MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(3x) - f(x) = x$. If $f(8) = 7$, then $f(14)$ is equal to

(A) 4 (B) 10

(C) 11 (D) 16

Answer (B)

Sol. $f(3x) - f(x) = x$ ...(1)

$x \to \frac{x}{3}$

$f(x) - f\left(\frac{x}{3}\right) = \frac{x}{3}$ ...(2)

Again $x \to \frac{x}{3}$

$f\left(\frac{x}{3}\right) - f\left(\frac{x}{9}\right) = \frac{x}{3^2}$ ...(3)

Similarly

$f\left(\frac{x}{3^{n-2}}\right) - f\left(\frac{x}{3^{n-1}}\right) = \frac{x}{3^{n-1}}$ ..... (n)

Adding all these and applying $n \to \infty$

$$\lim_{n\to\infty}\left(f(3x) - f\left(\frac{x}{3^{n-1}}\right)\right) = x\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right)$$

$f(3x) - f(0) = \frac{3x}{2}$

Putting $x = \frac{8}{3}$

$f(8) - f(0) = 4$

$\Rightarrow f(0) = 3$

Putting $x = \frac{14}{3}$

$f(14) - 3 = 7 \Rightarrow f(14) = 0$

2. Let O be the origin and A be the point $z_1 = 1 + 2i$. If B is the point z_2, $\text{Re}(z_2) < 0$, such that OAB is a right angled isosceles triangle with OB as hypotenuse, then which of the following is NOT true?

(A) $\arg z_2 = \pi - \tan^{-1}3$

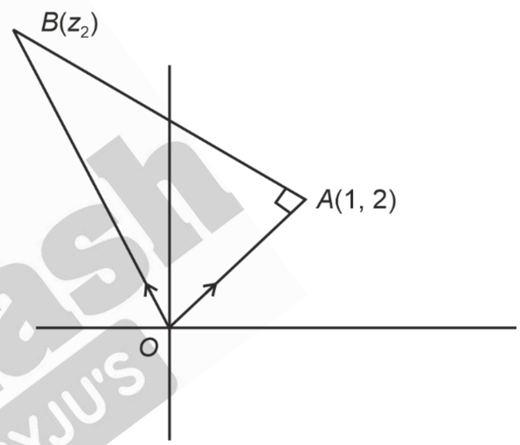
(B) $\arg(z_1 - 2z_2) = -\tan^{-1}\frac{4}{3}$

(C) $|z_2| = \sqrt{10}$

(D) $|2z_1 - z_2| = 5$

Answer (D)

Sol.



$$\frac{z_2 - 0}{(1+2i) - 0} = \frac{|OB|}{|OA|}e^{\frac{i\pi}{4}}$$

$$\Rightarrow \frac{z_2}{1+2i} = \sqrt{2}e^{\frac{i\pi}{4}}$$

OR $z_2 = (1 + 2i)(1 + i)$

$= -1 + 3i$

$\arg z_2 = \pi - \tan^{-1}3$

$|z_2| = \sqrt{10}$

$z_1 - 2z_2 = (1 + 2i) + 2 - 6i = 3 - 4i$

$\arg(z_1 - 2z_2) = -\tan^{-1}\frac{4}{3}$

$|2z_1 - z_2| = |2 + 4i + 1 - 3i| = |3 + i|$

$= \sqrt{10}$

3. If the system of linear equations.

$8x + y + 4z = -2$

$x + y + z = 0$

$\lambda x - 3y = \mu$

has infinitely many solutions, then the distance of the point $\left(\lambda, \mu, -\frac{1}{2}\right)$ from the plane $8x + y + 4z + 2 = 0$ is

(A) $3\sqrt{5}$ (B) 4

(C) $\frac{26}{9}$ (D) $\frac{10}{3}$

Answer (D)

Sol. $\Delta = \begin{vmatrix} 8 & 1 & 4 \\ 1 & 1 & 1 \\ \lambda & -3 & 0 \end{vmatrix}$

$= 8(3) - 1(-\lambda) + 4(-3 - \lambda)$

$= 24 + \lambda - 12 - 4\lambda$

$= 12 - 3\lambda$

So for $\lambda = 4$, it is having infinitely many solutions.

$\Delta_x = \begin{vmatrix} -2 & 1 & 4 \\ 0 & 1 & 1 \\ \mu & -3 & 0 \end{vmatrix}$

$= -2(3) - 1(-\mu) + 4(-\mu)$

$= -6 - 3\mu = 0$

For $\mu = -2$

Distance of $(4, -2, \frac{-1}{2})$ from $8x + y + 4z + 2 = 0$

$= \frac{32 - 2 - 2 + 2}{\sqrt{64 + 1 + 16}} = \frac{10}{3}$ units

4. Let A be a 2 × 2 matrix with det $(A) = -1$ and det $((A + I)$ (Adj $(A) + I)) = 4$. Then the sum of the diagonal elements of A can be

(A) –1 (B) 2

(C) 1 (D) $-\sqrt{2}$

Answer (B)

Sol. $|(A + I)(\text{adj } A + I)| = 4$

$\Rightarrow |A \text{ adj } A + A + \text{adj } A + I| = 4$

$\Rightarrow |(A)I + A + \text{adj } A + I| = 4$

$|A| = -1 \Rightarrow |A + \text{adj } A| = 4$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\text{adj } A = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$

$\Rightarrow \begin{vmatrix} (a+d) & 0 \\ 0 & (a+d) \end{vmatrix} = 4$

$\Rightarrow a + d = \pm 2$

5. The odd natural number a, such that the area of the region bounded by $y = 1$, $y = 3$, $x = 0$, $x = y^a$ is $\frac{364}{3}$, is equal to

(A) 3 (B) 5

(C) 7 (D) 9

Answer (B)

Sol. a is a odd natural number and

$\left|\int_1^3 y^a dy\right| = \frac{364}{3}$

$\Rightarrow \left|\frac{1}{a+1}\left(y^{a+1}\right)_1^3\right| = \frac{364}{3}$

$\Rightarrow \frac{3^{a+1} - 1}{a+1} = \pm\frac{364}{3}$

Solving with (–) sign,

$\frac{3^{a+1} - 1}{a+1} = \frac{364}{3} \Rightarrow (a = 5)$

Solving with (+) sign,

$\frac{3^{a+1} - 1}{a+1} = \frac{-364}{3}$, No a exist

$\therefore (a = 5)$

6. Consider two G.Ps. 2, 2^2, 2^3, …. and 4, 4^2, 4^3, … of 60 and n terms respectively. If the geometric mean of all the 60 + n terms is $(2)^{\frac{225}{8}}$, then $\sum_{k=1}^{n} k(n-k)$ is equal to

(A) 560 (B) 1540

(C) 1330 (D) 2600

Answer (C)

Sol. Given G.P's 2, 2^2, 2^3, … 60 terms

4, 4^2, … n terms

Now, $\text{G.M} = 2^{\frac{225}{8}}$

$\left(2.2^2 \ldots 4.4^2 \ldots\right)^{\frac{1}{60+n}} = 2^{\frac{225}{8}}$

$$\left(2^{\frac{n^2+n+1830}{60+n}}\right)=2^{\frac{225}{8}}$$

$$\Rightarrow \frac{n^2+n+1830}{60+n}=\frac{225}{8}$$

$\Rightarrow$ $8n^2 - 217n + 1140 = 0$

$$n=\frac{57}{8},\ 20,\ \text{so } n=20$$

$$\therefore \sum_{k=1}^{20} k(20-k)=20\times\frac{20\times 21}{2}-\frac{20\times 21\times 41}{6}$$

$$=\frac{20\times 21}{2}\left[20-\frac{41}{3}\right]=1330$$

7. If the function

$$f(x)=\begin{cases}\dfrac{\log_e\left(1-x+x^2\right)+\log_e\left(1+x+x^2\right)}{\sec x-\cos x}, & x\in\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)-\{0\}\\ k, & x=0\end{cases}$$

is continuous at $x = 0$, then k is equal to

(A) 1

(B) –1

(C) e

(D) 0

Answer (A)

Sol. $$f(x)=\begin{cases}\dfrac{\log_e\left(1-x+x^2\right)+\log_e\left(1+x+x^2\right)}{\sec x-\cos x}, & x\in\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)-\{0\}\\ k & , x=0\end{cases}$$

for continuity at $x = 0$

$$\lim_{x\to 0} f(x)=k$$

$$\therefore\ k=\lim_{x\to 0}\frac{\log_e\left(x^4+x^2+1\right)}{\sec x-\cos x}\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\cos x\log_e\left(x^4+x^2+1\right)}{\sin^2 x}$$

$$=\lim_{x\to 0}\frac{\log_e\left(x^4+x^2+1\right)}{x^2}$$

$$=\lim_{x\to 0}\frac{\ln\left(1+x^2+x^4\right)}{x^2+x^4}\cdot\frac{x^2+x^4}{x^2}$$

$= 1$

8. If

$$f(x)=\begin{cases}x+a, & x\le 0\\ |x-4|, & x>0\end{cases}\text{ and } g(x)=\begin{cases}x+1, & x<0\\ (x-4)^2+b, & x\ge 0\end{cases}$$

are continuous on ***R***, then $(gof)(2) + (fog)(-2)$ is equal to

(A) –10 (B) 10

(C) 8 (D) –8

Answer (D)

Sol. $$f(x)=\begin{cases}x+a, & x\le 0\\ |x-4|, & x>0\end{cases}\text{ and } g(x)=\begin{cases}x+1, & x<0\\ (x-4)^2+b, & x\ge 0\end{cases}$$

$\because$ $f(x)$ and $g(x)$ are continuous on R

$\therefore$ $a = 4$ and $b = 1 - 16 = -15$

then $(gof)(2) + (fog)(-2)$

$= g(2) + f(-1)$

$= -11 + 3 = -8$

9. Let $f(x)=\begin{cases}x^3-x^2+10x-7, & x\le 1\\ -2x+\log_2\left(b^2-4\right), & x>1\end{cases}$.

Then the set of all values of b, for which $f(x)$ has maximum value at $x = 1$, is

(A) $(-6, -2)$

(B) $(2, 6)$

(C) $[-6, -2) \cup (2, 6]$

(D) $\left[-\sqrt{6},-2\right)\cup\left(2,\sqrt{6}\right]$

Answer (C)

Sol. $$f(x)=\begin{cases}x^3-x^2+10x-7, & x\le 1\\ -2x+\log_2(b^2-4), & x>1\end{cases}$$

If $f(x)$ has maximum value at $x = 1$ then $f(1+) \le f(1)$

$-2 + \log_2(b^2 - 4) \le 1 - 1 + 10 - 7$

$\log_2(b^2 - 4) \le 5$

$0 < b^2 - 4 \le 32$

(i) $b^2 - 4 > 0 \Rightarrow b \in (-\infty, -2) \cup (2, \infty)$

(ii) $b^2 - 36 \le 0 \Rightarrow b \in [-6, 6]$

Intersection of above two sets

$b \in [-6, -2) \cup (2, 6]$

10. If $a = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2n}{n^2+k^2}$ and

$f(x) = \sqrt{\frac{1-\cos x}{1+\cos x}}, x \in (0, 1)$, then

(A) $2\sqrt{2} f\left(\frac{a}{2}\right) = f'\left(\frac{a}{2}\right)$ (B) $f\left(\frac{a}{2}\right) f'\left(\frac{a}{2}\right) = \sqrt{2}$

(C) $\sqrt{2} f\left(\frac{a}{2}\right) = f'\left(\frac{a}{2}\right)$ (D) $f\left(\frac{a}{2}\right) = \sqrt{2} f'\left(\frac{a}{2}\right)$

Answer (C)

Sol. $a = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2n}{n^2+k^2}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{2}{1+\left(\frac{k}{n}\right)^2}$$

$$a = \int_0^1 \frac{2}{1+x^2} dx = 2\tan^{-1} x \Big]_0^1 = \frac{\pi}{2}$$

$$f(x) = \sqrt{\frac{1-\cos x}{1+\cos x}}, \quad x \in (0, 1)$$

$$f(x) = \frac{1-\cos x}{\sin x} = \operatorname{cosec} x - \cot x$$

$$f'(x) = \operatorname{cosec}^2 x - \operatorname{cosec} x \cot x$$

$$\left.\begin{array}{l} f\left(\frac{a}{2}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{2} - 1 \\ f'\left(\frac{a}{2}\right) = f'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2} \end{array}\right\} f'\left(\frac{a}{2}\right) = \sqrt{2}.f\left(\frac{a}{2}\right)$$

11. If $\frac{dy}{dx} + 2y\tan x = \sin x, 0 < x < \frac{\pi}{2}$ and $y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of $y(x)$ is:

(A) $\frac{1}{8}$ (B) $\frac{3}{4}$

(C) $\frac{1}{4}$ (D) $\frac{3}{8}$

Answer (A)

Sol. $\frac{dy}{dx} + 2y\tan x = \sin x$

which is a first order linear differential equation.

Integrating factor (I. F.) $= e^{\int 2\tan x\, dx}$

$$= e^{2\ln|\sec x|} = \sec^2 x$$

Solution of differential equation can be written as

$$y \cdot \sec^2 x = \int \sin x \cdot \sec^2 x\, dx = \int \sec x \cdot \tan x\, dx$$

$$y \sec^2 x = \sec x + C$$

$$y\left(\frac{\pi}{3}\right) = 0, 0 = \sec\frac{\pi}{3} + C \Rightarrow \quad C = -2$$

$$y = \frac{\sec x - 2}{\sec^2 x} = \cos x - 2\cos^2 x$$

$$= \frac{1}{8} - 2\left(\cos x - \frac{1}{4}\right)^2$$

$$y_{\max} = \frac{1}{8}$$

12. A point P moves so that the sum of squares of its distances from the points (1, 2) and (–2, 1) is 14. Let $f(x, y) = 0$ be the locus of P, which intersects the x-axis at the points A, B and the y-axis at the points C, D. Then the area of the quadrilateral $ACBD$ is equal to

(A) $\frac{9}{2}$ (B) $\frac{3\sqrt{17}}{2}$

(C) $\frac{3\sqrt{17}}{4}$ (D) 9

Answer (B)

Sol. Let point $P : (h, k)$

$(h - 1)^2 + (k - 2)^2 + (h + 2)^2 + (k - 1)^2 = 14$

$2h^2 + 2k^2 + 2h - 6k - 4 = 0$

Locus of $P : x^2 + y^2 + x - 3y - 2 = 0$

Intersection with x-axis,

$x^2 + x - 2 = 0$

$\Rightarrow \quad x = -2, 1$

Intersection with y-axis,

$y^2 - 3y - 2 = 0$

$$\Rightarrow \quad y = \frac{3 \pm \sqrt{17}}{2}$$

Area of the quadrilateral $ACBD$ is

$$= \frac{1}{2}(|x_1| + |x_2|)(|y_1| + |y_2|)$$

$$= \frac{1}{2} \times 3 \times \sqrt{17} = \frac{3\sqrt{17}}{2}$$

13. Let the tangent drawn to the parabola $y^2 = 24x$ at the point (α, β) is perpendicular to the line $2x + 2y = 5$. Then the normal to the hyperbola $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$ at the point $(\alpha + 4, \beta + 4)$ does **NOT** pass through the point

(A) (25, 10)   (B) (20, 12)

(C) (30, 8)   (D) (15, 13)

Answer (D)

Sol. Any tangent to $y^2 = 24x$ at (α, β)

$\beta y = 12(x + \alpha)$

Slope $= \frac{12}{\beta}$ and perpendicular to $2x + 2y = 5$

$\Rightarrow \frac{12}{\beta} = 1 \quad \Rightarrow \quad \beta = 12, \alpha = 6$

Hence hyperbola is $\frac{x^2}{36} - \frac{y^2}{144} = 1$ and normal is drawn at (10, 16)

Equation of normal $\frac{36 \cdot x}{10} + \frac{144 \cdot y}{16} = 36 + 144$

$\Rightarrow \frac{x}{50} + \frac{y}{20} = 1$

This does not pass though (15, 13) out of given option

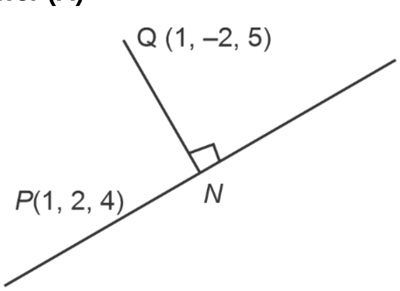
14. The length of the perpendicular from the point (1, –2, 5) on the line passing through (1, 2, 4) and parallel to the line $x + y - z = 0 = x - 2y + 3z - 5$ is

(A) $\sqrt{\frac{21}{2}}$ (B) $\sqrt{\frac{9}{2}}$

(C) $\sqrt{\frac{73}{2}}$ (D) 1

Answer (A)

Sol.



The line $x + y - z = 0 = x - 2y + 3z - 5$ is parallel to the vector

$$\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & -2 & 3 \end{vmatrix} = (1, 4, -3)$$

Equation of line through $P(1, 2, 4)$ and parallel to $\vec{b}$

$$\frac{x-1}{1} = \frac{y-2}{-4} = \frac{z-4}{-3}$$

Let $N \equiv (\lambda + 1, -4\lambda + 2, -3\lambda + 4)$

$\overrightarrow{QN} = (\lambda, -4\lambda + 4, -3\lambda - 1)$

$\overrightarrow{QN}$ is perpendicular to $\vec{b}$

$\Rightarrow (\lambda, -4\lambda + 4, -3\lambda - 1) \cdot (1, 4, -3) = 0$

$\Rightarrow \lambda = \frac{1}{2}$

Hence $\overrightarrow{QN} = \left(\frac{1}{2}, 2, \frac{-5}{2}\right)$ and $\left|\overrightarrow{QN}\right| = \sqrt{\frac{21}{2}}$

15. Let $\vec{a} = \alpha\hat{i} + \hat{j} - k$ and $\vec{b} = 2\hat{i} + \hat{j} - \alpha k, \alpha > 0$. If the projection of $\vec{a} \times \vec{b}$ on the vector $-\hat{i} + 2\hat{j} - 2k$ is 30, then α is equal to

(A) $\frac{15}{2}$ (B) 8

(C) $\frac{13}{2}$ (D) 7

Answer (D)

Sol. Given : $\vec{a} = (\alpha, 1, -1)$ and $\vec{b} = (2, 1, -\alpha)$

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 1 & -1 \\ 2 & 1 & -\alpha \end{vmatrix}$$

$$= (-\alpha + 1)\hat{i} + \left(\alpha^2 - 2\right)\hat{j} + (\alpha - 2)\hat{k}$$

Projection of $\vec{c}$ on $\vec{d} = -\hat{i} + 2\hat{j} - 2\hat{k}$

$$= \left|\vec{c} \cdot \frac{\vec{d}}{\left|\vec{d}\right|}\right| = 30 \text{ \{Given\}}$$

$$\Rightarrow \quad = \left|\frac{\alpha - 1 - 4 + 2\alpha^2 - 2\alpha + 4}{\sqrt{1 + 4 + 4}}\right| = 30$$

On solving $\alpha = \frac{-13}{2}$ (Rejected as $\alpha > 0$)

and $\alpha = 7$

16. The mean and variance of a binomial distribution are α and $\frac{\alpha}{3}$ respectively. If $P(X=1)=\frac{4}{243}$, then $P(X = 4 \text{ or } 5)$ is equal to :

(A) $\frac{5}{9}$ (B) $\frac{64}{81}$

(C) $\frac{16}{27}$ (D) $\frac{145}{243}$

Answer (C)

Sol. Given, mean = $np = \alpha$.

and variance $= npq = \frac{\alpha}{3}$

$\Rightarrow \quad q = \frac{1}{3}$ and $p = \frac{2}{3}$

$P(X=1) = n.p^1.q^{n-1} = \frac{4}{243}$

$\Rightarrow \quad n.\frac{2}{3}.\left(\frac{1}{3}\right)^{n-1} = \frac{4}{243}$

$\Rightarrow \quad n = 6$

$P(X = 4 \text{ or } 5) = {}^6C_4.\left(\frac{2}{3}\right)^4.\left(\frac{1}{3}\right)^2 + {}^6C_5.\left(\frac{2}{5}\right)^5.\frac{1}{3}$

$= \frac{16}{27}$

17. Let E_1, E_2, E_3 be three mutually exclusive events such that $P(E_1)=\frac{2+3p}{6}, P(E_2)=\frac{2-p}{8}$ and $P(E_3)=\frac{1-p}{2}$. If the maximum and minimum values of p are p_1 and p_2, then $(p_1 + p_2)$ is equal to :

(A) $\frac{2}{3}$ (B) $\frac{5}{3}$

(C) $\frac{5}{4}$ (D) 1

Answer (B)

Sol. $0 \le \frac{2+3P}{6} \le 1 \Rightarrow P \in \left[-\frac{2}{3}, \frac{4}{3}\right]$

$0 \le \frac{2-P}{8} \le 1 \Rightarrow P \in [-6, 2]$

$0 \le \frac{1-P}{2} \le 1 \Rightarrow P \in [-1, 1]$

$0 < P(E_1) + P(E_2) + P(E_3) \le 1$

$0 < \frac{13}{12} - \frac{P}{8} \le 1$

$P \in \left[\frac{2}{3}, \frac{26}{3}\right]$

Taking intersection of all

$P \in \left[\frac{2}{3}, 1\right)$

$P_1 + P_2 = \frac{5}{3}$

18. Let $S = \left\{\theta \in [0, 2\pi] : 8^{2\sin^2\theta} + 8^{2\cos^2\theta} = 16\right\}$. Then $n(S) + \sum_{\theta \in S}\left(\sec\left(\frac{\pi}{4}+2\theta\right)\text{cosec}\left(\frac{\pi}{4}+2\theta\right)\right)$ is equal to :

(A) 0 (B) – 2

(C) – 4 (D) 12

Answer (C)

Sol. $S = \left\{\theta \in [0, 2\pi] : 8^{2\sin^2\theta} + 8^{2\cos^2\theta} = 16\right\}$

Now apply AM $\ge$ GM for $8^{2\sin^2\theta}, 8^{2\cos^2\theta}$

$\frac{8^{2\sin^2\theta} + 8^{2\cos^2\theta}}{2} \ge \left(8^{2\sin^2\theta + 2\cos^2\theta}\right)^{\frac{1}{2}}$

$8 \ge 8$

$\Rightarrow \quad 8^{2\sin^2\theta} = 8^{2\cos^2\theta}$

or $\sin^2\theta = \cos^2\theta$

$\therefore \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$n(S) + \sum_{\theta \in S}\sec\left(\frac{\pi}{4}+2\theta\right)\text{cosec}\left(\frac{\pi}{4}+2\theta\right)$

$4 + \sum_{\theta \in S}\frac{2}{2\sin\left(\frac{\pi}{4}+2\theta\right)\cos\left(\frac{\pi}{4}+2\theta\right)}$

$= 4 + \sum_{\theta \in S}\frac{2}{\sin\left(\frac{\pi}{2}+4\theta\right)} = 4 + 2\sum_{\theta \in S}\text{cosec}\left(\frac{\pi}{2}+4\theta\right)$

$= 4 + 2\left[\text{cosec}\left(\frac{\pi}{2}+\pi\right) + \text{cosec}\left(\frac{\pi}{2}+3\pi\right) + \text{cosec}\left(\frac{\pi}{2}+5\pi\right) + \text{cosec}\left(\frac{\pi}{2}+7\pi\right)\right]$

$$= 4 + 2\left[-\text{cosec}\frac{\pi}{2} - \text{cosec}\frac{\pi}{2} - \text{cosec}\frac{\pi}{2} - \text{cosec}\frac{\pi}{2}\right]$$

= 4 – 2(4)

= 4 – 8

= – 4

19. $\tan\left(2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{\sqrt{5}}{2} + 2\tan^{-1}\frac{1}{8}\right)$ is equal to :

(A) 1

(B) 2

(C) $\frac{1}{4}$

(D) $\frac{5}{4}$

Answer (B)

Sol. $\tan\left(2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{\sqrt{5}}{2} + 2\tan^{-1}\frac{1}{8}\right)$

$$= \tan\left(2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\cdot\frac{1}{8}}\right) + \sec^{-1}\frac{\sqrt{5}}{2}\right)$$

$$= \tan\left[2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right]$$

$$= \tan\left[\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}} + \tan^{-1}\frac{1}{2}\right]$$

$$= \tan\left[\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2}\right]$$

$$= \tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{8}}\right] = \tan\left[\tan^{-1}\frac{\frac{5}{4}}{\frac{5}{8}}\right]$$

$$= \tan\left[\tan^{-1}2\right] = 2$$

20. The statement $(\sim(p \Leftrightarrow \sim q)) \wedge q$ is :

(A) a tautology

(B) a contradiction

(C) equivalent to $(p \Rightarrow q) \wedge q$

(D) equivalent to $(p \Rightarrow q) \wedge p$

Answer (D)

Sol. $\sim(p \Leftrightarrow \sim q) \wedge q$

$= (p \Leftrightarrow q) \wedge q$

p	q	$p\leftrightarrow q$	$(p\leftrightarrow q)\wedge q$	$(p\to q)$	$(p\to q)\wedge q$	$(p\to q)\wedge p$
T	T	T	T	T	T	T
T	F	F	F	F	F	F
F	T	F	F	T	T	F
F	F	T	F	T	F	F

$\therefore$ $(\sim(p \Leftrightarrow \sim q)) \wedge q$ is equivalent to $(p \Rightarrow q) \wedge p$.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. If for some $p, q, r \in \textbf{R}$, not all have same sign, one of the roots of the equation $(p^2 + q^2)x^2 - 2q(p + r)x + q^2 + r^2 = 0$ is also a root of the equation $x^2 + 2x - 8 = 0$, then $\frac{q^2 + r^2}{p^2}$ is equal to ________.

Answer (272)

Sol. Let roots of $(p^2 + q^2)x^2 - 2q(p + r)x + q^2 + r^2 = 0$ be α, β

$\therefore$ $\alpha + \beta > 0$ and $\alpha\beta > 0$

Also, it has a common root with $x^2 + 2x - 8 = 0$

$\therefore$ The common root between above two equations is 4.

$\Rightarrow$ $16(p^2 + q^2) - 8q(p + r) + q^2 + r^2 = 0$

$\Rightarrow$ $(16p^2 - 8pq + q^2) + (16q^2 - 8qr + r^2) = 0$

$\Rightarrow$ $(4p - q)^2 + (4q - r)^2 = 0$

$\Rightarrow$ $q = 4p$ and $r = 16p$

$$\therefore \frac{q^2 + r^2}{p^2} = \frac{16p^2 + 256p^2}{p^2} = 272$$

2. The number of 5-digit natural numbers, such that the product of their digits is 36, is ________.

Answer (180)

Sol. Factors of 36 = $2^2 \cdot 3^2 \cdot 1$

Five-digit combinations can be

(1, 2, 2, 3, 3) (1, 4, 3, 3, 1), (1, 9, 2, 2, 1)

(1, 4, 9, 11) (1, 2, 3, 6, 1) (1, 6, 6, 1, 1)

i.e., total numbers

$$\frac{5!}{2!2!}+\frac{5!}{2!2!}+\frac{5!}{2!2!}+\frac{5!}{3!}+\frac{5!}{2!}+\frac{5!}{3!2!}$$

= (30 × 3) + 20 + 60 + 10 = 180.

3. The series of positive multiples of 3 is divided into sets: {3}, {6, 9, 12}, {15, 18, 21, 24, 27},…… Then the sum of the elements in the 11th set is equal to ________.

Answer (6993)

Sol. Given series

$$\underbrace{\{3\times1\}}_{\text{1-term}},\underbrace{\{3\times2,3\times3,3\times4\}}_{\text{3-terms}},\underbrace{\{3\times5,3\times6,3\times7,3\times8,3\times9\}}_{\text{5-terms}},....$$

∴ 11th set will have 1 + (10)2 = 21 term

Also upto 10th set total 3 × k type terms will be 1 + 3 + 5 +……+19 = 100 – term

∴ Set 11 = {3 × 101, 3 × 102,……3 × 121}

∴ Sum of elements = 3 × (101 + 102 +…+121)

$$=\frac{3\times222\times21}{2}=6993$$

4. The number of distinct real roots of the equation $x^5(x^3 - x^2 - x + 1) + x(3x^3 - 4x^2 - 2x + 4) - 1 = 0$ is __________.

Answer (3)

Sol. $x^8 - x^7 - x^6 + x^5 + 3x^4 - 4x^3 - 2x^2 + 4x - 1 = 0$

⇒ $x^7(x-1) - x^5(x-1) + 3x^3(x-1) - x(x^2-1) + 2x(1-x) + (x-1) = 0$

⇒ $(x-1)(x^7 - x^5 + 3x^3 - x(x+1) - 2x + 1) = 0$

⇒ $(x-1)(x^7 - x^5 + 3x^3 - x^2 - 3x + 1) = 0$

⇒ $(x-1)(x^5(x^2-1) + 3x(x^2-1) - 1(x^2-1)) = 0$

⇒ $(x-1)(x^2-1)(x^5 + 3x - 1) = 0$

∴ $x = \pm 1$ are roots of above equation and $x^5 + 3x - 1$ is a monotonic term hence vanishes at exactly one value of x other then 1 or – 1.

∴ 3 real roots.

5. If the coefficients of x and x^2 – in the expansion of $(1+x)^p(1-x)^q$, p, $q \le 15$, are – 3 and – 5 respectively, then coefficient of x^3 is equal to ______.

Answer (23)

Sol. Coefficient of x in $(1+x)^p(1-x)^q$

$$-{}^pC_0\,{}^qC_1+{}^pC_1\,{}^qC_0=-3\Rightarrow \boxed{p-q=-3}$$

Coefficient of x^2 in $(1+x)^p(1-x)^q$

$${}^pC_0\,{}^qC_2-{}^pC_1\,{}^qC_1+{}^pC_2\,{}^qC_0=-5$$

$$\frac{q(q-1)}{2}-pq+\frac{p(p-1)}{2}=-5$$

$$\frac{q^2-q}{2}-(q-3)q+\frac{(q-3)(q-4)}{2}=-5$$

⇒ $q = 11$, $p = 8$

Coefficient of x^3 in $(1+x)^8(1-x)^{11}$ is

$$=-{}^{11}C_3+{}^8C_1\,{}^{11}C_2-{}^8C_2\,{}^{11}C_1+{}^8C_3=23$$

6. If $n(2n+1)\int_0^1\left(1-x^n\right)^{2n}dx=1177\int_0^1(1-x^n)^{2n+1}dx$, then $n\in\boldsymbol{N}$ is equal to ______.

Answer (24)

Sol. $\int_0^1(1-x^n)^{2n+1}dx=\int_0^1 1\cdot(1-x^n)^{2n+1}dx$

$$=\left[(1-x^n)^{2n+1}\cdot x\right]_0^1-\int_0^1 x\cdot(2n+1)(1-x^n)^{2n}\cdot -nx^{n-1}dx$$

$$=n(2n+1)\int_0^1\left(1-(1-x^n)\right)(1-x^n)^{2n}dx$$

$$=n(2n+1)\int_0^1(1-x^n)^{2n}dx-n(2n+1)\int_0^1(1-x^n)^{2n+1}dx$$

$$(1+n(2n+1))\int_0^1(1-x^n)^{2n+1}dx=n(2n+1)\int_0^1(1-x^n)^{2n}dx$$

$$(2n^2+n+1)\int_0^1(1-x^n)^{2n+1}dx=1177\int_0^1(1-x^n)^{2n+1}dx$$

∴ $2n^2 + n + 1 = 1177$

$2n^2 + n - 1176 = 0$

∴ $n = 24$ or $-\frac{49}{2}$

∴ $n = 24$

7. Let a curve $y = y(x)$ pass through the point (3, 3) and the area of the region under this curve, above the x-axis and between the abscissae 3 and $x(>3)$ be $\left(\frac{y}{x}\right)^3$. If this curve also passes through the point $\left(\alpha, 6\sqrt{10}\right)$ in the first quadrant, then α is equal to ________.

Answer (6)

Sol. $\int_3^x f(x)\,dx = \left(\frac{f(x)}{x}\right)^3$

$$x^3 \cdot \int_3^x f(x)dx = f^3(x)$$

Differentiate w.r.t. x

$$x^3 f(x) + 3x^2 \cdot \frac{f^3(x)}{x^3} = 3f^2(x)f'(x)$$

$$\Rightarrow \quad 3y^2\frac{dy}{dx} = x^3y + \frac{3y^3}{x}$$

$$3xy\frac{dy}{dx} = x^4 + 3y^2$$

Let $y^2 = t$

$$\frac{3}{2}\frac{dt}{dx} = x^3 + \frac{3t}{x}$$

$$\frac{dt}{dx} - \frac{2t}{x} = \frac{2x^3}{3}$$

$$I.F. = \cdot e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$$

Solution of differential equation

$$t \cdot \frac{1}{x^2} = \int \frac{2}{3}x\,dx$$

$$\frac{y^2}{x^2} = \frac{x^2}{3} + C$$

$$y^2 = \frac{x^4}{3} + Cx^2$$

Curve passes through (3, 3) $\Rightarrow C = -2$

$$y^2 = \frac{x^4}{3} - 2x^2$$

Which passes through $\left(\alpha, 6\sqrt{10}\right)$

$$\frac{\alpha^4 - 6\alpha^2}{3} = 360$$

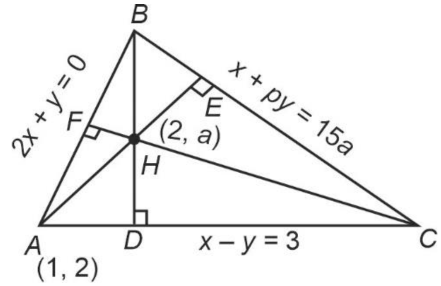
$$\alpha^4 - 6\alpha^2 - 1080 = 0$$

$\alpha = 6$

8. The equations of the sides AB, BC and CA of a triangle ABC are $2x + y = 0$, $x + py = 15a$ and $x - y = 3$ respectively. If its orthocentre is $(2, a), -\frac{1}{2} < a < 2$, then p is equal to _______.

Answer (3)

Sol.



Slope of $AH = \frac{a+2}{1}$

Sloe of $BC = -\frac{1}{p}$

$\therefore \quad p = a + 2 \qquad \ldots$(i)

Coordinate of $C = \left(\frac{18p-30}{p+1}, \frac{15p-33}{p+1}\right)$

Slope of HC

$$= \frac{\frac{15p-33}{p+1} - a}{\frac{18p-30}{p+1} - 2} = \frac{15p-33-(p-2)(p+1)}{18p-30-2p-2}$$

$$= \frac{16p - p^2 - 31}{16p - 32}$$

$$\because \quad \frac{16p-p^2-31}{16p-32} \times -2 = -1$$

$\therefore \quad p^2 - 8p + 15 = 0$

$\therefore \quad p = 3$ or 5

But if $p = 5$ then $a = 3$ not acceptable

$\therefore \quad p = 3$

9. Let the function $f(x) = 2x^2 - \log_e x$, $x > 0$, be decreasing in (0, a) and increasing in (a, 4). A tangent to the parabola $y^2 = 4ax$ at a point P on it passes through the point (8a, 8a –1) but does not pass through the point $\left(-\frac{1}{a}, 0\right)$. If the equation of the normal at P is $\frac{x}{\alpha} + \frac{y}{\beta} = 1$, then $\alpha + \beta$ is equal to _________.

Answer (45)

Sol. $\delta'(x) = \dfrac{4x^2 - 1}{x}$ so f(x) is decreasing in $\left(0, \dfrac{1}{2}\right)$ and increasing in $\left(\dfrac{1}{2}, \infty\right) \Rightarrow a = \dfrac{1}{2}$

Tangent at $y^2 = 2x \Rightarrow y = mx + \dfrac{1}{2m}$

It is passing through (4, 3)

$$3 = 4m + \frac{1}{2m} \Rightarrow m = \frac{1}{2} \text{ or } \frac{1}{4}$$

So tangent may be

$$y = \frac{1}{2}x + 1 \text{ or } y = \frac{1}{4}x + 2$$

But $y = \dfrac{1}{2}x + 1$ passes through (–2, 0) so rejected.

Equation of Normal

$$y = -4x - 2\left(\frac{1}{2}\right)(-4) - \frac{1}{2}(-4)^3$$

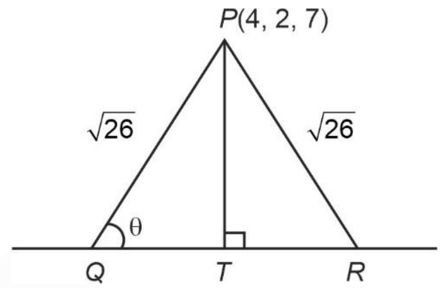
or $y = -4x + 4 + 32$

or $\dfrac{x}{9} + \dfrac{y}{36} = 1$

10. Let Q and R be two points on the line $\dfrac{x+1}{2} = \dfrac{y+2}{3} = \dfrac{z-1}{2}$ at a distance $\sqrt{26}$ from the point P(4, 2, 7). Then the square of the area of the triangle PQR is ________.

Answer (153)

Sol. $L : \dfrac{x+1}{2} = \dfrac{y+2}{3} = \dfrac{2-1}{2}$



Let $T(2t – 1, 3t – 2, 2t + 1)$

$\because$ $PT \perp^r QR$

$\therefore$ $2(2t – 5) + 3 (3t – 4) + 2 (2t – 6) = 0$

$17t = 34$ $\therefore$ $\boxed{t = 2}$ So $T(3, 4, 5)$

$\therefore$ $PT = \sqrt{1 + 4 + 4} = 3$

$\therefore$ $QT = \sqrt{26 - 9} = \sqrt{17}$

$\therefore$ Area of $\Delta PQR = \dfrac{1}{2} \times 2\sqrt{17} \times 3 = 3\sqrt{17}$

$\therefore$ Square of ar(ΔPQR) = 153.

❑ ❑ ❑