MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let R be a relation from the set {1, 2, 3, ..., 60} to itself such that $R = \{(a, b) : b = pq$, where $p, q \geq 3$ are prime numbers}. Then, the number of elements in R is :

(A) 600

(B) 660

(C) 540

(D) 720

Answer (B)

Sol. b can take its values as 9, 15, 21, 33, 39, 51, 57, 25, 35, 55, 49

b can take these 11 values

and a can take any of 60 values

So, number of elements in R = 60 × 11

= 660

2. If $z = 2 + 3i$, then $z^5 + (\bar{z})^5$ is equal to :

(A) 244 (B) 224

(C) 245 (D) 265

Answer (A)

Sol. $z = (2 + 3i)$

$$\Rightarrow\ z^5 = (2+3i)\left((2+3i)^2\right)^2$$

$$= (2+3i)(-5+12i)^2$$

$$= (2+3i)(-119-120i)$$

$$= -238 - 240i - 357i + 360$$

$$= 122 - 597i$$

$$\bar{z}^5 = 122 + 597i$$

$$z^5 + \bar{z}^5 = 244$$

3. Let A and B be two 3 × 3 non-zero real matrices such that AB is a zero matrix. Then

(A) the system of linear equations $AX = 0$ has a unique solution

(B) the system of linear equations $AX = 0$ has infinitely many solutions

(C) B is an invertible matrix

(D) adj(A) is an invertible matrix

Answer (B)

Sol. AB is zero matrix

$\Rightarrow$ $|A| = |B| = 0$

So neither A nor B is invertible

If $|A| = 0$

$\Rightarrow$ $|\text{adj } A| = 0$ so adj A is not invertible

$AX = 0$ is homogeneous system and $|A| = 0$

So, it is having infinitely many solutions

4. If $\frac{1}{(20-a)(40-a)} + \frac{1}{(40-a)(60-a)} + \ldots + \frac{1}{(180-a)(200-a)} = \frac{1}{256}$, then the maximum value of a is :

(A) 198 (B) 202

(C) 212 (D) 218

Answer (C)

Sol. $$\frac{1}{20}\left(\frac{1}{20-a} - \frac{1}{40-a} + \frac{1}{40-a} - \frac{1}{60-a} + \ldots.. + \frac{1}{180-a} - \frac{1}{200-a}\right) = \frac{1}{256}$$

$$\Rightarrow\ \frac{1}{20}\left(\frac{1}{20-a} - \frac{1}{200-a}\right) = \frac{1}{256}$$

$$\Rightarrow\ \frac{1}{20}\left(\frac{180}{(20-a)(200-a)}\right) = \frac{1}{256}$$

$\Rightarrow$ $(20 - a)(200 - a) = 9.256$

OR $a^2 - 220a + 1696 = 0$

$\Rightarrow$ $a = 212, 8$

5. If $\lim\limits_{x\to 0} \frac{\alpha e^x + \beta e^{-x} + \gamma \sin x}{x \sin^2 x} = \frac{2}{3}$, where $\alpha, \beta, \gamma \in \mathbf{R}$, then which of the following is **NOT** correct?

(A) $\alpha^2 + \beta^2 + \gamma^2 = 6$ (B) $\alpha\beta + \beta\gamma + \gamma\alpha + 1 = 0$

(C) $\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + 3 = 0$ (D) $\alpha^2 - \beta^2 + \gamma^2 = 4$

Answer (C)

Sol. $\lim_{x\to 0}\frac{\alpha e^x+\beta e^{-x}+\gamma\sin x}{x\sin^2 x}=\frac{2}{3}$

$\Rightarrow$ $\alpha + \beta = 0$ (to make indeterminant form) ...(i)

Now,

$\lim_{x\to 0}\frac{\alpha e^x-\beta e^{-x}+\gamma\cos x}{3x^2}=\frac{2}{3}$ (Using L-H Rule)

$\Rightarrow$ $\alpha - \beta + \gamma = 0$ (to make indeterminant form) ...(ii)

Now,

$\lim_{x\to 0}\frac{\alpha e^x+\beta e^{-x}-\gamma\sin x}{6x}=\frac{2}{3}$ (Using L-H Rule)

$\Rightarrow$ $\frac{\alpha-\beta-\gamma}{6}=\frac{2}{3}$

$\Rightarrow$ $\alpha - \beta - \gamma = 4$ ...(iii)

$\Rightarrow$ $\gamma = -2$

and (i) + (ii)

$2\alpha = -\gamma$

$\Rightarrow$ $\alpha = 1$ and $\beta = -1$

and $\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + 3 = 1 - 4 - 2 + 3 = -2$

6. The integral $\int_0^{\frac{\pi}{2}}\frac{1}{3+2\sin x+\cos x}dx$ is equal to

(A) $\tan^{-1}(2)$

(B) $\tan^{-1}(2)-\frac{\pi}{4}$

(C) $\frac{1}{2}\tan^{-1}(2)-\frac{\pi}{8}$

(D) $\frac{1}{2}$

Answer (B)

Sol. $I=\int_0^{\pi/2}\frac{1}{3+2\sin x+\cos x}dx$

$$=\int_0^{\pi/2}\frac{(1+\tan^2 x/2)dx}{3(1+\tan^2 x/2)+2(2\tan x/2)+(1-\tan^2 x/2)}$$

Let $\tan x/2=t \Rightarrow \sec^2 x/2dx=2dt$

$$I=\int_0^1\frac{2dt}{4+2t^2+4t}$$

$$=\int_0^1\frac{dt}{t^2+2t+2}=\int_0^1\frac{dt}{(t+1)^2+1}$$

$$=\tan^{-1}(t+1)\Big|_0^1=\tan^{-1}2-\frac{\pi}{4}$$

7. Let the solution curve $y = y(x)$ of the differential equation $(1+e^{2x})\left(\frac{dy}{dx}+y\right)=1$ pass through the point $\left(0,\frac{\pi}{2}\right)$. Then, $\lim_{x\to\infty}e^x y(x)$ is equal to

(A) $\frac{\pi}{4}$

(B) $\frac{3\pi}{4}$

(C) $\frac{\pi}{2}$

(D) $\frac{3\pi}{2}$

Answer (B)

Sol. D.E. $(1+e^{2x})\left(\frac{dy}{dx}+y\right)=1$

$\Rightarrow$ $\frac{dy}{dx}+y=\frac{1}{1+e^{2x}}$

I.F. = $e^{\int 1.dx}=e^x$

$\therefore$ Solution

$$e^x y(x)=\int\frac{e^x}{1+e^{2x}}dx$$

$\Rightarrow$ $e^x y(x)=\tan^{-1}(e^x)+C$

$\because$ It passes through $\left(0,\frac{\pi}{2}\right)$, $C=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

$\therefore$ $\lim_{x\to\infty}e^x y(x)=\lim_{x\to\infty}\tan^{-1}(e^x)+\frac{\pi}{4}$

$=\frac{3\pi}{4}$

8. Let a line L pass through the point intersection of the lines $bx + 10y - 8 = 0$ and $2x - 3y = 0$, $b\in \mathbf{R}-\left\{\frac{4}{3}\right\}$. If the line L also passes through the point (1, 1) and touches the circle $17(x^2 + y^2) = 16$, then the eccentricity of the ellipse $\frac{x^2}{5}+\frac{y^2}{b^2}=1$ is

(A) $\frac{2}{\sqrt{5}}$

(B) $\sqrt{\frac{3}{5}}$

(C) $\frac{1}{\sqrt{5}}$

(D) $\sqrt{\frac{2}{5}}$

Answer (B)

Sol. $L_1 : bx + 10y - 8 = 0$, $L_2 : 2x - 3y = 0$

then $L : (bx + 10y - 8) + \lambda(2x - 3y) = 0$

$\because$ It passes through (1, 1)

$\therefore$ $b + 2 - \lambda = 0 \Rightarrow \lambda = b + 2$

and touches the circle $x^2 + y^2 = \dfrac{16}{17}$

$$\left|\frac{8^2}{(2\lambda+b)^2+(10-3\lambda)^2}\right| = \frac{16}{17}$$

$$\Rightarrow\ 4\lambda^2 + b^2 + 4b\lambda + 100 + 9\lambda^2 - 60\lambda = 68$$

$$\Rightarrow\ 13(b+2)^2 + b^2 + 4b(b+2) - 60(b+2) + 32 = 0$$

$$\Rightarrow\ 18b^2 = 36\ \therefore\ b^2 = 2$$

$\therefore$ Eccentricity of ellipse : $\dfrac{x^2}{5} + \dfrac{y^2}{b^2} = 1$ is

$$\therefore\ e = \sqrt{1-\frac{2}{5}} = \sqrt{\frac{3}{5}}$$

9. If the foot of the perpendicular from the point $A(-1, 4, 3)$ on the plane $P : 2x + my + nz = 4$, is $\left(-2, \dfrac{7}{2}, \dfrac{3}{2}\right)$, then the distance of the point A from the plane P, measured parallel to a line with direction ratios 3, –1, –4, is equal to

(A) 1 (B) $\sqrt{26}$

(C) $2\sqrt{2}$ (D) $\sqrt{14}$

Answer (B)

Sol. $\left(-2, \dfrac{7}{2}, \dfrac{3}{2}\right)$ satisfies the plane $P : 2x + my + nz = 4$

$$-4 + \frac{7m}{2} + \frac{3n}{2} = 4\ \Rightarrow\ 7m + 3n = 16 \qquad \text{(i)}$$

Line joining $A(-1, 4, 3)$ and $\left(-2, \dfrac{7}{2}, \dfrac{3}{2}\right)$ is perpendicular to $P : 2x + my + nz = 4$

$$\frac{1}{2} = \frac{\frac{1}{2}}{m} = \frac{\frac{3}{2}}{n}\ \Rightarrow\ m = 1\ \&\ n = 3$$

Plane $P : 2x + y + 3z = 4$

Distance of P from $A(-1, 4, 3)$ parallel to the line

$$\frac{x+1}{3} = \frac{y-4}{-1} = \frac{z-3}{-4} : L$$

for point of intersection of P & L

$2(3r - 1) + (-r + 4) + 3(-4r + 3) = 4 \Rightarrow r = 1$

Point of intersection : (2, 3, –1)

Required distance $= \sqrt{3^2 + 1^2 + 4^2}$

$= \sqrt{26}$

10. Let $\vec{a} = 3\hat{i} + \hat{j}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$. Let $\vec{c}$ be a vector satisfying $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} + \lambda\vec{c}$. If $\vec{b}$ and $\vec{c}$ are non-parallel, then the value of λ is

(A) –5 (B) 5

(C) 1 (D) –1

Answer (A)

Sol. $\vec{a} = 3\hat{i} + \hat{j}$ & $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \vec{b} + \lambda\vec{c}$$

If $\vec{b}$ & $\vec{c}$ are non-parallel

then $\vec{a}\cdot\vec{c} = 1$ & $\vec{a}\cdot\vec{b} = -\lambda$

but $\vec{a}\cdot\vec{b} = 5 \Rightarrow \lambda = -5$

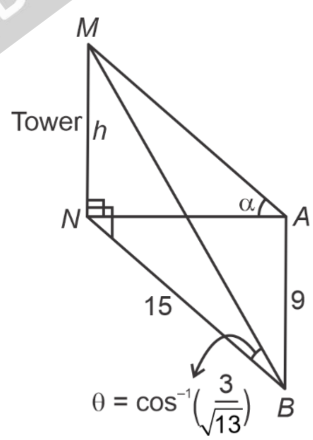
11. The angle of elevation of the top of a tower from a point A due north of it is α and from a point B at a distance of 9 units due west of A is $\cos^{-1}\left(\dfrac{3}{\sqrt{13}}\right)$. If the distance of the point B from the tower is 15 units, then $\cot\alpha$ is equal to :

(A) $\dfrac{6}{5}$ (B) $\dfrac{9}{5}$

(C) $\dfrac{4}{3}$ (D) $\dfrac{7}{3}$

Answer (A)

Sol.



$$NA = \sqrt{15^2 - 9^2} = 12$$

$$\frac{h}{15} = \tan\theta = \frac{2}{3}$$

$h = 10$ units

$$\cot\alpha = \frac{12}{10} = \frac{6}{5}$$

12. The statement $(p \wedge q) \Rightarrow (p \wedge r)$ is equivalent to :

(A) $q \Rightarrow (p \wedge r)$ (B) $p \Rightarrow (p \wedge r)$

(C) $(p \wedge r) \Rightarrow (p \wedge q)$ (D) $(p \wedge q) \Rightarrow r$

Answer (D)

Sol.

p	q	r	$p \wedge q$ (A)	$p \wedge r$ (B)	$A \to B$	$q \to B$	$p \to B$	$B \to A$	$A \to r$
T	T	T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	F	T
F	T	T	F	F	T	F	T	T	T
F	F	T	F	F	T	T	T	T	T
T	T	F	T	F	F	F	F	T	F
T	F	F	F	F	T	T	F	T	T
F	T	F	F	F	T	F	T	T	T
F	F	F	F	F	T	T	T	T	T

$(p \wedge q) \Rightarrow (p \wedge r)$ is equivalent to $(p \wedge q) \Rightarrow r$

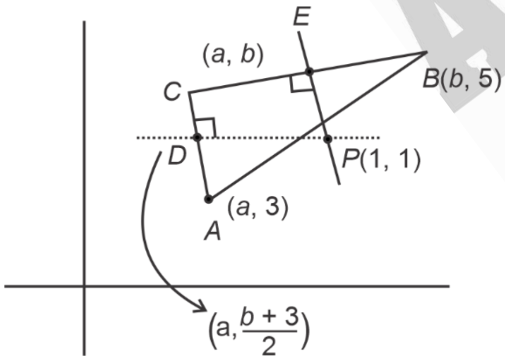
13. Let the circumcentre of a triangle with vertices $A(a, 3)$, $B(b, 5)$ and $C(a, b)$, $ab > 0$ be $P(1, 1)$. If the line AP intersects the line BC at the point $Q(k_1, k_2)$, then $k_1 + k_2$ is equal to :

(A) 2 (B) $\frac{4}{7}$

(C) $\frac{2}{7}$ (D) 4

Answer (B)

Sol.



Let D be mid-point of AC, then

$$\frac{b+3}{2} = 1 \Rightarrow b = -1$$

Let E be mid-point of BC,

$$\frac{5-b}{b-a} \cdot \frac{\frac{(3+b)}{2}}{\frac{a+b}{2} - 1} = -1$$

On Putting $b = -1$, we get $a = 5$ or -3

But $a = 5$ is rejected as $ab > 0$

$A(-3, 3)$, $B(-1, 5)$, $C(-3, -1)$, $P(1, 1)$

Line $BC \Rightarrow y = 3x + 8$

Line $AP \Rightarrow y = \frac{3-x}{2}$

Point of intersection $\left(\frac{-13}{7}, \frac{17}{7}\right)$

14. Let $\hat{a}$ and $\hat{b}$ be two unit vectors such that the angle between them is $\frac{\pi}{4}$. If θ is the angle between the vectors $(\hat{a} + \hat{b})$ and $(\hat{a} + 2\hat{b} + 2(\hat{a} \times \hat{b}))$, then the value of $164 \cos^2\theta$ is equal to :

(A) $90 + 27\sqrt{2}$

(B) $45 + 18\sqrt{2}$

(C) $90 + 3\sqrt{2}$

(D) $54 + 90\sqrt{2}$

Answer (A)

Sol. $\hat{a} \cdot \hat{b} = \frac{1}{\sqrt{2}}$ and $|\vec{a} \times \vec{b}| = \frac{1}{\sqrt{2}}$

$$\frac{(\hat{a} + \hat{b}) \cdot (\hat{a} + 2\hat{b} + 2(\hat{a} \times \hat{b}))}{|\hat{a} + \hat{b}||\hat{a} + 2\hat{b} + 2(\hat{a} \times \hat{b})|} = \cos\theta$$

$$\Rightarrow \cos\theta = \frac{1 + 3\hat{a}\hat{b} + 2}{|\hat{a} + \hat{b}||\hat{a} + 2\hat{b} + 2(\hat{a} \times \hat{b})|}$$

$$|\hat{a} + \hat{b}|^2 = 2 + \sqrt{2}$$

$$|\hat{a} + 2\hat{b} + 2(\hat{a} \times \hat{b})|^2 = 1 + 4 + 4|\hat{a} \times \hat{b}|^2 + 4\hat{a}\hat{b}$$

$$= 5 + 4 \cdot \frac{1}{2} + \frac{4}{\sqrt{2}} = 7 + 2\sqrt{2}$$

$$\text{So, } \cos^2\theta = \frac{\left(3 + \frac{3}{\sqrt{2}}\right)^2}{(2 + \sqrt{2})(7 + 2\sqrt{2})} = \frac{9\sqrt{2}(5\sqrt{2} + 3)}{164}$$

$$\Rightarrow 164\cos^2\theta = 90 + 27\sqrt{2}$$

15. If $f(\alpha) = \int_1^{\alpha} \frac{\log_{10} t}{1+t} dt$, $\alpha > 0$, then $f(e^3) + f(e^{-3})$ is equal to :

(A) 9 (B) $\frac{9}{2}$

(C) $\frac{9}{\log_e(10)}$ (D) $\frac{9}{2\log_e(10)}$

Answer (D)

Sol. $f(\alpha)=\int_1^{\alpha}\frac{\log_{10}t}{1+t}\,dt$ ...(i)

$$f\left(\frac{1}{\alpha}\right)=\int_1^{\frac{1}{\alpha}}\frac{\log_{10}t}{1+t}\,dt$$

Substituting $t\to\frac{1}{p}$

$$f\left(\frac{1}{\alpha}\right)=\int_1^{\alpha}\frac{\log_{10}\left(\frac{1}{p}\right)}{1+\frac{1}{p}}\left(\frac{-1}{p^2}\right)dp$$

$$=\int_1^{\alpha}\frac{\log_{10}p}{p(p+1)}dp=\int_1^{\alpha}\left(\frac{\log_{10}t}{t}-\frac{\log_{10}t}{t+1}\right)dt$$

...(ii)

By (i) + (ii)

$$f(\alpha)+f\left(\frac{1}{\alpha}\right)=\int_1^{\alpha}\frac{\log_{10}t}{t}\,dt=\int_1^{\alpha}\frac{\ln t}{t}\cdot\log_{10}e\,dt$$

$$=\frac{(\ln\alpha)^2}{2\log_e 10}$$

$$\alpha=e^3\Rightarrow f\left(e^3\right)+f\left(e^{-3}\right)=\frac{9}{2\log_e 10}$$

16. The area of the region $\left\{(x,y);|x-1|\le y\le\sqrt{5-x^2}\right\}$ is equal to

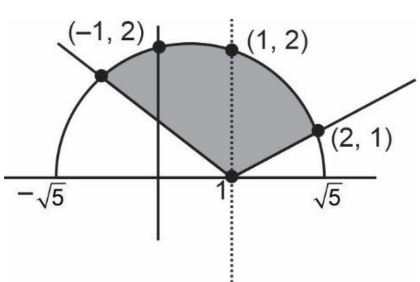
(A) $\frac{5}{2}\sin^{-1}\left(\frac{3}{5}\right)-\frac{1}{2}$ (B) $\frac{5\pi}{4}-\frac{3}{2}$

(C) $\frac{3\pi}{4}+\frac{3}{2}$ (D) $\frac{5\pi}{4}-\frac{1}{2}$

Answer (D)

Sol. $A=\int\limits_{-1}^{1}\left(\sqrt{5-x^2}-(1-x)\right)dx$

$$+\int\limits_{1}^{2}\left(\sqrt{5-x^2}-(x-1)\right)dx$$



$$A=2\left(\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right)-2x\Bigg|_0^1$$

$$+\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}-\frac{x^2}{2}+x\Bigg|_1^2$$

$$=\left(\frac{5\pi}{4}-\frac{1}{2}\right)\text{ sq. units}$$

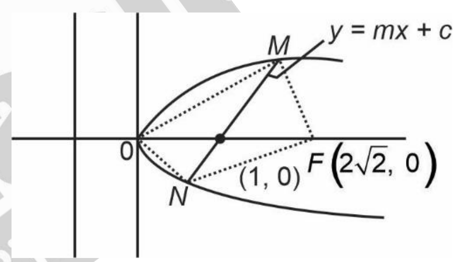
17. Let the focal chord of the parabola $P: y^2 = 4x$ along the line $L: y = mx + c$, $m > 0$ meet the parabola at the points M and N. Let the line L be a tangent to the hyperbola $H: x^2 - y^2 = 4$. If O is the vertex of P and F is the focus of H on the positive x-axis, then the area of the quadrilateral $OMFN$ is

(A) $2\sqrt{6}$ (B) $2\sqrt{14}$

(C) $4\sqrt{6}$ (D) $4\sqrt{14}$

Answer (B)

Sol. $H:\frac{x^2}{4}-\frac{y^2}{4}=1$



Focus $(ae, 0)$

$F\left(2\sqrt{2}, 0\right)$

$y = mx + c$ passes through (1, 0)

$0 = m + C$ ...(i)

L is tangent to hyperbola

$C=\pm\sqrt{4m^2-4}$

$-m=\pm\sqrt{4m^2-4}$

$m^2 = 4m^2 - 4$

$m=\frac{2}{\sqrt{3}}$

$C=\frac{-2}{\sqrt{3}}$

$T: y=\frac{2}{\sqrt{3}}x-\frac{2}{\sqrt{3}}$

$P: y^2 = 4x$

$$y^2 = 4\left(\frac{\sqrt{3}y+2}{2}\right)$$

$$y^2 - 2\sqrt{3}y - 4 = 0$$

Area

$$\frac{1}{2}\begin{vmatrix} 0 & 0 \\ x_1 & y_1 \\ 2\sqrt{2} & 0 \\ x_2 & y_2 \\ 0 & 0 \end{vmatrix}$$

$$= \left|\frac{1}{2}\left(-2\sqrt{2}y_1 + 2\sqrt{2}y_2\right)\right|$$

$$= \sqrt{2}\left|y_2 - y_1\right| = \sqrt{2}\sqrt{(y_1+y_2)^2 - 4y_1y_2}$$

$$= \sqrt{56}$$

$$= 2\sqrt{14}$$

18. The number of points, where the function $f : \mathbb{R} \to \mathbb{R}$,

$f(x) = |x-1|\cos|x-2|\sin|x-1| + (x-3)|x^2-5x+4|$, is **NOT** differentiable, is

(A) 1 (B) 2

(C) 3 (D) 4

Answer (B)

Sol. $f : R \to R$.

$f(x) = |x-1| \cos |x-2| \sin |x-1| + (x-3) |x^2 - 5x + 4|$

$= |x-1| \cos |x-2| \sin |x-1| + (x-3) |x-1| |x-4|$

$= |x-1| [\cos |x-2| \sin |x-1| + (x-3) |x-4|]$

Sharp edges at $x = 1$ and $x = 4$

$\therefore$ Non-differentiable at $x = 1$ and $x = 4$

19. Let $S = \{1, 2, 3, \ldots, 2022\}$. Then the probability, that a randomly chosen number n from the set S such that HCF $(n, 2022) = 1$, is

(A) $\frac{128}{1011}$ (B) $\frac{166}{1011}$

(C) $\frac{127}{337}$ (D) $\frac{112}{337}$

Answer (D)

Sol. $S = \{1, 2, 3, \ldots\ldots 2022\}$

HCF $(n, 2022) = 1$

$\Rightarrow$ n and 2022 have no common factor

Total elements = 2022

$2022 = 2 \times 3 \times 337$

M : numbers divisible by 2.

$\{2, 4, 6, \ldots.., 2022\}$ $n(M) = 1011$

N : numbers divisible by 3.

$\{3, 6, 9, \ldots.., 2022\}$ $n(N) = 674$

L : numbers divisible by 6.

$\{6, 12, 18, \ldots.., 2022\}$ $n(L) = 337$

$n(M \cup N) = n(M) + n(N) - n(L)$

$= 1011 + 674 - 337$

$= 1348$

0 = Number divisible by 337 but not in $M \cup N$

$\{337, 1685\}$

Number divisible by 2, 3 or 337

$= 1348 + 2 = 1350$

Required probability = $\frac{2022-1350}{2022}$

$$= \frac{672}{2022}$$

$$= \frac{112}{337}$$

20. Let $f(x) = 3^{(x^2-2)^3+4}, x \in \mathbb{R}$. Then which of the following statements are true?

$P : x = 0$ is a point of local minima of f

$Q : x = \sqrt{2}$ is a point of inflection of f

$R : f'$ is increasing for $x > \sqrt{2}$

(A) Only P and Q (B) Only P and R

(C) Only Q and R (D) All P, Q and R

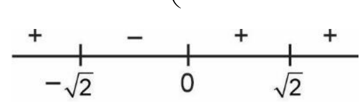
Answer (D)

Sol. $f(x) = 3^{(x^2-2)^3+4}$, $x \in R$

$$f(x) = 81.3^{(x^2-2)^3}$$

$$f'(x) = 81.3^{(x^2-2)^3}\ \ln 2.3(x^2-2)2x$$

$$= (486\ \ln 2)\left(3^{(x^2-2)^3}(x^2-2)x\right)$$



$\Rightarrow$ $x = 0$ is the local minima.

$$f''(x) = (486\ln 2)\left(3^{\left(x^2-2\right)^3}\cdot\left(x^2-2\right)\left(5x^2-2+6x^2\ln 3\left(x^2-2\right)\right)\right)$$

$$f''(x) = 0 \qquad x = \sqrt{2}$$

$$f''\left(\sqrt{2}^{+}\right) > 0$$

$$f''\left(\sqrt{2}^{-}\right) < 0$$

$\Rightarrow$ $x = \sqrt{2}$ is point of inflection

$f''(x) > 0 \ \forall\ x > \sqrt{2}$

$\Rightarrow$ $f'(x)$ is increasing for $x > \sqrt{2}$

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

1. Let $S = \{\theta \in (0, 2\pi) : 7\cos^2\theta - 3\sin^2\theta - 2\cos^2 2\theta = 2\}$. Then, the sum of roots of all the equations $x^2 - 2(\tan^2\theta + \cot^2\theta)x + 6\sin^2\theta = 0$, $\theta \in S$, is ________.

Answer (16)

Sol. $7\cos^2\theta - 3\sin^2\theta - 2\cos^2 2\theta = 2$

$$\Rightarrow\ 4\left(\frac{1+\cos 2\theta}{2}\right) + 3\cos 2\theta - 2\cos^2 2\theta = 2$$

$\Rightarrow$ $2 + 5\cos^2\theta - 2\cos^2 2\theta = 2$

$\Rightarrow$ $\cos 2\theta = 0$ or $\frac{5}{2}$ (rejected)

$$\Rightarrow\ \cos 2\theta = 0 = \frac{1-\tan^2\theta}{1+\tan^2\theta} \Rightarrow \tan^2\theta = 1$$

$\therefore$ Sum of roots $= 2(\tan^2\theta + \cot^2\theta) = 2 \times 2 = 4$

But as $\tan\theta = \pm 1$ for $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ in the interval $(0, 2\pi)$

$\therefore$ Four equations will be formed

Hence sum of roots of all the equations

$= 4 \times 4 = 16$.

2. Let the mean and the variance of 20 observations $x_1, x_2, \ldots, x_{20}$ be 15 and 9, respectively. For $a \in \mathbf{R}$, if the mean of $(x_1 + \alpha)^2, (x_2 + \alpha)^2, \ldots, (x_{20} + \alpha)^2$ is 178, then the square of the maximum value of α is equal to ____________.

Answer (4)

Sol. Given $\dfrac{\sum_{i=1}^{20} x_i}{20} = 15 \ \Rightarrow\ \sum_{i=1}^{20} x_i = 300$ …(1)

and $\dfrac{\sum_{i=1}^{20} x_i^2}{20} - \left(\bar{x}\right)^2 = 9 \ \Rightarrow\ \sum_{i=1}^{20} x_i^2 = 4680$ …(2)

$$\text{Mean} = \frac{(x_i+\alpha)^2 + (x_2+\alpha)^2 + \ldots\ldots + (x_{20}+\alpha)^2}{20}$$

$= 178$

$$\Rightarrow\ \frac{\sum_{i=1}^{20} x_i^2 + 2\alpha\sum_{i=1}^{20} x_i + 20\alpha^2}{20} = 178$$

$\Rightarrow$ $4680 + 600\alpha + 20\alpha^2 = 3560$

$\Rightarrow$ $\alpha^2 + 30\alpha + 56 = 0$

$\Rightarrow$ $\alpha^2 + 28\alpha + 2\alpha + 56 = 0$

$\Rightarrow$ $(\alpha + 28)(\alpha + 2) = 0$

$\alpha_{max} = -2 \ \Rightarrow\ \alpha_{max}^2 = 4$.

3. Let a line with direction ratios $a, -4a, -7$ be perpendicular to the lines with direction ratios $3, -1, 2b$ and $b, a, -2$. If the point of intersection of the line $\dfrac{x+1}{a^2+b^2} = \dfrac{y-2}{a^2-b^2} = \dfrac{z}{1}$ and the plane $x - y + z = 0$ is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to ______.

Answer (10)

Sol. Given $a.3 + (-4a)(-1) + (-7)\,2b = 0$ …(1)

and $ab - 4a^2 + 14 = 0$ …(2)

$\Rightarrow$ $a^2 = 4$ and $b^2 = 1$

$\therefore$ Line $L \equiv \dfrac{x+1}{5} = \dfrac{y-2}{3} = \dfrac{z}{1} = \lambda$ (say)

$\Rightarrow$ General point on line is $(5\lambda - 1, 3\lambda + 2, \lambda)$

for finding point of intersection with $x - y + z = 0$

we get $(5\lambda - 1) - (3\lambda + 2) + (\lambda) = 0$

$\Rightarrow$ $3\lambda - 3 = 0 \ \Rightarrow \lambda = 1$

$\therefore$ Point at intersection $(4, 5, 1)$

$\therefore$ $\alpha + \beta + \gamma = 4 + 5 + 1 = 10$

4. Let $a_1, a_2, a_3, \ldots$ be an A.P. If $\sum_{r=1}^{\infty} \frac{a_r}{2^r} = 4$, then $4a_2$ is equal to ________.

Answer (16)

Sol. Given

$$S = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + \ldots\ldots\infty$$

$$\frac{1}{2}S = \frac{a_1}{2^2} + \frac{a_2}{2^3} + \ldots\ldots\ldots\infty$$

$$\frac{S}{2} = \frac{a_1}{2} + \frac{(a_2 + a_1)}{2^2} + \frac{(a_3 + a_2)}{2^3} + \ldots\infty$$

$$\Rightarrow \frac{S}{2} = \frac{a_1}{2} + \frac{d}{2}$$

$\Rightarrow$ $a_1 + d = a_2 = 4$ $\Rightarrow 4a_2 = 16$

5. Let the ratio of the fifth term from the beginning to the fifth term from the end in the binomial expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$, in the increasing powers of $\frac{1}{\sqrt[4]{3}}$ be $\sqrt[4]{6} : 1$. If the sixth term from the beginning is $\frac{\alpha}{\sqrt[4]{3}}$, then α is equal to ___________.

Answer (84)

Sol. Fifth term from beginning $= {}^nC_4 \left(2^{\frac{1}{4}}\right)^{n-4} \left(3^{\frac{-1}{4}}\right)^4$

Fifth term from end = $(n - 5 + 1)^{\text{th}}$ term from begin

$$= {}^nC_{n-4} \left(2^{\frac{1}{4}}\right)^3 \left(3^{\frac{-1}{4}}\right)^{n-4}$$

Given $\dfrac{{}^nC_4 2^{\frac{n-4}{4}} \cdot 3^{-1}}{{}^nC_{n-3} 2^{\frac{4}{4}} \cdot 3^{-\left(\frac{n-4}{4}\right)}} = 6^{\frac{1}{4}}$

$$\Rightarrow 6^{\frac{n-8}{4}} = 6^{\frac{1}{4}}$$

$$\Rightarrow \frac{n-8}{4} = \frac{1}{4} \quad \Rightarrow n = 9$$

$$T_6 = T_{5+1} = {}^9C_5 \left(2^{\frac{1}{4}}\right)^4 \left(3^{\frac{-1}{4}}\right)^5$$

$$= \frac{{}^9C_5 \cdot 2}{3^{\frac{1}{4}} \cdot 3} = \frac{84}{3^{\frac{1}{4}}} = \frac{\alpha}{3^{\frac{1}{4}}}$$

$\Rightarrow$ $\alpha = 84$.

6. The number of matrices of order 3 × 3, whose entries are either 0 or 1 and the sum of all the entries is a prime number, is _________.

Answer (282)

Sol. In a 3 × 3 order matrix there are 9 entries.

These nine entries are zero or one.

The sum of positive prime entries are 2, 3, 5 or 7.

Total possible matrices $= \frac{9!}{2! \cdot 7!} + \frac{9!}{3! \cdot 6!} + \frac{9!}{5! \cdot 4!} + \frac{9!}{7! \cdot 2!}$

= 36 + 84 + 126 + 36

= 282

7. Let p and $p + 2$ be prime numbers and let

$$\Delta = \begin{vmatrix} p! & (p+1)! & (p+2)! \\ (p+1)! & (p+2)! & (p+3)! \\ (p+2)! & (p+3)! & (p+4)! \end{vmatrix}$$

Then the sum of the maximum values of α and β, such that p^α and $(p + 2)^\beta$ divide Δ, is _______.

Answer (04)

Sol. $\Delta = \begin{vmatrix} p! & (p+1)! & (p+2)! \\ (p+1)! & (p+2)! & (p+3)! \\ (p+2)! & (p+3)! & (p+4)! \end{vmatrix}$

$$= p! \cdot (p+1)! \cdot (p+2)! \begin{vmatrix} 1 & p+1 & (p+1)(p+2) \\ 1 & (p+2) & (p+2)(p+3) \\ 1 & (p+3) & (p+3)(p+4) \end{vmatrix}$$

$$= p! \cdot (p+1)! \cdot (p+2)! \begin{vmatrix} 1 & p+1 & p^2+3p+2 \\ 0 & 1 & 2p+4 \\ 0 & 1 & 2p+6 \end{vmatrix}$$

$$= 2(p!) \cdot ((p+1)!) \cdot ((p+2)!) \cdot$$

$$= 2(p+1) \cdot (p!)^2 \cdot ((p+2)!) \cdot$$

$$= 2(p+1)^2 \cdot (p!)^3 \cdot ((p+2)!) \cdot$$

$\therefore$ Maximum value of α is 3 and β is 1.

$\therefore$ $\alpha + \beta = 4$

8. If $\frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \frac{1}{4 \times 5 \times 6} + \ldots\ldots,$

$$+ \frac{1}{100 \times 101 \times 102} = \frac{k}{101}$$

then 34 k is equal to _____________.

Answer (286)

Sol. $S = \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \frac{1}{4\times5\times6}$

$$+\ldots+\frac{1}{100\times101\times102}$$

$$= \frac{1}{(3-1)\cdot1}\left[\frac{1}{2\times3} - \frac{1}{101\times102}\right]$$

$$= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{101\times102}\right)$$

$$= \frac{143}{102\times101} = \frac{k}{101}$$

$\therefore$ $34k = 286$

9. Let $S = \{4, 6, 9\}$ and $T = \{9, 10, 11, \ldots, 1000\}$. If $A = \{a_1 + a_2 + \ldots + a_k : k \in \mathbf{N}, a_1, a_2, a_3, \ldots, a_k \in S\}$, then the sum of all the elements in the set $T - A$ is equal to ____________.

Answer (11.00)

Sol. Here $S = \{4, 6, 9\}$

And $T = \{9, 10, 11, \ldots\ldots, 1000\}$.

We have to find all numbers in the form of

$4x + 6y + 9z$, where $x, y, z \in \{0, 1, 2, \ldots..\}$.

If a and b are coprime number then the least number from which all the number more than or equal to it can be express as $ax + by$ where $x, y \in \{0, 1, 2, \ldots.\}$ is $(a - 1) \cdot (b - 1)$.

Then for $6y + 9z = 3(2y + 3z)$

All the number from $(2 - 1) \cdot (3 - 1) = 2$ and above can be express as $2x + 3z$ (say t).

Now $4x + 6y + 9z = 4x + 3(t + 2)$

$= 4x + 3t + 6$

again by same rule $4x + 3t$, all the number from $(4 - 1)(3 - 1) = 6$ and above can be express from $4x + 3t$.

Then $4x + 6y + 9z$ express all the numbers from 12 and above.

again 9 and 10 can be express in form $4x + 6y + 9z$.

Then set $A = \{9, 10, 12, 13, \ldots., 1000\}$.

Then $T - A = \{11\}$

Only one element 11 is there.

Sum of elements of $T - A = 11$

10. Let the mirror image of a circle $c_1 : x^2 + y^2 - 2x - 6y + \alpha = 0$ in line $y = x + 1$ be $c_2 : 5x^2 + 5y^2 + 10gx + 10fy + 38 = 0$. If r is the radius of circle c_2, then $\alpha + 6r^2$ is equal to _________.

Answer (12)

Sol. c_1: $x^2 + y^2 - 2x - 6y + \alpha = 0$

Then centre = (1, 3) and radius $(r) = \sqrt{10-\alpha}$

Image of (1, 3) w.r.t. line $x - y + 1 = 0$ is (2, 2)

c_2: $5x^2 + 5y^2 + 10gx + 10fy + 38 = 0$

or $x^2 + y^2 + 2gx + 2fy + \frac{38}{5} = 0$

Then $(-g, -f) = (2, 2)$

$\therefore$ $g = f = -2$ …(i)

Radius of $c_2 = r = \sqrt{4+4-\frac{38}{5}} = \sqrt{10-\alpha}$

$\Rightarrow$ $\frac{2}{5} = 10 - \alpha$

$\therefore$ $\alpha = \frac{48}{5}$ and $r = \sqrt{\frac{2}{5}}$

$\therefore$ $\alpha + 6r^2 = \frac{48}{5} + \frac{12}{5}$

$= 12$

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