MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let $R_1 = \{(a, b) \in \mathbf{N} \times \mathbf{N} : |a - b| \le 13\}$ and $R_2 = \{(a, b) \in \mathbf{N} \times \mathbf{N} : |a - b| \ne 13\}$. Then on $\mathbf{N}$:

(A) Both R_1 and R_2 are equivalence relations

(B) Neither R_1 nor R_2 is an equivalence relation

(C) R_1 is an equivalence relation but R_2 is not

(D) R_2 is an equivalence relation but R_1 is not

Answer (B)

Sol. $R_1 = \{(a, b) \in N \times N : |a - b| \le 13\}$ and

$R_2 = \{(a, b) \in N \times N : |a - b| \ne 13\}$

In R_1: $\because$ $|2 - 11| = 9 \le 13$

$\therefore$ $(2, 11) \in R_1$ and $(11, 19) \in R_1$ but $(2, 19) \notin R_1$

$\therefore$ R_1 is not transitive

Hence R_1 is not equivalence

In R_2 : $(13, 3) \in R_2$ and $(3, 26) \in R_2$ but

$(13, 26) \notin R_2$ $\quad$ ($\because$ $|13 - 26| = 13$)

$\therefore$ R_2 is not transitive

Hence R_2 is not equivalence.

2. Let $f(x)$ be a quadratic polynomial such that $f(-2) + f(3) = 0$. If one of the roots of $f(x) = 0$ is -1, then the sum of the roots of $f(x) = 0$ is equal to:

(A) $\frac{11}{3}$ $\quad$ (B) $\frac{7}{3}$

(C) $\frac{13}{3}$ $\quad$ (D) $\frac{14}{3}$

Answer (A)

Sol. $\because$ $x = -1$ be the roots of $f(x) = 0$

$\therefore$ let $f(x) = A(x + 1)(x - b)$ $\quad$ ...(i)

Now, $f(-2) + f(3) = 0$

$\Rightarrow$ $A[-1(-2 - b) + 4(3 - b)] = 0$

$$b = \frac{14}{3}$$

$\therefore$ Second root of $f(x) = 0$ will be $\frac{14}{3}$

$\therefore$ Sum of roots $= \frac{14}{3} - 1 = \frac{11}{3}$

3. The number of ways to distribute 30 identical candies among four children C_1, C_2, C_3 and C_4 so that C_2 receives atleast 4 and atmost 7 candies, C_3 receives atleast 2 and atmost 6 candies, is equal to:

(A) 205 $\quad$ (B) 615

(C) 510 $\quad$ (D) 430

Answer (D)

Sol. By multinomial theorem, no. of ways to distribute 30 identical candies among four children C_1, C_2 and C_3, C_4

$= $ Coefficient of x^{30} in $(x^4 + x^5 + \ldots + x^7)(x^2 + x^3 + \ldots + x^6)(1 + x + x^2 \ldots)^2$

$$= \text{Coefficient of } x^{24} \text{ in } \left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^5}{1-x}\right)\frac{\left(1-x^{31}\right)^2}{(1-x)^2}$$

$= $ Coefficient of x^{24} in $(1 - x^4 - x^5 + x^9)(1 - x)^{-4}$

$= {}^{27}C_{24} - {}^{23}C_{20} - {}^{22}C_{19} + {}^{18}C_{15} = 430$

4. The term independent of x in the expansion of $\left(1 - x^2 + 3x^3\right)\left(\frac{5}{2}x^3 - \frac{1}{5x^2}\right)^{11}, x \ne 0$ is:

(A) $\frac{7}{40}$ $\quad$ (B) $\frac{33}{200}$

(C) $\frac{39}{200}$ $\quad$ (D) $\frac{11}{50}$

Answer (B)

Sol. $\left(1 - x^2 + 3x^3\right)\left(\frac{5}{2}x^3 - \frac{1}{5x^2}\right)^{11}, x \ne 0$

General term of $\left(\frac{5}{2}x^3 - \frac{1}{5x^2}\right)^{11}$ is

$$T_{r+1} = {}^{11}C_r \left(\frac{5}{2}x^3\right)^{11-r}\left(\frac{-1}{5x^2}\right)^r$$

$$= {}^{11}C_r \left(\frac{5}{2}\right)^{11-r}\left(\frac{-1}{5}\right)^r x^{33-5r}$$

So, term independent from x in given expression

$$= -{}^{11}C_7\left(\frac{5}{2}\right)^4\left(\frac{-1}{5}\right)^7 = \frac{11\times10\times9\times8}{24}\times\frac{1}{16\times125}$$

$$= \frac{33}{200}$$

5. If n arithmetic means are inserted between a and 100 such that the ratio of the first mean to the last mean is 1 : 7 and $a + n = 33$, then the value of n is:

(A) 21 (B) 22

(C) 23 (D) 24

Answer (C)

Sol. a, A_1, A_2 …….. A_n, 100

Let d be the common difference of above A.P. then

$$\frac{a+d}{100-d} = \frac{1}{7}$$

$\Rightarrow$ $7a + 8d = 100$ …(i)

and $a + n = 33$ …(ii)

and $100 = a + (n + 1)d$

$$\Rightarrow \quad 100 = a + (34 - a)\frac{(100-7a)}{8}$$

$\Rightarrow$ $800 = 8a + 7a^2 - 338a + 3400$

$\Rightarrow$ $7a^2 - 330a + 2600 = 0$

$$\Rightarrow \quad a = 10,\ \frac{260}{7},\ \text{but } a \neq \frac{260}{7}$$

$\therefore$ $n = 23$

6. Let $f, g : R \to R$ be functions defined by

$$f(x) = \begin{cases} [x] & , \ x<0 \\ |1-x| & , \ x \ge 0 \end{cases} \text{ and } g(x) = \begin{cases} e^x - x & , \ x<0 \\ (x-1)^2 - 1 & , \ x \ge 0 \end{cases}$$

Where [x] denote the greatest integer less than or equal to x. Then, the function fog is discontinuous at exactly :

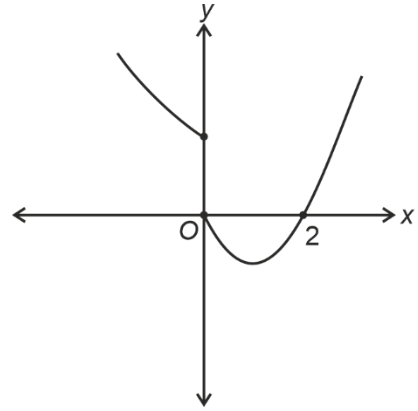
(A) one point (B) two points

(C) three points (D) four points

Answer (B)

Sol. $$f(x) = \begin{cases} [x] & , \ x<0 \\ |1-x| & , \ x \ge 0 \end{cases} \text{ and } g(x) = \begin{cases} e^x - x & , \ x<0 \\ (x-1)^2 - 1 & , \ x \ge 0 \end{cases}$$

$$fog(x) = \begin{cases} [g(x)] & , \ g(x)<0 \\ |1-g(x)| & , \ g(x) \ge 0 \end{cases}$$



(graph of $y = g(x)$)

$$= \begin{cases} |1+x-e^x| & , \quad x<0 \\ 1 & , \quad x=0 \\ \left[(x-1)^2-1\right] & , \quad 0<x<2 \\ \left|2-(x-1)^2\right| & , \quad x \ge 2 \end{cases}$$

So, $x = 0, 2$ are the two points where fog is discontinuous.

7. Let $f : R \to R$ be a differentiable function such that $f\left(\frac{\pi}{4}\right) = \sqrt{2},\ f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = 1$ and let $g(x) = \int_x^{\frac{\pi}{4}}\left(f'(t)\sec t + \tan t\sec t\, f(t)\right)dt$ for $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^-} g(x)$ is equal to

(A) 2 (B) 3

(C) 4 (D) –3

Answer (B)

Sol. Given : $f\left(\frac{\pi}{4}\right) = \sqrt{2},\ f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = 1$

$$g(x) = \int_x^{\frac{\pi}{4}}\left(f'(t)\sec t + \tan t\sec t\, f(t)\right)dt$$

$$= \left[\sec t + f(t)\right]_x^{\frac{\pi}{4}} = 2 - \sec x\, f(x)$$

$$\text{Now, } \lim_{x \to \frac{\pi}{2}^-} g(x) = \lim_{h \to 0} g\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \to 0} 2 - (\operatorname{cosec} h)\, f\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h\to 0}\left[2 - \frac{f\left(\frac{\pi}{2}-h\right)}{\sin h}\right]$$

$$= \lim_{h\to 0}\left[2 + \frac{f'\left(\frac{\pi}{2}-h\right)}{\cos h}\right]$$

$$= 3$$

8. Let $f : R \to R$ be a continuous function satisfying $f(x) + f(x + k) = n$, for all $x \in R$ where $k > 0$ and n is a positive integer. If $I_1 = \int_0^{4nk} f(x)dx$ and $I_2 = \int_{-k}^{3k} f(x)dx$, then

(A) $I_1 + 2I_2 = 4nk$ (B) $I_1 + 2I_2 = 2nk$

(C) $I_1 + nI_2 = 4n^2k$ (D) $I_1 + nI_2 = 6n^2k$

Answer (C)

Sol. $f : R \to R$ and $f(x) + f(x+k) = n \quad \forall x \in R$

$x \to x + k$

$f(x+k) + f(x+2k) = n$

$\therefore \quad f(x+2k) = f(x)$

So, period of $f(x)$ is $2k$

Now, $I_1 = \int_0^{4nk} f(x)dx = 2n\int_0^{2k} f(x)dx$

$$= 2n\left[\int_0^{k} f(x)dx + \int_k^{2k} f(x)dx\right]$$

$x = t + k \Rightarrow dx = dt$ (in second integral)

$$= 2n\left[\int_0^{k} f(x)dx + \int_0^{k} f(t+k)dt\right]$$

$$= 2n^2k$$

Now, $I_2 = \int_{-k}^{3k} f(x)dx = 2\int_0^{2k} f(x)dx$

$I_2 = 2(nk)$

$\therefore \quad I_1 + nI_2 = 4n^2k$

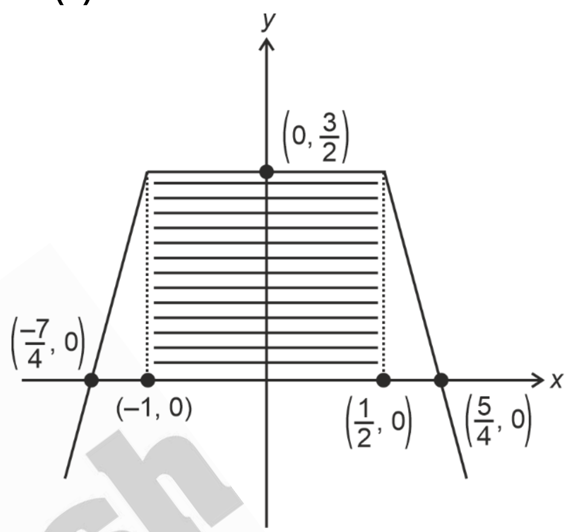
9. The area of the bounded region enclosed by the curve $y = 3 - \left|x - \frac{1}{2}\right| - |x+1|$ and the x-axis is

(A) $\frac{9}{4}$ (B) $\frac{45}{16}$

(C) $\frac{27}{8}$ (D) $\frac{63}{16}$

Answer (C)

Sol.



$$y = \begin{cases} 2x - \frac{7}{2} & x < -1 \\ \frac{3}{2} & -1 \le x \le \frac{1}{2} \\ \frac{5}{2} - 2x & x > \frac{1}{2} \end{cases}$$

$y = 3 - \left|x - \frac{1}{2}\right| - |x+1|$

Area of shaded region (required area)

$$= \frac{1}{2}\left(3 + \frac{3}{2}\right)\cdot\frac{3}{2} = \frac{27}{8}$$

10. Let $x = x(y)$ be the solution of the differential equation $2ye^{\frac{x}{y^2}}dx + \left(y^2 - 4xe^{\frac{x}{y^2}}\right)dy = 0$ such that $x(1) = 0$. Then, $x(e)$ is equal to

(A) $e\log_e(2)$ (B) $-e\log_e(2)$

(C) $e^2\log_e(2)$ (D) $-e^2\log_e(2)$

Answer (D)

Sol. Given differential equation

$$2ye^{\frac{x}{y^2}}dx + \left(y^2 - 4xe^{\frac{x}{y^2}}\right)dy = 0,\; x(1) = 0$$

$$\Rightarrow\ e^{\frac{x}{y^2}}\left[2ydx-4xdy\right]=-y^2dy$$

$$\Rightarrow\ e^{\frac{x}{y^2}}\left[\frac{2y^2dx-4xydy}{y^4}\right]=\frac{-1}{y}dy$$

$$\Rightarrow\ 2e^{\frac{x}{y^2}}d\left(\frac{x}{y^2}\right)=-\frac{1}{y}dy$$

$$\Rightarrow\ 2e^{\frac{x}{y^2}}=-\ln y+c \qquad \ldots\text{(i)}$$

Now, using $x(1) = 0$, $c = 2$

So, for $x(e)$, Put $y = e$ in (i)

$$2e^{\frac{x}{e^2}}=-1+2$$

$$\Rightarrow\ \frac{x}{e^2}=\ln\left(\frac{1}{2}\right)\ \Rightarrow x(e)=-e^2\ln 2$$

11. Let the slope of the tangent to a curve $y = f(x)$ at (x, y) be given by 2 tan$x$(cos$x$ – y). If the curve passes through the point $\left(\frac{\pi}{4},0\right)$, then the value of $\int_0^{\pi/2} y\,dx$ is equal to :

(A) $(2-\sqrt{2})+\frac{\pi}{\sqrt{2}}$ (B) $2-\frac{\pi}{\sqrt{2}}$

(C) $(2+\sqrt{2})+\frac{\pi}{\sqrt{2}}$ (D) $2+\frac{\pi}{\sqrt{2}}$

Answer (B)

Sol. $\frac{dy}{dx}=2\tan x(\cos x-y)$

$$\Rightarrow\ \frac{dy}{dx}+2\tan xy=2\sin x$$

I.F. = $e^{\int 2\tan x\,dx}=\sec^2 x$

$\therefore$ Solution of D.E. will be

$$y(x)\sec^2 x=\int 2\sin x\sec^2 x\,dx$$

$$y\sec^2 x=2\sec x+c$$

$\because$ Curve passes through $\left(\frac{\pi}{4},0\right)$

$\therefore\ c=-2\sqrt{2}$

$\therefore\ y=2\cos x-2\sqrt{2}\cos^2 x$

$\therefore\ \int_0^{\pi/2} ydx=\int_0^{\pi/2}\left(2\cos x-2\sqrt{2}\cos^2 x\right)dx$

$$=2-2\sqrt{2}\cdot\frac{\pi}{4}=2-\frac{\pi}{\sqrt{2}}$$

12. Let a triangle be bounded by the lines L_1 : $2x + 5y = 10$; L_2 : $-4x + 3y = 12$ and the line L_3, which passes through the point P(2, 3), intersects L_2 at A and L_1 at B. If the point P divides the line-segment AB, internally in the ratio 1 : 3, then the area of the triangle is equal to

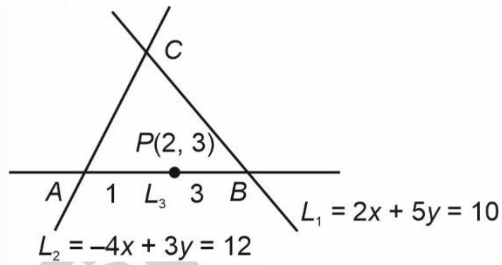
(A) $\frac{110}{13}$ (B) $\frac{132}{13}$

(C) $\frac{142}{13}$ (D) $\frac{151}{13}$

Answer (B)

Sol. $L_1 : 2x + 5y = 10$

$L_2 : -4x + 3y = 12$



Solving L_1 and L_2 we get

$$C\equiv\left(\frac{-15}{13},\frac{32}{13}\right)$$

Now, Let $A\left(x_1,\frac{1}{3}(12+4x_1)\right)$ and

$B\left(x_2,\frac{1}{5}(10-2x_2)\right)$

$$\therefore\ \frac{3x_1+x_2}{4}=2$$

and $\dfrac{(12+4x_1)+\dfrac{10-2x_2}{5}}{4}=3$

So, $3x_1 + x_2 = 8$ and $10x_1 - x_2 = -5$

So, $(x_1,x_2)=\left(\frac{3}{13},\frac{95}{13}\right)$

$A=\left(\frac{3}{13},\frac{56}{13}\right)$ and $B=\left(\frac{95}{13},\frac{-12}{13}\right)$

$$=\left|\frac{1}{2}\left(\frac{3}{13}\left(\frac{-44}{13}\right)\frac{-56}{13}\left(\frac{110}{13}\right)+1\left(\frac{2860}{169}\right)\right)\right|$$

$$=\frac{132}{13}\text{ sq. units}$$

13. Let a > 0, b > 0. Let *e* and *l* respectively be the eccentricity and length of the latus rectum of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Let e' and l' respectively be the eccentricity and length of the latus rectum of its conjugate hyperbola. If $e^2=\frac{11}{14}l$ and $(e')^2=\frac{11}{8}l'$, then the value of $77a + 44b$ is equal to :

(A) 100 (B) 110

(C) 120 (D) 130

Answer (D)

Sol. H : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

$e^2=\frac{11}{14}l$ (*l* be the length of LR)

$\Rightarrow\ a^2+b^2=\frac{11}{7}b^2a$ …(i)

and $e'^2=\frac{11}{8}l'$

(l' be the length of LR of conjugate hyperbola)

$\Rightarrow\ a^2+b^2=\frac{11}{4}a^2b$ …(ii)

By (i) and (ii)

$\boxed{7a=4b}$

then by (i)

$\frac{16}{49}b^2+b^2=\frac{11}{7}b^2\cdot\frac{4b}{7}$

$\Rightarrow$ $44b = 65$ and $77a = 65$

$\therefore$ $77a + 44b = 130$

14. Let, $\vec{a}=\alpha\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=-2\hat{i}+\alpha\hat{j}+\hat{k}$, where $\alpha\in\mathbf{R}$. If the area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $\sqrt{15(\alpha^2+4)}$, then the value of $2|\vec{a}|^2+(\vec{a}\cdot\vec{b})\,|\vec{b}|^2$ is equal to :

(A) 10 (B) 7

(C) 9 (D) 14

Answer (D)

Sol. $\vec{a}=\alpha\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=-2\hat{i}+\alpha\hat{j}+\hat{k}$

$$\therefore\quad \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \alpha & 2 & -1\\ -2 & \alpha & 1\end{vmatrix}=(2+\alpha)\hat{i}-(\alpha-2)\hat{j}+(\alpha^2+4)\hat{k}$$

Now $|\vec{a}\times\vec{b}|=\sqrt{15(\alpha^2+4)}$

$\Rightarrow$ $(2+\alpha)^2+(\alpha-2)^2+(\alpha^2+4)^2=15(\alpha^2+4)$

$\Rightarrow$ $\alpha^4-5\alpha^2-36=0$

$\therefore$ $\alpha=\pm 3$

Now, $2|\vec{a}|^2+(\vec{a}-\vec{b})|\vec{b}|^{-2}=2.14-14=14$

15. If vertex of a parabola is (2, –1) and the equation of its directrix is $4x - 3y = 21$, then the length of its latus rectum is :

(A) 2 (B) 8

(C) 12 (D) 16

Answer (B)

Sol. Vertex of Parabola : (2, –1)

and directrix : $4x - 3y = 21$

Distance of vertex from the directrix

$$a=\left|\frac{8+3-21}{\sqrt{25}}\right|=2$$

$\therefore$ length of latus rectum = $4a = 8$

16. Let the plane $ax + by + cz = d$ pass through (2, 3, –5) and is perpendicular to the planes $2x + y - 5z = 10$ and

$3x + 5y - 7z = 12$.

If *a*, *b*, *c*, *d* are integers $d > 0$ and *gcd* (|*a*|, |*b*|, |*c*|, *d*) = 1, then the value of $a + 7b + c + 20d$ is equal to :

(A) 18 (B) 20

(C) 24 (D) 22

Answer (D)

Sol. Equation of plane through point (2, 3, –5) and perpendicular to planes $2x + y - 5z = 10$ and $3x + 5y - 7z = 12$ is

$$\begin{vmatrix}x-2 & y-3 & z+5\\ 2 & 1 & -5\\ 3 & 5 & -7\end{vmatrix}=0$$

$\therefore$ Equation of plane is $(x - 2)(-7 + 25) - (y - 3)(-14 + 15) + (z + 5)\cdot 7 = 0$

$\therefore$ $18x - y + 7z + 2 = 0$

$\Rightarrow$ $18x - y + 7z = -2$

$\therefore$ $-18x + y - 7z = 2$

On comparing with $ax + by + cz = d$ where $d > 0$ is

$a = -18$, $b = 1$, $c = -7$, $d = 2$

$\therefore$ $a + 7b + c + 20d = 22$

17. The probability that a randomly chosen one-one function from the set {a, b, c, d} to the set {1, 2, 3, 4, 5} satisfies $f(a) + 2f(b) - f(c) = f(d)$ is :

(A) $\frac{1}{24}$ (B) $\frac{1}{40}$

(C) $\frac{1}{30}$ (D) $\frac{1}{20}$

Answer (D)

Sol. Number of one-one function from {a, b, c, d} to set {1, 2, 3, 4, 5} is ${}^5P_4 = 120\ n(s)$.

The required possible set of value

($f(a)$, $f(b)$, $f(c)$, $f(d)$) such that $f(a) + 2f(b) - f(c) = f(d)$ are (5, 3, 2, 1), (5, 1, 2, 3), (4, 1, 3, 5), (3, 1, 4, 5), (5, 4, 3, 2) and (3, 4, 5, 2)

$\therefore$ $n(E) = 6$

$\therefore$ Required probability = $\frac{n(E)}{n(S)} = \frac{6}{120} = \frac{1}{20}$

18. The value of $\lim\limits_{n\to\infty} 6\tan\left\{\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{r^2+3r+3}\right)\right\}$ is equal to :

(A) 1

(B) 2

(C) 3

(D) 6

Answer (C)

Sol. $\lim\limits_{n\to\infty} 6\tan\left\{\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{r^2+3r+3}\right)\right\}$

$= \lim\limits_{n\to\infty} 6\tan\left\{\sum_{r=1}^{n}\tan^{-1}\left(\frac{(r+2)-(r+1)}{1+(r+2)(r+1)}\right)\right\}$

$= \lim\limits_{n\to\infty} 6\tan\left\{\sum_{r=1}^{n}(\tan^{-1}(r+2)-\tan^{-1}(r+1))\right\}$

$= \lim\limits_{n\to\infty} 6\tan\left\{\tan^{-1}(n+2)-\tan^{-1}2\right\}$

$= 6\tan\left\{\frac{\pi}{2}-\cot^{-1}\left(\frac{1}{2}\right)\right\}$

$= 6\tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$

$= 3$

19. Let $\vec{a}$ be a vector which is perpendicular to the vector $3\hat{i}+\frac{1}{2}\hat{j}+2\hat{k}$. If $\vec{a}\times\left(2\hat{i}+\hat{k}\right) = 2\hat{i}-13\hat{j}-4\hat{k}$, then the projection of the vector $\vec{a}$ on the vector $2\hat{i}+2\hat{j}+\hat{k}$ is :

(A) $\frac{1}{3}$ (B) 1

(C) $\frac{5}{3}$ (D) $\frac{7}{3}$

Answer (C)

Sol. Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

and $\vec{a}\cdot\left(3\hat{i}-\frac{1}{2}\hat{j}+2\hat{k}\right) = 0 \Rightarrow 3a_1+\frac{a_2}{2}+2a_3 = 0$ ...(i)

and $\vec{a}\times\left(2\hat{i}+\hat{k}\right) = 2\hat{i}-13\hat{j}-4\hat{k}$

$\Rightarrow$ $a_2\hat{i}+(2a_3-a_1)\hat{j}-2a_2\hat{k} = 2\hat{i}-13\hat{j}-4\hat{k}$

$\therefore$ $a_2 = 2$ ...(ii)

and $a_1 - 2a_3 = 13$ ...(iii)

From eq. (i) and (iii) : $a_1 = 3$ and $a_3 = -5$

$\therefore$ $\vec{a} = 3\hat{i}+2\hat{j}-5\hat{k}$

$\therefore$ projection of $\vec{a}$ on $2\hat{i}+2\hat{j}+\hat{k} = \frac{6+4-5}{3} = \frac{5}{3}$

20. If $\cot\alpha = 1$ and $\sec\beta = -\frac{5}{3}$, where $\pi < \alpha < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$, then the value of $\tan(\alpha + \beta)$ and the quadrant in which $\alpha + \beta$ lies, respectively are :

(A) $-\frac{1}{7}$ and IVth quadrant

(B) 7 and Ist quadrant

(C) – 7 and IVth quadrant

(D) $\frac{1}{7}$ and Ist quadrant

Answer (A)

Sol. $\because$ $\cot\alpha = 1$, $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$

then $\tan\alpha = 1$

and $\sec\beta = -\frac{5}{3}$, $\beta \in \left(\frac{\pi}{2}, \pi\right)$

then $\tan\beta = -\frac{4}{3}$

$$\therefore \quad \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$$

$$=\frac{1-\frac{4}{3}}{1+\frac{4}{3}}$$

$$=-\frac{1}{7}$$

$\alpha+\beta\in\left(\frac{3\pi}{2},2\pi\right)$ *i.e.* fourth quadrant

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let the image of the point $P(1, 2, 3)$ in the line $L: \frac{x-6}{3}=\frac{y-1}{2}=\frac{z-2}{3}$ be Q. Let R (α, β, γ) be a point that divides internally the line segment PQ in the ratio 1 : 3. Then the value of $22(\alpha+\beta+\gamma)$ is equal to ________.

Answer (125)

Sol. The point dividing PQ in the ratio 1 : 3 will be mid-point of P & foot of perpendicular from P on the line.

$\therefore$ Let a point on line be λ

$$\Rightarrow \frac{x-6}{3}=\frac{y-1}{2}=\frac{z-2}{3}=\lambda$$

$$\Rightarrow \quad P'(3\lambda+6,\ 2\lambda+1,\ 3\lambda+2)$$

as P' is foot of perpendicular

$$(3\lambda+5)3+(2\lambda-1)2+(3\lambda-1)3=0$$

$$\Rightarrow 22\lambda+15-2-3=0$$

$$\Rightarrow \lambda=\frac{-5}{11}$$

$$\therefore \quad P'\left(\frac{51}{11},\frac{1}{11},\frac{7}{11}\right)$$

$$\text{Mid-point of } PP'\equiv\left(\frac{\frac{51}{11}+1}{2},\frac{\frac{1}{11}+2}{2},\frac{\frac{7}{11}+3}{2}\right)$$

$$\equiv\left(\frac{62}{22},\frac{23}{22},\frac{40}{22}\right)\equiv(\alpha,\beta,\gamma)$$

$$\Rightarrow 22(\alpha+\beta+\gamma)=62+23+40=125$$

2. Suppose a class has 7 students. The average marks of these students in the mathematics examination is 62, and their variance is 20. A student fails in the examination if he/she gets less than 50 marks, then in worst case, the number of students can fail is __________.

Answer (0)

Sol. According to given data

$$\frac{\sum_{i=1}^{7}(x_i-62)^2}{7}=20$$

$$\Rightarrow \sum_{i=1}^{7}(x_i-62)^2=140$$

So for any x_i, $(x_i-62)^2\le 140$

$\Rightarrow$ $x_i > 50\ \forall\ i = 1, 2, 3, \ldots 7$

So no student is going to score less than 50.

3. If one of the diameters of the circle $x^2+y^2-2\sqrt{2}x-6\sqrt{2}y+14=0$ is a chord of the circle $\left(x-2\sqrt{2}\right)^2+\left(y-2\sqrt{2}\right)^2=r^2$, then the value of r^2 is equal to _______.

Answer (10)

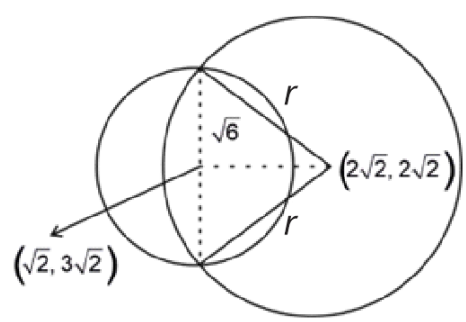
Sol. For $x^2+y^2-2\sqrt{2}x-6\sqrt{2}y+14=0$

$$\text{Radius}=\sqrt{\left(\sqrt{2}\right)^2+\left(3\sqrt{2}\right)^2-14}=\sqrt{6}$$

$\Rightarrow$ Diameter = $2\sqrt{6}$

If this diameter is chord to

$\left(x-2\sqrt{2}\right)^2+\left(y-2\sqrt{2}\right)^2=r^2$ then



$$\Rightarrow \quad r^2=6+\left(\sqrt{\left(\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2}\right)^2$$

$\Rightarrow$ $r^2 = 6 + 4 = 10$

$\Rightarrow$ $r^2 = 10$

4. If $\lim\limits_{x\to1}\dfrac{\sin(3x^2-4x+1)-x^2+1}{2x^3-7x^2+ax+b}=-2$, then the value of $(a-b)$ is equal to ________.

Answer (11)

Sol. $\lim\limits_{x\to1}\dfrac{\left(\dfrac{\sin\left(3x^2-4x+1\right)}{3x^2-4x+1}\right)\left(3x^2-4x+1\right)-x^2+1}{2x^3-7x^2+ax+b}=-2$

$\Rightarrow \lim\limits_{x\to1}\dfrac{3x^2-4x+1-x^2+1}{2x^3-7x^2+ax+b}=-2$

$\Rightarrow \lim\limits_{x\to1}\dfrac{2(x-1)^2}{2x^3-7x^2+ax+b}=-2$

So $f(x) = 2x^3 - 7x^2 + ax + b = 0$ has $x = 1$ as repeated root, therefore $f(1) = 0$ and $f'(1) = 0$ gives

$a + b + 5$ and $a = 8$

So, $a - b = 11$

5. Let for $n = 1, 2, \ldots, 50$, S_n be the sum of the infinite geometric progression whose first term is n^2 and whose common ratio is $\dfrac{1}{(n+1)^2}$. Then the value of

$\dfrac{1}{26}+\sum\limits_{n=1}^{50}\left(S_n+\dfrac{2}{n+1}-n-1\right)$ is equal to ________.

Answer (41651)

Sol. $S_n=\dfrac{n^2}{1-\dfrac{1}{(n+1)^2}}=\dfrac{n(n+1)^2}{n+2}=\left(n^2+1\right)-\dfrac{2}{n+2}$

Now $\dfrac{1}{26}+\sum\limits_{n=1}^{50}\left(S_n+\dfrac{2}{n+1}-n-1\right)$

$=\dfrac{1}{26}+\sum\limits_{n=1}^{50}\left\{\left(n^2-n\right)+2\left(\dfrac{1}{n+1}-\dfrac{1}{n+2}\right)\right\}$

$=\dfrac{1}{26}+\dfrac{50\times51\times101}{6}-\dfrac{50\times51}{2}+2\left(\dfrac{1}{2}-\dfrac{1}{52}\right)$

$=1+25\times17(101-3)$

$= 41651$

6. If the system of linear equations

$2x - 3y = \gamma + 5$,

$\alpha x + 5y = \beta + 1$, where $\alpha, \beta, \gamma \in R$ has infinitely many solutions, then the value of $|9\alpha + 3\beta + 5\gamma|$ is equal to ________.

Answer (58)

Sol. If $2x - 3y = \gamma + 5$ and $\alpha x + 5y = \beta + 1$ have infinitely many solutions then

$\dfrac{2}{\alpha}=\dfrac{-3}{5}=\dfrac{\gamma+5}{\beta+1}$

$\Rightarrow \alpha=-\dfrac{10}{3}$ and $3\beta + 5\gamma = -28$

So $|9\alpha + 3\beta + 5\gamma| = |-30 - 28| = 58$

7. Let $A=\begin{pmatrix}1+i & 1\\ -i & 0\end{pmatrix}$ where $i=\sqrt{-1}$. Then, the number of elements in the set $\left\{n\in\{1, 2, \ldots, 100\} : A^n = A\right\}$ is ________.

Answer (25)

Sol. $\therefore \quad A^2=\begin{bmatrix}1+i & 1\\ -i & 0\end{bmatrix}\begin{bmatrix}1+i & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}i & 1+i\\ 1-i & -i\end{bmatrix}$

$A^4=\begin{bmatrix}i & 1+i\\ 1-i & -i\end{bmatrix}\begin{bmatrix}i & 1+i\\ 1-i & -i\end{bmatrix}=I$

So $A^5 = A$, $A^9 = A$ and so on.

Clearly $n = 1, 5, 9, \ldots\ldots, 97$

Number of values of $n = 25$

8. Sum of squares of modulus of all the complex numbers z satisfying $\bar{z}=iz^2+z^2-z$ is equal to ________.

Answer (2)

Sol. Let $z = x + iy$

So $2x = (1 + i)(x^2 - y^2 + 2xyi)$

$\Rightarrow 2x = x^2 - y^2 - 2xy$ ...(i) and

$x^2 - y^2 + 2xy = 0$ ...(ii)

From (i) and (ii) we get

$x = 0$ or $y=-\dfrac{1}{2}$

When $x = 0$ we get $y = 0$

When $y=-\dfrac{1}{2}$ we get $x^2-x-\dfrac{1}{4}=0$

$\Rightarrow x=\dfrac{-1\pm\sqrt{2}}{2}$

So there will be total 3 possible values of z, which are 0, $\left(\dfrac{-1+\sqrt{2}}{2}\right)-\dfrac{1}{2}i$ and $\left(\dfrac{-1-\sqrt{2}}{2}\right)-\dfrac{1}{2}i$

Sum of squares of modulus

$=0+\left(\dfrac{\sqrt{2}-1}{2}\right)^2+\dfrac{1}{4}+\left(\dfrac{\sqrt{2}+1}{2}\right)^2=+\dfrac{1}{4}$

$= 2$

9. Let S = {1, 2, 3, 4}. Then the number of elements in the set $\{f : S \times S \to S : f$ is onto and $f(a, b) = f(b, a) \geq a \ \forall (a, b) \in S \times S\}$ is ______.

Answer (37)

Sol. There are 16 ordered pairs in $S \times S$. We write all these ordered pairs in 4 sets as follows.

$A = \{(1, 1)\}$

$B = \{(1, 4), (2, 4), (3, 4) (4, 4), (4, 3), (4, 2), (4, 1)\}$

$C = \{(1, 3), (2, 3), (3, 3), (3, 2), (3, 1)\}$

$D = \{(1, 2), (2, 2), (2, 1)\}$

All elements of set B have image 4 and only element of A has image 1.

All elements of set C have image 3 or 4 and all elements of set D have image 2 or 3 or 4.

We will solve this question in two cases.

Case I : When no element of set C has image 3.

Number of onto functions = 2 (when elements of set D have images 2 or 3)

Case II : When atleast one element of set C has image 3.

Number of onto functions = $(2^3 - 1)(1 + 2 + 2)$

= 35

Total number of functions = 37

10. The maximum number of compound propositions, out of $p \vee r \vee s$, $p \vee r \vee \sim s$, $p \vee \sim q \vee s$, $\sim p \vee \sim r \vee s$, $\sim p \vee \sim r \vee \sim s$, $\sim p \vee q \vee \sim s$, $q \vee r \vee \sim s$, $q \vee \sim r \vee \sim s$, $\sim p \vee \sim q \vee \sim s$ that can be made simultaneously true by an assignment of the truth values to p, q, r and s, is equal to ____________ .

Answer (9)

Sol. There are total 9 compound propositions, out of which 6 contain $\sim s$. So if we assign s as false, these 6 propositions will be true.

In remaining 3 compound propositions, two contain p and the third contains $\sim r$. So if we assign p and r as true and false respectively, these 3 propositions will also be true.

Hence maximum number of propositions that can be true are 9.

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