MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. The total number of functions,

$f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4, 5, 6\}$

such that $f(1) + f(2) = f(3)$, is equal to

(A) 60 (B) 90

(C) 108 (D) 126

Answer (B)

Sol. **Case 1:** If $f(3) = 3$ then $f(1)$ and $f(2)$ take 1 OR 2

No. of ways = $2 \cdot 6 = 12$

Case 2: If $f(3) = 5$ then $f(1)$ and $f(2)$ take 2 OR 3 OR 1 and 4

No. of ways = $2 \cdot 6 \cdot 2 = 24$

Case 3: If $f(3) = 2$ then $f(1) = f(2) = 1$

No. of ways = 6

Case 4: If $f(3) = 4$ then $f(1) = f(2) = 2$

No. of ways = 6

OR $f(1)$ and $f(2)$ take 1 and 3

No. of ways = 12

Case 5: If $f(3) = 6$ then $f(1) = f(2) = 3 \Rightarrow 6$ ways

OR $f(1)$ and $f(2)$ take 1 and 5 $\Rightarrow$ 12 ways

OR $f(2)$ and $f(1)$ take 2 and 4 $\Rightarrow$ 12 ways

2. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$, then $\alpha^{2021} + \beta^{2021} + \gamma^{2021} + \delta^{2021}$ is equal to

(A) –4 (B) –1

(C) 1 (D) 4

Answer (B)

Sol. $x^4 + x^3 + x^2 + x + 1 = 0$ OR $\dfrac{x^5-1}{x-1} = 0 \ (x \neq 1)$

So roots are $e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}$

i.e. α, β, γ and δ

From properties of n^{th} root of unity

$$1^{2021} + \alpha^{2021} + \beta^{2021} + \gamma^{2021} + \delta^{2021} = 0$$

$$\Rightarrow \alpha^{2021} + \beta^{2021} + \gamma^{2021} + \delta^{2021} = -1$$

3. For $n \in N$, let $S_n = \left\{z \in C : |z - 3 + 2i| = \dfrac{n}{4}\right\}$ and $T_n = \left\{z \in C : |z - 2 + 3i| = \dfrac{1}{n}\right\}$. Then the number of elements in the set $\{n \in N : S_n \cap T_n = \phi\}$ is

(A) 0 (B) 2

(C) 3 (D) 4

Answer (*)

Sol. $S_n = \left\{z \in C : |z - 3 + 2i| = \dfrac{n}{4}\right\}$ represents a circle with centre $C_1(3, -2)$ and radius $r_1 = \dfrac{n}{4}$

Similarly T_n represents circle with centre $C_2(2, -3)$ and radius $r_2 = \dfrac{1}{n}$

As $S_n \cap T_n = \phi$

$C_1C_2 > r_1 + r_2$ OR $C_1C_2 < |r_1 - r_2|$

$\sqrt{2} > \dfrac{n}{4} + \dfrac{1}{n}$ OR $\sqrt{2} < \left|\dfrac{n}{4} - \dfrac{1}{n}\right|$

$n = 1, 2, 3, 4$ $\quad$ n may take infinite values

4. The number of q $\in (0, 4\pi)$ for which the system of linear equations

$3(\sin 3\theta)\, x - y + z = 2$

$3(\cos 2\theta)\, x + 4y + 3z = 3$

$6x + 7y + 7z = 9$

has no solution, is

(A) 6

(B) 7

(C) 8

(D) 9

Answer (B)

Sol. $\Delta = \begin{vmatrix} 3\sin 3\theta & -1 & 1 \\ 3\cos 2\theta & 4 & 3 \\ 6 & 7 & 7 \end{vmatrix}$

$= 3\sin 3\theta(7) + 1(21\cos 2\theta - 18) + 1(21\cos 2\theta - 24)$

$\Delta = 21\sin 3\theta + 42\cos 2\theta - 42$

For no solution

$\sin 3\theta + 2\cos 2\theta = 2$

$\Rightarrow \sin 3\theta = 2 \cdot 2\sin^2\theta$

$\Rightarrow$ $3\sin\theta - 4\sin^3\theta = 4\sin^2\theta$

$\Rightarrow$ $\sin\theta(3 - 4\sin\theta - 4\sin^2\theta) = 0$

$\sin\theta = 0$ OR $\sin\theta = \frac{1}{2}$

$\theta = \pi, 2\pi, 3\pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

5. If $\lim_{n\to\infty}\left(\sqrt{n^2-n-1}+n\alpha+\beta\right)=0$, then $8(\alpha + \beta)$ is equal to

(A) 4 (B) –8

(C) –4 (D) 8

Answer (C)

Sol. $\lim_{n\to\infty}\left(\sqrt{n^2-n-1}+n\alpha+\beta\right)=0$

$$=\lim_{n\to\infty} n\left[\sqrt{1-\frac{1}{n}-\frac{1}{n^2}}+\alpha+\frac{\beta}{n}\right]=0$$

$\therefore$ $\alpha = -1$

Now,

$$\lim_{n\to\infty} n\left[\left\{1-\left(\frac{1}{n}+\frac{1}{n^2}\right)\right\}^{1/2}+\frac{\beta}{n}-1\right]=0$$

$$=\lim_{n\to\infty}\frac{\left(1-\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n^2}\right)+.....\right)+\frac{\beta}{n}-1}{\frac{1}{n}}=0$$

$\Rightarrow$ $\beta-\frac{1}{2}=0$

$\therefore$ $\beta=\frac{1}{2}$

Now, $8(\alpha + \beta) = 8\left(-\frac{1}{2}\right)=-4$

6. If the absolute maximum value of the function $f(x) = (x^2 - 2x + 7)\, e^{\left(4x^3-12x^2-180x+31\right)}$ in the interval $[-3, 0]$ is $f(\alpha)$, then

(A) $\alpha = 0$ (B) $\alpha = -3$

(C) $\alpha \in (-1, 0)$ (D) $\alpha \in (-3, -1]$

Answer (B)

Sol. Given, $f(x)=\underbrace{\left(x^2-2x+7\right)}_{f_1(x)}\underbrace{e^{\left(4x^3-12x^2-180x+31\right)}}_{f_2(x)}$

$f_1(x) = x^2 - 2x + 7$

$f_1'(x) = 2x - 2$, so $f(x)$ is decreasing in $[-3, 0]$ and positive also

$f_2(x)=e^{4x^3-12x^2-180x+31}$

$f_2'(x)=e^{4x^3-12x^2-180x+31}.\ 12x^2 - 24x - 180$

$=12(x-5)(x+3)e^{4x^3-12x^2-180x+31}$

So, $f_2(x)$ is also decreasing and positive in $\{-3, 0\}$

$\therefore$ absolute maximum value of $f(x)$ occurs at $x = -3$

$\therefore$ $\boxed{\alpha = -3}$

7. The curve $y(x) = ax^3 + bx^2 + cx + 5$ touches the x-axis at the point $P(-2, 0)$ and cuts the y-axis at the point Q, where y' is equal to 3. Then the local maximum value of $y(x)$ is

(A) $\frac{27}{4}$ (B) $\frac{29}{4}$

(C) $\frac{37}{4}$ (D) $\frac{9}{2}$

Answer (A)

Sol. $f(x) = y = ax^3 + bx^2 + cx + 5$ …(i)

$\frac{dy}{dx} = 3ax^2 + 2bx + c$ …(ii)

Touches x-axis at $P(-2, 0)$

$\Rightarrow$ $y|_{x=-2}=0 \Rightarrow -8a+4b-2c+5=0$ …(iii)

Touches x-axis at $P(-2, 0)$ also implies

$\frac{dy}{dx}|_{x=-2}=0 \Rightarrow 12a-4b+c=0$ …(iv)

$y = f(x)$ cuts y-axis at $(0, 5)$

Given, $\frac{dy}{dx}|_{x=0}=c=3$ …(v)

From (iii), (iv) and (v)

$a=-\frac{1}{2}, b=-\frac{3}{4}, c=3$

$\Rightarrow$ $f(x)=\frac{-x^2}{2}-\frac{3}{4}x^2+3x+5$

$f'(x)=\frac{-3}{2}x^2-\frac{3}{2}x+3$

$=\frac{-3}{2}(x+2)(x-1)$

$f'(x) = 0$ at $x = -2$ and $x = 1$

By first derivative test $x = 1$ in point of local maximum

Hence local maximum value of $f(x)$ is $f(1)$

i.e., $\frac{27}{4}$

8. The area of the region given by

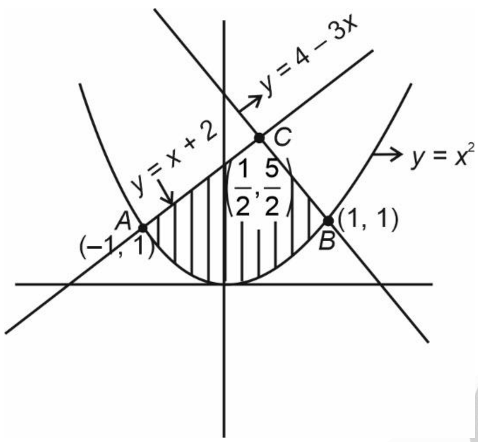
$A = \{(x, y); x^2 \le y \le \min\{x+2, 4-3x\}\}$ is

(A) $\frac{31}{8}$ (B) $\frac{17}{6}$

(C) $\frac{19}{6}$ (D) $\frac{27}{8}$

Answer (B)

Sol. $A = \{(x, y) : x^2 \le y \le \min\{x + 2, 4 - 3x\}$



So area of required region

$$A = \int_{-1}^{\frac{1}{2}} \left(x + 2 - x^2\right) dx + \int_{\frac{1}{2}}^{1} \left(4 - 3x - x^2\right) dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{\frac{1}{2}} + \left[4x - \frac{3x^2}{2} - \frac{x^3}{3}\right]_{\frac{1}{2}}^{1}$$

$$= \left(\frac{1}{8} + 1 - \frac{1}{24}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) + \left(4 - \frac{3}{2} - \frac{1}{3}\right) - \left(2 - \frac{3}{8} - \frac{1}{24}\right)$$

$$= \frac{17}{6}$$

9. For any real number x, let $[x]$ denote the largest integer less than equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}.$$

Then the value of $\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx$ is

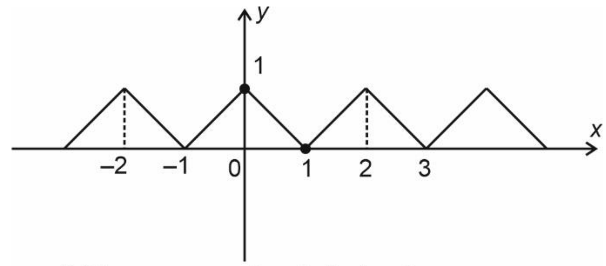
(A) 4 (B) 2

(C) 1 (D) 0

Answer (A)

Sol. $f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}$

Graph of $f(x)$



$f(x)$ is an even and periodic function

So, $\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx = \frac{\pi^2}{10} . 20 \int_0^1 f(x) \cos \pi x \, dx$

$$= 2\pi^2 \int_0^1 (1 - x) \cos \pi x \, dx$$

$$= 2\pi^2 \left\{ (1 - x) \frac{\sin \pi x}{\pi} \Big|_0^1 - \frac{\cos \pi x}{\pi^2} \Big|_0^1 \right\} = 4$$

10. The slope of the tangent to a curve $C : y = y(x)$ at any point (x, y) on it is $\frac{2e^{2x} - 6e^{-x} + 9}{2 + 9e^{-2x}}$. If C passes through the points $\left(0, \frac{1}{2} + \frac{\pi}{2\sqrt{2}}\right)$ and $\left(\alpha, \frac{1}{2} e^{2\alpha}\right)$, then e^{α} is equal to

(A) $\frac{3 + \sqrt{2}}{3 - \sqrt{2}}$ (B) $\frac{3}{\sqrt{2}} \left(\frac{3 + \sqrt{2}}{3 - \sqrt{2}}\right)$

(C) $\frac{1}{\sqrt{2}} \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right)$ (D) $\frac{\sqrt{2} + 1}{\sqrt{2} - 1}$

Answer (B)

Sol. $\frac{dy}{dx} = \frac{2e^{2x} - 6e^{-x} + 9}{2 + 9e^{-2x}} = e^{2x} - \frac{6e^{-x}}{2 + 9e^{-2x}}$

$$\int dy = \int e^{2x} dx - 3 \underbrace{\int \frac{e^{-x}}{1 + \left(\frac{3e^{-x}}{\sqrt{2}}\right)^2} dx}_{\text{put } e^{-x} = t}$$

$$= \frac{e^{2x}}{2} + 3 \int \frac{dt}{1 + \left(\frac{3t}{\sqrt{2}}\right)^2}$$

$$= \frac{e^{2x}}{2} + \sqrt{2} \tan^{-1} \frac{3t}{\sqrt{2}} + C$$

$$y=\frac{e^{2x}}{2}+\sqrt{2}\tan^{-1}\left(\frac{3e^{-x}}{\sqrt{2}}\right)+C$$

It is given that the curve passes through

$$\left(0,\frac{1}{2}+\frac{\pi}{2\sqrt{2}}\right)$$

$$\frac{1}{2}+\frac{\pi}{2\sqrt{2}}=\frac{1}{2}+\sqrt{2}\tan^{-1}\left(\frac{3}{\sqrt{2}}\right)+C$$

$$\Rightarrow\quad C=\frac{\pi}{2\sqrt{2}}-\sqrt{2}\tan^{-1}\left(\frac{3}{\sqrt{2}}\right)$$

Now if $\left(\alpha,\frac{1}{2}e^{2\alpha}\right)$ satisfies the curve, then

$$\frac{1}{2}e^{2\alpha}=\frac{e^{2\alpha}}{2}+\sqrt{2}\tan^{-1}\left(\frac{3e^{-\alpha}}{\sqrt{2}}\right)+\frac{\pi}{2\sqrt{2}}-\sqrt{2}\tan^{-1}\left(\frac{3}{\sqrt{2}}\right)$$

$$\tan^{-1}\left(\frac{3}{\sqrt{2}}\right)-\tan^{-1}\left(\frac{3e^{-\alpha}}{\sqrt{2}}\right)=\frac{\pi}{2\sqrt{2}}\times\frac{1}{\sqrt{2}}=\frac{\pi}{4}$$

$$\frac{\frac{3}{\sqrt{2}}-\frac{3e^{-\alpha}}{\sqrt{2}}}{1+\frac{9}{2}e^{-\alpha}}=1$$

$$\frac{3}{\sqrt{2}}e^{\alpha}-\frac{3}{\sqrt{2}}=e^{\alpha}+\frac{9}{2}$$

$$e^{\alpha}=\frac{\frac{9}{2}+\frac{3}{\sqrt{2}}}{\frac{3}{\sqrt{2}}-1}=\frac{3}{\sqrt{2}}\left(\frac{3+\sqrt{2}}{3-\sqrt{2}}\right)$$

11. The general solution of the differential equation $(x-y^2)dx+y(5x+y^2)dy=0$ is :

(A) $\left(y^2+x\right)^4=C\left|\left(y^2+2x\right)^3\right|$

(B) $\left(y^2+2x\right)^4=C\left|\left(y^2+x\right)^3\right|$

(C) $\left|\left(y^2+x\right)^3\right|=C\left(2y^2+x\right)^4$

(D) $\left|\left(y^2+2x\right)^3\right|=C\left(2y^2+x\right)^4$

Answer (A)

Sol. $\left(x-y^2\right)dx+y\left(5x+y^2\right)dy=0$

$$y\frac{dy}{dx}=\frac{y^2-x}{5x+y^2}$$

Let $y^2=t$

$$\frac{1}{2}\cdot\frac{dt}{dx}=\frac{t-x}{5x+t}$$

Now substitute, $t=vx$

$$\frac{dt}{dx}=v+x\frac{dv}{dx}$$

$$\frac{1}{2}\left\{v+x\frac{dv}{dx}\right\}=\frac{v-1}{5+v}$$

$$x\frac{dv}{dx}=\frac{2v-2}{5+v}-v=\frac{-3v-v^2-2}{5+v}$$

$$\int\frac{5+v}{v^2+3v+2}dv=\int-\frac{dx}{x}$$

$$\int\frac{4}{v+1}dv-\int\frac{3}{v+2}dv=-\int\frac{dx}{x}$$

$$4\ln|v+1|-3\ln|v+2|=-\ln x+\ln C$$

$$\left|\frac{(v+1)^4}{(v+2)^3}\right|=\frac{C}{x}$$

$$\left|\frac{\left(\frac{y^2}{x}+1\right)^4}{\left(\frac{y^2}{x}+2\right)^3}\right|=\frac{C}{x}$$

$$\left|\left(y^2+x\right)^4\right|=C\left|\left(y^2+2x\right)^3\right|$$

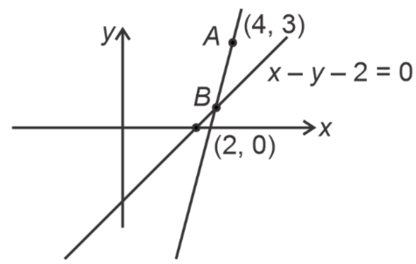
12. A line, with the slope greater than one, passes through the point $A(4, 3)$ and intersects the line $x-y-2=0$ at the point B. If the length of the line segment AB is $\frac{\sqrt{29}}{3}$, then B also lies on the line :

(A) $2x+y=9$ (B) $3x-2y=7$

(C) $x+2y=6$ (D) $2x-3y=3$

Answer (C)

Sol.



Let inclination of required line is θ,

So the coordinates of point B can be assumed as

$$\left(4-\frac{\sqrt{29}}{3}\cos\theta, 3-\frac{\sqrt{29}}{3}\sin\theta\right)$$

Which satisfices $x - y - 2 = 0$

$$4-\frac{\sqrt{29}}{3}\cos\theta-3+\frac{\sqrt{29}}{3}\sin\theta-2=0$$

$$\sin\theta-\cos\theta=\frac{3}{\sqrt{29}}$$

By squaring

$$\sin 2\theta=\frac{20}{29}=\frac{2\tan\theta}{1+\tan^2\theta}$$

$$\tan\theta=\frac{5}{2}\text{ only}(\text{because slope is greater than }1)$$

$$\sin\theta=\frac{5}{\sqrt{29}},\cos\theta=\frac{2}{\sqrt{29}}$$

Point $B:\left(\frac{10}{3},\frac{4}{3}\right)$

Which also satisfies $x + 2y = 6$

13. Let the locus of the centre (α, β), $\beta > 0$, of the circle which touches the circle $x^2 + (y - 1)^2 = 1$ externally and also touches the x-axis be L. Then the area bounded by L and the line $y = 4$ is :

(A) $\frac{32\sqrt{2}}{3}$

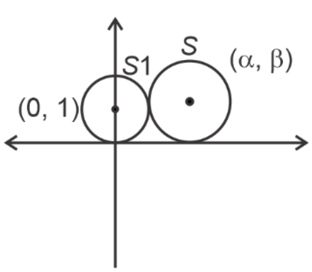
(B) $\frac{40\sqrt{2}}{3}$

(C) $\frac{64}{3}$

(D) $\frac{32}{3}$

Answer (C)

Sol.



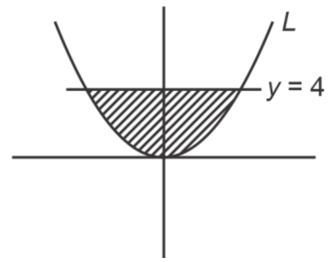
Radius of circle S touching x-axis and centre (α, β) is $|\beta|$. According to given conditions

$\alpha^2 + (\beta - 1)^2 = (|\beta| + 1)^2$

$\alpha^2 + \beta^2 - 2\beta + 1 = \beta^2 + 1 + 2|\beta|$

$\alpha^2 = 4\beta$ as $\beta > 0$

$\therefore$ Required louse is $L : x^2 = 4y$



The area of shaded region $=2\int_0^4 2\sqrt{y}\,dy$

$$=4.\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$=\frac{64}{3}\text{ square units.}$$

14. Let P be the plane containing the straight line $\frac{x-3}{9}=\frac{y+4}{-1}=\frac{z-7}{-5}$ and perpendicular to the plane containing the straight lines $\frac{x}{2}=\frac{y}{3}=\frac{z}{5}$ and $\frac{x}{3}=\frac{y}{7}=\frac{z}{8}$. If d is the distance P from the point $(2, -5, 11)$, then d^2 is equal to :

(A) $\frac{147}{2}$

(B) 96

(C) $\frac{32}{3}$

(D) 54

Answer (C*)

Sol. Let $<a, b, c>$ be direction ratios of plane containing lines $\frac{x}{2}=\frac{y}{3}=\frac{z}{5}$ and $\frac{x}{3}=\frac{y}{7}=\frac{z}{8}$.

$\therefore$ $2a + 3b + 5c = 0$ …(i)

and $3a + 7b + 8c = 0$ …(ii)

from eq. (i) and (ii) : $\frac{a}{24-35}=\frac{b}{15-16}=\frac{c}{14-9}$

$\therefore$ D.R^S. of plane are < 11, 1, –5>

Let D.R^S of plane P be $<a_1, b_1, c_1>$ then.

$11a_1 + b_1 - 5c_1 = 0$ …(iii)

and $9a_1 - b_1 - 5c_1 = 0$ …(iv)

From eq. (iii) and (iv) :

$$\frac{a_1}{-5-5}=\frac{b_1}{-45+55}=\frac{c_1}{-11-9}$$

$\therefore$ D.A^5. of plane P are < 1, –1, 2>

Equation plane P is : $1(x-3)-1(y+4)+2(z-7)=0$

$\Rightarrow\ x-y+2z-21=0$

Distance from point (2, –5, 11) is $d=\dfrac{|2+5+22-2|}{\sqrt{6}}$

$\therefore\ d^2=\dfrac{32}{3}$

15. Let ABC be a triangle such that $\overrightarrow{BC}=\vec{a}, \overrightarrow{CA}=\vec{b}, \overrightarrow{AB}=\vec{c}, |\vec{a}|=6\sqrt{2}, |\vec{b}|=2\sqrt{3}$ and $\vec{b}\cdot\vec{c}=12$. Consider the statements :

$(S1): \left|\left(\vec{a}\times\vec{b}\right)+\left(\vec{c}\times\vec{b}\right)\right|-|\vec{c}|=6\left(2\sqrt{2}-1\right)$

$(S2): \angle ACB=\cos^{-1}\left(\sqrt{\dfrac{2}{3}}\right)$

Then

(A) Both ($S1$) and ($S2$) are true

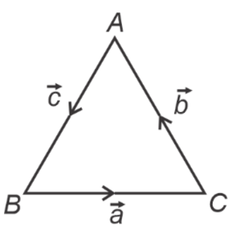
(B) Only ($S1$) is true

(C) Only ($S2$) is true

(D) Both ($S1$) and (S2) are false

Answer (C*)

Sol.



$\because\ \vec{a}+\vec{b}+\vec{c}=0$ …(i)

then $\vec{a}+\vec{c}=-\vec{b}$

then $(\vec{a}+\vec{c})\times\vec{b}=-\vec{b}\times\vec{b}$

$\therefore\ \vec{a}\times\vec{b}+\vec{c}\times\vec{b}=\vec{0}$ …(i)

For $(S1): \left|\vec{a}\times\vec{b}+\vec{c}\times\vec{b}\right|-|\vec{c}|=6\left(2\sqrt{2}-1\right)$

$\left|(\vec{a}+\vec{c})\times\vec{b}\right|-|\vec{c}|=6\left(2\sqrt{2}-1\right)$

$|\vec{c}|=6-12\sqrt{2}$ (not possible)

Hence ($S1$) is not correct

For ($S2$) : from (i) $\vec{b}+\vec{c}=-\vec{a}$

$\Rightarrow\ \vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}=-\vec{a}\cdot\vec{b}$

$\Rightarrow\ 12+12=-6\sqrt{2}\cdot2\sqrt{3}\cos(\pi-\angle ACB)$

$\therefore\ \cos(\angle ACB)=\sqrt{\dfrac{2}{3}}$

$\therefore\ \angle ACB=\cos^{-1}\sqrt{\dfrac{2}{3}}$

$\therefore$ S(2) is correct.

16. If the sum and the product of mean and variance of a binomial distribution are 24 and 128 respectively, then the probability of one or two successes is :

(A) $\dfrac{33}{2^{32}}$ (B) $\dfrac{33}{2^{29}}$

(C) $\dfrac{33}{2^{28}}$ (D) $\dfrac{33}{2^{27}}$

Answer (C)

Sol. If n is number of trails, p is probability of success and q is probability of unsuccess then,

Mean = np and variance = npq.

Here $np+npq=24$ …(i)

$np.npq=128$ …(ii)

and $q=1-p$ …(iii)

from eq. (i), (ii) and (iii) : $p=q=\dfrac{1}{2}$ and n = 32.

$\therefore$ Required probability $=p(X=1)+p(X=2)$

$={}^{32}C_1\cdot\left(\dfrac{1}{2}\right)^{32}+{}^{32}C_2\cdot\left(\dfrac{1}{2}\right)^{32}$

$=\left(32+\dfrac{32\times31}{2}\right)\cdot\dfrac{1}{2^{32}}$

$=\dfrac{33}{2^{28}}$

17. If the numbers appeared on the two throws of a fair six faced die are α and β, then the probability that $x^2+\alpha x+\beta>0$, for all $x\in R$, is :

(A) $\dfrac{17}{36}$ (B) $\dfrac{4}{9}$

(C) $\dfrac{1}{2}$ (D) $\dfrac{19}{36}$

Answer (A)

Sol. For $x^2+\alpha x+\beta>0\ \forall\ x\in R$ to hold, we should have $\alpha^2-4\beta<0$

If $\alpha=1$, β can be 1, 2, 3, 4, 5, 6 *i.e.*, 6 choices

If $\alpha=2$, β can be 2, 3, 4, 5, 6 *i.e.*, 5 choices

If $\alpha=3$, β can be 3, 4, 5, 6 *i.e.*, 4 choices

If $\alpha=4$, β can be 5 or 6 *i.e.*, 2 choices

If $\alpha=6$, No possible value for β *i.e.*, 0 choices

Hence total favourable outcomes

$= 6 + 5 + 4 + 2 + 0 + 0$

$= 17$

Total possible choices for α and $\beta = 6 \times 6 = 36$

Required probability $= \frac{17}{36}$

18. The number of solutions of $|\cos x| = \sin x$, such that $-4\pi \le x \le 4\pi$ is :

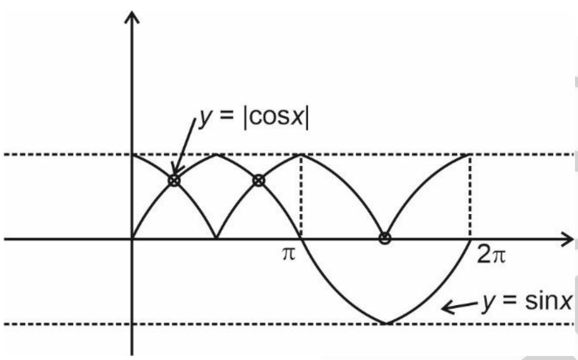
(A) 4 (B) 6

(C) 8 (D) 12

Answer (C)

Sol. Number of solutions of the equation $|\cos x| = \sin x$ for $x \in [-4\pi, 4\pi]$ will be equal to 4 times the number of solutions of the same equation for $x \in [0, 2\pi]$.

Graphs of $y = |\cos x|$ and $y = \sin x$ are as shown below.



Hence, two solutions of given equation in $[0, 2\pi]$

$\Rightarrow$ Total of 8 solutions in $[-4\pi, 4\pi]$

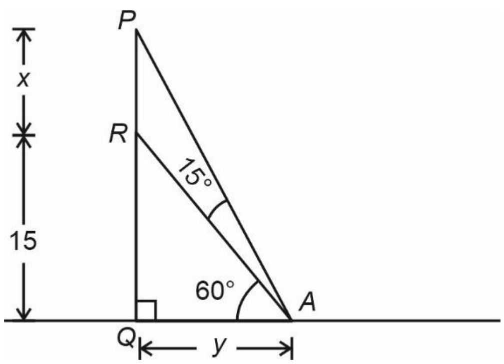
19. A tower PQ stands on a horizontal ground with base Q on the ground. The point R divides the tower in two parts such that QR = 15 m. If from a point A on the ground the angle of elevation of R is 60° and the part PR of the tower subtends an angle of 15° at A, then the height of the tower is :

(A) $5\left(2\sqrt{3}+3\right)$ m (B) $5\left(\sqrt{3}+3\right)$ m

(C) $10\left(\sqrt{3}+1\right)$ m (D) $10\left(2\sqrt{3}+1\right)$ m

Answer (A)

Sol.



From ΔAPQ

$$\frac{x+15}{y} = \tan 75° \qquad \ldots(i)$$

From ΔRQA,

$$\frac{15}{y} = \tan 60° \qquad \ldots(ii)$$

From (i) and (ii)

$$\frac{x+15}{15} = \frac{\tan 75°}{\tan 60°} = \frac{\tan(45°+30°)}{\tan 60°} = \frac{\sqrt{3}+1}{\left(\sqrt{3}-1\right).\sqrt{3}}$$

On simplification,

$x = 10\sqrt{3}$ m

Hence height of the tower $= (15 + 10\sqrt{3})$ m

$= 5(2\sqrt{3}+3)$ m

20. Which of the following statements is a tautology?

(A) $\left((\sim p) \vee q\right) \Rightarrow p$

(B) $p \Rightarrow \left((\sim p) \vee q\right)$

(C) $\left((\sim p) \vee q\right) \Rightarrow q$

(D) $q \Rightarrow \left((\sim p) \vee q\right)$

Answer (D)

Sol. Truth Table

					A	B	C	D
p	q	$\sim p$	$\sim q$	$(\sim p)\vee q$	$((\sim p)\vee q) \to p$	$p \to ((\sim p)\vee q)$	$(\sim p)\vee q \to q$	$q \to ((\sim p)\vee q)$
T	T	F	F	T	T	T	T	T
T	F	F	T	F	T	F	T	T
F	T	T	F	T	F	T	T	T
F	F	T	T	T	F	T	F	T

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let $A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ and $B = A - I$. If $\omega = \frac{\sqrt{3}i - 1}{2}$, then the number of elements in the set $\{n \in \{1, 2, ..., 100\} : A^n + (\omega B)^n = A + B\}$ is equal to _______.

Answer (17)

Sol. Here $A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

We get $A^2 = A$ and similarly for

$$B = A - I = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

We get $B^2 = -B \Rightarrow B^3 = B$

$\therefore$ $A^n + (\omega B)^n = A + (\omega B)^n$ for $n \in N$

For ω^n to be unity n shall be multiple of 3 and for B^n to be B. n shell be 3, 5, 7, … 99

$\therefore$ $n = \{3, 9, 15, ….. 99\}$

Number of elements = 17.

2. The letters of the work 'MANKIND' are written in all possible orders and arranged in serial order as in an English dictionary. Then the serial number of the word 'MANKIND' is ______.

Answer (1492)

Sol. Arranging letter in alphabetical order A D I K M N N for finding rank of MANKIND making arrangements of dictionary we get

A ....→ $\frac{6!}{2!} = 360$

D ....→ 360

I ....→ 360

K ....→ 360

$M\,A\,D$ ....→ $\frac{4!}{2!} = 12$

$M\,A\,I$ ....→ 12

$M\,A\,K$ ....→ 12

$M\,A\,N\,D$ ....→ 3! = 6

$M\,A\,N\,I$ ....→ 6

$M\,A\,N\,K\,D$ ....→ 2

$M\,A\,N\,K\,I\,D$ ....→ 1

$M\,A\,N\,K\,I\,N\,D$ ....→ 1

$\therefore$ Rank of MANKIND = 1440 + 36 + 12 + 2 + 2

= 1492

3. If the maximum value of the term independent of t in the expansion of $\left(t^2 x^{\frac{1}{5}} + \frac{(1-x)^{\frac{1}{10}}}{t}\right)^{15}$, $x \geq 0$ is K, then 8 K is equal to __________.

Answer (6006)

Sol. General Term = $15C_r \left(t^2 x^{\frac{1}{5}}\right)^{15-r} \left(\frac{(1-x)^{\frac{1}{10}}}{t}\right)^r$

for term independent on t

$2(15 - r) - r = 0$

$\Rightarrow$ $r = 10$

$\therefore$ $T_{11} = {}^{15}C_{10}\, x(1 - x)$

Maximum value of $x\,(1 - x)$ occur at $x = \frac{1}{2}$

i.e., $(x(1 - x))_{max} = \frac{1}{4}$

$\Rightarrow$ $K = {}^{15}C_{10} \times \frac{1}{4}$

$\Rightarrow$ $8\,K = 2({}^{15}C_{10}) = 6006$

4. Let a, b be two non-zero real numbers. If p and r are the roots of the equation $x^2 - 8ax + 2a = 0$ and q and s are the roots of the equation $x^2 + 12bx + 6b = 0$, such that $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}$ are in A.P., then $a^{-1} - b^{-1}$ is equal to _________.

Answer (38)

Sol. $\because$ Roots of $2ax^2 - 8ax + 1 = 0$ are $\frac{1}{p}$ and $\frac{1}{r}$ and roots of $6bx^2 + 12bx + 1 = 0$ are $\frac{1}{q}$ and $\frac{1}{s}$.

Let $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}$ as $\alpha - 3\beta$, $\alpha - \beta$, $\alpha + \beta$, $\alpha + 3\beta$

So sum of roots $2\alpha - 2\beta = 4$ and $2\alpha + 2\beta = -2$

Clearly $\alpha = \frac{1}{2}$ and $\beta = -\frac{3}{2}$

Now product of roots, $\frac{1}{p}\cdot\frac{1}{r}=\frac{1}{2a}=-5 \Rightarrow \frac{1}{a}=-10$

and $\frac{1}{q}\cdot\frac{1}{x}=\frac{1}{6b}=-8 \Rightarrow \frac{1}{b}=-48$

So, $\frac{1}{a}-\frac{1}{b}=38$

5. Let $a_1 = b_1 = 1$, $a_n = a_{n-1} + 2$ and $b_n = a_a + b_{n-1}$ for every natural number $n \geq 2$. Then $\sum_{n=1}^{15} a_n \cdot b_n$ is equal to ______.

Answer (27560)

Sol. $a_1 = b_1 = 1$

$a_n = a_{n-1} + 2$ (for $n \geq 2$) $\qquad b_n = a_n + b_{n-1}$

$a_2 = a_1 + 2 = 1 + 2 = 3 \qquad b_2 = a_2 + b_1 = 3 + 1 = 4$

$a_3 = a_2 + 2 = 3 + 2 = 5 \qquad b_3 = a_3 + b_2 = 5 + 4 = 9$

$a_4 = a_3 + 2 = 5 + 2 = 7 \qquad b_4 = a_4 + b_3 = 7 + 9 = 16$

$a_{15} = a_{14} + 2 = 29 \qquad b_{15} = 225$

$$\sum_{n=1}^{15} a_n\, b_n = 1\times1+3\times4+5\times9+......29\times225$$

$$\therefore \sum_{n=1}^{11} a_n\, b_n = \sum_{n=1}^{15}(2n-1)n^2 = \sum_{n=1}^{15}2n^3 - \sum_{n=1}^{15}n^2$$

$$= 2\left[\frac{15\times16}{2}\right]^2 - \left[\frac{15\times16\times31}{6}\right] = 27560.$$

6. Let $f(x)=\begin{cases}\left|4x^2-8x+5\right|, & \text{if } 8x^2-6x+1\geq 0\\ \left[4x^2-8x+5\right], & \text{if } 8x^2-6x+1<0\end{cases}$

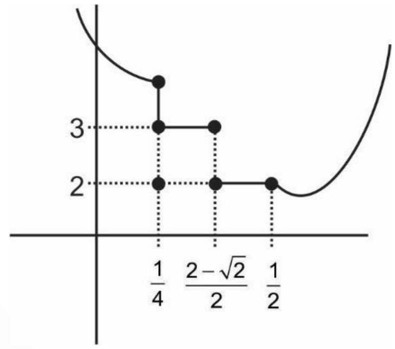
where $[\alpha]$ denotes the greatest integer less than or equal to α. Then the number of points in **R** where f is not differentiable is

Answer (3)

Sol. $f(x)=\begin{cases}\left|4x^2-8x+5\right|, & \text{if } 8x^2-6x+1\geq 0\\ \left[4x^2-8x+5\right], & \text{if } 8x^2-6x+1<0\end{cases}$

$$=\begin{cases}4x^2-8x+5, & \text{if } x\in\left[-\infty,\frac{1}{4}\right]\cup\left[\frac{1}{2},\infty\right)\\ \left[4x^2-8x+5\right] & \text{if } x\in\left(\frac{1}{4},\frac{1}{2}\right)\end{cases}$$

$$f(x)=\begin{cases}4x^2-8x+5 & \text{if } x\in\left(-\infty,\frac{1}{4}\right]\cup\left[\frac{1}{2},\infty\right)\\ 3 & x\in\left(\frac{1}{4},\frac{2-\sqrt{2}}{2}\right)\\ 2 & x\in\left[\frac{2-\sqrt{2}}{2},\frac{1}{2}\right)\end{cases}$$



$\therefore$ Non-diff at $x=\frac{1}{4},\frac{2-\sqrt{2}}{2},\frac{1}{2}$

7. If $\lim\limits_{n\to\infty}\frac{(n+1)^{k-1}}{n^{k+1}}[(nk+1)+(nk+2)+...+(nk+n)]$ $=33\cdot\lim\limits_{n\to\infty}\frac{1}{n^{k+1}}\cdot\left[1^k+2^k+3^k+...+n^k\right]$, then the integral value of k is equal to _______.

Answer (5)

Sol. $\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{k-1}\frac{1}{n}\sum_{r=1}^{n}\left(k+\frac{r}{n}\right)=33\lim\limits_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{r}{n}\right)^k$

$\Rightarrow \int_0^1 (k+x)dx = 33\int_0^1 x^k dx$

$\Rightarrow \frac{2k+1}{2}=\frac{33}{k+1}$

$\Rightarrow k = 5$

8. Let the equation of two diameters of a circle $x^2 + y^2 - 2x + 2fy + 1 = 0$ be $2px - y = 1$ and $2x + py = 4p$. Then the slope $m\in(0, \infty)$ of the tangent to the hyperbola $3x^2 - y^2 = 3$ passing through the centre of the circle is equal to _______.

Answer (2)

Sol. $x^2 + y^2 - 2x + 2fy + 1 = 0$ [entre = (1, –f]

Diameter $2px - y = 1$ …(i)

$2x + py = 4p$ …(ii)

$x=\frac{5P}{2P^2+2} \qquad y=\frac{4P^2-1}{1+P^2}$

$\because \; x=1 \qquad f=0 \qquad$ [for $P=\frac{1}{2}$]

$$\frac{5P}{2P^2+2}=1 \qquad f=3 \qquad [\text{for } P=2]$$

$\therefore \quad P=\frac{1}{2},\ 2$

Centre can be $\left(\frac{1}{2},0\right)$ or $(1,3)$

$\left(\frac{1}{2},0\right)$ will not satisfy

$\therefore$ Tangent should pass through

$(2, 3)$ for $3x^2-y^2=3$

$$\frac{x^2}{1}-\frac{y^2}{3}=1$$

$y=mx\pm\sqrt{m^2-3}$

substitute (2, 3)

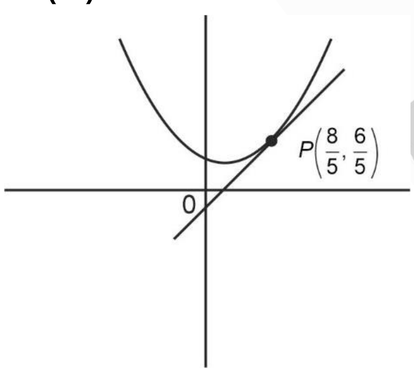
$3=m\pm\sqrt{m^2-3}$

$\therefore \quad \boxed{m=2}$

9. The sum of diameters of the circles that touch (i) the parabola $75x^2 = 64(5y - 3)$ at the point $\left(\frac{8}{5},\frac{6}{5}\right)$ and (ii) the y-axis, is equal to ________.

Answer (10)

Sol.



Equation of tangent to the parabola at $P\left(\frac{8}{5},\frac{6}{5}\right)$

$$75x\cdot\frac{8}{5}=160\left(y+\frac{6}{5}\right)-192$$

$\Rightarrow \quad 120x = 160y$

$\Rightarrow \quad 3x = 4y$

Equation of circle touching the given parabola at P can be taken as

$$\left(x-\frac{8}{5}\right)^2+\left(y-\frac{6}{5}\right)^2+\lambda(3x-4y)=0$$

If this circle touches y-axis then

$$\frac{64}{25}+\left(y-\frac{6}{5}\right)^2+\lambda(-4y)=0$$

$$\Rightarrow \quad y^2-2y\left(2\lambda+\frac{6}{5}\right)+4=0$$

$\Rightarrow \quad D=0$

$$\Rightarrow \quad \left(2\lambda+\frac{6}{6}\right)^2=4$$

$$\Rightarrow \quad \lambda=\frac{2}{5} \text{ or } -\frac{8}{5}$$

Radius = 1 or 4

Sum of diameter = 10

10. The line of shortest distance between the lines $\frac{x-2}{0}=\frac{y-1}{1}=\frac{z}{1}$ and $\frac{x-3}{2}=\frac{y-5}{2}=\frac{z-1}{1}$ makes an angle of $\cos^{-1}\left(\sqrt{\frac{2}{27}}\right)$ with the plane P : $ax-y-z=0$, $(a>0)$. If the image of the point $(1, 1, -5)$ in the plane P is (α, β, γ), then $\alpha+\beta-\gamma$ is equal to ________.

Answer (*)

Sol. Line of shortest distance will be along $\overline{b_1}\times\overline{b_2}$

Where, $\overline{b_1}=\hat{j}+\hat{k}$ and $\vec{b}_2=2\hat{i}+2\hat{j}+\hat{k}$

$$\overline{b_1}\times\overline{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 1\\ 2 & 2 & 1\end{vmatrix}=-\hat{i}+2\hat{j}-2\hat{k}$$

Angle between $\overline{b_1}\times\overline{b_2}$ and plane P,

$$\sin\theta=\left|\frac{-a-2+2}{3\cdot\sqrt{a^2+2}}\right|=\frac{5}{\sqrt{27}}\Rightarrow\frac{|a|}{\sqrt{a^2+2}}=\frac{5}{\sqrt{3}}$$

$$\Rightarrow \quad a^2=-\frac{25}{11}\ \text{(not possible)}$$

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