Question 1: Let [t] denote the greatest integer ≤ t and $\lim_{x\to 0} x[4/x] = A$. Then the function, $f(x) = [x^2] \sin \pi x$ is discontinuous, when x is equal to

(a) $\sqrt{(A+1)}$

(b) $\sqrt{(A+5)}$

(c) $\sqrt{(A+21)}$

(d) $\sqrt{A}$

Answer: a

Solution:

$\lim_{x\to 0} x[4/x] = A$

$\lim_{x\to 0} x[4/x - \{4/x\}] = A$

=> $\lim_{x\to 0} 4 - x\{4/x\} = A$

=> $4-0 = A$

As, $f(x) = [x^2]\sin(\pi x)$ will be discontinuous at non-integers

And when $x = \sqrt{(A+1)}$

=> $x = \sqrt{5}$, which is not an integer.

Hence f(x) is discontinuous when x is equal to $\sqrt{(A+1)}$.

Hence option a is the answer.

Question 2: Let f be a twice differentiable function such that $g'(x) = -f(x)$ and $f'(x) = g(x)$, $h(x) = \{f(x)\}^2 + \{g(x)\}^2$. If $h(5) = 11$, then $h(10)$ is equal to

(a) 22

(b) 11

(c) 0

(d) 20

Answer: b

Solution:

Given g'(x) = -f(x) and f'(x) = g(x)

$h(x) = \{f(x)\}^2 + \{g(x)\}^2$

Differentiate w.r.t x

h'(x) = 2f(x) f'(x) + 2g(x)g'(x)

= 2f(x) g(x) + 2g(x)(-f(x))

= 0

So h(x) is a constant.

h(5) = 11

So h(10) = 11

Hence option (b) is the answer.

Question 3: A differentiable function f(x) is defined for all x>0, and satisfies $f(x^3) = 4x^4$, for all x> 0. The value of f'(8) is

(a) 16/3

(b) 32/3

(c) 16√2/3

(d) 32√2/3

Answer: b

Solution:

$f(x^3) = 4x^4$

Differentiate w.r.t.x

$f'(x^3)\, 3x^2 = 16x^3$

$f'(x^3) = 16x/3$

To find f'(8) put x = 2

$f'(2^3) = 16 \times 2/3$

$= 32/3$

Hence option (b) is the answer.

Question 4: Let f: R→ R and g: R→ R be two non constant differentiable functions. If $f'(x) = (e^{(f(x) - g(x))}) g'(x)$ for all x ∈R, and f(1) = g(2) = 1, then which of the following statement (s) is (are) true?

(a) $f(2) < 1 - \log_e 2$

(b) $f(2) > 1 - \log_e 2$

(c) $g(1) > 1 - \log_e 2$

(d) $g(1) < 1 - \log_e 2$

Answer: b, c

Solution:

Given $f'(x) = (e^{(f(x) - g(x))}) g'(x)$

$\Rightarrow e^{-f(x)} f'(x) = e^{-g(x)} g'(x)$

Integrating both sides, we get

$-e^{-f(x)} = -e^{-g(x)} + c$

$\Rightarrow -e^{-f(x)} + e^{-g(x)} = c$

$\Rightarrow -e^{-f(1)} + e^{-g(1)} = -e^{-f(2)} + e^{-g(2)}$

So $-e^{-1} + e^{-g(1)} = -e^{-f(2)} + e^{-1}$ (Since f(1) = g(2) = 1)

$e^{-f(2)} + e^{-g(1)} = 2/e$

$\Rightarrow e^{-f(2)} < 2/e$ and $e^{-g(1)} < 2/e$

$\Rightarrow -f(2) < \ln 2 - 1$ and $-g(1) < \ln 2 - 1$

$\Rightarrow f(2) > 1 - \ln 2$ and $g(1) > 1 - \ln 2$

Question 5: Let a, b ∈R and f: R→ R be defined by $f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$. Then f is

(a) differentiable at x= 0 if a = 0 and b = 1

(b) differentiable at x= 1 if a = 1 and b = 0

(c) not differentiable at x= 0 if a = 1 and b = 0

(d) not differentiable at x= 1 if a = 1 and b = 1

Answer: a, b

Solution:

$f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$

If a = 0, b = 1

=> $f(x) = |x| \sin(|x^3 + x|)$

= $x \sin(x^3 + x)$, which is differentiable everywhere.

If a = 1, b = 0

=> $f(x) = \cos(|x^3 - x|)$

= $\cos(x^3-x)$, which is differentiable everywhere.

When a = 1, b = 1, $f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$, which is differentiable at x = 1.

Question 6: The derivative of $\tan^{-1}(\sqrt{(1+x^2)} - 1)/x)$ with respect to $\tan^{-1}(2x\sqrt{(1-x^2)}/(1-2x^2))$ at x = 1/2 is

(a) 2√3/5

(b) √3/12

(c) 2√3/3

(d) √3/10

Answer: d

Solution:

Let $u = \tan^{-1}((\sqrt{(1+x^2)} - 1)/x)$

Put $x = \tan\theta$

=> $\theta = \tan^{-1} x$

So $u = \tan^{-1}(\sec\theta - 1)/\tan\theta$

$= \tan^{-1} \tan (\theta/2)$

$= \theta/2$

$= ½ \tan^{-1} x$

$du/dx = ½ (1/(1+x^2))$

Let $v = \tan^{-1}(2x\sqrt{(1-x^2)}/(1-2x^2))$

Put $x = \sin \varphi$

$\varphi = \sin^{-1}x$

$v = \tan^{-1}(2 \sin \varphi \cos \varphi/\cos 2\varphi)$

$= \tan^{-1} \tan 2\varphi$

$= 2\varphi$

$= 2\sin^{-1}x$

$dv/dx = 2/\sqrt{(1-x^2)}$

$du/dv = (du/dx)/(dv/dx)$

$= \sqrt{(1-x^2)}/4(1+x^2)$

du/dv at $x = ½ = \sqrt{3}/10$

Hence option d is the answer.

Question 7: If $(a +\sqrt{2}\ b \cos x)(a - \sqrt{2}\ b \cos y) = a^2-b^2$, where $a>b>0$, then dx/dy at $(\pi/4, \pi/4)$ is

(a) (a-2b)/(a+2b)

(b) (a-b)/(a+b)

(c) (a+b)/(a-b)

(d) (2a+b)/(2a-b)

Answer: c

Solution:

$(a+\sqrt{2}\ b \cos x)(a-\sqrt{2}\ b \cos y) = a^2 - b^2$

Differentiating both sides

$-\sqrt{2}b \sin x(a-\sqrt{2}b \cos y) + (a+\sqrt{2}b \cos x)\sqrt{2}b \sin y\ (dy/dx) = 0$

$dy/dx = (\sqrt{2}b \sin x)(a-\sqrt{2}\ b \cos y)/(a+\sqrt{2}\ b \cos x)\sqrt{2}b \sin y$

So dy/dx at (π/4, π/4) = (a-b)/(a+b)

Therefore dx/dy = (a+b)/(a-b).

Hence option c is the answer.

Question 8: Let f: [0, 2] → R be a function which is continuous on [0, 2] and its differentiable on (0, 2) with f(0) = 1. Let F(x) =

$$\int_0^{x^2} f(\sqrt{t})dt$$

for x∈ [0, 2]. If F(x) = f'(x) for all x belongs to (0, 2), then F(2) equals

(a) $e^2 - 1$

(b) $e^4 - 1$

(c) e - 1

(d) e^4

Answer: b

Solution:

F(x) =

$$\int_0^{x^2} f(\sqrt{t})dt$$

F'(x) = f(x) 2x

F'(x) = f'(x) for all x belongs to (0, 2)

=> f(x) 2x = f'(x)

=> f'(x)/f(x) = 2x

On integrating w.r.t.x,we get

ln f(x) = $x^2 + c$

f(x) =

$$e^{x^2+c}$$

=

$$e^{x^2}.e^c$$

Since f(0) = 1

=> e^c = 1

=> f(x) =

$$e^{x^2}$$

Hence F(x) =

$$\int_0^{x^2} e^x dx$$

=

$$e^{x^2} - 1$$

F(2) = e^4 - 1

Hence option b is the answer.

Question 9: If y is a function of x and log(x+y)-2xy = 0, then the value of y'(0) is equal to

(a) 1

(b) -1

(c) 2

(d) 0

Answer: a

Solution:

log(x+y) - 2xy = 0

When x = 0, y = 1

Differentiating w.r.t.x

[1/(x+y)](1+ dy/dx) -2y - 2x dy/dx = 0

=> dy/dx = ([1/(x+y)] - 2y)/(2x - 1/(x+y))

y'(0) = (1-2)/(0-1)

= 1

Hence option a is the answer.

Question 10: If y = $(\sin x)^{\tan x}$, then dy/dx is equal to

(a) $(\sin x)^{\tan x}(1 + \sec^2 x \log \sin x)$

(b) $\tan x(\sin x)^{\tan x - 1} \cos x$

(c) $(\sin x)^{\tan x} \sec^2 x \log \sin x$

(d) $\tan x (\sin x)^{\tan x - 1}$

Answer: a

Solution:

Given y = $(\sin x)^{\tan x}$

log y = tan x log sin x

Differentiate w.r.t.x

(1/y) dy/dx = $\sec^2 x$ log sin x + tan x .(1/sin x) cos x

dy/dx = $(\sin x)^{\tan x}$ [1+ $\sec^2 x$ log sin x]

Hence option a is the answer.