3}dx + c$$

$$\Rightarrow y = -\frac{1}{2}\left(x^2+4\right) + c\left(x^2+4\right)^3$$

When $x = 0$, $y = 0$ gives $c = \frac{1}{32}$,

So, for $x = 2$, $y = 12$

6. If $\sin^2(10°)\sin(20°)\sin(40°)\sin(50°)\sin(70°)$ $= \alpha - \frac{1}{16}\sin(10°)$, then $16 + \alpha^{-1}$ is equal to ________.

Answer (80)

Sol: $(\sin 10° \cdot \sin 50° \cdot \sin 70°) \cdot (\sin 10° \cdot \sin 20° \cdot \sin 40°)$

$$= \left(\frac{1}{4}\sin 30°\right) \cdot \left[\frac{1}{2}\sin 10°\left(\cos 20° - \cos 60°\right)\right]$$

$$= \frac{1}{16}\left[\sin 10°\left(\cos 20° - \frac{1}{2}\right)\right]$$

$$= \frac{1}{32}\left[2\sin 10° \cdot \cos 20° - \sin 10°\right]$$

$$= \frac{1}{32}\left[\sin 30° - \sin 10° - \sin 10°\right]$$

$$= \frac{1}{64} - \frac{1}{16}\sin 10°$$

Clearly $\alpha = \frac{1}{64}$

Hence $16 + \alpha^{-1} = 80$

7. Let $A = \{n \in \mathbf{N} : \text{H.C.F.}\,(n, 45) = 1\}$ and

Let $B = \{2k : k \in \{1, 2, \ldots, 100\}\}$. Then the sum of all the elements of $A \cap B$ is ___________.

Answer (5264)

Sol: Sum of all elements of $A \cap B$ = 2 [Sum of natural numbers upto 100 which are neither divisible by 3 nor by 5]

$$= 2\left[\frac{100\times 101}{2} - 3\left(\frac{33\times 34}{2}\right) - 5\left(\frac{20\times 21}{2}\right) + 15\left(\frac{6\times 7}{2}\right)\right]$$

= 10100 – 3366 – 2100 + 630

= 5264

8. The value of the integral

$$\frac{48}{\pi^4}\int_0^{\pi}\left(\frac{3\pi x^2}{2} - x^3\right)\frac{\sin x}{1+\cos^2 x}dx \text{ is equal to}$$

________.

Answer (6)

Sol: $I = \frac{48}{\pi^4}\int_0^{\pi}\left[\left(\frac{\pi}{2}-x\right)^3 - \frac{3\pi^2}{4}\left(\frac{\pi}{2}-x\right) + \frac{\pi^3}{4}\right]\frac{\sin x\, dx}{1+\cos^2 x}$

Using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx$ we get

$$I = \frac{48}{\pi^4}\int_0^{\pi}\left[-\left(\frac{\pi}{2}-x\right)^3 + \frac{3\pi^2}{4}\left(\frac{\pi}{2}-x\right) + \frac{\pi^3}{4}\right]\frac{\sin x\, dx}{1+\cos^2 x}$$

Adding these two equations, we get

$$2I = \frac{48}{\pi^4}\int_0^{\pi}\frac{\pi^3}{2}\cdot\frac{\sin x\, dx}{1+\cos^2 x}$$

$$\Rightarrow I = \frac{12}{\pi}\left[-\tan^{-1}(\cos x)\right]_0^{\pi} = \frac{12}{\pi}\cdot\frac{\pi}{2} = 6$$

9. Let $A = \sum_{i=1}^{10}\sum_{j=1}^{10}\min\{i,j\}$ and $B = \sum_{i=1}^{10}\sum_{j=1}^{10}\max\{i,j\}$.

Then $A + B$ is equal to ____________.

Answer (1100)

Sol: Each element of ordered pair $\{i, j\}$ is either present in A or in B.

So, $A + B$ = Sum of all elements of all ordered pairs $\{i, j\}$ for $1 \le i \le 10$ and $1 \le j \le 10$

= 20 (1 + 2 + 3 +…. + 10)

= 1100

10. Let $S = (0, 2\pi) - \left\{\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\right\}$. Let $y = y(x)$, $x \in S$, be the solution curve of the differential equation $\frac{dy}{dx} = \frac{1}{1+\sin 2x}$, $y\left(\frac{\pi}{4}\right) = \frac{1}{2}$. If the sum of abscissas of all the points of intersection of the curve $y = y(x)$ with the curve $y = \sqrt{2}\sin x$ is $\frac{k\pi}{12}$, then k is equal to ________.

Answer (42)

Sol: $\frac{dy}{dx} = \frac{1}{1+\sin 2x}$

$$\Rightarrow \quad dy = \frac{\sec^2 x\, dx}{(1+\tan x)^2}$$

$$\Rightarrow \quad y = -\frac{1}{1+\tan x} + c$$

When $x = \frac{\pi}{4}$, $y = \frac{1}{2}$ gives $c = 1$

So $y = \frac{\tan x}{1+\tan x} \Rightarrow y = \frac{\sin x}{\sin x + \cos x}$

Now, $y = \sqrt{2}\sin x \quad \Rightarrow \sin x = 0$

or $\sin x + \cos x = \frac{1}{\sqrt{2}}$

$\sin x = 0$ gives $x = \pi$ only.

and $\sin x + \cos x = \frac{1}{\sqrt{2}} \Rightarrow \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{2}$

So $x + \frac{\pi}{4} = \frac{5\pi}{6}$ or $\frac{13\pi}{6} \Rightarrow x = \frac{7\pi}{12}$ or $\frac{23\pi}{12}$

Sum of all solutions = $\pi + \frac{7\pi}{12} + \frac{23\pi}{12} = \frac{42\pi}{12}$

Hence $k = 42$.

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