PART-III : MATHEMATICS

SECTION – 1 (Maximum marks : 24)

- This section contains **EIGHT (08)** questions.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 **ONLY** if the correct numerical value is entered;

Zero Marks : 0 In all other cases.

1. Considering only the principal values of the inverse trigonometric functions, the value of

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}+\frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2}+\tan^{-1}\frac{\sqrt{2}}{\pi}$$

is ______.

Answer ($\simeq 2.36$)

Sol. $\frac{3}{2}\tan^{-1}\frac{\pi}{\sqrt{2}}+\frac{1}{4}\tan^{-1}\left(\frac{2\sqrt{2}\pi}{\pi^2-2}\right)+\tan^{-1}\frac{\sqrt{2}}{\pi}$

$$=\frac{\pi}{2}+\frac{1}{2}\tan^{-1}\frac{\pi}{\sqrt{2}}-\frac{1}{4}\tan^{-1}\left(\frac{2\sqrt{2}\pi}{2-\pi^2}\right)$$

$$=\frac{\pi}{2}+\frac{1}{2}\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)-\frac{1}{4}\tan^{-1}\left(\frac{2\cdot\left(\frac{\pi}{\sqrt{2}}\right)}{1-\left(\frac{\pi}{\sqrt{2}}\right)^2}\right)$$

$$=\frac{\pi}{2}+\frac{1}{2}\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)-\frac{1}{4}\left(-\pi+2\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)\right)$$

$$=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$$

$$\simeq 2.36$$

2. Let α be a positive real number. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:(\alpha,\infty)\to\mathbb{R}$ be the functions defined by

$$f(x)=\sin\left(\frac{\pi x}{12}\right) \text{ and } g(x)=\frac{2\log_e(\sqrt{x}-\sqrt{\alpha})}{\log_e(e^{\sqrt{x}}-e^{\sqrt{\alpha}})}.$$

Then the value of $\lim\limits_{x\to\alpha^+} f(g(x))$ is ______.

Answer (00.50)

Sol. $$\lim_{x\to\alpha^+} g(x)=\lim_{x\to\alpha^+}\frac{\dfrac{2}{\sqrt{x}-\sqrt{\alpha}}\left(\dfrac{1}{2\sqrt{x}}\right)}{\dfrac{1}{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}\left(\dfrac{1}{2\sqrt{x}}\cdot e^{\sqrt{x}}\right)}$$

$$=\lim_{x\to\alpha^+}\frac{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}{\sqrt{x}-\sqrt{\alpha}}\cdot\frac{1}{e^{\sqrt{x}}}\cdot 2$$

$$=\lim_{x\to\alpha^+}\frac{e^{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}}}{\dfrac{1}{2\sqrt{x}}}\cdot\frac{2}{e^{\sqrt{x}}}=2$$

$$\lim_{x\to\alpha^+} f(g(x))=f\left(\lim_{x\to\alpha^+} g(x)\right)=\sin\frac{\pi}{6}=\frac{1}{2}=00.50$$

3. In a study about a pandemic, data of 900 persons was collected. It was found that

190 persons had symptom of fever,

220 persons had symptom of cough,

220 persons had symptom of breathing problem,

330 persons had symptom of fever or cough or both,

350 persons had symptom of cough or breathing problem or both,

340 persons had symptom of fever or breathing problem or both,

30 persons had all three symptoms (fever, cough and breathing problem).

If a person is chosen randomly from these 900 persons, then the probability that the person has at most one symptom is ____________.

Answer (0.8)

Sol. We denote the set of people having symptoms of fever, cough and breathing problem by F, C and B respectively.

Given that n(F) = 190, n(B) = 220 and n(C) = 220

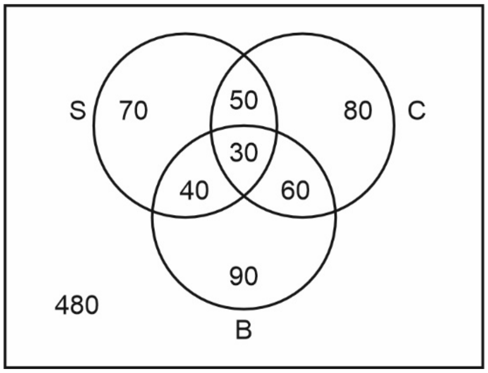
Also, $n(F\cup C)$ = 330, $n(C\cup B)$ = 350, $n(F\cup B)$ = 340 and $n(F\cap C\cap B)$ = 30

So $n(F\cap C)=n(F)+n(C)-n(F\cup C)$

$= 80$

Similarly, $n(F\cap B)=70$ and $n(C\cap B)=90$

So refer to Venn diagram



Number of people having at most one symptom

= 70 + 80 + 90 + 480 = 720

Required probability $= \frac{720}{900} = 0.8$.

4. Let z be a complex number with non-zero imaginary part. If

$$\frac{2+3z+4z^2}{2-3z+4z^2}$$

is a real number, then the value of $|z|^2$ is ________.

Answer (0.50)

Sol. Let $w = \frac{4z^2+3z+2}{4z^2-3z+2} = 1 + \frac{6z}{4z^2-3z+2}$

$$\Rightarrow \quad w = 1 + \frac{6}{2\left(2z+\frac{1}{z}\right)-3}$$

$$\because \quad w \in R \text{ then } 2z + \frac{1}{z} \in R$$

$$\Rightarrow \quad 2z + \frac{1}{z} = 2\bar{z} + \frac{1}{\bar{z}}$$

$$\Rightarrow \quad 2(z-\bar{z}) - \frac{z-\bar{z}}{|z|^2} = 0$$

$$\Rightarrow \quad (z-\bar{z})\left(2 - \frac{1}{|z|^2}\right) = 0$$

$\because \quad z \neq \bar{z}$ (given)

So $|z|^2 = \frac{1}{2}$

5. Let $\bar{z}$ denote the complex conjugate of a complex number z and let $i = \sqrt{-1}$. In the set of complex numbers, the number of distinct roots of the equation

$$\bar{z} - z^2 = i(\bar{z} + z^2)$$

is ________.

Answer (4)

Sol. Given $\bar{z}(1-i) = z^2(1+i)$ ...(i)

So $|\bar{z}||1-i| = |z|^2|1+i|$

$\Rightarrow$ $|z| = |z|^2$ $\Rightarrow$ $|z| = 0$ or $|z| = 1$

Let arg $(z) = \theta$

So from (i) we get

$$2n\pi - \theta - \frac{\pi}{4} = 2\theta + \frac{\pi}{4}$$

$$\Rightarrow \quad \theta = \frac{1}{3}\left(\frac{4n-1}{2}\right)\pi = \frac{(4n-1)\pi}{6}$$

So we will get 3 distinct values of θ. Hence there will be total 4 possible values of complex number z.

6. Let $l_1, l_2, \ldots, l_{100}$ be consecutive terms of an arithmetic progression with common difference d_1, and let $w_1, w_2, \ldots, w_{100}$ be consecutive terms of another arithmetic progression with common difference d_2, where $d_1d_2 = 10$. For each $i = 1, 2, \ldots, 100$, let R_i be a rectangle with length l_i, width w_i and area A_i. If $A_{51} - A_{50} = 1000$, then the value of $A_{100} - A_{90}$ is ____________.

Answer (18900)

Sol. For A.P. $l_1, l_2, \ldots l_{100}$

Let $T_1 = a$ and common difference $= d_1$ and similarly for A.P. $w_1, w_2, \ldots w_{100}$

$T_1 = b$ and common difference $= d_2$

$A_{51} - A_{50} = l_{51}w_{51} - l_{50}w_{50}$

$= (a + 50d_1)(b + 50d_2) - (a + 49d_1)(b + 49d_2)$

$= 50bd_1 + 50ad_2 + 2500d_1d_2 - 49ad_2 - 49bd_1 - 2401d_1d_2$

$= bd_1 + ad_2 + 99d_1d_2 = 1000$

$\therefore$ $bd_1 + ad_2 = 10$ ...(i) (As $d_1d_2 = 10$)

$\therefore$ $A_{100} - A_{90} = l_{100}w_{100} - l_{90}w_{90}$

$= (a + 99d_1)(b + 99d_2) - (a + 89d_1)(b + 89d_2)$

$= 99bd_1 + 99ad_2 + 99^2d_1d_2 - 89bd_1 - 89ad_2 - 89^2d_1d_2$

$= 10(bd_1 + ad_2) + 1880d_1d_2$

$= 10(10) + 18800$

$= 18900$

7. The number of 4-digit integers in the closed interval [2022, 4482] formed by using the digits 0, 2, 3, 4, 6, 7 is ______.

Answer (569)

Sol. Counting integers starting from 2

Case-I: if zero on 2nd place

i.e., $2\,0\,\underline{2}\,\underset{5}{\uparrow}$ $\rightarrow$ 5 cases

or $2\,0\,\underset{4}{\uparrow}\,\underset{6}{\uparrow}$ $\rightarrow$ 24 cases

(Numbers except 0 or 2 in 3rd place)

Case-II: If non-zero number on 2nd place

i.e., $2\,\underset{5}{\uparrow}\,\underset{6}{\uparrow}\,\underset{6}{\uparrow}$ = 180 cases

Counting integers starting from 3

$\underline{3}\ \underset{6}{\uparrow}\ \underset{6}{\uparrow}\ \underset{6}{\uparrow}$ = 216 cases

Counting integers starting from 4

Case-I: If 0, 2 or 3 on 2^{nd} place

i.e., $4\ \underset{3}{\uparrow}\ \underset{6}{\uparrow}\ \underset{6}{\uparrow}$ = 108 cases

Case II: If 4 on 2^{nd} place

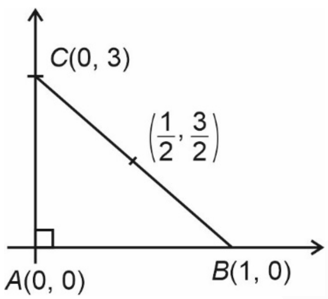
i.e., $4\ 4\ \underset{6}{\uparrow}\ \underset{6}{\uparrow}$ = 36 cases

$\therefore$ Total 5 + 24 + 180 + 216 + 108 + 36 = 569 numbers

8. Let ABC be the triangle with $AB = 1$, $AC = 3$ and $\angle BAC = \frac{\pi}{2}$. If a circle of radius $r > 0$ touches the sides AB, AC and also touches internally the circumcircle of the triangle ABC, then the value of r is ____________.

Answer (0.84)

Sol. Let A be the origin B on x-axis, C on y-axis as shown below



$\therefore$ Equation of circumcircle is

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2=\frac{5}{2} \quad \ldots(1)$$

Required circle touches AB and AC, have radius r

$\therefore$ Equation be $(x-r)^2+(y-r)^2=r^2 \quad \ldots(2)$

If circle in equation (2) touches circumcircle internally, we have

$$d_{c_1c_2}=|r_1-r_2|$$

$$\Rightarrow \left(\frac{1}{2}-r\right)^2+\left(\frac{3}{2}-r\right)^2=\left(\left|\sqrt{\frac{5}{2}}-r\right|\right)^2$$

$$\Rightarrow \frac{1}{4}+r^2-r+\frac{9}{4}+r^2-3r=\left(\sqrt{\frac{5}{2}}-r\right)^2 \text{ or } \left(r-\sqrt{\frac{5}{2}}\right)^2$$

$$\Rightarrow 2r^2-4r+\frac{5}{2}=\frac{5}{2}+r^2-\sqrt{10}\,r$$

$$\Rightarrow r=0 \text{ or } 4-\sqrt{10}$$

$\Rightarrow r = 0.837$

= 0.84 (on rounding off)

SECTION – 2 (Maximum marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	**ONLY** if (all) the correct option(s) is(are) chosen;
Partial Marks	:	+3	If all the four options are correct but **ONLY** three options are chosen;
Partial Marks	:	+ 2	If three or more options are correct but **ONLY** two options are chosen, both of which are correct;
Partial Marks	:	+1	If two or more options are correct but **ONLY** one option is chosen and it is a correct option;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	–2	In all other cases.

9. Consider the equation

$$\int_1^e \frac{(\log_e x)^{1/2}}{x(a-(\log_e x)^{3/2})^2}\,dx = 1,\ a \in (-\infty, 0) \cup (1, \infty).$$

Which of the following statements is/are TRUE?

(A) No a satisfies the above equation

(B) An integer a satisfies the above equation

(C) An irrational number a satisfies the above equation

(D) More than one a satisfy the above equation

Answer (C, D)

Sol. Let $I = \int_1^e \frac{(\ln x)^{1/2}dx}{x(a-(\ln x)^{3/2})^2}$

Put $a - (\ln x)^{3/2} = t$

$\Rightarrow\ -\frac{3}{2}(\ln x)^{1/2}\cdot\frac{1}{x}dx = dt$

$\therefore\ I = \int_a^{a-1} \frac{\left(-\frac{2}{3}\right)dt}{t^2}$

$= \left(-\frac{2}{3}\right)\frac{t^{-2+1}}{-2+1}\Big|_a^{a-1}$

$= \frac{2}{3t}\Big|_a^{a-1} = \frac{2}{3}\left(\frac{1}{a-1}-\frac{1}{a}\right)$

$\therefore\ I = \left(\frac{2}{3}\right)\frac{1}{a(a-1)} = 1$

$\Rightarrow\ 2 = 3a^2 - 3a$

$\Rightarrow\ 3a^2 - 3a - 2 = 0$

$\Rightarrow\ a = \frac{3 \pm \sqrt{9-4(3)(-2)}}{6}$

$a = \frac{3+\sqrt{33}}{6}, \frac{3-\sqrt{33}}{6}$

10. Let a_1, a_2, a_3,… be an arithmetic progression with $a_1 = 7$ and common difference 8. Let T_1, T_2, T_3, … be such that $T_1 = 3$ and $T_{n+1} - T_n = a_n$ fo $n \geq 1$. Then, which of the following is/are TRUE ?

(A) $T_{20} = 1604$

(B) $\sum_{k=1}^{20} T_k = 10510$

(C) $T_{30} = 3454$

(D) $\sum_{k=1}^{30} T_k = 35610$

Answer (B, C)

Sol. Here $a_n = 7 + (n - 1)\,8$ and $T_1 = 3$

Also $T_{n+1} = T_n + a_n$

$T_n = T_{n-1} + a_{n-1}$

$\vdots$

$T_2 = T_1 + a_1$

$\therefore \quad T_{n+1} = (T_{n-1} + a_{n-1}) + a_n$

$= T_{n-2} + a_{n-2} + a_{n-1} + a_n$

$\vdots$

$\Rightarrow \quad T_{n+1} = T_1 + a_1 + a_2 + \ldots. + a_n$

$\Rightarrow \quad T_{n+1} = T_1 + \frac{n}{2}\left[2(7) + (n-1)8\right]$

$\Rightarrow \quad T_{n+1} = T_1 + n(4n+3) \quad \ldots (1)$

$\therefore$ For $n = 19$ $\quad T_{20} = 3 + (19)(79) = 1504$

For $n = 29$ $\quad T_{30} = 3 + (29)(119) = 3454 \rightarrow$ (C)

$$\sum_{k=1}^{20} T_k = 3 + \sum_{k=2}^{20} T_k = 3 + \sum_{k=1}^{19}\left(3 + 4n^2 + 3n\right)$$

$$= 3 + 3(19) + \frac{3(19)(20)}{2} + \frac{4(19)(20)(39)}{6}$$

$= 3 + 10507 = 10510 \rightarrow$ (B)

And Similarly $\sum_{k=1}^{30} T_k = 3 + \sum_{k=1}^{29}\left(4n^2 + 3n + 3\right) = 35615$

11. Let P_1 and P_2 be two planes given by

P_1: $10x + 15y + 12z - 60 = 0$,

P_2: $-2x + 5y + 4z - 20 = 0$.

Which of the following straight lines can be an edge of some tetrahedron whose two faces lie on P_1 and P_2?

(A) $\frac{x-1}{0} = \frac{y-1}{0} = \frac{z-1}{5}$

(B) $\frac{x-6}{-5} = \frac{y}{2} = \frac{z}{3}$

(C) $\frac{x}{-2} = \frac{y-4}{5} = \frac{z}{4}$

(D) $\frac{x}{1} = \frac{y-4}{-2} = \frac{z}{3}$

Answer (A, B)

Sol. Equation of pair of planes is

S: $(10x + 15y + 12z - 60)(-2x + 5y + 4z - 20) = 0$

We will find a general point of each line and we will solve it with S. If we get more than one value of variable λ, then the line can be the edge of given tetrahedron.

(A) Point is (1, 1, 5λ + 1)

So, (60λ – 23) (20λ – 17) = 0

$\lambda = \frac{23}{60}$ and $\frac{17}{20}$

So, it can be the edge of tetrahedron.

(B) Point is (–5λ + 6, 2λ, 3λ)

So, (16λ) (32λ – 32) = 0

⇒ λ = 0 and 1

So, it can be the edge of tetrahedron.

(C) Point is (–2λ, 5λ + 4, 4λ)

So, (103λ) (45λ) = 0

λ = 0 only

So, it cannot be the edge of tetrahedron.

(D) Point is (λ, –2λ + 4, 3λ)

⇒ (16λ) (–2λ) = 0

⇒ λ = 0 only

Hence, it cannot be the edge of tetrahedron.

12. Let *S* be the reflection of a point *Q* with respect to the plane given by

$\vec{r} = -(t+p)\hat{i} + t\hat{j} + (1+p)\hat{k}$

where *t*, *p* are real parameters and $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the three positive coordinate axes. If the position vectors of *Q* and *S* are $10\hat{i} + 15\hat{j} + 20\hat{k}$ and $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ respectively, then which of the following is/are TRUE ?

(A) 3(α + β) = –101

(B) 3(β + γ) = –71

(C) 3(γ + α) = –86

(D) 3(α + β + γ) = –121

Answer (A, B, C)

Sol. Equation of plane is

$$\vec{r} = -(t+p)\hat{i} + t\hat{j} + (1+p)\hat{k}$$

$$\vec{r} = \hat{k} + t(-\hat{i} + \hat{j}) + p(-\hat{i} + \hat{k})$$

Equation of plane in standard form is

$$\left[\vec{r} - \hat{k} \quad -\hat{i} + \hat{j} \quad -\hat{i} + \hat{k}\right] = 0$$

∴ $x + y + z = 1$ ...(1)

Coordinate of *Q* = (10, 15, 20)

Coordinate of *S* = (α, β, γ)

$$\therefore \quad \frac{\alpha - 10}{1} = \frac{\beta - 15}{1} = \frac{\gamma - 20}{1} = \frac{-2(10 + 15 + 20 - 1)}{3}$$

$$\therefore \quad \alpha - 10 = \beta - 15 = \gamma - 20 = -\frac{88}{3}$$

$$\therefore \quad \alpha = -\frac{58}{3},\ \beta = -\frac{43}{3},\ \gamma = -\frac{28}{3}$$

$\therefore$ $3(\alpha + \beta) = -101$, $3(\beta + \gamma) = -71$

$3(\gamma + \alpha) = -86$ and $3(\alpha + \beta + \gamma) = -129$

$\therefore$ Ans. A, B, C

13. Consider the parabola $y^2 = 4x$. Let S be the focus of the parabola. A pair of tangents drawn to the parabola from the point $P = (-2, 1)$ meet the parabola at P_1 and P_2. Let Q_1 and Q_2 be points on the lines SP_1 and SP_2 respectively such that PQ_1 is perpendicular to SP_1 and PQ_2 is perpendicular to SP_2. Then, which of the following is/are TRUE ?

(A) $SQ_1 = 2$

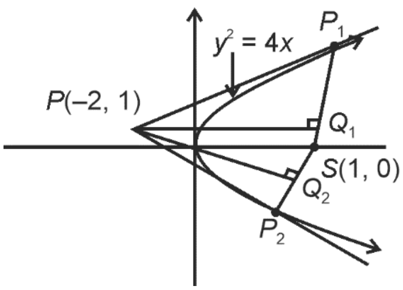
(B) $Q_1Q_2 = \frac{3\sqrt{10}}{5}$

(C) $PQ_1 = 3$

(D) $SQ_2 = 1$

Answer (A, B, C, D)

Sol.



Let $P_1(t^2, 2t)$ then tangent at P_1

$ty = x + t^2$

Since it passes through $(-2, 1)$

$\therefore$ $t^2 - t - 2 = 0$

$\therefore$ $t = 2, -1$

$\therefore$ $P_1(4, 4)$ and $P_2(1, -2)$

$\therefore$ $SP_1 : 4x - 3y - 4 = 0$

and $SP_2 : x - 1 = 0$

and for Q_1 : $\frac{x_1+2}{4} = \frac{y_1-1}{-3} = \frac{-(-8-3-4)}{25} = \frac{3}{5}$

$\therefore$ $x_1 = \frac{2}{5}, y_1 = \frac{-4}{5}$

and $Q_2 = (1, 1)$

So, $SQ_1 = \sqrt{\left(1-\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$

$Q_1Q_2 = \sqrt{\frac{9}{25} + \frac{81}{25}} = \sqrt{\frac{90}{25}} = \frac{3\sqrt{10}}{5}$

$PQ_1 = \sqrt{\frac{144}{25} + \frac{81}{25}} = 3$

$SQ_2 = 1$

14. Let $|M|$ denote the determinant of a square matrix M. Let $g:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}$ be the function defined by

$g(\theta)=\sqrt{f(\theta)-1}+\sqrt{f\left(\frac{\pi}{2}-\theta\right)-1}$ where

$$f(\theta)=\frac{1}{2}\begin{vmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{vmatrix}+\begin{vmatrix}\sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right)\\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi\end{vmatrix}.$$

Let p(x) be a quadratic polynomial whose roots are the maximum and minimum values of the function $g(\theta)$, and $p(2)=2-\sqrt{2}$. Then, which of the following is/are TRUE ?

(A) $p\left(\frac{3+\sqrt{2}}{4}\right)<0$

(B) $p\left(\frac{1+3\sqrt{2}}{4}\right)>0$

(C) $p\left(\frac{5\sqrt{2}-1}{4}\right)>0$

(D) $p\left(\frac{5-\sqrt{2}}{4}\right)<0$

Answer (A and C)

Sol. $\because$ $$f(\theta)=\frac{1}{2}\begin{vmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{vmatrix}+\begin{vmatrix}\sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right)\\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi\end{vmatrix}.$$

Here $\cos\left(\theta+\frac{\pi}{4}\right)=-\sin\left(\theta-\frac{\pi}{4}\right)$

and $\tan\left(\theta-\frac{\pi}{4}\right)=-\cot\left(\theta+\frac{\pi}{4}\right)$

and $\log_e\left(\frac{4}{\pi}\right)=-\log_e\left(\frac{\pi}{4}\right)$

Also $\sin\pi=-\cos\frac{\pi}{2}=\tan\pi=0$

$\therefore$ $f(\theta)=1+\sin^2\theta$

$\therefore$ $g(\theta)=|\sin\theta|+|\cos\theta|$

$\therefore$ maximum and minimum values are $\sqrt{2}$ and 1 respectively.

$\therefore$ $P(x)=a(x-\sqrt{2})(x-1)$, where $a\in R-\{0\}$,

But $P(2)=2-\sqrt{2}$ then $a=1$.

$\therefore$ $P(x)=(x-\sqrt{2})(x-1)$

$\therefore$ $P\left(\frac{3+\sqrt{2}}{4}\right)=\left(\frac{3-3\sqrt{2}}{4}\right)\cdot\left(\frac{\sqrt{2}-1}{4}\right)<0$

$P\left(\frac{1+3\sqrt{2}}{4}\right)=\left(\frac{1-\sqrt{2}}{4}\right)\cdot\left(\frac{3\sqrt{2}-3}{4}\right)<0$

$$P\left(\frac{5\sqrt{2}-1}{4}\right)=\left(\frac{\sqrt{2}-1}{4}\right)\cdot\left(\frac{5\sqrt{2}-5}{4}\right)>0$$

$$P\left(\frac{5-\sqrt{2}}{4}\right)=\left(\frac{5-5\sqrt{2}}{4}\right)\left(\frac{1-\sqrt{2}}{4}\right)>0$$

$\therefore$ (A) and (C) are correct.

SECTION – 3 (Maximum marks : 12)

- This section contains **FOUR (04)** Matching List Sets.
- Each set has **ONE** Multiple Choice Question.
- Each set has **TWO** lists: **List-I** and **List-II**.
- **List-I** has **Four** entries (I), (II), (III) and (IV) and **List-II** has **Five** entries (P), (Q), (R), (S) and (T).
- FOUR options are given in each Multiple Choice Question based on **List-I** and **List-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3	**ONLY** if the option corresponding to the correct combination is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	−1	In all other cases.

15. Consider the following lists:

List-I	**List-II**
(I) $\left\{x\in\left[-\frac{2\pi}{3},\frac{2\pi}{3}\right]:\cos x+\sin x=1\right\}$	(P) has two elements
(II) $\left\{x\in\left[-\frac{5\pi}{18},\frac{5\pi}{18}\right]:\sqrt{3}\tan 3x=1\right\}$	(Q) has three elements
(III) $\left\{x\in\left[-\frac{6\pi}{5},\frac{6\pi}{5}\right]:2\cos(2x)=\sqrt{3}\right\}$	(R) has four elements
(IV) $\left\{x\in\left[-\frac{7\pi}{4},\frac{7\pi}{4}\right]:\sin x-\cos x=1\right\}$	(S) has five elements
	(T) has six elements

The correct option is:

(A) (I) → (P); (II) → (S); (III) → (P); (IV) → (S)

(B) (I) → (P); (II) → (P); (III) → (T); (IV) → (R)

(C) (I) → (Q); (II) → (P); (III) → (T); (IV) → (S)

(D) (I) → (Q); (II) → (S); (III) → (P); (IV) → (R)

Answer (B)

Sol. (i) $\left\{x\in\left[\frac{-2\pi}{3},\frac{2\pi}{3}\right],\cos x+\sin x=1\right\}$

$\cos x + \sin x = 1$

$$\sin\left(\frac{\pi}{4}+x\right)=\frac{1}{\sqrt{2}}$$

$$\frac{\pi}{4}+x=n\pi+(-1)^n\frac{\pi}{4}$$

$$x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$$

$\therefore$ x has 2 elements. $\to P$

(ii) $\left\{x \in \left[\frac{-5\pi}{18}, \frac{5\pi}{18}\right] : \sqrt{3}\tan 3x = 1\right\}$

$\sqrt{3}\tan 3x = 1$

$\tan 3x = \frac{1}{\sqrt{3}}$

$3x = n\pi + \frac{\pi}{6}$

$x = \frac{n\pi}{3} + \frac{\pi}{18}$

$\therefore$ x has 2 elements. $\rightarrow P$

(iii) $\left\{x \in \left[\frac{-6\pi}{5}, \frac{6\pi}{5}\right] : 2\cos 2x = \sqrt{3}\right\}$

$2\cos 2x = \sqrt{3}$

$\cos 2x = \frac{\sqrt{3}}{2}$

$2x = 2n\pi \pm \frac{\pi}{6}$

$x = n\pi \pm \frac{\pi}{12}$

$\therefore$ x has 6 elements. $\rightarrow T$

(iv) $\left\{x \in \left[\frac{-7\pi}{4}, \frac{7\pi}{4}\right] : \sin x - \cos x = 1\right\}$

$\sin x - \cos x = 1$

$\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$x - \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4}$

$x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4}$

$\therefore$ x has 4 elements. $\rightarrow R$

$\therefore$ option B is correct.

16. Two players, P_1 and P_2, play a game against each other. In every round of the game, each player rolls a fair die once, where the six faces of the die have six distinct numbers. Let x and y denote the readings on the die rolled by P_1 and P_2, respectively. If $x > y$, then P_1 scores 5 points and P_2 scores 0 point. If $x = y$, then each player scores 2 points. If $x < y$, then P_1 scores 0 point and P_2 scores 5 points. Let X_i and Y_i be the total scores of P_1 and P_2, respectively, after playing the i^{th} round.

	List-I		**List-II**
(I)	Probability of ($X_2 \geq Y_2$) is	(P)	$\frac{3}{8}$
(II)	Probability of ($X_2 > Y_2$) is	(Q)	$\frac{11}{16}$
(III)	Probability of ($X_3 = Y_3$) is	(R)	$\frac{5}{16}$
(IV)	Probability of ($X_3 > Y_3$) is	(S)	$\frac{355}{864}$
		(T)	$\frac{77}{432}$

The correct option is:

(A) (I) → (Q); (II) → (R); (III) → (T); (IV) → (S)

(B) (I) → (Q); (II) → (R); (III) → (T); (IV) → (T)

(C) (I) → (P); (II) → (R); (III) → (Q); (IV) → (S)

(D) (I) → (P); (II) → (R); (III) → (Q); (IV) → (T)

Answer (A)

Sol. $P(X_i > Y_i) + P(X_i < Y_i) + P(X_i = Y_i) = 1$

and $P(X_i > Y_i) = P(X_i < Y_i) = p$

for $i = 2$

$P(X_2 = Y_2) = 2p(x > y).\ p(x < y) + (p(x = y))^2$

$$= 2.\frac{{}^6C_2}{36}.\frac{{}^6C_2}{36} + \left(\frac{{}^6C_1}{36}\right)^2$$

$$= \frac{25}{72} + \frac{1}{36} = \frac{27}{72} = \frac{3}{8}$$

$$P(X_2 > Y_2) = \frac{1}{2}\left(1 - \frac{3}{8}\right) = \frac{5}{16}$$

$$P(X_2 \geq Y_2) = \frac{5}{16} + \frac{3}{8} = \frac{11}{16}$$

I → Q, II → R

for $i = 3$

$P(X_3 = Y_3) = 6.\ p(x > y).\ p(x < y)\ p(x = y) + (p(x = y))^3$

$$= 6.\frac{{}^6C_2}{36}.\frac{{}^6C_2}{36}.\frac{{}^6C_1}{36} + \left(\frac{{}^6C_1}{36}\right)^3$$

$$= \frac{77}{432}$$

$$P(X_3 > Y_3) = \frac{1}{2}\left(1 - \frac{77}{432}\right)$$

$$= \frac{355}{864}$$

III → T, IV → S

17. Let p, q, r be non-zero real numbers that are, respectively, the 10th, 100th and 1000th terms of a harmonic progression. Consider the system of linear equations

$x + y + z = 1$

$10x + 100y + 1000z = 0$

$qr\,x + pr\,y + pq\,z = 0$

	List-I		List-II
(I)	If $\frac{q}{r} = 10$, then the system of linear equations has	(P)	$x = 0,\ y = \frac{10}{9},\ z = -\frac{1}{9}$ as a solution
(II)	If $\frac{p}{r} \neq 100$, then the system of linear equations has	(Q)	$x = \frac{10}{9},\ y = -\frac{1}{9},\ z = 0$ as a solution
(III)	If $\frac{p}{q} \neq 10$, then the system of linear equations has	(R)	infinitely many solutions
(IV)	If $\frac{p}{q} = 10$, then the system of linear equations has	(S)	no solution
		(T)	at least one solution

The **correct** option is:

(A) (I) → (T); (II) → (R); (III) → (S); (IV) → (T)

(B) (I) → (Q); (II) → (S); (III) → (S); (IV) → (R)

(C) (I) → (Q); (II) → (R); (III) → (P); (IV) → (R)

(D) (I) → (T); (II) → (S); (III) → (P); (IV) → (T)

Answer (B)

Sol. $x + y + z = 1$ ___(1)

$10x + 100y + 1000z = 0$ ___(2)

$qrx + pry + pqz = 0$ ___(3)

Equation (3) can be re-written as

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 0 \qquad (\because p, q, r \neq 0)$$

Let $p = \frac{1}{a+9d},\ q = \frac{1}{a+99d},\ r = \frac{1}{a+999d}$

Now, equation (3) is

$(a + 9d)x + (a + 99d)y + (a + 999d)z = 0$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 100 & 1000 \\ a+9d & a+99d & a+999d \end{vmatrix} = 0$$

$$\Delta_x = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 100 & 1000 \\ 0 & a+99d & a+999d \end{vmatrix} = 900(d-a)$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 0 & 1000 \\ a+9d & 0 & a+999d \end{vmatrix} = 990(a-d)$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 100 & 0 \\ a+9d & a+99d & 0 \end{vmatrix} = 90(d-a)$$

Option I: If $\frac{q}{r} = 10 \Rightarrow a = d$

$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$

And eq. (1) and eq. (2) represents non-parallel planes and eq. (2) and eq. (3) represents same plane

$\Rightarrow$ Infinitely many solutions

I $\rightarrow$ *P*, *Q*, *R*, *T*

Option II : $\frac{p}{r} \neq 100 \Rightarrow a \neq d$

$\Delta = 0$, $\Delta_x, \Delta_y, \Delta_z \neq 0$

No solution

II $\rightarrow$ S

Option III: $\frac{p}{q} \neq 10 \Rightarrow a \neq d$

No solution

III $\rightarrow$ S

Option IV: If $\frac{p}{q} = 10 \Rightarrow a = d$

Infinitely many solution

IV $\rightarrow$ *P*, *Q*, *R*, *T*

18. Consider the ellipse

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Let H(α, 0), 0< α < 2, be a point. A straight line drawn through *H* parallel to the *y*-axis crosses the ellipse and its auxiliary circle at points *E* and *F* respectively, in the first quadrant. The tangent to the ellipse at the point *E* intersects the positive *x*-axis at a point *G*. Suppose the straight line joining *F* and the origin makes an angle ϕ with the positive *x*-axis.

	List-I		List-II
(I)	If $\phi = \frac{\pi}{4}$, then the area of the triangle *FGH* is	(P)	$\frac{(\sqrt{3}-1)^4}{8}$
(II)	If $\phi = \frac{\pi}{3}$, then the area of the triangle *FGH* is	(Q)	1
(III)	If $\phi = \frac{\pi}{6}$, then the area of the triangle *FGH* is	(R)	$\frac{3}{4}$
(IV)	If $\phi = \frac{\pi}{12}$, then the area of the triangle *FGH* is	(S)	$\frac{1}{2\sqrt{3}}$
		(T)	$\frac{3\sqrt{3}}{2}$

The correct option is:

(A) (I) → (R); (II) → (S); (III) → (Q); (IV) → (P)

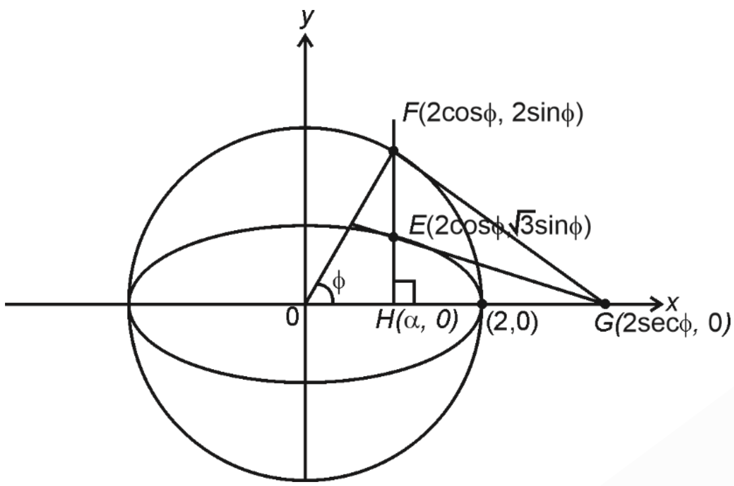
(B) (I) → (R); (II) → (T); (III) → (S); (IV) → (P)

(C) (I) → (Q); (II) → (T); (III) → (S); (IV) → (P)

(D) (I) → (Q); (II) → (S); (III) → (Q); (IV) → (P)

Answer (C)

Sol.



$\alpha \equiv 2\cos\phi$

Tangent at $E\left(2\cos\phi, \sqrt{3}\sin\phi\right)$ to ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$

i.e. $\frac{x\cos\phi}{2}+\frac{y\sin\phi}{\sqrt{3}}=1$ intersect x-axis at $G(2\sec\phi, 0)$

Area of triangle FGH $=\frac{1}{2}(2\sec\phi-2\cos\phi)2\sin\phi$

$\Delta = 2\sin^2\phi.\tan\phi$

$\Delta = (1-\cos 2\phi).\tan\phi$

I. If $\phi=\frac{\pi}{4}, \Delta = 1 \rightarrow (Q)$

II. If $\phi=\frac{\pi}{3}, \Delta = 2.\left(\frac{\sqrt{3}}{2}\right)^2.\sqrt{3}=\frac{3\sqrt{3}}{2}\rightarrow (T)$

III. If $\phi=\frac{\pi}{6}, \Delta = 2.\left(\frac{1}{2}\right)^2.\frac{1}{\sqrt{3}}=\frac{1}{2\sqrt{3}}\rightarrow (S)$

IV. If $\phi=\frac{\pi}{12}, \Delta = \left(1-\frac{\sqrt{3}}{2}\right).\left(2-\sqrt{3}\right)=\frac{\left(2-\sqrt{3}\right)^2}{2}\rightarrow (P)$

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