Question 1: Let [x] be the greatest integer less than or equal to x. Then at which of the following point(s) the function f(x) = x cos (π(x + [x])) is discontinuous?

(a) x = 2

(b) x = 0

(c) x = 1

(d) x = -1

Answer: a, c, d

Solution:

Given f(x) = x cos (π(x + [x]))

At x = 2

$\lim_{x\to 2^-} x \cos(\pi(x + [x])) = 2 \cos(\pi+2\pi)$

= 2 cos 3π

= -2

$\lim_{x\to 2^+} x \cos(\pi(x + [x])) = 2 \cos(2\pi+2\pi)$

= 2 cos 4π

= 2

LHL ≠ RHL

So f(x) is discontinuous at x = 2.

At x = 0

$\lim_{x\to 0^-} x \cos(\pi(x + [x])) = 0 \cos(-\pi+0)$

= 0

$\lim_{x\to 0^+} x \cos(\pi(x + [x])) = 0$

LHL = RHL

So f(x) is continuous at x = 0.

At x = 1

$\lim_{x\to 1^-} x \cos(\pi(x + [x])) = \cos(\pi)$

= -1

$\lim_{x\to 1^+} x \cos(\pi(x + [x])) = \cos 2\pi$

= 1

LHL ≠ RHL

So f(x) is discontinuous at x = 1.

At x = -1

$\lim_{x\to -1^-} x \cos(\pi(x + [x])) = -\cos(-3\pi)$

= 1

$\lim_{x\to -1^+} x \cos(\pi(x + [x])) = -\cos 2\pi$

= -1

LHL ≠ RHL

So f(x) is discontinuous at x = 1.

The function is discontinuous at x = 2, 1, -1

Question 2: For every pair of continuous functions f, g: [0, 1] → R such that max {f(x): x∈[0, 1] = max {g(x) : x∈ [0, 1]}, the correct statements is (are)

(a) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some c ∈ [0,1]

(b) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some c ∈ [0,1]

(c) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some c ∈ [0,1]

(d) $(f(c))^2 = (g(c))^2$ for some c∈ [0,1]

Answer: a, d

Solution:

Let f and g be maximum at c_1 and c_2 respectively.

$c_1, c_2 \in [0, 1]$

Then, $f(c_1) = g(c_2)$

Let h(x) = f(x) - g(x)

Then $h(c_1) = f(c_1) - g(c_1) > 0$

And $h(c_2) = f(c_2) - g(c_2) < 0$

So $h(x) = 0$ has atleast one root in $[c_1, c_2]$

i.e $f(c) = g(c)$ for $c \in [c_1, c_2]$

which shows that options a and d are correct.

Question 3: Suppose that f(x) is a differentiable function such that f '(x) is continuous f '(0) = 1 and f '' (0) does not exist. If g(x) = x f ' (x). Then,

(a) g'(0) does not exist

(b) g'(0) = 0

(c) g'(0) = 1

(d) g'(0) = 2

Answer: c

Solution:

Given f '(0) = 1

g(x) = x f ' (x)

g'(x) = f'(x) + x f''(x)

g'(0) = f'(0) + 0 f''(0)

So g'(0) = 1

Hence option (c) is the answer.

Question 4: If a function f is such that f(0) = 2, f(1) = 3, f(x+2) = 2f(x) - f(x+1) for x ≥ 0, then f(5) is equal to

(a) -7

(b) -3

(c) 17

(d) 13

Answer: d

Solution:

Given f(0) = 2, f(1) = 3

f(x+2) = 2f(x) - f(x+1)

Put x = 0,

Then f(2) = 2f(0) - f(1)

= 4-3

= 1

Put x = 1

f(3) = 2f(1) - f(2)

= 6-1

= 5

Put x = 2

f(4) = 2f(2) - f(3)

= 2 - 5

= -3

f(5) = 2f(3) - f(4)

= 10 - -3

= 13

Hence option d is the answer.

Question 5: Let [t] denote the greatest integer ≤ t and $\lim_{x \to 0} x[4/x] = A$. Then the function, $f(x) = [x^2] \sin \pi x$ is discontinuous, when x is equal to

(a) √(A+1)

(b) √(A+5)

(c) √(A+21)

(d) √A

Answer: a

Solution:

$\lim_{x\to 0} x[4/x] = A$

$\lim_{x\to 0} x[4/x - \{4/x\}] = A$

=> $\lim_{x\to 0} 4 - x\{4/x\} = A$

=> 4-0 = A

As, $f(x) = [x^2]\sin(\pi x)$ will be discontinuous at non-integers

And when $x = \sqrt{(A+1)}$

=> $x = \sqrt{5}$, which is not an integer.

Hence f(x) is discontinuous when x is equal to $\sqrt{(A+1)}$.

Hence option a is the answer.

Question 6: Let f: [0, 2] → R be a function which is continuous on [0, 2] and its differentiable on (0, 2) with f(0) = 1. Let F(x) =

$\int_0^{x^2} f(\sqrt{t})dt$

for x∈ [0, 2]. If F(x) = f'(x) for all x belongs to (0, 2), then F(2) equals

(a) $e^2 - 1$

(b) $e^4 - 1$

(c) $e - 1$

(d) e^4

Answer: b

Solution:

F(x) =

$$\int_0^{x^2} f(\sqrt{t})dt$$

F'(x) = f(x) 2x

F'(x) = f'(x) for all x belongs to (0, 2)

=> f(x) 2x = f'(x)

=> f'(x)/f(x) = 2x

On integrating w.r.t.x,we get

ln f(x) = x^2 + c

f(x) =

$$e^{x^2+c}$$

=

$$e^{x^2}.e^c$$

Since f(0) = 1

=> e^c = 1

=> f(x) =

$$e^{x^2}$$

Hence F(x) =

$$\int_0^{x^2} e^x dx$$

=

$$e^{x^2} - 1$$

F(2) = e^4 - 1

Hence option b is the answer.

Question 7: If f(x) = cos x cos 2x cos 4x cos 8x cos 16x, then f '(π/4) is equal to

(a) √2

(b) 1/√2

(c) 0

(d) √3/2

Answer: a

Solution:

f(x) = cos x cos 2x cos 4x cos 8x cos 16x

Multiply and divide by 2 sin x

f(x) = (2 sin x cos x cos 2x cos 4x cos 8x cos 16x)/2 sin x

= 2 sin 2x cos 2x cos 4x cos 8x cos 16x)/2^2 sin x

= 2 sin 4x cos 4x cos 8x cos 16x)/2^3 sin x

= 2 sin 8x cos 8x cos 16x)/2^4 sin x

= 2 sin 16 x cos 16x)/2^5 sin x

= sin 32x/2^5 sin x

f'(x) = (1/2^5)[32 sin x cos 32 x - sin 32x cos x]/$\sin^2$ x

f '(π/4) = (1/2^5)[32 sin π/4 cos 8π - sin 8π cos π/4]/$\sin^2$ π/4

= (1/2^5)[(32×1/√2 - 0)/(½)]

= √2

Hence option (a) is the answer.

Question 8: If f(x) = [x] - [x/4], x∈R, where [x] denotes the greatest integer function, then

(a) f is continuous at x = 4

(b) $\lim_{x\to 4+}$ f(x) exists but $\lim_{x\to 4-}$ f(x) does not exist

(c) Both $\lim_{x\to 4-}$f(x) and $\lim_{x\to 4+}$f(x) exist but are not equal

(d) $\lim_{x\to 4-}$ f(x) exists but $\lim_{x\to 4+}$ f(x) does not exist

Answer: a

Solution:

$\lim_{x\to 4-}$ ([x] - [x/4]) = 3-0 = 3

$\lim_{x\to 4+}$ ([x] - [x/4]) = 4-1 = 3

RHL = LHL

f(4) = [4] - [4/4]

= 3

LHL = RHL = f(4)

So f(x) is continuous at x = 4.

Hence option a is the answer.

Question 9: Let f be a composite function of x defined by f(u) = $1/(u^2+u-2)$, u(x) = 1/(x-1). Then the number of points x where f is discontinuous is

(a) 4

(b) 3

(c) 2

(d) 1

Answer: b

Solution:

Given f(u) = $1/(u^2+u-2)$

= 1/(u+2)(u-1), is discontinuous at x = -2, 1

u(x) = 1/(x-1), is discontinuous at x = 1

When u = -2,

Then 1/(x-1) = -2

=> x = ½

When u = 1, then 1/(x-1) = 1

=> x = 2

Hence the given composite function is discontinuous at three points x = 1, ½ and 2.

Hence option b is the answer.