Question 1: Let a_1, a_2, a_3,...be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n<0$ is

(a) 22

(b) 23

(c) 24

(d) 25

Solution:

Given a_1, a_2, a_3,…are in HP.

Then $1/a_1, 1/a_2, 1/a_3$ are in AP.

Take d as the common difference,

Given $a_1 = 5$ and $a_{20} = 25$

$1/a_{20} = 1/a_1 + 19d$

$1/a_{20} - 1/a_1 = 19d$

$1/25 - 1/5 = 19d$

$-4/25 = 19d$

$d = -4/25\times19$

$a+(n-1)d < 0$

$\Rightarrow (1/5) + (n-1)(-4/25\times19) < 0$

$4(n-1)/19\times5 > 1$

$n-1 > 19\times5/4$

$n > 19\times5/4 + 1$

$n = 25$

So the least positive value of n is 25.

Hence option d is the answer.

Question 2: Let a, b, c be positive integers such that b/a is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is b+2, then the value of $(a^2 + a - 14)/(a+1)$ is

Solution:

Given that a, b, c are in geometric progression.

Let a, b, c be a, ar, ar^2 (where r is the common ratio)

Given (a+b+c)/3 = b+2

=> $(a + ar + ar^2) = 3(ar) + 6$ (since b = ar)

=> $ar^2 - 2ar + a = 6$

=> $a(r^2 - 2r + 1) = 6$

=> $a(r-1)^2 = 6$

=> $(r-1)^2 = 6/a$

6/a should be a perfect square.

So the possible value for a is 6.

$(a^2 + a - 14)/(a+1) = (36 + 6 - 14)/7$

= 28/7

= 4

Hence the value of $(a^2 + a - 14)/(a+1)$ is 4.

Question 3: A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller to the numbers on the removed card is k, then k-20 equals

(a) 5

(b) 10

(c) 15

(d) 50

Solution:

Sum of n natural numbers = n(n+1)/2

n(n+1)/2 - (k+k+1) = 1224

n(n+1)/2 - 2k = 1225

$n^2 + n - 2450 = 4k$

=> $k = (n+50)(n-49)/4$

The smaller to the numbers on the removed card is k.

So $1 \leq k \leq n-1$

=> $1 \leq (n+50)(n-49)/4 \leq n-1$

$49 < n < 51$

=> $n = 50$

=> $k = 100/4 = 25$

$k-20 = 25-20 = 5$

Hence option a is the answer.

Question 4: The sides of the right-angled triangle are in AP. If the triangle has an area 24, then what is the length of its smallest side?

Solution:

Let a-d, a, a+d be the sides of the right angled triangle.

Using pythagoras theorem, we can write

$(a+d)^2 = a^2 + (a-d)^2$

$a^2+2ad+d^2 = a^2 + a^2- 2ad+d^2$

$a^2-4ad = 0$

$a(a-4d) = 0$

=> $a = 4d$ or $a = 0$ (rejected)

=> $a = 4d$ ..(i)

Area of triangle = ½ bh

$24 = ½ (a-d)a$

$48 = 3d \times 4d$

$48/12 = d^2$

=> $d^2 = 4$

=> $d = 2$

So a = 8

Length of smallest side = a-d

= 8-2

= 6

Hence the length of smallest side is 6 units.

Question 5: If the sum of the first n terms of an AP is cn^2, then the sum of squares of these n terms is

(a) $n(4n^2-1)c^2/6$

(b) $n(4n^2+1)c^2/3$

(c) $n(4n^2-1)c^2/3$

(d) $n(4n^2+1)c^2/6$

Solution:

Let the sum of the first n terms of an AP be S_n.

Given $S_n = cn^2$

$S_{n-1} = c(n-1)^2$

$T_n = S_n - S_{n-1}$

$= cn^2 - c(n-1)^2$

$= cn^2 - cn^2 + 2cn - c$

$= c(2n-1)$

Sum of squares = $\sum Tn^2$

$= c^2[4\sum n^2 + \sum 1 - 4\sum n]$

$= [4c^2(n(n+1)(2n+1)/6] + nc^2 - (2c^2n(n+1))$

$= nc^2(4n^2 + 6n + 2 + 3 - 6n - 6)/3$

$= nc^2(4n^2-1)/3$

$= n(4n^2-1)c^2/3$

Hence option (c) is the answer.

Question 6: The third term of a geometric progression is 4. The product of the first five terms is

(a) 4^3

(b) 4^4

(c) 4^5

(d) 4

Solution:

Let a be the first term and r be the common ratio.

Third term = $ar^2 = 4$

Product of first five terms = $a(ar)(ar^2)(ar^3)(ar^4)$

$= a^5r^{10}$

$= (ar^2)^5$

$= 4^5$

Hence option c is the answer.

Question 7: Let X be the set consisting of the first 2018 terms of an arithmetic progression, 1, 6, 11...., and Y be the set consisting of the first 2018 terms of arithmetic progression 9, 16, 23, ... Then, the number of elements in the set (X ∪ Y) is

Solution:

X = {1, 6, 11, …}

Here a = 1, d = 5

We know $t_n = a + (n-1)d$

2018^{th} term = 1 + 2017×5

= 10086

Y = { 9, 16, 23, ..}

Here a = 9, d = 7

2018^{th} term = 9 + 2017×7

= 14128

X∩Y = {16, 51, 86, …}

$10086 \geq 16 + (n-1)35$

=> $(n - 1)35 \leq 10070$

=> $n \leq 288.71$

=> $n = 288$

The number of elements in the set $X \cup Y$ is $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$= 2018 + 2018 - 288$

$= 3748$

The number of elements in the set $(X \cup Y)$ is 3748.

Question 8: The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} with $a > 0$ is

(a) 9

(b) 8

(c) 2

(d) 1

Solution:

Since $a>0$, a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} are positive.

So AM $\geq$ GM

=> $(1/8)\,[a^{-5}+ a^{-4}+ a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}] \geq [a^{-5} \times a^{-4} \times a^{-3} \times a^{-3} \times a^{-3} \times 1 \times a^8 \times a^{10}]^{1/8}$

=> $(1/8)\,[a^{-5}+ a^{-4}+ a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}] \geq 1^{1/8}$

$[a^{-5}+ a^{-4}+ a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}] \geq 8$

The minimum value of the sum is 8.

Hence option b is the answer.

Question 9: Let $S_n = \sum_{k=1}^{4n} (-1)^{k(k+1)/2} k^2$. Then S_n can take values

(a) 1056

(b) 1088

(c) 1120

(d) 1332

Solution:

$S_n = \sum_{k=1}^{4n} (-1)^{k(k+1)/2} k^2 = -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 - \ldots + (4n)^2$

$= (-1^2 + 3^2 - 5^2 + 7^2 - \ldots + (4n-1)^2) + (-2^2 + 4^2 - 6^2 + 8^2 - .. + (4n^2)$

= 2(4 + 12 + 20+…(8n-4)) + 2(6 + 14 + 22+…(8n-2)) (Sum of n terms in A.P)

= n(8n) + n(8n+4) … (Use equation for sum of terms of A.P)

= n(16n+4)

Put n = 8, S_n = 1056.

Put n = 9, S_n = 1332

Hence option (a) and (d) are correct.

Question 10: If m arithmetic means (A.Ms) and three geometric means (G.Ms) are inserted between 3 and 243 such that 4th A.M is equal to 2nd G.M, then m is equal to

Solution:

Let the AP be 3, a_1, a_2 ...a_m, 243.

Common difference, d = (243-3)/(m+1)

= 240/(m+1)

Let 3, G_1, G_2, G_3, 243 be the GP.

Let r be the common ratio.

$3r^4 = 243$

=> r^4 = 243/3 = 81

=> r = 3

4th A.M is equal to 2nd G.M

$G_2 = a_4$

$ar^2 = a + 4d$

=>27 = 3 + 4×240/(m+1)

=> 24 = 960/(m+1)

=>960/24 = 1/(m+1)

=> 1/40 = 1/(m+1)

$\Rightarrow m+1 = 40$

So $m = 39$

Hence the value of m is 39.

Question 11: Let $b_i > 1$ for i = 1,2, ...101. Suppose $\log_e b_1, \log_e b_2, ...\log_e b_{10}$ are in AP with the common difference log 2. Suppose $a_1, a_2, ..a_{101}$ are in AP such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + …+ b_{51}$ and $s = a_1 + a_2 + …+ a_{51}$ then

(a) $s>t$ and $a_{101}>b_{101}$

(b) $s>t$ and $a_{101}<b_{101}$

(c) $s<t$ and $a_{101}>b_{101}$

(d) $s<t$ and $a_{101}<b_{101}$

Solution:

Given $\log_e b_1, \log_e b_2, ...\log_e b_{10}$ are in AP.

$\log_e b_2 - \log_e b_1 = \log_e 2$

$\log_e (b_2/b_1) = \log_e 2$

$\Rightarrow (b_2/b_1) = 2$

Similarly $b_3/b_2 = 2$

$b_1, b_2, ...b_{101}$ are in GP with common ratio $r = 2$.

$b_n = 2^{n-1}b_1$ …(i)

Given $a_1, a_2, ..a_{101}$ are in AP.

$t = b_1 + b_2 + …+ b_{51}$

$s = a_1 + a_2 + …+ a_{51}$

t = sum of 51 terms of G.P

$= b_1(r^{51}-1)/(r-1)$

$= b_1(2^{51}-1)$

s = sum of 51 terms of A.P

= (n/2)(first term+ last term)

$= (51/2)(a_1 + a_{51})$

$= (51/2)(b_1 + b_{51})$

$= (51/2)(b_1 + 2^{50}b_1)$

$= b_1(51/2)(1 + 2^{50})$

It is clear that s> t.

Given $a_{51} = b_{51}$amd $a_1 = b_1$.

$a_{51} = a_1 +50d$

$b_{51} = b_1 + 50d$

=> $d = (b_{51} - b_1)/50$

$a_{101} = a_1+100d$

$= b_1+ 100 (b_{51} - b_1)/50$

$= b_1+ 2(b_{51} - b_1)$

$= 2b_{51} - b_1$

$= 2 \times 2^{50}b_1 - b_1$ (from (i))

$= b_1(2^{51} -1)$

$b_{101} = 2^{100}b_1$

It is clear that $2^{100}b_1 > b_1(2^{51} -1)$

=> $b_{101} > a_{101}$

Hence option b is the answer.

Question 12: The harmonic mean and geometric mean of two positive numbers be the ratio 4:5. Then the two numbers are in the ratio

(a) 1 : 4

(b) 4 : 1

(c) 3 : 4

(d) 4 : 3

Solution:

Let a and b be the two positive numbers.

HM = 2ab/(a+b)

GM = √(ab)

HM/GM = 4/5

2ab/(a+b)÷√(ab) = 4/5

=> 2√(ab)/(a+b) = 4/5

√ab/(a+b) = 2/5

Cross multiply

5√ab = 2(a+b)

Squaring both sides

$25ab = 4(a+b)^2$

$25ab = 4(a^2+2ab+b^2)$

$4a^2 - 17ab + 4b^2 = 0$

(a-4b) (4a-b) = 0

=> (a-4b) = 0 or (4a-b) = 0

=> a/b = 4/1 or a/b = ¼

Hence option a and option b are correct.

Question 13: If the sum of the first 40 terms of the series, 3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + …. is (102)m, then m is equal to

(a) 20

(b) 5

(c) 10

(d) 25

Solution:

Given 3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + …. = 102m

=> (3 + 4) + (8 + 9) + (13 + 14) + (18 + 19) + …. = 102m

=> 7 + 17 + 27 + ...20 terms = 102m

Here a = 7, d = 10, n = 20

Sum of n terms of AP = (n/2)(2a+(n-1)d)

= 10(14 + 190)

= 2040

= 102 m

So m = 2040/102

= 20

Hence option a is the answer.

Question 14: The number of terms in an AP is even; the sum of the odd terms in it is 24 and that the even terms is 30. If the last term exceeds the first term by

$10\frac{1}{2}$ **then the number of terms in the AP is:**

(a) 4

(b) 8

(c) 12

(d) 16

Solution:

Let 2n be the number of terms in the AP.

Sum of odd terms = $(a_1 + a_3 + ...a_{2n-1}) = 24$ …(i)

Sum of even terms = $(a_2 + a_4 + ...a_{2n}) = 30$…(ii)

$a_{2n} = a_1 + (2n-1)d$

$a_{2n} - a_1 = 2nd - d$

$10\frac{1}{2}$ = 2nd - d

21/2 = 2nd - d… (iii)

(ii) - (i)

$a_2 - a_1 + a_4 - a_3 + ...a_{2n} - a_{2n-1} = 30-24$

d + d + …. n times = 6

nd = 6 …(iv)

Substitute (iv) in (iii)

21/2 = 2× 6 - d

12 - d = 21/2

=> d = (24 - 21)/2

= 3/2

Put d in (iv)

3n/2 = 6

=>n = 12/3 = 4

So 2n = 8

Hence option b is the answer.

Question 15: The value of $1^2 + 3^2 + 5^2 + \ldots + 25^2$ is

(a) 2925

(b) 1469

(c) 1728

(d) 1456

Solution:

$1^2 + 3^2 + 5^2 + \ldots + 25^2 = (1^2 + 2^2 + 3^2 + \ldots + 25^2) - (2^2 + 4^2 + \ldots + 24^2)$

$= (1^2 + 2^2 + 3^2 + \ldots + 25^2) - 2^2(1^2 + 2^2 + \ldots + 12^2)$

= 25×26×51/6 - 4×12×13×25/6

= 5525 - 2600

= 2925

Hence option a is the answer.