Question 1: If $(\sin^4 x)/2 + (\cos^4 x)/3 = 1/5$, then

(a) $\tan^2 x = 2/3$

(b) $(\sin^8 x)/8 + (\cos^8 x)/27 = 1/125$

(c) $\tan^2 x = 1/3$

(d) $(\sin^8 x)/8 + (\cos^8 x)/27 = 2/125$

Solution:

Given $(\sin^4 x)/2 + (\cos^4 x)/3 = 1/5$

$\Rightarrow (\sin^4 x)/2 + (1 - \sin^2 x)^2/3 = 1/5$

$\Rightarrow (\sin^4 x)/2 + (1 - 2\sin^2 x + \sin^4 x)/3 = 1/5$

Put $t = \sin^2 x$

$\Rightarrow t^2/2 + (1-2t+t^2)/3 = 1/5$

$\Rightarrow 3t^2 + 2 - 4t + 2t^2 = 6/5$

$\Rightarrow 5t^2 - 4t + 2 = 6/5$

$\Rightarrow 25t^2 - 20t + 4 = 0$

$\Rightarrow (5t-2)^2 = 0$

$\Rightarrow t = 2/5$

$\sin^2 x = 2/5$

$\cos^2 x = 1-\sin^2 x$

$= 1 - 2/5$

$= 3/5$

$\tan^2 x = 2/3$

$(\sin^8 x)/8 + (\cos^8 x)/27 = (2/5)^4/8 + (3/5)^4/27$

$= (2/5^4) + (3/5^4)$

$= 5/5^4$

$= 1/125$

Hence option a and b are correct.

Question 2: The maximum value of the expression $1/(\sin^2 x + 3 \sin x \cos x + 5 \cos^2 x)$ is

(a) 2

(b) 1

(c) 3

(d) 4

Solution:

Let $f(x) = \sin^2 x + 3 \sin x \cos x + 5 \cos^2 x$

$= (\sin^2 x + \cos^2 x) + (3/2) (2 \sin x \cos x) + 4 \cos^2 x$

$= 1 + (3/2) \sin 2x + 2(2 \cos^2 x)$

$= 1 + (3/2) \sin 2x + 2(1 + \cos 2x)$

$= 3 + (3/2) \sin 2x + 2 \cos 2x$

$= 3 + (3 \sin 2x + 4 \cos 2x)/2$ …(i)

Use $-\sqrt{(a^2+b^2)} \leq a \sin x + b \cos x \leq \sqrt{(a^2+b^2)}$

$\Rightarrow -(5/2) \leq (3 \sin 2x + 4 \cos 2x)/2 \leq (5/2)$

(i) $\Rightarrow 3 - (5/2) \leq f(x) \leq 3 + 5/2$

$\Rightarrow \frac{1}{2} \leq f(x) \leq 11/2$

$\Rightarrow 2/11 \leq 1/f(x) \leq 2$

Hence the maximum value of the expression is 2.

Hence option a is the answer.

Question 3: Let α and β be non zero real numbers such that $2(\cos \beta - \cos \alpha) + \cos \alpha \cos \beta = 1$. Then which of the following is/are true.

(a) $\tan (\alpha/2) - \sqrt{3} \tan (\beta/2) = 0$

(b) $\sqrt{3} \tan (\alpha/2) - \tan (\beta/2) = 0$

(c) $\tan (\alpha/2) + \sqrt{3} \tan (\beta/2) = 0$

(d) $\sqrt{3}\tan(\alpha/2) + \tan(\beta/2) = 0$

Solution:

Given that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$

=> $2(\cos\beta - \cos\alpha) = 1 - \cos\alpha\cos\beta$

=> $(\cos\beta - \cos\alpha) = (1 - \cos\alpha\cos\beta)/2$

$\cos 2x = (1 - \tan^2 x)/(1 + \tan^2 x)$

Put $\tan(\alpha/2) = x$ and $\tan(\beta/2) = y$

So $\cos\alpha = (1 - x^2)/(1 + x^2)$

And $\cos\beta = (1-y^2)/(1+y^2)$

=> $2[(1-y^2)/(1+y^2) - (1-x^2)/(1+x^2)] = 1 - [(1-x^2)(1-y^2)/(1+x^2)(1+y^2)]$

=> $2[(1+x^2)(1-y^2) - (1-x^2)(1+y^2)] = (1+x^2)(1+y^2) - (1-x^2)(1-y^2)$

=> $4(x^2 - y^2) = 2(x^2+y^2)$

=> $x^2 = 3y^2$

=> $x = \pm\sqrt{3}y$

=> $\tan(\alpha/2) = \pm\sqrt{3}\tan(\beta/2)$

=> $\tan(\alpha/2) \pm\sqrt{3}\tan(\beta/2) = 0$

Hence option a and c are correct.

Question 4: Let P = {x: sin x – cos x = √2 cos x} and Q = {x: sin x + cos x = √2 sin x} be two sets. Then

(a) $P \subset Q$ and $Q - P \neq \emptyset$

(b) $Q \not\subset P$

(c) $P \not\subset Q$

(d) $P = Q$

Solution:

Given $P = \sin x - \cos x = \sqrt{2}\cos x$

Divide by cos x

=> tan x – 1 = √2

=> tan x = √2+1 …(i)

Given Q = sin x + cos x = √2 sin x

Divide by cos x

tan x + 1 = √2 tan x

tan x (√2-1) = 1

tan x = 1/(√2-1)

Multiply numerator and denominator by (√2+1)

=> tan x = (√2+1)/(√2-1)(√2+1)

= (√2+1)/1

= (√2+1) …(ii)

From (i) and (ii), we get P = Q.

Hence option d is the answer.

Question 5: If tan A = n/(n+1) and tan B = 1/(2n+1), 0 < A+B < 2 π, then A+B equals

(a) π/4

(b) π/2

(c) π/3

(d) π/4

Solution:

Given tan A = n/(n+1) and tan B = 1/(2n+1)

We know tan (A + B) = (tan A + tan B)/(1 – tan A tan B)

= [n/(n+1) + 1/(2n+1)]/(1 – n/(n+1)(2n+1)]

= $(2n^2 + n + n+1)/(2n^2+2n+n+1 – n)$

= $(2n^2 + 2n+1)/(2n^2 + 2n+1)$

= 1

=> A+B = $\tan^{-1} 1$

= π/4

Hence option a is the answer.

Question 6: The value of cos (π/4 – A) cos (π/4 – B) – sin (π/4 – A) sin (π/4 – B) is

(a) sin (A+B)

(b) cos (A+B)

(c) 0

(d) 1

Solution:

We know that cos X cos Y – sin X sin Y = cos (X+Y)

(take X = π/4 – A and Y = π/4 – B)

So cos (π/4 – A) cos (π/4 – B) – sin (π/4 – A) sin (π/4 – B) = cos (π/4 – A + π/4 – B)

= cos (π/2 – (A+B))

= sin (A+B) (since cos (90-x) = sin x)

Hence option a is the answer.

Question 7: If sin (A-B) = 1/√10, cos (A+B) = 2/√29, find the value of tan 2A where A and B lie between 0 and π/4.

(a) 1/17

(b) 17

(c) 17/6

(d) 1

Solution:

tan 2A = tan (A+B + A-B)

= [tan (A+B) + tan (A-B)]/(1 – tan (A+B) tan (A-B)) …(i)

Given that $\sin(A-B) = 1/\sqrt{10}$

=> $\tan(A-B) = \frac{1}{3}$ (Use Pythagoras theorem)

Given that $\cos(A+B) = 2/\sqrt{29}$

=> $\tan(A+B) = 5/2$ (Use Pythagoras theorem)

Substitute tan (A-B) and tan (A+B) in (i)

$\tan 2A = [(5/2) + \frac{1}{3}]/(1-(5/2)(\frac{1}{3}))$

$= (17/6)/(\frac{1}{6})$

$= 17$

Hence option b is the answer.

Question 8: The value of $\tan(\pi/4 + x).\tan(3\pi/4 + x)$ is

(a) 1

(b) 0

(c) 2

(d) -1

Solution:

We know $\tan(A+B) = (\tan A + \tan B)/(1-\tan A \tan B)$

$\tan(\pi/4 + x) = (1 + \tan x)/(1 - \tan x)$ (since $\tan \pi/4 = 1$)

$\tan(3\pi/4 + x) = (-1 + \tan x)/(1 + \tan x)$ (since $\tan 3\pi/4 = -1$)

$\tan(\pi/4 + x)\tan(3\pi/4 + x) = (1 + \tan x)(-1 + \tan x)/(1 - \tan x)(1 + \tan x)$

$= (-1 + \tan x)/(1 - \tan x)$

$= -1(1 - \tan x)/(1 - \tan x)$

$= -1$

Hence option d is the answer.

Question 9: If $\cos(x-y) + \cos(y-z) + \cos(z-x) = -3/2$, then find the value of $\sin x + \sin y + \sin z$

(a) 0

(b) 1

(c) 3

(d) -1

Solution:

We know cos (A-B) = cos A cos B + sin A sin B

Given that cos (x-y) + cos (y-z) + cos (z-x) = -3/2

=> cos x cos y + sin x sin y + cos y cos z + sin y sin z + cos z cos x + sin z sin x = -3/2

=> 2 cos x cos y + 2 sin x sin y + 2 cos y cos z + 2 sin y sin z + 2 cos z cos x + 2 sin z sin x = -3

=> 2 cos x cos y + 2 sin x sin y + 2 cos y cos z + 2 sin y sin z + 2 cos z cos x + 2 sin z sin x + 3 = 0

=> 2 cos x cos y + 2 sin x sin y + 1 + 2 cos y cos z + 2 sin y sin z + 1 + 2 cos z cos x + 2 sin z sin x + 1 = 0

Instead of 1, put $\sin^2 x + \cos^2 x$, $\sin^2 y + \cos^2 y$, and $\sin^2 z + \cos^2 z$.

=> $2 \cos x \cos y + 2 \sin x \sin y + \sin^2 x + \cos^2 x + 2 \cos y \cos z + 2 \sin y \sin z + \sin^2 y + \cos^2 y + 2 \cos z \cos x + 2 \sin z \sin x + \sin^2 z + \cos^2 z = 0$

=> $(\sin x + \sin y + \sin z)^2 + (\cos x + \cos y + \cos z)^2 = 0$

Sum of perfect squares = 0 implies both quantities will be zero.

=> sin x + sin y + sin z = 0

Hence option a is the answer.

Question 10: If tan a and tan b are the roots of the equation $x^2 + px + q = 0$, $p \neq 0$, then

(a) $\sin^2(a+b) + p.\sin(a+b).\cos(a+b) + q \cos^2(a+b) = q$

(b) tan (a+b) = p/(q-1)

(c) cos (a+b) = 1-q

(d) sin (a+b) = -p

Solution:

Given that tan a and tan b are the roots of the equation $x^2 + px + q = 0$.

Sum of roots = tan a + tan b = -p

Product of roots = tan a tan b = q

tan (a+b) = (tan a + tan b)/(1 – tan a tan b)

= -p/(1-q)

= p/(q-1) …(i)

Option b is correct.

Check option a.

$\sin^2(a+b) + p. \sin(a+b). \cos(a+b) + q \cos^2(a+b)$

Multiply and divide by $\cos^2(a+b)$

=> $\cos^2(a+b)[\tan^2(a+b) + p.\tan(a+b) + q]$

= $[\tan^2(a+b) + p.\tan(a+b) + q]/\sec^2(a+b)$

= $[\tan^2(a+b) + p.\tan(a+b) + q]/(1+\tan^2(a+b))$

Put (i) in above equation

= $[p^2/(q-1)^2 + p^2/(q-1) + q)]/(1 + p^2/(q-1)^2)$

= $q(p^2+q^2-2q+1)/(p^2+q^2-2q+1)$

= q

Option a is also correct.

Hence option a and b is the answer.

Question 11: Let a, b, c be three non zero real numbers such that the equation √3a cos x + 2b sin x = c, x belongs to [-π/2, π/2] has two distinct roots α and β with α + β = π/3. Then the value of b/a is

(a) 1

(b) 4/3

(c) 1/2

(d) 0

Solution:

Given that $\sqrt{3}a \cos x + 2b \sin x = c$

Divide by a, we get

$\sqrt{3} \cos x + (2b/a) \sin x = c/a$ …(i)

Since α and β are the roots of (i)

$\sqrt{3} \cos \alpha + (2b/a) \sin \alpha = c/a$ …(ii)

$\sqrt{3} \cos \beta + (2b/a) \sin \beta = c/a$ …(iii)

Subtract (iii) from (ii)

=> $\sqrt{3} (\cos \alpha - \cos \beta) + (2b/a) (\sin \alpha - \sin \beta) = 0$

We know $\cos A - \cos B = -2 \sin (A+B)/2 \sin (A-B)/2$

Also $\sin A - \sin B = 2 \cos (A+B)/2 \sin (A-B)/2$

=> $\sqrt{3} (-2 \sin ((\alpha+\beta)/2) \sin ((\alpha-\beta)/2)) + (2b/a) (2 \cos ((\alpha+\beta)/2) \sin ((\alpha-\beta)/2)) = 0$

Given $\alpha + \beta = \pi/3$

=> $\sqrt{3} (-2 \sin (\pi/6) \sin (\alpha-\beta)/2) + (2b/a) (2 \cos (\pi/6) \sin (\alpha-\beta)/2) = 0$

=> $\sqrt{3} (-2 \times (½) \times \sin (\alpha-\beta)/2) + (2b/a) (2\times(\sqrt{3}/2) \sin (\alpha-\beta)/2) = 0$

=> $\sqrt{3} (\sin (\alpha-\beta)/2) = (2b/a) (\sqrt{3} \sin (\alpha-\beta)/2)$

=> $1 = 2b/a$

=> $b/a = 1/2$

Hence option c is the answer.

Question 12: Suppose θ and φ (≠ 0) are such that sec (θ + φ), sec θ, and sec (θ – φ) are in AP. If cos θ = k cos (φ/2) for some k, then k is equal to

(a) $\pm\sqrt{2}$

(b) ± 1

(c) $\pm 1/\sqrt{2}$

(d) ± 2

Solution:

Given that sec $(\theta + \varphi)$, sec θ, and sec $(\theta - \varphi)$ are in AP.

So $2 \sec \theta = \sec (\theta + \varphi) + \sec (\theta - \varphi)$

=> $2/\cos \theta = [1/\cos (\theta + \varphi)] + [1/\cos (\theta - \varphi)]$

Take LCM and add

$2/\cos \theta = [\cos (\theta - \varphi) + \cos (\theta + \varphi)]/\cos (\theta - \varphi).\cos (\theta + \varphi)$

$2/\cos \theta = 2 \cos \theta \cos \varphi/(\cos^2\theta - \sin^2\varphi)$

$(\cos^2\theta - \sin^2\varphi) = \cos^2\theta \cos \varphi$

$(\cos^2\theta - \cos^2\theta \cos \varphi) = \sin^2\varphi$

$\cos^2\theta (1 - \cos \varphi) = \sin^2\varphi$

$\cos^2\theta (1 - \cos \varphi) = 1 - \cos2\varphi$

$\cos^2\theta (1 - \cos \varphi) = (1 - \cos \varphi)(1 + \cos \varphi)$

$\cos^2\theta = (1 + \cos \varphi)$

$\cos^2\theta = 2 \cos^2 (\varphi/2)$

So $\cos \theta = \pm\sqrt{2} \cos (\varphi/2)$

Given that $\cos \theta = k \cos (\varphi/2)$

=> $k = \pm\sqrt{2}$

Hence option a is the answer.

Question 13: In a triangle PQR, let $\angle PQR = 30^0$ and the sides PQ and QR have length $10\sqrt{3}$ and 10, respectively. Then which of the following statement(s) is (are) true?

(a) $\angle QPR = 45^0$

(b) Area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^0$

(c) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(d) The area of the circumcircle of the triangle is PQR is 100π

Solution:

Given that $\angle PQR = 30^0$

PQ = $10\sqrt{3}$

QR = 10

Area of triangle PQR = ½ ab sin C

$= \frac{1}{2} \times 10\sqrt{3} \times 10 \times \sin 30$

$= \frac{1}{2} \times 10\sqrt{3} \times 10 \times 1/2$

$= 25\sqrt{3}$ …(i)

Let PR = x

Use cosine rule

$\cos 30 = ((10\sqrt{3})^2 + (10)^2 - x^2)/2\times10\sqrt{3}\times10$

$\sqrt{3}/2 = (300 + 100 - x^2)/200\sqrt{3}$

$(300 + 100 - x^2) = 300$

$100 - x^2 = 0$

=> x = 10

So triangle PQR is isosceles.

So $\angle QPR = 30^0$

$\angle QRP = 180 - (30+30)$

$\angle QRP = 120^0$ …(ii)

Radius of incircle = Δ/s

$= 25\sqrt{3}/(10+5\sqrt{3})$ (semi perimeter = $10+5\sqrt{3}$)

$= 25\sqrt{3}(10-5\sqrt{3})/(10+5\sqrt{3})(10-5\sqrt{3})$

$= (250\sqrt{3} - 375)/(100 - 75)$

$= (250\sqrt{3} - 375)/25$

$= 10\sqrt{3} - 15$ …(iii)

Area of circumcircle = πR^2, where R is the circum radius

sin A/a = 1/2R

(½)/10 = 1/2R

=> R = 10

So area = $\pi(10)^2$

= 100π …(iv)

From (i), (ii), (iii) and (iv), we can say that options b, c, d are correct.

Question 14: If tan A and tan B are the roots of the quadratic equation $x^2 - px + q = 0$, then $\sin^2(A+B)$ equals

(a) $p^2/[p^2 + (1- q)^2]$

(b) $p^2/[p^2 - (1- q)^2]$

(c) $q^2/[p^2 + (1- q)^2]$

(d) none of these

Solution:

Given tan A and tan B are the roots of the quadratic equation $x^2 - px + q = 0$.

Sum of roots = tan A + tan B = p

Product of roots = tan A tan B = q

We know

tan(A + B) = (tan A + tan B)/ (1 – tan A tan B)

= p/(1 – q)

$\tan^2(A + B) = p^2/(1 - q)^2$

$\cot^2(A + B) = (1 - q)^2/p^2$

Using the identity $\operatorname{cosec}^2x - \cot^2x = 1$,

$\operatorname{cosec}^2(A + B) = 1 + \cot^2(A + B)$

$= 1 + [(1 - q)^2/p^2]$

$= [p^2 +(1- q)^2]/p^2$

So, $\sin^2(A + B) = 1/\operatorname{cosec}^2(A + B) = p^2/[p^2 +(1- q)^2]$

Hence option a is the answer.

Question 15: If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of cos 4x is :

(a) 1/3

(b) -7/9

(c) 2/9

(d) -3/5

Solution:

Given that $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$

(We use $\tan^2 x = \sec^2 x - 1$ and $\cos 2x = 2\cos^2 x - 1$)

$5(\sec^2 x - 1 - \cos^2 x) = 2(2\cos x - 1) + 9$

$5\sec^2 x - 5 = 9\cos^2 x + 7$

Put $\cos^2 x = t$

$(5 / t) = 9t + 12$

$9t^2 + 12t - 5 = 0$

$t = 1/3$ or $-5/3$

$t = 1/3$ as $t \neq -5/3$

$\cos^2 x = 1/3$, $\cos 2x = 2\cos^2 x - 1 = -1/3$

$\cos 4x = 2\cos^2 2x - 1$

$= (2/9) - 1$

$= -7/ 9$

Hence option b is the answer.