



- 2) (i) p is T, q is F, r is F      (ii) p is T, q is F, r is F  
(iii) p is T, q is T, r is F      (iv) p is T, q is T, r is F, s is T

3) (i)

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(ii)

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim q$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

(iii)

p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

(iv)

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

(v)

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(vi)

p	q	$p \vee q$	$q \wedge p$	$(p \vee q) \leftrightarrow (q \wedge p)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(vii)

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(viii)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \leftrightarrow (p \wedge \sim q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	F	F

4) True

**Five mark questions:**

1) (i)

p	q	r	$\sim r$	$q \wedge \sim r$	$p \vee (q \wedge \sim r)$
T	T	T	F	F	T
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	T	F	F

(ii)

p	q	r	$\sim p$	$\sim r$	$\sim p \wedge q$	$(\sim p \wedge q) \wedge \sim r$
T	T	T	F	F	F	F
T	T	F	F	T	F	F
T	F	T	F	F	F	F
T	F	F	F	T	F	F
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	T	T	F	F
F	T	T	T	F	F	F

(iii)

p	q	r	$\sim q$	$\textcircled{x}$ $p \wedge q$	$\textcircled{y}$ $r \vee \sim q$	$\textcircled{x} \rightarrow \textcircled{y}$
T	T	T	F	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	T	F	F	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

(iv)

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

(v)

p	q	r	$\sim r$	$p \rightarrow q$	$(p \rightarrow q) \rightarrow \sim r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	T	F	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	T	F
F	F	F	T	T	T

(vi)

p	q	r	$\textcircled{x}$ $p \rightarrow r$	$\textcircled{y}$ $p \rightarrow q$	$\textcircled{x} \wedge \textcircled{y}$
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	T	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

(vii)

p	q	r	$q \wedge r$	$p \rightarrow (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

(viii)

p	q	r	$\sim q$	$p \rightarrow \sim q$	$(p \rightarrow \sim q) \wedge r$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	T	F

## 6.4 Tautology and Contradiction

**Definition 1 :** A compound proposition is said to be a Tautology if it is true for all possible combinations of the truth values of its components.

**Definition 2 :** A compound proposition is said to be a contradiction if it is false for all possible combinations of the truth values of its components.

### WORKED EXAMPLES

#### Example 7 :

Show that the proposition  $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$  is a Tautology.

**Solution:**

		(A)		(B)	
p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$	$(A) \leftrightarrow (B)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

The last column indicates that the given proposition is a Tautology.

#### Example 8:

Show that the proposition  $(p \wedge q) \wedge \sim (p \vee q)$  is a contradiction.

**Solution:**

		(A)		(B)	
p	q	$p \wedge q$	$p \vee q$	$\sim (p \vee q)$	$(A) \wedge (B)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

The last column indicates that the given proposition is a contradiction.

#### Example 9 :

Verify whether the proposition  $(p \wedge \sim q) \wedge (\sim p \vee q)$  is a contradiction or not.

**Solution:**

			(A)		(B)	
p	q	$\sim q$	$p \wedge \sim q$	$\sim p$	$\sim p \vee q$	$A \wedge B$
T	T	F	F	F	T	F
T	F	T	T	F	F	F
F	T	F	F	T	T	F
F	F	T	F	T	T	F

Since the last column is all 'F', the given proposition is a contradiction.

**Example 10: Prove that  $[p \vee (p \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)]$  is a tautology.**

**Solution:**

p	q	r	$q \wedge r$	$\textcircled{A}$ $p \vee (q \wedge r)$	$\textcircled{x}$ $p \vee q$	$\textcircled{y}$ $p \vee r$	$\textcircled{B}$ $\textcircled{x} \wedge \textcircled{y}$	$\textcircled{A} \leftrightarrow \textcircled{B}$
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

The last column indicates that the given proposition is a Tautology.

**Example 11: Find whether  $p \rightarrow (\sim p \vee q)$  is a tautology or a contradiction.**

**Solution:**

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow (\sim p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The last column suggests that the proposition is neither a Tautology nor a contradiction.

## 6.5 Logical Equivalence

Two compound propositions 'p' and 'q' involving the same components are said to be logically equivalent, if their truth values are the same for each different combinations of the truth values of their components. We then write ' $p \equiv q$ ' (read as 'p is logically equivalent to q').

Some of the important logically equivalent propositions are as follows:

- 1) Idempotent Laws : (i)  $p \vee p \equiv p$  (ii)  $p \wedge p \equiv p$
- 2) Law of double negation :  $\sim(\sim p) \equiv p$
- 3) Commutative Laws : (i)  $p \vee q \equiv q \vee p$  (ii)  $p \wedge q \equiv q \wedge p$
- 4) Associative Laws : (i)  $p \vee (q \vee r) \equiv (p \vee q) \vee r$  (ii)  $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
- 5) Distributive Laws : (i)  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$   
(ii)  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- 6) De'Morgans Laws : (i)  $\sim(p \wedge q) \equiv \sim p \vee \sim q$  (ii)  $\sim(p \vee q) \equiv \sim p \wedge \sim q$
- 7)  $\sim(p \rightarrow q) \equiv p \wedge \sim q$
- 8)  $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

All the above results may be verified by constructing their respective truth tables.

### WORKED EXAMPLES

**Example 12 :** Show that (i)  $\sim (p \wedge q) \equiv \sim p \vee \sim q$       (ii)  $\sim (p \vee q) \equiv \sim p \wedge \sim q$   
**(Verification of De'Morgans Laws)**

**Solution:**

(i)

1	2	3	4	5	6	7
p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

4th and 7th columns are identical.

$$\therefore \sim (p \wedge q) \equiv \sim p \vee \sim q$$

**Solution:**

(ii)

1	2	3	4	5	6	7
p	q	$p \vee q$	$\sim (p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

4th and 7th columns are identical.

$$\therefore \sim (p \vee q) \equiv \sim p \wedge \sim q$$

**Example 13 :**

**Prove :**  $\sim (p \rightarrow q) \equiv p \wedge \sim q$

**Note :** This result is useful in writing the negation of the given implication.

**Solution:**

1	2	3	4	5	6
p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Columns 4 and 6 are identical. Hence proved.

**Example 14:**

**Prove :  $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$**

**Solution:**

1	2	3	4	5	6	7	8
p	q	r	$q \wedge r$	$p \rightarrow (q \wedge r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Columns 5 and 8 are identical, proving the logical equivalence of the given propositions.

**Example 15 :**

**Negate the following propositions:**

**(i)  $\sim p \vee q$       (ii)  $p \rightarrow (q \wedge r)$       (iii)  $p \wedge \sim q$       (iv)  $p \rightarrow \sim q$**

**Solution:**

$$(i) \sim(\sim p \vee q) \equiv \sim(\sim p) \wedge \sim q \text{ [using De'Morgans law]}$$

$$\equiv p \wedge \sim q \text{ [using } \sim(\sim p) = p]$$

$$\therefore \sim(\sim p \vee \sim q) \equiv p \wedge \sim q$$

$$(ii) \sim[p \rightarrow (q \wedge r)] \equiv p \wedge \sim(q \wedge r) \text{ [using } \sim(p \rightarrow q) \equiv p \wedge \sim q]$$

$$\equiv p \wedge (\sim q \vee \sim r) \text{ [using De'Morgan's laws]}$$

$$\therefore \sim[p \rightarrow (q \wedge r)] \equiv p \wedge (\sim q \vee \sim r)$$

$$(iii) \sim(p \wedge \sim q) \equiv \sim p \vee \sim(\sim q) \text{ [using De'Morgan's law]}$$

$$\equiv \sim p \vee q \text{ [using } \sim(\sim p) \equiv p]$$

$$\therefore \sim(p \wedge \sim q) \equiv \sim p \vee q$$

$$(iv) \sim(p \rightarrow \sim q) \equiv p \wedge \sim(\sim q) \text{ [using } \sim(p \rightarrow q) \equiv p \wedge \sim q]$$

$$\equiv p \wedge q \text{ [using } \sim(\sim p) \equiv p]$$

$$\therefore \sim(p \rightarrow \sim q) \equiv p \wedge q$$

**Example 16:**

**Negate the proposition:**

**“cow is big and it is black”.**

**Solution:**

Let  $p$  : cow is big

$q$  : cow is black

Given proposition is “ $p \wedge q$ ”

we know that:  $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Therefore, negation of the given proposition is “cow is not big or it is not black”.

**Example 17:**

**Negate the proposition “If the number is real then it is either rational or irrational”**

**Solution:**

Let  $p$  : The number is real.

$q$  : The number is rational.

$r$  : The number is irrational.

Then, given proposition is  $p \rightarrow (q \vee r)$

$\sim [p \rightarrow (q \vee r)] \equiv p \wedge (\sim q \wedge \sim r)$

Thus, the negation of the given proposition is “A number is real and it is not rational and not irrational.”

This may also be stated as “A number is real but it is neither rational nor irrational”.

**Example 18:**

**Negate : “If he is rich then he is happy”.**

**Solution:**

Let  $p$  : He is rich,  $q$  : He is happy.

we are given :  $p \rightarrow q$

$\sim (p \rightarrow q) \equiv p \wedge \sim q$

we therefore get the negation as: “He is rich and he is not happy”.

**Example 19:**

**Negate : “6 is an even number or  $\sqrt{5}$  is not rational”.**

**Solution:**

Let  $p$  : 6 is an even number.

$q$  :  $\sqrt{5}$  is rational.

The given proposition is  $p \vee \sim q$

Then,  $\sim(p \vee \sim q) \equiv \sim p \wedge q$

$\therefore$  the negation of the given statement is “6 is not an even number and  $\sqrt{5}$  is rational”.

**Example 20 :**

**Negate :** “A triangle is equiangular if and only if the corresponding sides are equal”.

**Solution:**

Consider  $p$  : A triangle is equiangular.

$q$  : It's corresponding sides are equal.

we are given  $p \leftrightarrow q$

we know that  $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$ . Hence, the negation of the proposition becomes “a triangle is equiangular and the corresponding sides are not equal or the triangle is not equiangular and the corresponding sides are equal”.

**6.6 Converse, Inverse and Contrapositive of a Conditional:**

Given a conditional  $p \rightarrow q$ ,

- (i) the conditional  $q \rightarrow p$  is called the converse of  $p \rightarrow q$
- (ii) the conditional  $\sim p \rightarrow \sim q$  is called the inverse of  $p \rightarrow q$
- (iii) the conditional  $\sim q \rightarrow \sim p$  is called the contrapositive of  $p \rightarrow q$

It may easily be verified that  $q \rightarrow p \equiv \sim p \rightarrow \sim q$  and  $p \rightarrow q \equiv \sim q \rightarrow \sim p$

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**WORKED EXAMPLES**

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**Example 21 :**

**Write the converse, inverse and contrapositive of “If I work hard then I get the grade.**

**Solution:**

$p$  : I work hard

$q$  : I get the grade

$p \rightarrow q$  is the given proposition.

**converse :**  $q \rightarrow p$

*ie.*, If I get the grade then I work hard.

**inverse :**  $\sim p \rightarrow \sim q$

*ie.*, If I do not work hard then I do not get the grade.

**contrapositive :**  $\sim q \rightarrow \sim p$

*ie.*, If I do not get the grade then I do not work hard.

**Example 22 :**

**Write the converse, inverse and contrapositive of the implication “If  $x \in (A \cup B)$  then  $x \in A$  or  $x \in B$ ”**

**Solution:**

Let  $p : x \in A \cup B$

$q : x \in A$

$r : x \in B$

$p \rightarrow (q \vee r)$  is the given proposition.

**converse :**  $(q \vee r) \rightarrow p$

ie., If  $x \in A$  or  $x \in B$  then  $x \in (A \cup B)$

**inverse :**  $\sim p \rightarrow \sim (q \vee r) \equiv \sim p \rightarrow (\sim q \wedge \sim r)$

ie., If  $x \notin (A \cup B)$  then  $x \notin A$  and  $x \notin B$

**contrapositive :**  $\sim (q \vee r) \rightarrow \sim p \equiv (\sim q \wedge \sim r) \rightarrow \sim p$

ie., If  $x \notin A$  and  $x \notin B$  then  $x \notin (A \cup B)$ .

**Example 23 :**

**Write the converse, inverse and contrapositive of “If  $x$  is less than 1 then it is a prime number”.**

**Solution:**

$p : x$  is less than 1

$q : x$  is a prime number

The given proposition is  $p \rightarrow q$

**converse :**  $q \rightarrow p$

ie., If  $x$  is a prime number then it is less than 1.

**inverse :**  $\sim p \rightarrow \sim q$

ie., If  $x$  is not less than 1 then it is not a prime number.

**contrapositive :**  $\sim q \rightarrow \sim p$

ie., If  $x$  is not a prime number then it is not less than 1.

**EXERCISE 6.2****One mark questions:**

1) Negate the following propositions:

(i)  $p \vee \sim q$

(ii)  $\sim p \rightarrow q$

(iii)  $\sim p \wedge \sim q$

(iv)  $p \wedge \sim q$

(v)  $\sim p \rightarrow \sim q$

2) Negate the following:

(i) 4 is an even integer or 7 is a prime number.

(ii) He likes to run and he does not like to sit.

(iii) He likes Mathematics and he does not like Logic.

(iv) If 6 is a divisor of 120 then 486 is not divisible by 6.

(v) If 2 triangles are similar then their areas are equal.

(vi) It is cold or it is raining.

**Two marks questions:**

1) Negate :

(i)  $p \rightarrow (q \wedge r)$

(ii)  $q \vee [\sim (p \wedge r)]$

(iii)  $(p \rightarrow q) \wedge (q \rightarrow p)$

(iv)  $p \rightarrow (q \wedge \sim r)$

2) Negate the following:

(i) If an integer is greater than 3 and less than 5 then it is a multiple of 5.

(ii) If 'x' is divisible by 'y' then it is divisible by 'a' and 'b'.

(iii) Weather is fine and my friends are not coming or we do not go to a movie.

(iv) If a triangle is equilateral then its sides are equal and angles are equal.

(v) 14 is a divisor of 48 and 28 is not divisible by 82.

3) Determine whether the following propositions is a Tautology or a contradiction or neither.

(i)  $(p \wedge q) \wedge \sim p$

(ii)  $[\sim p \wedge (p \vee q)]$

(iii)  $(p \wedge q) \rightarrow (p \vee q)$

(iv)  $(p \wedge q) \rightarrow p$

(v)  $\sim p \wedge \sim q$

**Three marks questions:**

1) Write the converse, inverse and contrapositive of the implications given below:

(i) If  $x(x - 2) = 0$  then  $x = 2$

(ii) If  $x \in A \cap B$  then  $x \in A$  and  $x \in B$

(iii) If the questions are easy then students score better marks.

(iv) If I get a seat then I will watch a cinema and have fun.

- (v) If oxygen is a gas then accountancy is easy or the child is brave.  
(vi) If  $x^2 = y^2$  then  $x = y$   
(vii) If 2 straight lines are parallel then they do not intersect.

**Five marks questions:**

- 1) Check whether the following propositions is a Tautology or a contradiction:  
(i)  $(p \wedge \sim q) \rightarrow (p \wedge q)$  (ii)  $[\sim p \wedge (p \vee q)] \rightarrow q$   
(iii)  $(p \rightarrow q) \leftrightarrow (\sim p \rightarrow \sim q)$  (iv)  $[\sim (p \rightarrow \sim q)] \vee (\sim p \leftrightarrow q)$   
(v)  $(\sim p \vee q) \leftrightarrow (p \vee \sim q)$
- 2) Show that  $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$  is a Tautology.
- 3) Show that  $\sim(p \vee q) \rightarrow (\sim p \wedge \sim q)$  is a Tautology.
- 4) Prove :  $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$  is a Tautology.
- 5) Prove that  $(p \vee q) \wedge (\sim p \wedge \sim q)$  is a contradiction.
- 6) Show that  $(\sim p \wedge q) \wedge (q \wedge r) \wedge (\sim q)$  is a contradiction.
- 7) Examine whether the following are logically equivalent:  
(i)  $p \leftrightarrow q$  and  $(p \rightarrow q) \wedge (q \rightarrow p)$  (ii)  $p \rightarrow (q \rightarrow r)$  and  $(p \rightarrow q) \rightarrow r$   
(iii)  $(p \wedge \sim q) \vee q$  and  $p \vee q$  (iv)  $p \leftrightarrow q$  and  $(\sim p \vee q) \wedge (\sim q \vee p)$   
(v)  $p \wedge q$  and  $\sim(p \rightarrow \sim q)$  (vi)  $\sim(p \leftrightarrow q)$  and  $(p \wedge \sim q) \vee (q \wedge \sim p)$   
(vii)  $p \vee (q \wedge r)$  and  $(p \vee q) \wedge (p \vee r)$

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**ANSWERS 6.2**

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**One mark questions:**

- 1) (i)  $\sim p \wedge q$  (ii)  $\sim p \wedge \sim q$  (iii)  $p \vee q$   
(iv)  $\sim p \vee q$  (v)  $\sim p \wedge q$
- 2) (i) 4 is not an even integer and 7 is not a prime number.  
(ii) He does not like to run or he likes to sit.  
(iii) He does not like Mathematics or he likes Logic.  
(iv) 6 is a divisor of 120 and 486 is divisible by 6.  
(v) 2 triangles are similar and their areas are not equal.  
(vi) It is not cold and it is not raining. (It is neither cold nor raining)

**Two marks questions:**

- 1) (i)  $p \wedge (\sim q \vee \sim r)$                       (ii)  $\sim q \wedge (p \wedge r)$   
       (iii)  $(p \wedge \sim q) \vee (q \wedge \sim p)$             (iv)  $p \wedge (\sim q \vee r)$
- 2) (i) An integer is greater than 3 and less than 5 but is not a multiple of 5.  
       (ii) 'x' is divisible by 'y' and it is not divisible by 'a' and 'b'.  
       (iii) Weather i not fine or my friends are coming and we go to a movie.  
       (iv) A triangle is equilateral and it's sides are not equal or angles are not equal.  
           (A triangle is equilateral and neither it's sides nor angles are equal).  
       (v) 14 is not a divisor of 48 or 28 is divisible by 82.
- 3) (i) contradiction                      (ii) neither                      (iii) tautology  
       (iv) tautology                          (v) neither

**Three marks questions:**

- 1) (i) converse : If  $x = 2$  then  $x(x - 2) = 0$   
       inverse : If  $x(x - 2) \neq 0$  then  $x \neq 2$   
       contrapositive: If  $x \neq 2$  then  $x(x - 2) \neq 0$
- (ii) converse : If  $x \in A$  and  $x \in B$  then  $x \in A \cap B$   
       inverse : If  $x \notin A \cap B$  then  $x \notin A$  or  $x \notin B$   
       contrapositive: If  $x \notin A$  or  $x \notin B$  then  $x \notin A \cap B$
- (iii) converse : If the students score better marks then questions are easy.  
       inverse : If the questions are not easy then students do not score better marks.  
       contrapositive: If the students do not score better marks then questions are not easy.
- (iv) converse : If I watch a cinema and have fun then I get a seat.  
       inverse : If I do not get a seat then I will not watch a cinema or not have fun.  
       contrapositive: If I will not watch a cinema or not have fun then I do not get a seat.
- (v) converse : If accountancy is easy or the child is brave then oxygen is a gas.  
       inverse : If oxygen is not a gas then accountancy is not easy and the child is not brave.  
       contrapositive: If accountancy is not easy and the child is not brave then oxygen is not a gas.

- (vi) converse : If  $x = y$  then  $x^2 = y^2$   
inverse : If  $x^2 \neq y^2$  then  $x \neq y$   
contrapositive: If  $x \neq y$  then  $x^2 \neq y^2$
- (vii) converse : If 2 straight lines do not intersect then they are parallel.  
inverse : If 2 straight lines are not parallel then they intersect.  
contrapositive: If 2 straight lines intersect then they are not parallel.

**Five marks questions:**

- 1) (i) Neither (ii) Tautology (iii) neither  
(iv) neither (v) neither
- 7) (i) Logically equivalent (ii) Not Logically equivalent  
(iii) Logically equivalent (iv) Logically equivalent  
(v) Logically equivalent (vi) Logically equivalent  
(vii) Logically equivalent

\* \* \* \* \*



## UNIT II - COMMERCIAL ARITHMETIC

Chapter	Title	No. of Teaching hrs.
7.	RATIOS AND PROPORTIONS	10 hrs
8.	BILL DISCOUNTING	06 hrs
9.	STOCKS AND SHARES	04 hrs
10.	LEARNING CURVE	04 hrs
11.	LINEAR PROGRAMMING PROBLEMS	06 hrs
12.	SALES TAX AND VALUE ADDED TAX	04 hrs
	<b>TOTAL TEACHING HOURS</b>	<b>34 hrs</b>

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## Chapter

# 7

## RATIO AND PROPORTIONS

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### 7.1 Introduction:

Ratio, and proportions are extensively used in many branches of Science like Physics, Chemistry and Mathematics etc.

### 7.2 Ratio

**Ratio is a relationship between two quantities of the same kind** with respect to their magnitude and denotes how many times one of the quantities is contained in the other.

#### Expression of a ratio:

If there are two quantities  $a$  and  $b$ , the relationship between them can be expressed as a fraction  $\frac{a}{b}$  where in the relationship between ' $a$ ' and ' $b$ ' is expressed as 'how many times ' $a$ ' is to ' $b$ ' =  $a : b$

#### Antecedent and Consequent

If  $\frac{p}{q}$  is a ratio then  $p$  is called as **antecedent** and  $q$  is called as the **consequent**.

#### Example 1 :

If  $\frac{2}{3}$  is a ratio. Find the antecedent and consequent.

#### Solution :

If  $\frac{2}{3}$  is a ratio

2 = antecedent

3 = consequent

#### Example 2 :

$x$  gets a salary of ₹15000,  $y$  gets a salary of ₹5000. Find the ratio of their salaries.

#### Solution :

$$\frac{x}{y} = \frac{15000}{5000} = \frac{3}{1} = 3:1$$

## Rules Related to Ratios:

### I. Inverse Ratio (Reciprocal Ratio)

If  $\frac{p}{q}$  is a ratio then  $\frac{q}{p}$  is called as the inverse ratio of  $\frac{p}{q}$ .

#### Example 3 : Find the inverse ratio of 2:3

**Solution :**

Inverse Ratio of 2:3 = 3 :2

#### Example 4 :

**A house consumes 20 kgs of rice and 5 kgs of wheat. Compare the consumption of rice and wheat in the form of the ratio.**

**Solution :**

$$\frac{\text{Rice}}{\text{Wheat}} = \frac{20}{5} = \frac{4}{1} = 4:1$$

#### Example 5 :

**Mr. x completes a job in 2 hours and Mr. y completes the same job in 40 minutes. Find the ratio in time taken.**

**Solution :**

x has taken 2 hrs = 120 minutes

y has taken 40 minutes

$$\frac{x}{y} = \frac{120}{40} = \frac{3}{1} = 3:1$$

### II. Ratio of greater inequality:

A ratio of greater inequality is one in which the value of antecedent is greater than the value of consequent. Thus  $\frac{p}{q}$  would be a ratio of greater inequality if  $p > q$ .

**Ex :** 3:2, 5:4.

### III. Ratio of lesser inequality:

A ratio of lesser inequality is one in which the value of antecedent is lesser than the value of the consequent. Thus  $\frac{p}{q}$  would be a ratio of lesser inequality if  $p < q$  **Ex :** 2:3, 4:5

### IV. Compound ratio:

A compound ratio is one, which is obtained by multiplying the antecedent term with the antecedent term, and the subsequent consequent term with the consequent terms of the given ratios.

**Example 6 :**

**Find the compound ratio of 3:5 and 4:7**

**Solution :**

$$\frac{3}{5} \times \frac{4}{7} = \frac{12}{35} = 12:35$$

**Example 7 :**

**Find the compound ratio of the ratios 1 : 2, 2 : 3 and 3 : 4**

**Solution :**

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4} \text{ or } 1 : 4$$

**Example 8 :**

**Determine the compound ratio of**

**(i) 1:2, 2:3, 3:4 and 4:5**

**(ii) 3:5, 2:7, 1:3 and 4:9**

**(iii)  $6 : \frac{1}{3}, \frac{1}{3} : 3, 5 : 3$  and  $7 : 8$**

**Solution :**

$$(i) \quad \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5} \text{ or } 1:5$$

$$(ii) \quad \frac{3}{5} \times \frac{2}{7} \times \frac{1}{3} \times \frac{4}{9} = \frac{8}{315} \text{ or } 8:315$$

$$(iii) \quad \frac{6}{\frac{1}{3}} \times \frac{\frac{1}{3}}{3} \times \frac{5}{3} \times \frac{7}{8}$$

$$= \frac{18}{1} \times \frac{1}{9} \times \frac{5}{3} \times \frac{7}{8}$$

$$= \frac{35}{12} \text{ or } 35 : 12$$

**V. Duplicate ratio:**

A duplicate ratio is one which is obtained by taking the square of the respective terms of the given ratio. i.e., duplicate ratio of 3 : 4 is  $3^2 : 4^2$  or 9 : 16

### VI. Subduplicate ratio:

A Subduplicate ratio is the one which is obtained by taking the square roots of the respective terms of a given ratio. Thus if the given ratio is 4 : 9, then the sub-duplicate ratio is  $\sqrt{4} : \sqrt{9}$  or 2 : 3

### VII. Triplicate ratio:

A triplicate ratio is the one which is obtained by taking the cubes of the respective terms of a ratio. Thus if the given ratio is 1 : 2 then the triplicate ratio would be  $1^3 : 2^3$  or 1 : 8

### VIII. Subtriplicate ratio:

A Subtriplicate ratio is the one which is obtained by taking the cube roots of the respective terms in the given ratio. Thus if the given ratio is 8 : 27 then its sub-triplicate ratio would be  $\sqrt[3]{8} : \sqrt[3]{27} = 2 : 3$

### IX. Continued ratio:

A continued ratio is the one which is obtained by taking the relation between the magnitude of three or more quantities of the same kind. Thus the ratio of x : y : z and 3 : 4 : 5 are examples of a continued ratio.

For example the monthly incomes of A, B and C are ₹300, ₹500 and ₹800 respectively. The ratio of incomes is 300 : 500 : 800 or 3 : 5 : 8 which is a matter of continued ratio.

#### Example 9 :

Find the duplicate and triplicate ratios of the following:

- (i) 2 : 3      (ii) 5 : 3      (iii) 9 : 4

Solution:

	Given ratio	Duplicate	Triplicate
(i)	2 : 3	4 : 9	8 : 27
(ii)	5 : 3	25 : 9	125 : 27
(iii)	9 : 4	81 : 16	729 : 64

#### Example 10 :

Find the sub-duplicate and sub-triplicate ratio of 1 : 64

Solution:

Given ratio	Sub-duplicate	sub-triplicate
1 : 64	1 : 8	1 : 4

#### Example 11 :

Find the ratio between two numbers such as their sum is 50 and their difference is 8.

**Solution:**

Let the two numbers be  $x$  &  $y$

$$x + y = 50 \quad (1)$$

$$x - y = 8 \quad (2)$$

Adding eq<sup>n</sup> (1) & (2)

$$2x = 58$$

$$\therefore x = \frac{58}{2} = 29$$

substituting in eq<sup>n</sup> (1) we have

$$29 + y = 50$$

$$y = 50 - 29$$

$$y = 21$$

$$x : y = 29 : 21$$

**Example 12 :**

**A ratio in the lowest terms is 3 : 7. If the difference between the quantities is 24. Find the quantities.**

**Solution:**

Let the two quantities be  $x$  &  $y$

$$\frac{x}{y} = \frac{3a}{7a} \Rightarrow x = 3a, y = 7a$$

$$7a - 3a = 4a = 24$$

$$4a = 24$$

$$a = 6$$

$$\therefore x = 3 \times 6 = 18 \quad y = 7a = 7 \times 6 = 42$$

$\therefore$  the 2 quantities are 18 and 42.

**Example 13 :**

**Two numbers are in the ratio 3 : 5 If 5 is added to each, they are in the ratio 22:35. Find the numbers.**

**Solution:**

Let the two numbers be  $x$  and  $y$

$$\frac{x}{y} = \frac{3}{5}$$

$$x = 3a \quad y = 5a$$

if 5 is added to each, the new ratio is

$$\frac{3a+5}{5a+5} = \frac{22}{35}$$

$$105a + 175 = 110a + 110$$

$$65 = 5a$$

$$a = \frac{65}{5}$$

$$a = 13$$

$$\therefore x = 3a = 3 \times 13 = 39$$

$$y = 5a = 5 \times 13 = 65$$

$\therefore$  the two numbrs are 39 and 65.

**Example 14 :**

**What must be added to each term in the ratio 5 : 6, so that it becomes 8 : 9?**

**Solution:**

Let the number  $x$  be added to each term in the ratio 5 : 6

$$\therefore \frac{5+x}{6+x} = \frac{8}{9}$$

$$45 + 9x = 48 + 8x$$

$$x = 48 - 45$$

$$x = 3$$

$\therefore$  Number 3 must be added to each term in the ratio 5 : 6 so that it becomes 8 : 9

**Example 15 :**

**Monthly incomes of A and B are in the ratio 2 : 3 and their monthly expenditure are in the ratio 3 : 5. If each save ₹100 per month. Find the monthly incomes of A and B.**

**Solution:**

Let the monthly incomes be  $2x$  and  $3x$  respectively, since their savings are ₹ 100 each their expenditure would be  $(2x - 100)$  and  $(3x - 100)$

$$\frac{2x-100}{3x-100} = \frac{3}{5}$$

$$5(2x - 100) = 3(3x - 100)$$

$$10x - 500 = 9x - 300$$

$$x = ₹ 200$$

Hence the monthly income of A =  $200 \times 2 = ₹400$

the monthly income of B =  $200 \times 3 = ₹600$

**Example 16 :**

If  $a : b = 2 : 3$

$x : y = 4 : 5$

Find  $(5ax + 3by) = (10ax + 4by)$

**Solution :**

$$\frac{a}{b} = \frac{2}{3} \Rightarrow a = \frac{2b}{3} \text{ and } \frac{x}{y} = \frac{4}{5} \therefore x = \frac{4y}{5}$$

$$\frac{5ax + 3by}{10ax + 4by} = \frac{\left(5 \times \frac{2b}{3} \times \frac{4y}{5}\right) + 3by}{\left(10 \times \frac{2b}{3} \times \frac{4y}{5}\right) + 4by}$$

$$= \frac{\frac{8by}{3} + 3by}{\frac{16by}{3} + 4by}$$

$$= \frac{\frac{17by}{3}}{\frac{28by}{3}} = \frac{17}{28} \text{ or } 17 : 28$$

**Example 17 :**

Find the value of  $x$  if  $32 : x = 75 : 50$

**Solution :**  $\frac{32}{x} = \frac{75}{50}$

$$x = \frac{32 \times 50}{75} = \frac{64}{3} = 21.33$$

**Example 18 :**

If  $a : b = 2 : 3$  and  $b : c = 5 : 7$  and  $c : d = 3 : 1$  find  $a : d$

**Solution :**

$$a : b = 2 : 3 \quad b : c = 5 : 7 \quad c : d = 3 : 1$$

$$\frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{2}{3} \times \frac{5}{7} \times \frac{3}{1} = \frac{10}{7} \therefore a : d = 10 : 7$$

**Example 19 :**

**The angles of a triangle in the ratio 2 : 3 : 4. Find the angles.**

**Solution :**

Let the angles be  $2x$ ,  $3x$  and  $4x$

$$2x + 3x + 4x = 180$$

$$9x = 180$$

$$x = 20^\circ$$

$$\therefore \text{the 1st angle} = 2 \times 20^\circ = 40^\circ$$

$$\text{2nd angle} = 3 \times 20^\circ = 60^\circ$$

$$\text{3rd angle} = 4 \times 20^\circ = 80^\circ$$

**Example 20 :**

**An article is sold at 20% gain on the cost price. Find the ratio of the selling price and cost price.**

**Solution :** Let the cost price of the article =  $x$

$$\text{Profit in Rs} = 20\% \text{ of } x = \frac{20}{100} x = \frac{x}{5}$$

$$\text{Selling price} = \text{cost price} + \text{profit}$$

$$\text{Selling price} = x + \frac{x}{5} = \frac{6x}{5}$$

$$\text{SP} : \text{CP} = \frac{6x}{5} : x$$

Ratio of selling price to cost price

$$= \frac{\left(\frac{6x}{5}\right)}{x} = \frac{6}{5} = 6:5$$

**Example 21 :**

**Divide ₹ 6000 in the ratio 3 : 4 : 5**

**Solution :**

Let the three amounts are  $3x$ ,  $4x$  and  $5x$

$$3x + 4x + 5x = 6000$$

$$12x = 6000$$

$$\therefore x = \frac{6000}{12} = 500$$

The three amounts are

$$3 \times 500 = ₹1500$$

$$4 \times 500 = ₹2000$$

$$5 \times 500 = ₹2500$$

**Example 22 :**

If  $x : y = 2 : 3$  find  $\frac{2x^2 + 5y^2}{x^2 + y^2}$

**Solution :**

$$\frac{x}{y} = \frac{2}{3}$$

$$\frac{x^2}{y^2} = \frac{4}{9} \quad \therefore x^2 = \frac{4}{9} y^2$$

$$\therefore \frac{2x^2 + 5y^2}{x^2 + y^2} = \frac{\left(2 \times \frac{4}{9} y^2\right) + 5y^2}{\left(\frac{4}{9} y^2\right) + y^2}$$

$$= \frac{\frac{8}{9} y^2 + 5y^2}{\frac{4}{9} y^2 + y^2}$$

$$= \frac{53y^2}{13y^2} = \frac{53}{13}$$

$$\therefore \frac{2x^2 + 5y^2}{x^2 + y^2} = \frac{53}{13} \text{ or } 53 : 13$$

**Example 23 :**

**Distribute 632 amongst A, B and C in such a way that 'B' will have 20% more than 'A' and 'C' has 20% less than 'B'.**

**Solution :**

Express the money received by B and C as ratio

$$\text{i.e., B receives} = \frac{120}{100} A, \text{ If } A = 100 \text{ then } B = 120$$

$$\therefore \frac{A}{B} = \frac{100}{120} = \frac{5}{6}$$

$$C \text{ receives} = \frac{80}{100} B \text{ If } B = 100, \text{ then } C = 80$$

$$\frac{B}{C} = \frac{100}{80} = \frac{5}{4}$$

$$A : B = 5 : 6, B : C = 5 : 4$$

$$A : B, B : C$$

$$5 : 6, 5 : 4$$

Multiply by 5, Multiply by 6

$$25 : 30, 30 : 24$$

$$\therefore A : B : C = 25 : 30 : 24$$

$$25x + 30x + 24x = 632$$

$$79x = 632$$

$$x = \frac{632}{79} \quad \therefore x = 8$$

$$\therefore A \text{ receives } 25 \times 8 = ₹ 200$$

$$A \text{ receives } 30 \times 8 = ₹ 240$$

$$C \text{ receives } 24 \times 8 = ₹ 192$$

**Example 24 :**

Rajeev planned his journey to Mumbai as follows. He will travel  $\frac{5}{9}$  th of the total distance

by an aeroplane  $\frac{3}{4}$  th of the remaining by Train and the remaining distance of 200 km by

a car. What is the total distance to Mumbai?

**Solution :**

Let the total distance be  $x$

Aeroplane travel  $\frac{5}{9} x \text{ km}$

$$\text{Remaining distance} = \left( x - \frac{5}{9} x \right) \text{ km}$$

$$= \frac{4}{9} x \text{ km}$$

He covers  $\left(\frac{3}{4} \times \frac{4}{9} x\right) km$  by train  $= \frac{1}{3} x km$  by train

He covers 200 km by car

Total Distance travelled by (Aeroplane + train + car)

$$\therefore \text{Total distance } x = \frac{5}{9}x + \frac{x}{3} + 200$$

$$x - 200 = \frac{5x + 3x}{9}$$

$$9x - 1800 = 8x$$

$$x = 1800 \text{ km}$$

$\therefore$  The total distance to Mumbai 1800 km.

**Example 25 :**

Divide 1,647 into three parts such that  $\frac{3}{7}$  th of the first,  $\frac{2}{3}$  rd of the second and  $\frac{4}{5}$  th of the third are equal.

**Solution :**

Let the 3 parts be A, B and C

$$\frac{3}{7} A = \frac{2}{3} B$$

$$\frac{A}{B} = \frac{2}{3} \times \frac{7}{3}$$

$$\frac{A}{B} = \frac{14}{9}$$

$$\therefore A : B = 14 : 9$$

$$\text{and } \frac{2}{3} B = \frac{4}{5} C$$

$$\frac{B}{C} = \frac{4}{5} \times \frac{3}{2} = \frac{12}{10} = \frac{6}{5} = 6 : 5$$

$$A : B = 14 : 9 \text{ multiple by 2 } \therefore A : B = 28 : 18$$

$$B : C = 6 : 5 \text{ multiple by 3 } \therefore B : C = 18 : 15$$

$$\therefore A : B : C = 28 : 18 : 15$$

$$\therefore 28x + 18x + 15x = 1647$$

$$61x = 1647$$

$$x = \frac{1647}{61}$$

$$\therefore x = 27$$

$$\therefore \text{A receives} = 28 \times 27 = ₹ 756$$

$$\text{B receives} = 18 \times 27 = ₹ 486$$

$$\text{C receives} = 15 \times 27 = ₹ 405$$

**Example 26 :**

**$x$ ,  $y$  and  $z$  play cricket. The runs scored by  $x$  and  $y$  are in the ratio of 3 : 2.  $y$ 's runs to  $z$ 's runs are in the ratio 3 : 2. Together they all score 342 runs. How many runs did each score?**

**Solution :**

$$x : y = 3 : 2$$

$$y : z = 3 : 2$$

$$x : y : z = 9 : 6 : 4$$

$$\therefore 9x + 6x + 4x = 342$$

$$19x = 342$$

$$x = \frac{342}{19}$$

$$x = 18$$

$$\therefore x \text{ scores} = 9 \times 18 = 162 \text{ runs}$$

$$y \text{ scores} = 6 \times 18 = 108 \text{ runs}$$

$$z \text{ scores} = 4 \times 18 = 72 \text{ runs}$$

**Example 27 :**

**Three numbers are in the ratio 2 : 3 : 4. If the sum of their squares is 1856. Find the numbers.**

**Solution :**

Let the 3 numbers be  $2x$ ,  $3x$  and  $4x$

$$(2x)^2 + (3x)^2 + (4x)^2 = 1856 \quad \therefore 29x^2 = 1856$$

$$x^2 = \frac{1856}{29}$$

$$x^2 = 64 \quad \therefore x = 8$$

∴ the 3 numbers are

1st number  $2 \times 8 = 16$

2nd number  $3 \times 8 = 24$

3rd number  $4 \times 8 = 32$

**Example 28 :**

**₹ 5625 is divided among A, B and C so that A receives one half as much as B and C together receive and B receives one fourth of what A and C together receive.**

**Find the share of A, B and C**

**Solution:**

Given  $A + B + C = 5625$  ..... (1)

and  $A = \frac{1}{2} (B + C)$

$2A = B + C$  ..... (2)

$B = \frac{1}{4} (A + C)$

$4B = A + C$  ..... (3)

from (1) & (2) we have

But  $A + B + C = 5625$

$A + 2A = 5625$

$3A = 5625$

∴  $A = \frac{5625}{3}$

$A = ₹1875$

∴ A'S share is ₹1875

from (1) & (3) we have

$B + 4B = 5625$

$5B = 5625$

$B = \frac{5625}{5}$  ∴  $B = ₹1125$

∴ A'S share is ₹ 1875

B's share is ₹ 1125

C's share is ₹ 2625

**Example 29 :**

**The monthly incomes of A and B are in the ratio 9 : 7 and those of B and C are in the ratio 3 : 2. If 10% of A's income and 15% of C's incomes differ by Rs. 18.**

**Find the incomes of A, B and C**

**Solution :**

Let the income of A and B be  $9x$  and  $7x$

$$B : C = 3 : 2 \Rightarrow \frac{B}{C} = \frac{3}{2} \therefore C = \frac{2B}{3}$$

$$\therefore C's \text{ income} = \frac{2}{3} \times 7x = \frac{14x}{3}$$

$$10\% \text{ of A's income} = \frac{10}{100} \times 9x = 0.9x$$

$$15\% \text{ of C's income} = \frac{15}{100} \times \frac{14x}{3} = 0.7x$$

$$10\% \text{ of A's income} - 15\% \text{ C income} = ₹18$$

$$0.9x - 0.7x = 18$$

$$0.2x = 18$$

$$x = \frac{18}{0.2}$$

$$x = \frac{180}{2}$$

$$x = ₹90$$

$$\text{Hence A's income} = 9 \times 90 = ₹810$$

$$\text{B's income} = 7 \times 90 = ₹630$$

$$\text{C's income is} = \frac{14}{3} \times 90 = ₹420$$

**Example 30 :**

**The ratio of prices of two house was 16 : 23 two years later when the price of the first had risen by 10% and that of second by ₹ 477, the ratio of their prices becomes 11 : 20. Find the original prices of the two houses.**

**Solution :**

Let the prices of the two houses be  $16x$  and  $23x$

$$\therefore \frac{16x + 1.6x}{23x + 47} = \frac{11}{20} \quad (\because 10\% \text{ of } 16x = \frac{10}{100} \times 16x = 1.6x)$$

$$320x + 32x = 253x + 5247$$

$$352x = 253x + 5247$$

$$99x = 5247$$

$$x = \frac{5247}{99}$$

$$x = 53$$

The original price of 1st house =  $16 \times 53 = ₹ 848$

The original price of 2nd house =  $23 \times 53 = ₹ 1219$

### EXERCISE 7.1

#### One mark questions:

1. If 3 : 5 is a ratio, find the antecedent and consequent.
2.  $x$  gets a salary of ₹ 20000,  $y$  gets a salary of ₹ 5000. Find the ratio of their salaries.
3. Find the inverse ratio of 4 : 5
4. A house consumes 30 kgs of wheat and 4 kg of sugar compare the consumption of wheat and sugar in the form of ratio.
5. Mr.  $x$  completes a job in 3 hours and Mr  $y$  completes the same job in 45 minutes represent their time in ratio.
6. Find the compound ratio of 3 : 4 and 4 : 7
7. Find the compound ratio of 1 : 2, 2 : 3 and 3 : 5
8. Find the duplicate ratio of 5 : 4
9. Find the triplicate ratio of 3 : 5
10. Find the subduplicate ratio of 9 : 49
11. Find the subtriplicate ratio of 125 : 64
12. Find the value of  $x$  of  $5 : 20 = 3 : x$

#### 2 marks questions

13. Find the ratio between two numbers such that their sum is 40 and their difference is 8.
14. A ratio is in the lowest term is 3 : 8. If the difference between the quantities is 25. Find the quantities.
15. Two numbers are in the ratio 3 : 5. If 5 is added to each, they are in the ratio. 2 : 3 find the numbers.

16. What must be added to each term in the ratio 2 : 3 so that it becomes 5 : 6.
17. What must be added to each term in the ratio 4 : 5 so that it becomes 7 : 8.
18. What must be subtracted from each term in the ratio 7 : 4 so that it becomes 5 : 2.
19. What must be subtracted from each term in the ratio 8 : 7 so that it becomes 4 : 3.
20. If  $a : b = 2 : 3$ ,  $b : c = 3 : 5$  and  $c : d = 5 : 7$  find  $a : d$ .
21. If  $a : b = 2 : 3$  and  $b : c = 6 : 13$  Find  $a : b : c$ .
22. If  $a : b = 3 : 4$ ,  $b : c = 8 : 15$  Find  $a : b : c$ .

### 3 marks questions

23. Divide 1800 in the ratio 3 : 4 : 5

24. If  $x : y = 3 : 4$  find  $\frac{2x^2 + 3y^2}{x^2 + y^2}$

25. If  $a : b = 2 : 3$  and  $x : y = 4 : 7$

Find  $\frac{5ax + 4by}{8ax + 3by}$

26. The angles of a triangle are in the ratio 3 : 4 : 5. Find the angles.
27. An article is sold at 40% gain on the cost price. Find the ratio of the selling price and cost price.
28. If the monthly incomes of A and B are in the ratio 3 : 4 and their expenditures are in the ratio 1 : 2. If each saves ₹ 1000 find the monthly incomes.
29. If the monthly incomes of A and B are in the ratio 3 : 4 and their expenditure are in the ratio 1 : 2. If each saves ₹ 2000. Find their monthly incomes.
30. Two numbers are in the ratio 6 : 7. If the difference of their squares is 117. Find the numbers.
31. Two numbers are in the ratio 3 : 4. If the sum of their squares is 900 find the two numbers.
32. If  $x : y = 3 : 4$  and  $y : z = 7 : 9$  Find  $x : y : z$ .

### 5 marks questions

33. Divide ₹1890 in three parts such that three times of the first, five times of the second and six times the third are equal.
34. Divide ₹3262 among x, y and z such that if ₹ 35, ₹15 and ₹12 are deducted from their respective shares, the remainder are in the ratio 3 : 5 : 8.

35. If  $x : y = 2 : 3$  Find the value of  $\frac{2x^3 + 3y^3}{x^3 + y^3}$

36. Divide 5880 is to three parts, such that 'B' receive twice as 'A' and C receives  $\frac{5}{6}$  of what B receives.
37. If  $\frac{2x^2 + 3y^2}{x^3 + y^2} = \frac{2}{41}$  Find  $x : y$ .
38. Divide 6000 into three parts is the ratio  $\frac{1}{2} : \frac{1}{3} : \frac{1}{6}$
39. Divide 17,640 among P, Q, R and S such that Q gets  $\frac{2}{5}$  of P, R gets  $\frac{5}{8}$  of Q and S gets  $\frac{2}{13}$  of the sum of Q and R.
40. The Rajdhani express takes 18 hours to reach Delhi from Bhubaneswar while Nilachala Express takes 24 hours for the same of Delhi is 2880 kms from Bhubaneswar find the ratio between the average speeds of the two trains.

### ANSWERS 7.1

#### 1 mark question:

- 1) Antecedent = 3 , Consequent = 5      2) 4 : 1      3) 5 : 4      4) 15 : 2  
 5) 5 : 4      6) 3 : 7      7) 1 : 5      8) 25 : 16      9) 27 : 125      10) 3 : 7  
 11) 5 : 4      12)  $x = 12$

#### 2 marks questions:

13. 3 : 2
14. Quantities are 15 and 40.
15. The numbers are 15 and 25.
16. 3 must be added to each term is the ratio 2 : 3 so that it becomes 5 : 6.
17. 3 must be added to each term is the ratio 4 : 5 so that it becomes 7 : 8.
18. 2 must be subtracted from each term is the ratio 7 : 4 so that it becomes 5 : 2.
19. 4 must be subtracted from each term in the ratio 8 : 7 so that it becomes 4 : 3.
20.  $\frac{a}{d} = \frac{2}{3} \times \frac{3}{5} \times \frac{5}{7} = \frac{2}{7}$
21.  $a : b : c = 4 : 6 : 13$
22.  $a : b : c = 6 : 8 : 15$

**3 marks questions:**

23. 1st part = ₹ 450, 2nd part = ₹ 600, 3rd part = ₹ 750

24.  $\frac{2x^2 + 3y^2}{x^2 + y^2} = \frac{66}{25}$

25.  $\frac{124}{127}$

26. 1st angle  $3 \times 15^\circ = 45^\circ$ , 2nd angle  $4 \times 15^\circ = 60^\circ$ , 3rd angle  $5 \times 15^\circ = 75^\circ$

27. 7 : 5

28. Monthly income of A = ₹ 1500, Monthly income of B = ₹ 2000

29. Monthly income of A = ₹ 3000, Monthly income of B = ₹ 4000

30. 18 and 21 are the 2 numbers

31. 18 and 24 are the 2 numbers

32.  $x : y : z = 21 : 28 : 36$ .

**5 marks questions:**

33. 1st part ₹ 900, 2nd part ₹ 540, 3rd part ₹ 450

34. x receives ₹ 635, y receives ₹ 1015, z receives ₹ 1612

35.  $\frac{108}{35}$

36. A receives ₹ 1260, B receives ₹ 2520, C receives ₹ 2100

37.  $x : y = 5 : 4$

38. A receives ₹ 3000, B receives ₹ 2000, C receives ₹ 1000

39. P gets ₹ 10,080, Q gets ₹ 4032, R gets ₹ 2520, S gets ₹ 1008

40. 4 : 3.

**7.3 Proportion :**

The term proportion may be defined as equality of two ratios as  $a : b = c : d$  which is usually

expressed as  $\frac{a}{b} = \frac{c}{d}$  or  $ad = bc$  or  $a : b :: c : d$

**For example** the ratio 2 : 3 is equal to the ratio 6 : 9 and hence it can be said that 2, 3, 6 and 9 are in proportion.

i.e.  $2 \times 9 = 3 \times 6 = 18$

**Rules related to proportion**

**(i) Mean proportion :**

$a$ ,  $b$  and  $c$  are said to be in mean proportion if

$$a : b :: b : c$$

i.e.  $b^2 = ac$

ie  $b = \sqrt{ac}$

For ex : 1, 3 and 9 are in mean proportion as  $1 : 3 :: 3 : 9$

$$b = \sqrt{1 \times 9} = \sqrt{9} = 3$$

**(ii) Simple proportion :**

When the number of related terms remains within four it is a case of simple proportion.

i.e.  $a : b :: c : d$

For example :  $2 : 3 :: 8 : 12$

**(iii) Compound proportion :**

When the number of related terms exceed four, it is case of compound proportion

$a : b :: c : d :: e : f$

eg :  $2 : 3 :: 8 : 12 :: 6 : 9$

**(iv) Continued proportion :**

A continued proportion is the one in which there is a chain of ratios between the related terms.

i.e.  $a : b = b : c = c : d$  and so on

For example  $1 : 2 = 2 : 4 = 4 : 8 = 8 : 16$  and so on

**(v) Direct proportion :**

A direct proportion means a positive co-relation between the related terms where by an increase in the value of one is followed by a proportionate increase in the value of another and a decrease in the value of one is followed by a proportionate decrease in the value of another.

**Examples for**

- (i) The ratio between cost and quantity of articles.
- (ii) The height and weight of a person.

**(vi) Inverse proportion :**

An inverse proportion means a negative co-relation between the two related terms where by an increase in the value of one is followed by a proportionate decrease in the value of another and vice versa.

Examples for

- 1. The ratio between the number of workers and amount of time
- 2. The ratio between supply and price

## 7.4 Properties proportion :

### (i) Invertendo :

According to this property If  $a : b = c : d$  then invertendo is  $b : a = d : c$

**Proof :**  $\frac{a}{b} = \frac{c}{d}$

$$1 \div \frac{a}{b} = 1 \div \frac{c}{d}$$

$$\therefore \boxed{\frac{b}{a} = \frac{d}{c}}$$

### (ii) Alternendo :

According to this property If  $a : b = c : d$  then  $a : c = b : d$

**Proof :**  $\frac{a}{b} = \frac{c}{d}$  Multiplying both the side by  $\frac{b}{c}$

$$\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$$

$$\boxed{\frac{a}{c} = \frac{b}{d}}$$

### (iii) Componendo:

According this property  $a : b = c : d$

then  $a + b = b : : c + d = d$  or  $\frac{a+b}{b} = \frac{c+d}{d}$

**Proof :**  $\frac{a}{b} = \frac{c}{d}$

Adding 1 to both the sides we have

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

$$\boxed{\frac{a+b}{b} = \frac{c+d}{d}}$$

### (iv) Dividendo:

According to this property if  $a : b : : c : d$

then  $a - b = b : : c - d : d$  or  $\frac{a-b}{b} = \frac{c-d}{d}$

**Proof**  $\frac{a}{b} = \frac{c}{d}$

Subtracting 1 from both the sides

$$\frac{a}{b} - 1 = \frac{c}{d} - 1$$

$$\boxed{\frac{a-b}{b} = \frac{c-d}{d}}$$

**(v) Componendo and Dividendo:**

According to this property

If  $a : b :: c : d$  then

$$a + b : a - b :: c + d : c - d \text{ or } \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

**Proof** By componendo if  $a : b :: c : d$

$$\text{then } \frac{a+b}{b} = \frac{c+d}{d} \quad \text{----- (1)}$$

By dividendo if  $\frac{a}{b} = \frac{c}{d}$

$$\text{then } \frac{a-b}{b} = \frac{c-d}{d} \quad \text{----- (2)}$$

Dividing componendo by dividendo

We have  $\boxed{\frac{a+b}{a-b} = \frac{c+d}{c-d}}$

**Example 1 : Find the fourth proportional of 6, 14, 15**

**Solution :**  $6 : 14 :: 15 : x$

$$6x = 15 \times 14$$

$$x = \frac{15 \times 14}{6} = \frac{5 \times 14}{2}$$

$$x = 35$$

**Example 2 : Find the mean proportional of 9 and 16**

**Solution :**  $9 : x :: x : 16$

$$x^2 = 144 \quad \therefore x = 12$$

**Example 3 :**

**Find the 3rd proportional of 6 and 24**

**Solution :**  $6 : 24 :: 24 : x$

$$6x = 24 \times 24$$

$$x = 4 \times 24$$

$$x = 96$$

**Example 4 :**

**If  $a : b = 4 : 5$ . Find  $\frac{3a + 2b}{3a - 2b}$**

**Solution :**

$$\text{Let } a = 4k, b = 5k$$

$$\therefore \frac{3(4k) + 2(5k)}{3(4k) - 2(5k)} = \frac{12k + 10k}{12k - 10k} = \frac{22k}{2k} = 11$$

**Example 5 :**

**If  $a : 3 : 15 = 5 : b : 5$ . Find the values of  $a$  and  $b$**

**Solution :**

$$\frac{a}{5} = \frac{3}{b} = \frac{15}{5} = 3 \quad \therefore \frac{a}{5} = 3 \Rightarrow \therefore a = 15$$

$$\frac{3}{b} = 3 \Rightarrow b = 1$$

**Example 6 :**

**If  $a + b = a - b = 4 : 3$  Find the value of  $a$  and  $b$**

**Solution :**

$$\frac{a+b}{a-b} = \frac{4}{3}$$

Applying componendo and dividendo

$$\text{We have, } \frac{a+b+a-b}{a+b-a+b} = \frac{4+3}{4-3}$$

$$\frac{2a}{2b} = \frac{7}{1} \quad \therefore \frac{a}{b} = \frac{7}{1}$$

$$\therefore a = 7 \text{ and } b = 1$$

**Example 7 :**

If  $\frac{a}{b} = \frac{c}{d}$  prove that  $\frac{2a+7b}{2c+7d} = \frac{2a-7b}{2c-7d}$

**Solution :**

If  $\frac{a}{b} = \frac{c}{d}$

Multiplying by  $\frac{2}{7}$  on both the sides

We have  $\frac{2a}{7b} = \frac{2c}{7d}$

Applying componendo and dividendo we have

$$\frac{2a+7b}{2a-7b} = \frac{2c+7d}{2c-7d} \quad \therefore \frac{2a+7b}{2c+7d} = \frac{2a-7b}{2c-7d}$$

**Example 8 :**

If  $\frac{2a+2b-3c-3d}{2a-2b-3c+3d} = \frac{a+b-4c-4d}{a-b-4c+4d}$  the prove that  $a : b :: c : d$

**Solution :**

By applying componendo and dividendo we have

$$\frac{(2a+2b-3c-3d)+(2a-2b-3c+3d)}{(2a+2b-3c-3d)-(2a-2b-3c+3d)} = \frac{(a+b-4c-4d)+(a+b-4c+4d)}{(a+b-4c-4d)-(a-b-4c+4d)}$$

$$\frac{4a-6c}{4b-6d} = \frac{2a-8c}{2b-8d}$$

OR

$$\Rightarrow \frac{2a-3c}{2b-3d} = \frac{a-4c}{b-4d} \quad \text{By applying alternendo}$$

$$\Rightarrow \frac{2a-3c}{a-4c} = \frac{2b-3d}{b-4d}$$

Subtracting 2 from both the sides

We have  $\frac{2a-3c}{a-4c} - 2 = \frac{2b-3d}{b-4d} - 2$

$$\frac{\cancel{2a} - 3c - \cancel{2a} + 8c}{a - 4c} = \frac{\cancel{2b} - 3d - \cancel{2b} + 8d}{b - 4d}$$

$$\frac{\cancel{2}c}{a - 4c} = \frac{\cancel{2}d}{b - 4d}$$

$$\frac{c}{a - 4c} = \frac{d}{b - 4d}$$

by applying Inventendo

$$\frac{a - 4c}{c} = \frac{b - 4d}{d}$$

adding 4 to both the sides

$$\frac{a - 4c}{c} + 4 = \frac{b - 4d}{d} + 4$$

$$\frac{a - \cancel{4c} + 4c}{c} = \frac{b - \cancel{4d} + 4d}{d} \quad \therefore \frac{a}{c} = \frac{b}{d}$$

By applying alternendo we have  $\boxed{\frac{a}{b} = \frac{c}{d}}$

### Example 9 :

**Four number are in proportion. The sum of the extremes is 54 and the sum of the means is 36. If the ratio of their means is 2 : 1 Find the numbers.**

#### Solution :

Let the numbers be  $a, b, c, d$

$$a : b :: c : d$$

$$a + d = 54 \quad \dots\dots\dots(1)$$

$$b + c = 36 \quad \dots\dots\dots(2)$$

$$\text{Given } \frac{b}{c} = \frac{2}{1} \quad \therefore b = 2c \text{ in (2)}$$

$$3c = 36 \quad \therefore c = 12 \text{ and } b = 24$$

$$\therefore bc = 288 = ad$$

$$\therefore a = \frac{288}{d} \quad \dots\dots\dots(3)$$

$$(3) \text{ in (1), } \frac{288}{d} + d = 54$$

$$288 + d^2 = 54d$$

$$d^2 - 54d + 288 = 0$$

$$d^2 - 48d + 68d + 288 = 0$$

$$d(d - 48) - 6(d - 48) = 0$$

$$(d - 6)(d - 48) = 0$$

$$d = 6 \text{ or } 48$$

$$a + d = 54$$

$$\text{If } d = 6 \quad a = 48$$

$$\text{If } d = 48 \quad a = 6$$

$\therefore a, b, c, d$  are 6, 24, 12, 48

OR

$a, b, c, d$  are 48, 24, 12, 6

#### Example 10 :

Evaluate  $\frac{x+a}{x-a} + \frac{x+b}{x-b}$  where  $x = \frac{2ab}{a+b}$

#### Solution :

$$\frac{x+a}{x-a} + \frac{x+b}{x-b}$$

By applying componendo and dividendo

We have

$$\frac{x + \cancel{a} + x - \cancel{a}}{\cancel{x} + a - \cancel{x} + a} + \frac{x + \cancel{b} + x - \cancel{b}}{\cancel{x} + b - \cancel{x} + b}$$

$$\frac{2x}{2a} + \frac{2x}{2b} \quad \frac{x}{a} + \frac{x}{b}$$

Putting the value of  $x = \frac{2ab}{a+b}$  we have

$$\frac{2ab}{a(a+b)} + \frac{2ab}{b(a+b)}$$

$$\frac{2ab^2 + 2a^2b}{ab(a+b)} = \frac{2ab(\cancel{b+a})}{ab(\cancel{a+b})} = 2$$

**EXERCISE 7.2**

**One mark questions:**

1. State which of the following are proportions.
  - (i)  $2 : 3 :: 6 : 9$
  - (ii)  $1 : 3 :: 4 : 15$
  - (iii)  $3 : 5 :: 6 : 15$
  - (iv)  $7 : 9 :: 14 : 18$
2. Find the fourth proportional to each of the following
  - (i) 4, 5, 24
  - (ii) 6, 12, 15
  - (iii) 1.5, 4.5, 3.5
  - (iv)  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$
3. Find the mean proportional to each of the following
  - (i) 1 and 9
  - (ii) 2 and 8
  - (iii)  $\frac{1}{16}$  and  $\frac{1}{25}$
  - (iv) 0.8 and 1.8
4. Find the third proportional to each of the following
  - (i) 4, 6
  - (ii) 3, 12
  - (iii) 2.4, 3.6
  - (iv)  $2\frac{2}{3}, 4$

**3 marks questions**

5. If  $\frac{a}{b} = \frac{c}{d}$  then prove that  $\frac{3a+5b}{3c+5d} = \frac{3a-5b}{3c-5d}$
6. If  $\frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}} = \frac{a}{b}$  find  $x$
7. Find the numbers which added to the terms numerator and denominator of  $25 : 37$  make it  $5 : 6$ .
8. The ages of a father and his son in the ratio  $6 : 1$ . After 14 years their age will be in the ratio  $8 : 3$  what are their present ages?

**5 marks questions**

9. Four numbers formed by adding 1, 5, 10 and 15 to a certain number are in proportion. Find the number?
10. A concrete mixture was made of cement, chips and sand. The ratio of the cement and chip is the same as that of chips and the sand. If 144 bags of cement and 225 bags of sand were used is the mixture. Find the number of bags of chips used in the mixture.

## ANSWERS 7.2

### 1 Mark answers

1. (i) and (iv) are proportions
2. (i) 30 (ii) 30 (iii) 10.5 (iv)  $\frac{1}{6}$
3. (i) 3 (ii) 4 (iii)  $\frac{1}{20}$  (iv) 1.2
4. (i) 9 (ii) 48 (iii) 5.4 (iv) 6

### 3 Mark answers

6.  $x = \frac{2ab}{a^2 + b^2}$
7. 35
8. Father's age = 42 years, son's age = 7

### 5 Mark answers

9. 36, 40, 45, 50 are the four numbers
10. 180 bags of chips were used in the mixture.

### 7.5 Time and work, Time and distance and mixtures

**Note :**

- (i) Time to complete, a work varies directly as the amount of work and inversely as the number of workers employed.
- (ii) Time taken to travel a certain distance varies directly as the distance but varies inversely as the speed.

#### Example 1 :

**If ₹120 maintain a family of 4 persons for 30 days. How long ₹300 maintain a family of 6 persons?**

**Solution :**

Person	Money	Days
4	120	30
6	300	$x$

$$120 : 300 = 30 = x$$

Money and days varies directly  $120 : 300 = 30 : x$

Person & days are ..... inversly  $6 : 4 = 30 : x$

$$120 \times 6 \times x = 300 \times 4 \times 30 \Rightarrow \frac{120 \times 6}{300 \times 4} = \frac{30}{x}$$

$$x = \frac{300 \times 120}{120 \times 6} = 50 \text{ days}$$

**Example 2 :**

**500 workers can finish a work in 8 days. How many workers will finish the same work in 5 days.**

**Solution :**

Worker	Days
500	8
$x$	5

First ratio of workers is  $500 : x$

2nd ratio of days  $8 : 5$

Here the number of workers and number of days are inversely proportional

$$500 : x :: 5 : 8$$

$$5x = 500 \times 8$$

$$x = 800$$

$\therefore$  800 workers will be required to finish the same work in 5 days.

**Example 3 :**

**3 carpenters can earn ₹360 in 6 days working at 9 hours a day. How much will 8 carpenters can earn in 12 days working 6 hours a day?**

**Solution :**

Carpenter	days	hrs.	earning
3	6	9	360
8	12	6	$x$

1st ratio are carpenters  $3 : 8$ , 2nd ratio are days  $6 : 12$ , 3rd ratio are hours  $9 : 6$

Here all are direct proportion to  $360 : x$

$$3 \times 6 \times 9 \times x = 8 \times 12 \times 6 \times 360$$

$$x = \frac{\cancel{9}6^{\cancel{32}} \times \cancel{6} \times \cancel{3}60^{\cancel{40}}}{\cancel{54}_6 \times \cancel{6}}$$

$$x = ₹1280$$

Hence 8 carpenters can earn ₹1280 in 12 days working at the rate of 6 hours per day

**Example 4 :**

**A mixture contains milk and water in the ratio  $5 : 1$  on adding 5 litres of water, the ratio of milk and water becomes  $5 : 2$ , Find the quantity of milk in the original mixture.**

**Solution :**

Let the quantity of milk be  $5x$  and water be  $x$

$$\frac{5x}{x+5} = \frac{5}{2}$$

$$10x = 5x + 25$$

$$5x = 25$$

$$x = 5$$

$\therefore$  the quantity of milk =  $5 \times 5 = 25$  litres

**Example 5 :**

**A jar contains two liquids A and B in the ratio 7 : 5 when 9 litres of the mixture is drawn and the jar is filled with the same quantity of B, the ratio of A and B becomes 7 : 9. Find the quantity of A in the jar initially.**

**Solution :**

Let the quantities of the two liquids be  $7x$  and  $5x$

Quantity of A and B in 9 liter of mixture drawn in  $\left(\frac{7}{12} \times 9\right)$  and  $\left(\frac{5}{12} \times 9\right)$

$\therefore$  the new ratio of A and B

$$\frac{7x - \left(\frac{7}{12} \times 9\right)}{5x - \left(\frac{5}{12} \times 9\right) + 9} = \frac{7}{9}$$

$$\therefore \frac{84x - 63}{60x - 45 + 108} = \frac{7}{9}$$

$$\therefore x = 3 \text{ liters}$$

$$\therefore \text{Quantity of A} = 7x = 21 \text{ liters.}$$

**Example 6 :**

**Two taps fill a tank separately is 24 minutes and 40 minutes respectively and a drain pipe can drain off 30 litres per minutes. If all the three pipes are opened the tank fills in 60 minutes. What is the capacity of the tank?**

**Solution :**

Tap 1 can fill  $\frac{1}{24}$  th of the tank in 1 minute

Tap 2 can fill  $\frac{1}{40}$  th of the tank in 1 mintue

Let the drain pipe empty the tank in 'x' minute

$\frac{1}{x}$  In one minute the 3 pipes fill will be

$$\left( \frac{1}{24} + \frac{1}{40} - \frac{1}{x} \right) = \frac{1}{60}$$

$$\frac{1}{24} + \frac{1}{40} - \frac{1}{60} = \frac{1}{x}$$

$$\frac{10 + 6 - 4}{240} = \frac{1}{x}$$

$$\frac{12}{240} = \frac{1}{x} \quad \therefore x = 20$$

$\therefore$  the capacity of the tank =  $30 \times 20 = 600$  liters.

**Example 7 :**

**If ten persons can do a job in 30 days. In how many days can fifteen persons do the same job?**

**Solution :**

Person	Days
10	30
15	x

Ratio of persons = 10 : 15

Ratio of days = 30 : x  $\Rightarrow$  x : 30 (inverse)

It is inverse proportion because more men will do job in lesser time

$$10 : 15 :: x : 30$$

$$300 = 15x$$

$$x = \frac{300}{15}$$

$$x = 20 \text{ days}$$

15 persons can do the same job in 20 days

**Example 8 :**

**If 8 men and 16 boys can do a piece of work in 6 days and 12 men and 24 boys can do the same work in 8 days. In how many days can 16 men and 20 boys do it.**

**Solution :**

8 men + 16 boys can do the job in 6 days, Multiply by 6

48 men + 96 boys can do the Job in 1 day

Next 12 men and 24 boys can do the same job is 8 days

12 men + 24 boys can do it is 8 days  $\times^{\text{ly}} 8$

96 men + 192 boys can do the Same job is 1 day

Hence in doing one day's work it may be stated that

48 men + 96 boys = 96 men + 192 boys

1 man = 2 boys

$\therefore$  16 men + 20 boys = 32 boys + 20 boys = 52 boys can do the job in how many days.

This is the case of inverse proportion

Let the unknown quantity be  $x$  days

Boys	Days
32	6
52	$x$

$32 : 52 :: x : 6$

$32 \times 6 = 52 x$

$$x = \frac{192}{52}$$

$x = 3.7$  days.

### Example 9 :

**If two men and four women can do a work is 33 days and 3 men and 5 women can do the same work is 24 days. How long shall 5 men and 2 women take to do the same work?**

### Solution :

2 men and 4 women can do a work in 33 days

66 men and 132 women can do (by  $\times 33$ ), work in 1 day ..... (1)

Similarly 3 men and 5 women can do a work is 24 days ( $\times 24$ )

72 men and 120 women can do a work is 1 day ..... (2)

From (1) and (2) we have

66 men + 132 women = 72 men + 120 women

$\therefore$  1 man = 2 women

5 men + 2 women = 10 women + 2 women = 12 women ..... (3)

Let the required number of days be  $x$

$$2\text{men} + 4\text{ women} = 4\text{ women} + 4\text{ women} = 8\text{ women} \quad \dots\dots\dots(4)$$

$$8W : 12W = x : 33$$

$$12x = 33 \times 8$$

$$3x = 33 \times 2$$

$$x = 22\text{ days}$$

**Example 10 :**

**5 men each working 9 hours a day can finish a work in 30 days. How many men are required to finish eight times the work in 25 days each working 8 hours a day?**

**Solution :**

Men	Hours	days	work
5	9	30	1
$x$	8	25	8

Hours :  $8 : 9 = 5 : x$  ( $\therefore$  more men less hrs. Inverse proportion)  
Days  $25 : 30 = 5\text{ men} : x\text{ men}$  ( $\therefore$  more men less day Inverse proportion)  
work,  $1 : 8 = 5x$  ( $\therefore$  more work more men direct proportion)

$$x = \frac{\cancel{8} \times 9 \times \cancel{30}^6 \times 8}{\cancel{8} \times \cancel{25}_5 \times 1} = 54\text{ men}$$

$\therefore$  54 men are required to finish the work

**Example 11 :**

**If 10 men or 20 boys can do a piece of work in 30 days, how long will 30 boys and 5 men take to do the same work?**

**Solution :**

10 men can do a job in 30 days

1 man can do the job in  $30 \times 10 = 300$  days

$$20\text{ men can do the job in} = \frac{300}{20} = 15\text{ days}$$

$$\left[ \begin{array}{l} \therefore 20\text{ Boys work} = 10\text{ mens work} \\ \therefore 1\text{ Boy's work} = \frac{10}{20} = \frac{1}{2}\text{ men} \end{array} \right]$$

$$\therefore 30\text{ boys} = \frac{1}{2} \times 30 = 15\text{ men}$$

$$(5 + 15 = 20\text{ men})$$

**Example 12 :**

**Two taps can separately fill a tank in 12 min and 15 minutes respectively. The tank when full can be emptied by a drain pipe in 20 minutes. When the tank was empty, all the three were opened simultaneously. In what time will the tank be filled up?**

**Solution :**

1st tap can fill  $\frac{1}{12}$ <sup>th</sup> tank in 1 min.

2nd tap can fill  $\frac{1}{15}$  tank in 1 min.

drain pipe drain out  $\frac{1}{20}$ <sup>th</sup> tank in 1 min.

$$\begin{aligned} \text{In 1 min} &= \left( \frac{1}{12} + \frac{1}{15} - \frac{1}{20} \right) \\ &= \frac{15 + 12 - 9}{180} \\ &= \frac{18}{180} = \frac{1}{10} \text{ of tank will get filled} \end{aligned}$$

$\therefore$  the tank will get filled in 10 minutes

**Example 13 :**

**Walking 4 kmph a student reaches his college 5 minutes late and if he walks at 5 kmph, he reach is  $2\frac{1}{2}$  minutes early. What is the distance from his house to the college?**

**Solution :**

Let the distance from his house to the college =  $x$  km

Time taken to cover  $x$  kms at 4 km ph =  $\frac{x}{4}$  hrs.

Time to taken to cover  $x$  kms at 5 kmph =  $\frac{x}{5}$  hrs

From the given problem - Time taken to cover  $x$  km is he reach 5 min late  
Time taken to cover  $x$  km y he reach  $2\frac{1}{2}$  early

$$\begin{aligned} \therefore \frac{x}{4} - \frac{5}{60} &= \frac{x}{5} + \frac{2\frac{1}{2}}{60} \\ \frac{x}{4} - \frac{5}{60} &= \frac{x}{5} + \frac{5}{120} \end{aligned}$$

$$\frac{x}{4} - \frac{x}{5} = \frac{5}{60} + \frac{5}{120}$$

$$\frac{x}{20} = \frac{10+5}{120}$$

$$\frac{x}{20} = \frac{15}{120}$$

$$x = \frac{15 \times 2}{12} = \frac{30}{12}$$

$$x = 2.5 \text{ km}$$

$\therefore$  the distance from his house to the college = 2.5 km

**Example 14 :**

**Two men 'X' and 'Y' starts from a place 'A' walking at 5 kms and 6 kms per hour respectively. How many kilometer will they be apart at the end of 5 hours. If (a) they walk in opposite directions and (b) they walk in the same direction?**

**Solution :**

- (a) If they walk in opposite directions, their relative speed will be  $5 + 6 = 11$  km ph. In one hour, they will be apart 11 kms. In 5 hours they will be apart by  $5 \times 11 = 55$  kms.
- (b) If they walk in the same direction their relative speed =  $6 - 5 = 1$  kmph.  
In one hour, they will be 1 km apart  
In 5 hours, they will be  $5 \times 1 = 5$  kms.  
They will be 5 kms apart.

**Example 15 :**

**In a fort, there was ration for 560 soliders that would last the soldiers for 70 days After 20 days, 60 soldiers left the fort. For how many days the remaining ration can support the remaining soldiers?**

**Solution :**

The remaining soldiers =  $560 - 60 = 500$

The remaining days =  $70 - 20 = 50$  days

The remaining food would have been sufficient for the soldiers for 50 days.

Since the number of soldiers is reduced to 500, the remaining food would now support them for more than 50 days.

Soldiers	days
560	50
500	$x$

∴ It is an Inverse Ratio

$$560 : 500 = x : 50$$

$$x = \frac{560 \times 50}{500} \quad x = 56 \text{ days}$$

Hence the food would last for 56 days for the remaining soldiers.

### EXERCISE 7.3

#### 3 Mark Questions:

- Q1. If ₹ 150 maintains a family of 4 persons for 30 days. How long ₹ 600 maintain a family of 6 persons?
- Q2. 300 workers can finish a work in 8 days. How many workers will finish the same work in 5 days.
- Q3. 5 carpenters can earn ₹ 540 in 6 days working 9 hours a day. How much will 8 carpenters can earn in 12 days working 6 hours a day?
- Q4. A mixture contains milk and water in the ratio 6 : 1 on adding 5 litres of water, the ratio of milk and water becomes 7 : 2, find the quantity of milk in the original mixture.

#### 5 Mark Questions:

- Q5. A jar contains two liquids X and Y in the ratio 7 : 5. When 6 litres of the mixture is drawn and the jar is filled with the same quantity of Y, the ratio of X and Y becomes 7 : 9. Find the quantity X in the jar initially.
- Q6. Two taps fill a cistern separately in 20 minutes and 40 minutes respectively and a drain pipe can drain off 30 litres per minute. If all the three pipes are opened, the cistern fills in 72 minutes what is the capacity of the cistern?
- Q7. If ten persons can do a job in 60 days. In how many days can twenty persons do the same job?
- Q8. A can do a piece of work in 20 days, B in 30 days and C in 60 days. All of them began to work together. However A left the job after 6 days and B quit work 6 days before the completion of work. How many days did the work last?
- Q9. 8 men and 16 women can finish a job in 6 days but 12 men & 24 women can finish it in 8 days. How many days will 26 men and 20 women take to finish the job?
- Q10. 4 men or 12 boys can do a piece of work in 5 days by working 8 hours per day. In how many days 2 men & 4 boys can do the same piece of work working 12 hours a day.

- Q.11 A railway train 100 metres long is running at the speed of 30 kmph. In what time will it pass  
(i) a man standing near the line (ii) a bridge 100 metres long?
- Q.12 The driver of a car is travelling at a speed of 36 kmph and spots a bus 80 metres ahead of him. After 1 hour the bus is 120 metres behind the car. What is the speed of the bus?

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**ANSWERS 7.3**

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**3 Mark Answers**

- 1) 80 days                      2) 480 days                      3) ₹ 1152                      4) 42 liters

**5 Mark Answers**

- 5) 14 litres  
6) 490.9 liters  
7) 20 persons can do the same job in 30 days.  
8) 15.6 days  
9) 267 days  
10) 4 days  
11) (a) 12 sec                      (b) 24 sec  
12) Bus speed = 35.8 kmph.

\*\*\*\*\*

**8.1 Bill of Exchange:**

According to section 5 of the Indian Negotiable Instrument Act, 1881, “A Bill of Exchange is an instrument in writing containing an unconditional order, signed by the maker, directing a certain person to pay a certain sum of money only to or to the order of a certain person or the bearer of the instrument”. In essence, it is a negotiable instrument by which one person agrees to pay another a sum of money at a specified date.

**Specimen of a Bill of Exchange:****Example 1 :**

Stamp	Place : Bangalore Date : 06-07-2013
Three months after date, pay to me or my order, the sum of Rs.25,000 (Rupees Twenty Five Thousand only) for the value received.	
Rs. 25,000	
Accepted sd/- Subhash	sd/- Pranathi

**Fig. 1**

The negotiable instrument becomes receivable only after it is signed by the drawee or his agent on the face of the instrument as shown above or on its back. The bill of exchange should be written on a stamped paper of the court so that it becomes a legal document.

A Bill of exchange can be

- (i) realised by its holder on maturity or retirement of the same from the party liable on it.
- (ii) discounted by its holder with his banker before the due date.
- (iii) endorsed to a creditor by its holder before the due date.
- (iv) renewed for a further period by a mutual agreement between the parties.

We now discuss the various terms associated with this procedure of Bill discounting.

With respect to the bill of exchange in Example (1) [Refer Fig. (1)]

- 1) Creditor (or Drawer) is Ms. Pranathi
- 2) Debtor (or Drawee) is Mr. Subhash
- 3) Drawing Date : Date on which the bill is drawn  
For instance, 06-07-2013 in Example (1)
- 4) Bill period : The period after which the bill is due  
In Example (1), bill period is 3 months.
- 5) Grace period : The bill of exchange becomes legally due on a date 3 days after the due date. These 3 days is called the grace period.
- 6) Legally due date : Legally due date of a bill is calculated as follows:

$$\begin{aligned}\text{Legally due date} &= \text{Date of drawing} \\ &\quad + \text{Bill period} \\ &\quad + \text{Grace period}\end{aligned}$$

**Thus, in Example (1),**

$$\begin{aligned}\text{Legally due date} &= 06-07-2013 \rightarrow \text{Dt of drawing} \\ (+) 0 - 3 - 0 &\rightarrow \text{Bill period} \\ (+) 3 - 0 - 0 &\rightarrow \text{Grace period} \\ \hline &= 09-10-2013\end{aligned}$$

Note : Nominal due date = Date of drawing  
+ Bill period

- 7) Bill Amount : It is the amount due on the legal due date of a Bill. Bill amount is also called face value or nominal value of the bill. In Example (1), ₹ 25,000 is the face value.

## 8.2 Discount Date:

In order to meet one's financial urgency, a bill may be cashed from a bank or a broker before the bill falls legally due. This date on which the banker discounts the bill is the date of discount.

## 8.3 Discount Period

It means the period of early payment on the bill or the unexpired period of the bill. It is calculated by counting the number of days from the date of discount till the legal due date. Discount period is otherwise called the unexpired period, denoted as 't'.

Let the Bill in Example (1) be discounted on 30-08-2013.

Then the discounted period will be from 30-08-2013 to 09-10-2013.

The unexpired period is calculated as shown:

Discount Date : 30-08-2013, which is in August

$\therefore$  Number of days remaining in August =  $31 - 30$

$$= 1 \text{ day} \quad \dots (1)$$

Total Number of days in September = 30 days  $\dots (2)$

We have the legally due date as 09-10-2013, which is in October.

This means, the number of days considered in October = 9 days  $\dots (3)$

$$\begin{aligned} \therefore \text{Total number of days} &= (1) + (2) + (3) \\ &= 1 + 30 + 9 \text{ days} \\ &= 40 \text{ days} \end{aligned}$$

*ie.*, discount period = 40 days.

#### 8.4 Discount Rate:

It is the SI charged by the banker for discounting the bill. It is the rate at which the face value of the bill is discounted to its present value.

#### 8.5 True Present Value (or Present Worth):

Present value of a bill is the amount that would have to be invested today so as to obtain the maturity value of the bill on the due date. It is calculated using the formula

$$P = \frac{F}{1 + tr}$$

where  $t$  is the unexpired period in years,

$r$  is the discount rate in percentages

and  $F$  is the face value of the bill in rupees.

With reference to Example (1), we have already computed the discount period (in section 8.3) as 40 days

$$\text{ie., } t = 40 \text{ days} = \frac{40}{365} \text{ years}$$

$$F = ₹ 25,000$$

Assuming  $r = 15\% \equiv 0.15$

$$\text{we get } P = \frac{25000}{1 + \frac{40}{365} \times 0.15}$$

$$\therefore P = ₹ 24,596.61$$

## 8.6 Discount

Discount on a bill can be allowed either to the debtor or to a banker who advances money on the bill. It is classified into two types viz.,

- (i) True Discount and (ii) Banker's Discount
- (i) **True Discount** : It is basically the difference between face value and the present value of a bill. It is the SI calculated on the present value of a bill, which is given by

$$\boxed{TD = Ptr}$$

where  $t$  is the unexpired period in years

$r$  is the discount rate in percentages.

In Example (1), we get

$$TD = 24,596.61 \times \frac{40}{365} \times 0.15$$

$$\therefore TD = ₹ 404.36$$

- (ii) **Banker's Discount**: Bankers discount is the simple interest calculated on the face value of the bill, for the discount period, at the discount rate. It is calculated using

$$\boxed{BD = Ftr}$$

where  $F$  is the face value of the bill

$t$  is the discount period

$r$  is the discount rate.

Clearly, Banker's discount is always greater than the True Discount.

In Example (1),  $F = ₹ 25000$

$t = 40$  days

$r = 15\%$

$$\therefore BD = 25000 \times \frac{40}{365} \times 0.15$$

$$\therefore BD = ₹ 411$$

## 8.7 Banker's Gain:

The difference between Banker's discount and True discount is the Banker's gain.

$$\boxed{BG = BD - TD}$$

Thus, in Example (1),

$$BG = 411 - 404.36$$

$$\Rightarrow BG = ₹ 6.64$$

### 8.8 Discounted Value of the Bill:

The amount obtained after deducting the bankers discount from the face value of the bill is called the discounted cash value of the bill.

Hence, Discounted value,  $\boxed{DV = F - BD}$

In Example (1),  $F = ₹ 25,000$

$$BD = ₹ 411$$

$$\therefore \text{Discounted Value} = 25,000 - 411 = ₹ 24,589$$

Discounted value is also termed as the **Banker's present value**.

**We list below a few important formulae related to bill discounting:**

$1) \quad F = \frac{BD \times TD}{BG}$	$2) \quad BG = \begin{cases} \frac{F(tr)^2}{1 + tr} \\ TD \cdot tr \\ BD - TD \end{cases}$
$3) \quad TD = \frac{Ftr}{1 + tr}$	$4) \quad \text{Discounted Value} = F(1 - tr)$
$5) \quad \text{Legally due date : Draw date + Bill period + 3 days grace}$	
$6) \quad BD = Ftr$	

### WORKED EXAMPLES

#### Example 2:

**For ₹ 512.50 due 6 months at 15% p.a., find the true present worth and discounted cash value.**

#### Solution:

Given  $F = ₹ 512.50$

$$t = 6 \text{ m} = \frac{1}{2} \text{ yr}$$

$$r = 15\% = 0.15$$

$$\therefore P = \frac{F}{1 + tr} = \frac{512.50}{1 + \frac{1}{2} \times 0.15}$$

$$\text{i.e., } P = ₹ 476.44$$

$$\text{Discounted value} = F(1 - tr) = 512.50 \left( 1 - \frac{1}{2} \times 0.15 \right)$$

$$\text{i.e., Discounted value} = ₹ 474.06$$

**Example 3:**

**Banker's gain on a bill due after 6 months at 4% p.a. is ₹ 24. Find TD, BD, bill amount and discounted value of the bill.**

**Solution:**

Given BG = ₹ 24

$$t = 6 \text{ m} = \frac{1}{2} \text{ y}$$

$$r = 4\% = 0.04$$

We have  $BG = TD \text{ tr}$

$$\Rightarrow 24 = TD \times \frac{1}{2} \times 0.04$$

$$\therefore \text{TD} = ₹ 1200$$

Bankers gain, BG = BD – TD

$$\therefore BD = BG + TD = 24 + 1200$$

$$\text{ie., } \text{BD} = ₹ 1224$$

Further,  $BD = F \text{tr}$

$$\therefore 1224 = F \times \frac{1}{2} \times 0.04$$

$$\Rightarrow F = ₹ 61,200$$

$$\text{Discounted value} = F - BD = 61200 - 1224$$

$$\therefore \text{Discounted value} = ₹ 59,976$$

**Example 4:**

**A Banker discounts a bill for a certain amount having 73 days to run before it matures at 15% p.a. The discounted value of the bill is ₹970. What is the face value of the bill? Also find the banker's discount.**

**Solution:**

$$\text{Given: } t = 73 \text{ days} = \frac{73}{365} \text{ y}$$

$$\text{i.e., } t = \frac{1}{5} \text{ yr}$$

$$r = 15\% = 0.15$$

$$\text{Discounted value} = ₹ 970$$

$$\text{Discounted value} = F(1 - tr)$$

$$970 = F\left(1 - \frac{1}{5} \times 0.15\right)$$

$$\therefore F = ₹1000$$

$$BD = F - \text{Discounted value} = 1000 - 970$$

$$\therefore BD = ₹30$$

**Example 5:**

**A bill drawn for 3 months was legally due on 06-07-2009. Find the date of drawing of the bill.**

**Solution:**

Given Legally due date as 06-07-2007

Bill period = 3 months

We know that

Legally Due Date = Date of drawing

(+) Bill period

(+) 3 days grace

Now, working backwards, we get

Date of drawing = Legally due date

(-) Bill period

(-) 3 days grace period

$\therefore$  Date of drawing = 06-07-2009

(-) 0 - 3 - 0

(-) 3 - 0 - 0

= 03 - 04 - 2009

**Example 6 :**

**A bill of ₹1,460 was drawn on 1st April for 6 months after date and was discounted at 5% p.a. for ₹ 1451. On what date was the bill discounted?**

**Sol.** Legal due date = 01 - 04 → Date of drawing

(+) 0 - 6 → Bill period

+ 3 - 0 → Grace period

= 04 - 10

ie., Legal due date is 4 October.

Given that  $F = ₹1460$

Discounted value = ₹1451

$$\therefore BD = 1460 - 1451$$

$$BD = ₹9$$

But,  $BD = F t r$

$$9 = 1460 \times t \times 0.05$$

$$\Rightarrow t = \frac{9}{73} \text{ years}$$

$$t = \frac{9}{73} \times 365 \Rightarrow t = 45 \text{ days}$$

*ie.*, the bill was discounted 45 days before the legally due date. Hence, we calculate backwards to find the date of discount.

Legal due date : 4 October

*ie.*, In October : 4 days

In September : 30 days

In August : 11 days

45 days

which means the date of discount falls in the month of August.

Total number of days in August = 31

$$\begin{array}{r} (-) 11 \\ \hline 20 \end{array}$$

*ie.*, date of discount is 20 August.

### Example 7:

**A bill of ₹5000 drawn on 10-4-1998 at 3 months was discounted on 1-5-1998 at 12% p.a. For what sum was the bill discounted and how much has the banker gained in this?**

### Solution:

Here  $F = ₹ 5000$

Bill period = 3 months

$\therefore$  Legally due date = 10 - 4 - 98 (Date of drawing)

(+) 0 - 3 - 0 (Bill period)

+ 3 - 0 - 0 (Grace period)

= 13 - 7 - 98

Date of discount = 1 - 5 - 98

This means, the number of days from 1-5-98 to 13-7-98 becomes the unexpired period.

This may be calculated as follows:

1-5-98 to 13-7-98

∴ In may = 30 days (31 – 1)

In June = (+) 30 days

In July = (+) 13 (∴ 13-7-98 is the due date legally)

Total = 73 days

$$\text{ie., } t = 73 \text{ days} = \frac{73}{365} \text{ year}$$

$$t = \frac{1}{5} \text{ year}$$

Given  $r = 12\%$

$$\therefore \text{BD} = Ftr = 5000 \times \frac{1}{5} \times 0.12$$

$$\text{BD} = ₹120$$

$$\text{Discounted value} = F - \text{BD} = 5000 - 120$$

$$\therefore \text{Discounted value} = ₹4,880$$

To find BG, we must first calculate P and TD.

$$P = \frac{F}{1 + tr} = \frac{5000}{1 + \frac{1}{5} \times 0.12}$$

$$\text{ie., } P = ₹4882.81$$

$$\text{TD} = P t r = 4882.81 \times \frac{1}{5} \times 0.12$$

$$\text{TD} = ₹117.18$$

$$\therefore \text{BG} = \text{BD} - \text{TD} = 120 - 117.18$$

$$\text{ie., BG} = ₹2.82$$

**Example 8:**

The banker's gain on a bill is  $\frac{1}{5}$  th of the banker's discount and the rate of interest is 20% p.a. Find the unexpired period of the bill.

**Solution:**

Given:  $BG = \frac{1}{5} BD, \quad r = 20\%$

we know that  $F = \frac{BD \times TD}{BG} = \frac{BD \times TD}{\frac{1}{5} BD}$

$$\Rightarrow F = 5 TD$$

$$\text{ie., } F = 5 \left( \frac{Ftr}{1 + tr} \right)$$

$$\Rightarrow 1 + tr = 5 tr$$

$$\text{ie., } 1 + t \times 0.2 = 5t \times 0.2$$

solving for t, we get

$$t = 1.25 \text{ years.}$$

**Example 9 :**

A bill for ₹ 2920 drawn at 6 months was discounted on 10-4-97 for ₹ 2916. If the discounted rate is 5% p.a., on what date was the bill drawn?

**Solution:**

Given:  $F = ₹ 2920$

$$\text{Bill period} = 6m = \frac{1}{2} y$$

$$r = 5\%$$

$$BD = F - \text{discounted value} = 2920 - 2916$$

$$\therefore BD = ₹ 4$$

Since  $BD = Ftr$ , we get

$$4 = 2920 \times t \times 0.05$$

$$\Rightarrow t = 10 \text{ days}$$

ie., Legally due date is 10 days after 10-4-97

$\therefore$  Legally due date is 20-4-97

Thus, date of drawing = 20-4-97 (Legal due date)

(−) 0 - 6 - 0 (Bill Period)

(−) 3 - 0 - 0 (Grace period)

Date of drawing = 17-10-96

**Example 10 :**

**The difference between BD and TD on a certain sum of money due in 6 months is ₹ 27. Find the amount of the bill if the rate of interest is 6% p.a.**

**Solution:**

Given:  $BD - TD = 27$

ie.,  $BG = ₹ 27$

Since  $BG = TD$ . t.r, we get

$$27 = TD \frac{6}{12} \times 0.06$$

$$\Rightarrow TD = ₹ 900$$

Then  $BD = BG + TD$

$$\therefore BD = ₹ 927$$

Now,  $BD = F t r$

$$\Rightarrow 927 = F \times \frac{6}{12} \times 0.06$$

$$\therefore F = ₹ 30,900$$

**Example 11 :**

**A banker pays ₹ 4520 on a bill of ₹ 5000, 146 days before the legally due date. Find the rate of discount charged by the banker.**

**Solution:**

Given:  $F = ₹ 5000$

Discounted value = ₹ 4520

$$t = 146 \text{ days} = \frac{146}{365} \text{ year}$$

$$r = ?$$

$$BD = F - \text{Discounted value} = 5000 - 4520$$

$$\therefore BD = ₹ 480$$

$$BD = Ftr \Rightarrow 480 = 5000 \times \frac{146}{365} \times r$$

$$\Rightarrow r = 0.24$$

$$\Rightarrow r = 0.24 \times 100$$

$$\therefore \boxed{r = 24\%}$$

**Example 12 :**

**The present worth of a bill due sometime hence is ₹ 1100 and TD on the bill is ₹ 110. Find BD and BG.**

**Sol.** Given  $P = ₹ 1100$

$$TD = ₹ 110$$

To find BD and BG

We know that  $TD = P t r$

$$\therefore 110 = 1100 \times tr$$

$$\Rightarrow tr = 0.1 \quad \dots(1)$$

consider  $BG = TD \times tr$

$$\therefore BG = 110 \times 0.1 \text{ [using (1)]}$$

$$\therefore \mathbf{BG = ₹ 11}$$

$$BD = BG + TD$$

$$BD = ₹ 121$$

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**EXERCISE 8.1**

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**One mark questions:**

- 1) A bill was drawn on 14-3-2013 for 3 months. When does the bill fall legally due?
- 2) A bill drawn for 3 months was legally due on 18-8-2012. Find the date of drawing of the bill.
- 3) Find the present value of ₹ 750 due 4 months hence at 15% p.a.
- 4) Find the banker's discount on a bill of ₹ 415 due 9 months hence at 15% p.a.
- 5) Define Banker's Gain.
- 6) Define Banker's Discount.
- 7) Find the present value of ₹ 2320, due 2 years hence, at 8% per annum.
- 8) Find the BD on ₹ 1015, payable after 3 months at 6% p.a.
- 9) Find the true discount on ₹ 1380, due  $1\frac{1}{2}$  years after, at 10% p.a.

**Two marks questions:**

- 1) TD on a bill was ₹ 100 and BG was ₹ 10. What is the face value of the bill?
- 2) A banker pays ₹ 2380 on a bill of ₹ 2500, 73 days before the legal due date. Find the rate of discount charged by the banker.
- 3) The Banker's discount and true discount on a sum of money due 3 months hence are ₹ 154.50 and ₹ 150 respectively. Find the sum of money and the rate of interest.
- 4) BD and BG on a certain bill due after sometime are ₹ 1250 and ₹ 50 respectively. Find the face value of the bill.

**Five marks questions:**

- 1) The difference between banker's discount and true discount on a bill due after 6 months at 4% interest p.a. is ₹ 20. Find the true discount, banker's discount and face value of the bill.
- 2) The BG on a certain bill due 6 months hence is ₹ 10, the rate of interest being 10% p.a. Find the face value of the bill and the true present value.
- 3) A banker discounts a bill for a certain amount having 32 days to run before it matures at 15% p.a. The discounted value of the bill is ₹ 995.90. What is the face value of the bill, banker's discount, true discount and banker's gain?
- 4) A bill for ₹ 14,600 drawn at 3 months after date was discounted on 11-11-99 for ₹ 14,320. If the discount rate is 20% p.a., on what date was the bill drawn?
- 5) A bill for ₹ 2920 was drawn on September 11 for 3 months after date and was discounted at 16% p.a. for ₹ 2875.20. On what date was the bill discounted.
- 6) A bill for ₹ 3500 due for 3 months was drawn on 27 March 2012 and was discounted on 18 April 2012, at the rate of 7% p.a. Find the banker's discount and discounted value of the bill.
- 7) A bill for ₹ 12900 was drawn on 3 Feb 2004 at 6 months and discounted on 13 March 2004 at 8% p.a. For what sum was the bill discounted and how much did the banker gain in this transaction?
- 8) The banker's gain on a bill is  $\frac{1}{9}$  th of the banker's discount, rate of interest being 10% p.a. Find the unexpired period of the bill.
- 9) A bill for ₹ 2725.25 was drawn on 03-6-2010 and made payable 3 months after due date. It was discounted on 15-6-2010 at 16% per annum. What is the discounted value of the bill and how much did the banker gain?

**ANSWERS 8.1**

**One mark questions:**

- |                  |                  |             |
|------------------|------------------|-------------|
| 1) 17 - 6 - 2013 | 2) 15 - 5 - 2012 | 3) ₹ 714.28 |
| 4) ₹ 46.68       | 7) ₹ 2000        | 8) ₹ 15.22  |
| 9) ₹ 180         |                  |             |

**Two marks questions:**

- |             |        |                           |
|-------------|--------|---------------------------|
| 1) ₹ 1100   | 2) 24% | 3) $F = ₹ 5150, r = 12\%$ |
| 4) ₹ 30,000 |        |                           |

**Five marks questions:**

- |  |                                   |   |
|--|-----------------------------------|---|
| 1) $TD = 1,000$<br>$BD = 1,020$<br>$F = 51,000$            | 2) $F = ₹ 4,200$<br>$P = ₹ 4,000$ | 3) $F = ₹ 1009.17$<br>$BD = ₹ 13.27$<br>$TD = ₹ 13.09$<br>$BG = ₹ 0.18$ |
| 4) 13 September 99   | 5) 9 November                     |   |
| 6) $BD = ₹ 49$<br>Discounted value = ₹ 3451                |                                   |   |
| 7) Discounted value = ₹ 12,487.20<br>Bankers gain = ₹ 12.8 |                                   |   |
| 8) 15 months (1.25 years)                                  |                                   |   |
| 9) Discounted value = ₹ 2624.96<br>Banker's gain = ₹ 3.57  |                                   |   |

\* \* \* \* \*

**9.1 Introduction :**

Wouldn't you love to be a business owner without ever having to show up at work? Imagine if you could sit back, watch your company grow and collect the dividend cheques as the money rolls in. This situation might sound like a dream, but it's closer to reality than you might think.

As you've probably guessed we are talking about owning stocks or shares. This fabulous category of financial instruments is, without a doubt, one of the greatest tools ever invented for building wealth. Over the last few decades the average person's interest in the stock market has grown exponentially. What was once a toy of the rich has now turned into the vehicle of choice of growing wealth. This demand coupled with advances in trading technology has opened up the markets so that nowadays nearly anybody can own stocks.

**9.2 The definition of a stock**

**Stock is a share in the ownership of a company. A stock is represented by a stock certificate.** This is a fancy piece of paper that is proof of your ownership. In today's computer age, you won't actually get to see this document because your broker keeps these records electronically, which is also known as holding shares "in street name". This is done to make the shares easier to trade. In the past when a person wanted to sell his or her shares, that person physically took the certificates down to the broker. Now, trading with a click of the mouse or a phone call makes life easier for everybody.

Whether you say shares, equity or stock it all means the same thing. In today's financial markets, the distinction between stocks and shares has been somewhat blurred. Generally, these words are used interchangeably. However, the difference between the two words comes from the context in which they are used. For example, "Stock" is a general term used to describe the ownership certificates of any company, in general and 'shares' refers to the ownership certificates of a particular company. So, if investors say they own stocks, they are generally referring to their overall ownership in one or more companies technically if someone says that they own shares the question then becomes shares in which company? Bottom line, stocks and shares are the same thing.

**Being an Owner:** Holding a company's stock means that you are one of the many owners (shareholders) of a company and as such, you have a claim to everything the company owns. But, being a share holder of a public company does not mean you have a say in the day-to-day running of the business. Instead one vote per share to elect the board of directors at annual meetings is the extent to which you have a say in the company. For instance being a Microsoft shareholder does not mean you can call up Bill Gates and tell him how you think the company should be run.

### 9.3 Nominal Interest or dividend

**Profits are sometimes paid out in the form of dividends.** The more shares you own, the larger the portion of the profits you get. However some companies pay out dividends but many others do not. There is no obligation to pay out dividends even for those firms that have traditionally given them. Without dividends an investor can make money on a stock only through its appreciation in the open market. On the downside any stock may go bankrupt, in which case your investment is worth nothing.

**Limited Liability:** As an owner of a stock you are not personally liable if the company is not able to pay its debts. Owning a stock means that, no matter what, the maximum value you can lose is the value of your investment. Even if a company of which you are a shareholder goes bankrupt, you can never lose your personal assets.

Although risk might sound all negative, there is also a bright side Taking on greater risk demands a greater return on your investment. This is the reason why stocks have historically out performed, other investments such as bonds or saving accounts. Over the long term, an investment in stocks has historically had an average return of around 10-12%.

**Nominal and market value of a share:**

**The original value of a share is called its nominal value or face value and the price at which it is selling is called its market value.**

- a) When the market value of a share is higher than its nominal value, it is said to be at premium or above par
- b) When the market value of a share is equal to its nominal value the share is said to be at par
- c) When the market value of a share is less than its market value it is said to be at discount or below par.

#### Example

1. 8% stock at 125 means.

A stock of face value 100 is being sold at ₹125 (Market Value) and the nominal interest obtained on this share is ₹ 8.

2. 12% stock at 20 premium means face value is 100, Market Value is 120 and dividend is ₹12.
3. 7% stock at 10 discount means face value is 100, market value is 90 and interest is ₹ 7
4. 10% stock at par means

Face value is 100, market value is 100, interest is ₹10.

Note: (1) Nominal interest or dividend is always declared on face value of share.

(2) The face value of a share can be ₹ 2, ₹ 5, ₹ 10 etc. but if nothing is mentioned in the problem then it is understood that the face value of one share is ₹ 100.

(3) MV means market value and FV means face value.

**Example 1:**

**What income can be obtained from ₹ 8000 of 4% stock.**

**Solution:**

4% stock means if the face value stock is ₹100 income or dividend obtained is ₹4.

Face Value	Income
100	4
8000	?

$$\therefore \text{Income obtained is } \frac{8000 \times 4}{100} = ₹ 320$$

**Example 2:**

**How much stock can be purchased by investing ₹ 9600 in 3.5% stock at 120.**

**Solution:**

Note here that ₹ 9600 is market value and we are required to find face value.

Face Value	MV
100	120
?	9600

$$\therefore \text{Amount of stock} = \frac{100 \times 9600}{120} = ₹ 8000$$

**Example 3:**

**How much does Maya realize by selling ₹ 30,000 stock at 20 discount.**

**Solution:**

Note here that Maya is selling stock worth ₹ 30,000. So here 30,000 is face value. Also ₹ 20 discount means cost of ₹100 stock is ₹ 80.

FV	MV
100	80
30,000	?

$$\therefore \text{Money obtained} = \frac{30,000 \times 80}{100} = ₹ 24,000$$

**Example 4 :**

**Ayush buys ₹ 10,000 stock at 96 and sells when its price rises to ₹102. Find his gain.**

**Solution :**

While Buying

FV	MV
100	96
10,000	?

$$\text{Cost of stock} = \frac{10,000 \times 96}{100} = ₹ 9600 \text{ (Money spent)}$$

While Selling

FV	MV
100	102
10,000	?

$$\text{Money obtained} = \frac{102 \times 10,000}{100} = 10,200$$

$$\therefore \text{Gain} = 10,200 - 9,600 = ₹ 600$$

OR

$$\text{Gain on ₹ 100 share} = 102 - 96 = 6$$

$$\therefore \text{gain on ₹ 10,000 share} =$$

FV	gain
100	6
10,000	?

$$= \frac{6 \times 10,000}{100} = ₹ 600$$

**Example 5:**

**What is the market value of 12% stock when an investment of ₹ 6900 produces an income of ₹ 720.**

**Solution:**

Given for an investment of ₹ 6900 income is ₹ 720. We have to find the investment (market value) which can earn an income of ₹ 12.

MV	Income
6900	720
?	12

$$\therefore \text{Market value of share} = \frac{6900 \times 12}{720} = ₹ 115$$

**Example 6:**

**Sukanya holds ₹ 8000 of 3% stock. She sells it at ₹ 110 and invests the proceeds in 5% stock. Thereby her income increases by ₹ 260. Find the market price of 5% stock.**

**Solution: Case 1 : 3% stock**

FV	Income
100	3
8000	?

$$\text{Income obtained from 3\% stock} = \frac{8000 \times 3}{100} = ₹ 240$$

To calculate cash obtained by selling 3% stock

FV	MV
100	110
8000	?

$$\therefore \text{Cash obtained} = \frac{8000 \times 110}{100} = ₹ 8800$$

**Case 2 : 5% stock**

Money invested in 5% stock is 8800

$$\begin{aligned} \text{Income obtained from 5\% stock} \\ = 240 + 260 = ₹ 500 \end{aligned}$$

To calculate Market value of ₹ 100 at 5% stock

MV	Income
8800	500
?	5

$$\therefore \text{Market value} = \frac{5 \times 8800}{500} = ₹ 88$$

**Example 7:**

**Raksha sells her 9,600 4% stock at a premium of 6. How much 3.5% stock at 4 discount can she buy from the sale proceeds of the former stock. What will be the difference in her income.**

**Solution : Case 1 : 4% stock**

FV	Income
100	4
9600	?

$$\text{Income obtained} = \frac{4 \times 9600}{100} = ₹ 384$$

By selling this stock

FV	MV
100	106
9600	?

$$\text{Money obtained by selling} = \frac{9600 \times 106}{100} = ₹ 10,176$$

**Case 2 : 3.5% stock**

Money invested in 3.5% stock = 10,176

FV	MV
100	96
?	10,176

stock purchased from sale proceeds is

$$= \frac{10,176 \times 100}{96} = 10,600$$

To calculate income

FV	MV	Income
100	96	3.5
10,600	10,176	?

[Here 1st & 3rd column or 2nd & 3rd column may be used for calculation]

$$\therefore \text{Income from 3.5\% stock} = \frac{10,600 \times 3.5}{100} = 371$$

Difference in Income = 384 – 371

Hence income decreases by ₹13

**Example 8:**

**A man sells ₹5000, 4½% stock at 144 and invests the proceeds partly in 3% stock at 90 and partly in 4% stock at 108. He thereby increases his income by ₹25. How much of the proceeds were invested in each stock.**

**Solution: Case 1: 4½ stock**

FV	MV	Income
100	144	4.5
5000	?	?

To calculate Income consider first and 3rd column

$$\text{Income} = \frac{5000 \times 4.5}{100} = 225 \quad \dots\dots(1)$$

$$\text{Money obtained by selling at 144} = \frac{5000 \times 144}{100} = 7200$$

[consider first and second column]

Let the amount invested in 3% stock be  $x$

Let the amount invested in 4% stock be  $7200 - x$

$$\text{Income from 3\% stock } I_1 = \frac{3x}{90}$$

$$[\text{Note income} = \frac{\text{dividend of one share} \times \text{money invested}}{\text{market value of 1 share}}]$$

$$\text{Income from 4\% stock } I_2 = \frac{4}{108} (7200 - x)$$

given total income increases by ₹ 25

$$\therefore I_1 + I_2 = 225 + 25$$

$$\frac{3x}{90} + \frac{4(7200 - x)}{108} = 250$$

$$\frac{x}{30} + \frac{7200 - x}{27} = 250$$

$$\frac{27x + 2,16,000 - 30x}{810} = 250$$

$$2,16,000 - 3x = 2,02,500$$

$$2,16,000 - 2,02,500 = 3x$$

$$\frac{13500}{3} = x$$

$$\therefore x = 4500$$

Investment in 3% stock = ₹ 4,500

Investment in 4% stock =  $7200 - 4500 = ₹ 2700$

### Example 9:

**A person sells out ₹ 4000 of 6.25 Government of India stock at 112.5 and re-invests the proceeds in 8% railway debentures, thereby increasing his income by ₹50. At what price did he buy the debentures.**

**Solution: Case 1 : 6.25 Govt. of India stock**

To calculate income

FV	Income
100	6.25
4000	?

$$\text{Income} = \frac{4000 \times 6.25}{100} = ₹ 250$$

To calculate amount received by selling

FV	MV
100	112.5
4000	?

$$\text{Amount received} = \frac{4000 \times 112.5}{100} = ₹ 4500$$

**Case 2:** 8% railway debentures

given Income increases by 50

$$\therefore \text{New Income} = 250 + 50 = 300$$

To calculate market value of ₹ 100 debenture

MV	Income
?	8
4500	300

$$\text{Market value} = \frac{4500 \times 8}{300} = ₹ 120$$

**Example 10:**

**Rahul invests ₹ 1,74,000 partly in 12% stock at 110 and partly in 15% stock at 86. If the total income from both is 24,600, find the amount invested by Rahul in 12% stock.**

**Solution :** Let Amount invested in 12% stock be  $x$ . Then amount invested in 15% stock is  $1,74,000 - x$ .

Income from 12% stock

MV	Income
110	12
$x$	?

$$I_1 = \frac{12x}{110}$$

Income from 15% stock

MV	Income
86	15
$1,74,000 - x$	?

$$I_2 = \frac{(1,74,000 - x)15}{86}$$

Given total income = 24,600

$$I_1 + I_2 = 24,600$$

$$\frac{12x}{110} + \frac{(174000 - x) \times 15}{86} = 24,600$$

$$\frac{12x}{110} + \frac{174000 \times 15}{86} - \frac{15x}{86} = 24,600$$

$$\frac{174000 \times 15}{86} - 24,600 = \frac{15x}{86} - \frac{12x}{110}$$

$$\frac{2610000 - 2115600}{86} = \frac{1650x - 1032x}{86 \times 110}$$

$$\frac{494400 \times 86 \times 110}{86} = 618x$$

$$88000 = x$$

$\therefore$  Rahul invest 88000 in 12% stock.

Note : The nominal value or face value of a share is generally ₹ 10 or ₹ 25 or ₹ 100. If nothing is mentioned in the problem then we assume the face value is ₹ 100.

#### Example 11:

**Shreya holds 1200 shares of a company each of face value ₹ 10. She sells 400 shares at the rate of ₹ 12 per share and the remaining at par. She invests the proceeds in a share of ₹ 100 at ₹ 80. Find the number of shares purchased by Shreya.**

#### Solution :

Amount received by selling 400 shares =  $400 \times 12 = 4800$

Amount received by selling remaining 800 shares =  $800 \times 10 = 8000$ .

$\therefore$  Total amount received =  $4800 + 8000 = 12800$

Cost of one share = 80

$\therefore$  No. of shares purchased for ₹ 12800

$$= \frac{12800}{80} = 160 \text{ shares.}$$

#### 9.4 Yield :

An investor before purchasing a stock also looks at the return or yield or actual interest obtained on particular stock. **The actual percentage income on investment is called yield.**

$$\text{Yield} = \frac{\text{Interest}}{\text{Market Value}} \times 100$$

The yield on 6% stock at ₹ 143 is yield =  $\frac{6}{143} \times 100$

In brief a) Dividend — Annual income on ₹ 100 stock (face value)

b) Yield — Annual income on ₹ 100 investment (market value)

**Example 12:**

**Mr. Sandeep invests ₹15000 cash partly in 3% stock at 75 and partly in 6% debentures at 125 in such a way as to get a return on 4.5% for his money. How much does he invest his money. How much does he invest in each.**

**Solution:**

[Note here return of 4.5% on 15000 invested means yield is 4.5%. However here total income can be calculated without yield formula also]

$$\text{Total Income} = \frac{4.5}{100} \times 15000 = ₹ 675$$

Let Amount Invested in 3% stock be  $x$  then amount invested in 6% debentures is  $15000 - x$

$$\text{Income from 3% stock } I_1 = \frac{3}{75} \times x$$

$$\text{Income from 6% debenture } I_2 = \frac{6}{125} \times (15000 - x)$$

$$I_1 + I_2 = 675$$

$$\frac{3}{75}x + \frac{6}{125}(15000 - x) = 675$$

$$0.04x + 0.048(15000 - x) = 675$$

$$0.04x + 720 - 0.048x = 675$$

$$720 - 675 = 0.048x - 0.04x$$

$$45 = 0.008x$$

$$x = ₹ 5625$$

∴ Amount invested in 3% stock = ₹ 5625

Amount invested in 6% debentures = ₹ 9375

**Example 13:**

**What is the market value of 6% stock if it earns an interest of 4.5% after deducting the income tax of 4%**

**Solution:**

Let the Market value of ₹ 100 stock be  $x$

Then income = ₹ 6

$$\text{Tax deduction} = \frac{4}{100} \times 6 = ₹ 0.24$$

$$\text{Net Income} = 6 - 0.24 = ₹ 5.76$$

Given yield after tax deduction = 4.5%

$$\text{Yield} = \frac{\text{Income}}{\text{MV}} \times 100$$

$$4.5 = \frac{5.76}{x} \times 100$$

$$x = \frac{5.76 \times 100}{4.5} = ₹ 128$$

∴ Market value of ₹ 100 stock is ₹ 128

**Example 14:**

**Which is better investment 7.5% stock at 125 or 5% stock at 80.**

**Solution:**

Here we need to calculate the yield and see which stock gives higher returns.

$$\text{yield on 7.5\% stock} = \frac{7.5}{125} \times 100 = 6\%$$

$$\text{yield on 5\% stock} = \frac{5}{80} \times 100 = 6.25\%$$

∴ 5% stock at 80 is better.

**Example 15:**

**What rate of interest is obtained by investing in 9% stock at 180.**

**Solution :** Rate of interest on money invested is yield itself.

$$\therefore \text{yield} = \frac{\text{interest}}{\text{MV}} \times 100 = \frac{9}{180} \times 100 = 5\%$$

**EXERCISE 9.1**

**One mark questions:**

1. Find the income obtained by investing ₹ 3600 in 5% stock at 90.
2. How much stock at ₹ 75 can be bought for ₹ 3375
3. 6% stock is being sold at 15 discount. How much money is required to buy ₹ 6000 stock
4. Define yield.
5. What is the yield obtained when ₹ 5000, 3% stock is purchased at ₹ 125

**Two marks questions:**

1. What income can be obtained from an investment of ₹ 10,725 in 6.5% stock at 143. What is the amount of stock purchased.
2. 10% stock is quoted at ₹ 120. Find the rate of interest.  
[Hint: here rate of interest means yield]
3. How much of 8% stock at 96 can be purchased for ₹ 4800? Also find the income obtained.
4. How much money will a man get by selling ₹ 6000 stock at 5% stock at 20 premium?
5. Abhi purchased 850 shares of a company the nominal value being ₹ 10 each. Company declared an annual dividend of 12%. How much dividend did Abhi receive.

**Three marks questions:**

1. Rakshitha invests ₹ 15000 in a company paying 7% per annum, when a share of value ₹ 100 is selling for ₹ 150. What is her annual income and what percentage does she get on her money.
2. Which is a better investment
  - a) 8% stock at 110 or 9% stock at 98
  - b) 6½% stock at 84 or 4% stock at 102
  - c) 7% stock at 115 or 5% stock at 88
  - d) 4% stock at 118 or 6% stock at 124
3. A man invests equal sums of money in 4%, 5% and 6% stock, each stock being at par. If the total income of the man is ₹ 3,600. Find his total investment.
4. What is the market value of 9.5% stock when an investment of ₹ 12,400 produces an income of ₹ 1472.5
5. What is the quoted value of 12% stock if it earns an interest of 8% after deducting the income tax of 8%.
6. Venu invested ₹ 5,02,950 in Infosys when price of ₹ 5 shares was 3353. He sold shares worth ₹ 500 when the price went upto 3583 and the remaining he sold when the price was ₹ 3253. How much did Venu gain.

7. Ramesh has invested ₹ 4,300 partly in 4.5% stock at ₹ 72 and partly in 5% stock at ₹ 95. If the total income from both is ₹ 250, find the investment in both the types of stock
8. Ramesh holds ₹ 2,100 of 3% stock. He sells at ₹ 121 and invests the proceeds in 5% stock. Thereby his income increases by ₹ 14. Find the market price of 5% stock.
9. How much must be invested in 14.25% stock at 98 to produce the same income as would be obtained by investing ₹ 9,975 in 15% stock at 105.
10. A stock yields 5% to an investor. A fall of ₹ 5 in its price causes it to yield  $5\frac{1}{2}\%$ . What was its market value if the two income are equal.
11. Sanjana invests ₹ 3240 in a stock at 108 and sells when the price falls to 104. How much stock at 130 can Sanjana now buy.
12. A person invested 42000 partly in 5% stock at 125 and the remaining in 7.5% stock at 75. If income derived from the two stocks is the same. Find the respective investments in each stock. Also find the total income.
13. If a person wishes to obtain 18% yield from his investment at what price should he buy 13.5% stock.
14. a) A invests a sum of money in 5.5% stock at 90 and B an equal sum in 3.5% stock. If A's income is 10% more than B's find the price of 3.5% stock.  
b) How much money has to be invested in 11.5% stock at 73 to obtain an income of ₹ 150 after a tax deduction at source of 20%.
15. a) A person invests ₹ 15000 partly in 3% stock at 75 and partly in 6% stock at 125. If the income from both is ₹ 675 find his investment in the 2 types of stock.  
b) At what price a person has bought 25% shares with face value 100 which has given him 20% returns.
16. Rohan invested ₹ 55000 partly in 8% stock at 80 and partly in 12% stock at 150 in such a way as to get a return of 9% for his money. How much did Rohan invest in each.
17. Prathik sells out ₹ 6000 of 7.5% stock at 108 and reinvests the proceeds in 9% stock. If Prathik's income increases by 270. At what price did Prathik buy 9% stock.
18. Jane sells her ₹ 12500, 4.5% stock at 94. How much of 9% stock at 125 can Jane purchase from the sale proceeds of the former stock. what is the change in Jane's income.
19. A man sells ₹ 25000, 13.5% stock when the shares were selling at a premium of 20. He invests the proceeds partly in 15% stock at 75 and partly in 16% stock at 128. Find how much he has invested in each stock if his income increases by ₹ 1875.
20. Nihal has ₹ 5 Maruthi shares worth ₹ 1000. He sells 120 shares when the shares are selling at ₹ 1400 and the remaining shares when the price goes upto ₹ 1600. He invests the proceeds in ₹ 10 Maruthi shares selling at ₹ 1184 find the number of ₹ 10 Maruthi shares purchased by Nihal.

**ANSWERS 9.1**

**One mark questions:**

- 1) ₹ 200                      2) ₹ 4500 stock                      3) ₹ 5100                      4) 2.4%

**Two marks questions:**

- 1) ₹ 487.5, ₹ 7500 stock                      2) 8.33%  
3) ₹ 5000 stock, ₹ 400                      4) ₹ 7200  
5) ₹ 1020

**Three marks questions:**

- 1) ₹ 700, 4.66%  
2) a) 9% stock                      b) 6½% stock  
c) 7% stock                      d) 6% stock  
3) ₹ 72000                      4) ₹ 80                      5) ₹ 138  
6) ₹ 18000                      7) 2400, 1900                      8) ₹ 165  
9) ₹ 9800                      10) ₹ 55                      11) ₹ 2400  
12) ₹ 30,000 in 5% stock, ₹ 12,000 in 7.5% stock. Income is 1200 + 1200 = 2400  
13) ₹ 75  
14) a) ₹ 63                      b) ₹ 1190.21  
15) a) ₹ 5625, ₹ 9375                      b) ₹ 125  
16) ₹ 27,500 in 8%, ₹ 27,500 in 12%                      17) ₹ 81  
18) ₹ 9400 stock, ₹ 283.5 increase  
19) ₹ 20,000, ₹ 10,000  
20) ₹ 250 = No. of shares

**9.5 Brokerage :**

Stock exchange is the name of the market where the transfer of stock are done through brokers. The brokers charge a small fee as commission from both the buyers as well as from the sellers. This commission is called brokerage. Brokerage is always calculated on the market value.

- a) While selling shares, since you have to pay brokerage the net amount received will be market value – Brokerage.  
b) While buying shares, since you have to pay brokerage, the total amount to be paid is market value + brokerage.

**Example 16 :**

**Prashanth sold 25 Biocon shares when the market price was ₹ 400 per share. He then bought 10 Reliance shares which were selling at ₹ 800 per share. Brokerage for each transaction was 0.25%. The balance amount he gave to his daughter Sonu for shopping. How much did Sonu receive.**

**Solution :**

Amount received by selling

$$\text{Beocon shares} = 25 \times 400 = ₹ 10,000$$

$$\text{Brokerage} = \frac{0.25}{100} \times 10000 = ₹ 25$$

$$\text{Net Amount received} = 10,000 - 25 = ₹ 9975$$

$$\text{Amount paid for Reliance shares} = 800 \times 10 = ₹ 8000$$

$$\text{Brokerage} = \frac{0.25}{100} \times 8000 = ₹ 20$$

$$\text{Net Amount paid} = 8000 + 20 = ₹ 8020$$

$$\text{Balance Amount} = 9975 - 8020 = ₹ 1955$$

∴ Amount paid to Sonu for shopping is ₹ 1955

**Example 17:**

**If a client buys shares worth ₹ 1,25,000 and sells shares worth ₹ 75,000 through a broker assuming that the buying side brokerage is 0.5 and the selling side brokerage is 0.25%. Find the total brokerage paid to the broker.**

**Solution:**

Brokerage while buying

$$= \frac{0.25}{100} \times 1,25,000$$

$$= ₹ 625$$

Brokerage while selling

$$= \frac{0.25}{100} \times 75,000$$

$$= 187.5$$

$$\therefore \text{Total brokerage} = 625 + 187.5 = ₹ 812.5$$

**EXERCISE 9.2**

**Three marks questions:**

1. A man owns 50 SBI shares which are now selling at the rate of 1800. He needs 50,000 for his daughter's education. He decides to sell 25 SBI shares. The brokerage charged is 0.25%. How much more money does he need to arrange after selling the share.
2. Rakshith decides to invest in TCS shares which are selling at 2020 per share. How much money is required to purchase 10 shares if the brokerage is 0.5%
3. Ritu purchased 200 HDFC shares when the price was 625 and then sold all the shares when the price went upto 715. If the brokerage for each transaction was 1%. How much did Ritu gain.
4. If a client buys shares worth ₹ 90,000 and sells shares worth ₹ 1,10,000 through a stock broker calculate the brokerage payable to the stock broker if the brokerage rate is 0.5% each side.
5. Veena buys 100 shares of Karnataka bank at ₹ 101 per share. She pays ₹ 10,130.3 to her broker. What is the total brokerage she paid and calculate the percentage rate of brokerage.
6. Mr. Ravi sold ₹ 2,250 stock at 75 and bought stock at 88.5 with the proceeds. How much stock does he buy if the brokerage is 2% for selling and 1.5% for buying.

**ANSWERS 9.2**

- |             |              |            |           |
|-------------|--------------|------------|-----------|
| 1) ₹ 5112.5 | 2) ₹ 20301   | 3) ₹ 15320 | 4) ₹ 1000 |
| 5) 0.3%     | 6) ₹ 1841.02 |            |           |

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**10.1 Introduction :**

By studying learning curve, we acquire skill and knowledge of performing a particular task repeatedly, we perfect the task and also gain efficiency in doing the same. This is called as **learning process**. The theory of learning curve deals with labour efficiency.

**10.2 Learning curve :**

**Definition :** Learning curve is defined as the curvilinear relation between the decrease in average labour hours per unit with increase in the total output.

**10.3 Learning curve ratio :**

The slope of the learning curve is called as the learning curve ratio. The learning curve ratio is expressed as a percentage

$\text{Learning curve ratio} = \frac{\text{Average Labour cost of first } 2N \text{ units}}{\text{Average labour cost of } 1N \text{ units}}$
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**Example 1 :**

**The average labour cost of a production for 100 units is ₹75. If the learning curve ratio is 0.75. Find the average cost of production for 50 units.**

**Solution :**

Average labour cost of 100 units = ₹ 75

Learning curve ratio = 0.75

Let the average cost of 50 units = ₹  $x$

$$\text{Learning curve ratio} = \frac{\text{Average cost of } 2N \text{ units}}{\text{Average cost of } N \text{ units}}$$

$$0.75 = \frac{75}{x}$$

$$x = \frac{75}{0.75}$$

$$x = \frac{7500}{75}$$

$$x = 100$$

∴ the average labour cost of 50 units = ₹100

**Example 2 :**

**The average labour cost of 100 units is ₹ 50. The learning curve ratio is 0.5. What is the average labour cost of 200 units?**

**Solution :**

Let the average labour cost of 200 units = ₹x

$$\text{Learning curve ratio} = \frac{\text{Average labour cost of } 2N \text{ units}}{\text{Average labour cost } N \text{ units}}$$

$$0.5 = \frac{x}{50}$$

$$0.5 \times 50 = x$$

$$x = 25$$

∴ The average labour cost of 200 units = ₹ 25

**Example 3:**

**The average labour cost for the first 150 units of production by a factory is ₹ 50 when the output was doubled to 300 units, the average labour cost was ₹ 25. Find the learning curve ratio and the learning curve percent.**

**Solution :**

Average labour cost of 150 units = ₹ 50

Average labour cost of 300 units = ₹ 25

$$\begin{aligned}\text{Learning curve ratio} &= \frac{\text{Average cost of first } 2N \text{ units}}{\text{Average cost of first } N \text{ units}} \\ &= \frac{\text{₹}25}{\text{₹}50} = 0.5\end{aligned}$$

$$\text{Learning curve percent} = 0.5 \times 100 = 50\%$$

**10.4 Learning effect:**

The efficiency in performance of a labourer increases by repeatedly doing the task. This is due to the learning process and the learning effect improves the production. **For example** 80% learning effect means that when the cumulative output is doubled the cumulative average labour hours per unit will be 80% of the previous level.

**The following tables show the learning effect of 90%, 80%, 70%, 60% and 50% when the time required to produce 1 unit is 100 hrs.**

**Table 1 : 90% learning effect**

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100
1	2	90% of 100 $\frac{90 \times 100}{100} = 90$	180	$\frac{180 - 100}{2 - 1} = 80$
2	4	90% of 90 $\frac{90 \times 90}{100} = 81$	324	$\frac{324 - 180}{4 - 2} = 72$
4	8	90% of 81 = 72.9	583.2	$\frac{583.2 - 324}{8 - 4} = 64.8$
8	16	80% of 72.9 = 65.6	1104.76	$\frac{1104.76 - 583.2}{16 - 8} = 58.32$

**Table 2 : 80% learning effect**

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100.00
1	2	80% of 100 = 80	160	60.00
2	4	80% of 80 = 64	256	48.00
4	8	80% of 64 = 51.2	409.6	38.40
8	16	80% of 51.2 = 40.96	655.3	30.72

**Table 3 : 70% learning effect**

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100.00
1	2	70% of 100 = 70	140	40.00
2	4	70% of 70 = 49	196	28.00
4	8	70% of 49 = 34.30	274.4	19.60
8	16	70% of 34.30	384.16	13.72

**Table 4 : 60% learning effect**

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100
1	2	60% of 100 = 60	120	20
2	4	60% of 60 = 36	144	12
4	8	60% of 36 = 21.6	172.8	7.2
8	16	60% of 21.6 = 12.96	207.36	4.32

**Table 5 : 50% learning effect**

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100.00
1	2	50% of 100 = 50	100	00.00

**Note :** From this table it is clear that learning curve of 50% or less is mathematically not possible.

Generally 80% learning effect is commonly used in day to day life.

### 10.5 Learning curve equation

The equation  $y = a \cdot x^b$  represents learning curve equation. Here

$y$  = Cumulative average time per unit

$a$  = Time for first unit

$x$  = Cumulative total number of units produced

$$b = \text{index of learning} = \frac{\log(\text{learning effect})}{\log 2}$$

Taking log on both the sides of learning equation we have;

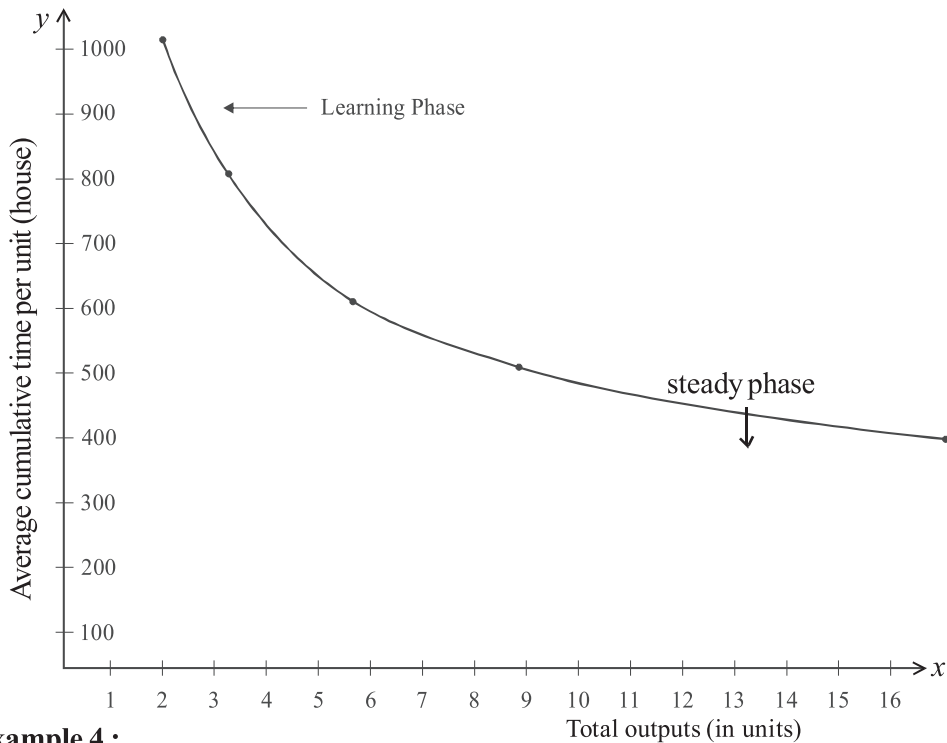
**Which is a Linear Equation**

$$\log y = \log a + b \log x$$

**Table for the Graph**

Total output in units	Cumulative and time per unit
1	1000
2	800
4	640
8	512
16	409.6

Graphical representation of the learning curve (80% learning effect)



**Example 4 :**

Give the formula for learning index.

**Solution :**

$$\text{Learning index} = \frac{\log (\text{learning effect})}{\log 2}$$

**Example 5 :**

The production manager of a company obtained the following equation for the learning effect.

$$y = 1356 x^{-0.3219}$$

This function is based on the company's experience for assembling the first 50 units of the product. Find the labour hours required to assemble 100 units.

**Solution :**

$$y = a x^b \quad \text{Apply log on both sides}$$

$$\log y = \log a + b \log x$$

$$y = \text{average cumulative time period}$$

$a$  = time taken for 1 lot / 1 unit

$x$  = cumulative total no. of lots

$b$  = Index of learning

where  $a = 1356.4$ ,  $x = 2$ ,  $b = -0.3219$

$\log y = \log a + b \log x$

$\log y = \log 1356 + -0.3219 \log 2$

$\log y = 3.1324 - 0.3219 (0.3010)$

$\log y = 3.1324 - 0.0969$

$\log y = 3.0355$

$y = \text{Anti log } 3.0355 = 1085.18 \text{ hrs}$

Total time for 2 units =  $2 \times 1085.18 \text{ hrs} = 2170.35 \text{ hrs}$ .

**Example 6 :**

**A Motor Company Ltd., has observed that a 90% learning effect applies to all labour related costs. Whenever a new product is taken up for production, the anticipated production to 320 units for the coming year. The production is done in lots of 10 units each. Each lot requires 1000 hours at ₹ 15/hour. Calculate the total labour hours and labour cost to manufacture 320 units.**

**Solution :** 1 lot = 10 unit

Lots produced	Total output in Lots	Cumulative average time per unit (hours)	Total hours
1	1	1000	1000
1	2	90% of 1000 = 900	1800
2	4	90% of 900 = 810	3240
4	8	90% of 810 = 729	5832
8	16	90% of 729 = 656.1	10,497.6
16	32	90% of 656.1 = 590.49	18,895.68

The total hours is 18,895.68 hrs. and the labour cost at

₹ 15 per hr =  $18,895.68 \times 15$

= ₹ 2,83,435.20

**Example 7 :**

**An Engineering company has 80% learning effect and spends 500 hours for the prototype. Estimate the labour cost of producing 7 engines of new order if the labour cost is ₹ 40 per hour.**

**Solution :**

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours
1	1	500	500
1	2	80% of 500 = 400	800
2	4	80% of 400 = 320	1280
4	8	80% 320 = 256	2048

Total hours of 8 engines = 2048

Total hours of 1 engine = 500

Total hours for 7 engines = 1548 hrs.

If the labour cost ₹ 40 per hour.

Total labour cost of producing 7 engines =  $1548 \times 40 = ₹ 61920$

**Example 8 :**

**The average labour cost for first 100 units produced by a firm is ₹ 25. When the output was doubled to 200 units, the average labour cost was ₹ 20. Determine learning curve ratio and learning curve percent.**

**Solution :** Learning curve ratio =  $\frac{\text{Average labour cost for } 2N \text{ units}}{\text{Average labour cost for } N \text{ units}}$

$$= \frac{20}{25} = \frac{4}{5} = 0.8$$

Learning curve percent =  $0.8 \times 100 = 80\%$

**Example 9 :**

**xyz company supplies water tankers to the Government. The first water tanker takes 20,000 labour hours. The government auditors suggest that there should be 90% learning effect rate. The management expects an order of 8 water tankers in the next year. what will be the labour cost the company will incur at the rate of ₹ 20 per hour ?**

**Solution :**

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours
1	1	20,000	20,000
1	2	90% of 20,000 = 18,000	36,000
2	4	90% of 18,000 = 16,200	64,800
4	8	90% of 16,200 = 14,580	1,16,640

Time taken for 8 water tankers = 116640 hrs.

Total cost the company will incur at the 20 per hour

$$= 116640 \times 20 = ₹ 23,32,800$$

**Example 10 :**

**A company requires 100 hours to produce the first 10 units at ₹ 15 per hour. The learning curve effect is 80%. Find the total labour cost to produce a total of 160 units.**

**Solution :** Assume 10 unit = 1 lot

Units produced	Total output in lots	Cumulative average time per lot (hours)	Total hours
1	1	100	100
1	2	80% of 100 = 80	160
2	4	80% of 80 = 64	256
4	8	80% of 64 = 51.2	409.6
8	16 (160 unit)	80% of 51.2 = 40.96	655.36

Time taken for 160 units = 655.36 hrs.

Total cost the company has to incur at the rate of 15 ₹ per hour

$$= 655.36 \times 15 = ₹ 9830.40$$

**EXERCISE 10.1**

**One mark questions:**

1. Define learning curve.
2. Write the formula for learning index.
3. Find the index of learning for 70% learning effect.
4. Find the index of learning for 80% learning effect.
5. Write the learning curve equation.

**Five mark questions:**

6. An engineering company has 80% learning effect and spends 1000 hours to produce 1 lot of the product. Estimate the labour cost of producing 8 lots of the product if the labour cost is ₹ 40 per hour.
7. ABC company required 1000 hours to produce first 30 engines. If the learning effect is 90%. Find the total labour cost at ₹ 20 per hour to produce total of 120 engines?
8. The production manager of a company obtained the following equation for the learning effect.

$$y = 1400 x^{-0.3}$$

This function is based on the company's experience for assembling the first 50 units of the product. The company was asked to bid a new order of 100 additional units and the labour cost for producing an additional 100 units at the rate of ₹ 20 per hour.

9. An aircraft manufacturer supplies aircraft engines to different airlines. They have just completed an initial order for 30 engines involving a total of 6000 direct labour hours at ₹ 20 per hour. They have been asked to bid for a prospective contract for a supply of 90 engines. It is expected that there will be 80% learning effect. Estimate the labour cost for the new order.
10. A first sample batch of 50 units of product A took 80 hours to make. The company now wishes to estimate the average time per unit will be if the total output of product A is 200 units and 80% learning rate applies.

### ANSWERS 10.1

1. Definition : Learning curve is the curvilinear relationship between the decrease in average labour hours per unit with increase in the total output

$$2. \text{ Learning index} = \frac{\text{Log (learning effect)}}{\log 2}$$

$$3. \text{ Learning index} = \frac{\text{Log (70\%)}}{\log 2} = -0.5146$$

$$4. \text{ Learning index} = \frac{\log (80\%)}{\log 2} = -0.3219$$

5. Learning curve eq is  $y = ax^b$

$y$  = Cumulative avg. time per unit

$a$  = Time for 1st unit

$x$  = Cumulative total number of units

$b$  = Index of learning

6. 

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours
1	1	1000	1000
1	2	80% of 1000 = 800	1600
2	4	80% of 800 = 640	2560
4	8	80% of 640 = 512	4096

Total hours to produce 8 lots = 4096 hrs

Total labour cost at ₹40 per hr = 4096 × 40

= ₹ 1,63,840

7) Assume 1 lot = 30 engine

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours
1	1	1000	1000
1	2	90% of 1000 = 900	1800
2	4	90% of 900 = 810	3240

The total number of hours to produce 4 lot (120 engine) = 3240

The lab cost at 20 per hour =  $3240 \times 20 = ₹ 64,800$

8)  $y = 1400 x^{-0.3}$

$$\log y = \log 1400 - 0.3 \log 2$$

$$\log y = 3.14612 - (0.3) 0.4471$$

$$\log y = 3.14612 - 0.14313$$

$$\log y = 3.05582$$

$$y = \text{Anti log } 3.00297$$

$$y = 1007$$

The total labour hours =  $1007 \times 3 = 3021$  hours

The labour cost at ₹ 20 per hr. =  $1621 \times 20 = ₹ 32,420$

Total labours hours for additional

100 unit =  $3021 - 1400 = 1621$

9)

1 lot = 30 engine

Lot produced	Total output in lot	Cumulative average time per lot (hours)	Total hours
1	1	6000	6000
1	2	80% of 6000 = 4800	9600
2	4	80% of 4800 = 3840	15360

The total hours to manufacture 4 lots = 15360

$$= \frac{6000}{9360}$$

Total lab cost for 9360 hrs at ₹ 20 per hour =  $9360 \times 20 = ₹ 1,87,200$

10)

Lot produced	Total output in Lot	Cumulative average time per Lot (hours)	Total hours
1	1	80	80
1	2	80% of 80 = 64	128
2	4	80% of 64 = 51.2	204.8

Total Labourhours = 204.8

(Assume 50 unit = 1 lot)

\* \* \* \* \*

**11.1 Introduction :**

In the field of Business and Industrial world, decision making is a very important task. Any organisation-big or small, has variety of resources like men, machine, money and material at their disposal. However the supply of these resources may be limited. In such situations, the management must find the best allocation of the available resources in order to maximize the profit or minimize the cost etc., utilize the resources to the best possible extent. In these contexts Linear Programming provides an optimal solution with the available information.

Linear Programming is a mathematical technique which deals with the optimization i.e. minimization or maximization of activities subject to the available resources.

***For instance :***

- (i) A biscuit manufacturing factory may be interested in taking decision regarding the type of biscuits to be manufactured and the quantity produced so as to earn maximum profit.
- (ii) A factory having some machines to manufacture cars would like to know the best way to utilize its machines so that maximum production is made possible in minimum time available.
- (iii) A dietician may want to suggest food with certain basic vitamins and proteins. She may like to know the best way to prescribe food with optimum requirement of vitamins and proteins at the minimum cost.

These are few examples where Linear Programming provides an optimal solution. Linear programming takes optimal decisions from the available set of choices. Programming (or planning) refers to the process of determining a particular plan of action.

**11.2 Definition:**

Linear programming deals with the optimization of a linear function of variables subject to a set of linear constraints.

**11.3 Formulation of Linear Programming problem**

Consider the following steps for the construction of L.P.P. in a given situation.

- Step 1 :** Identify the unknown variables-also known as decision variables and denote them in terms of algebraic symbols.
- Step 2 :** Identify the objective of the problem and represent it with Z.
- Step 3 :** Identify all the restrictions-also known as constraints. Represent them as equations or inequations in terms of symbols.

Let us consider some examples in order to explain the formulation of L.P.P.

**WORKED EXAMPLES**

**Example 1 :**

**Production allocation problem**

A manufacturer Akhil produces two types of models  $M_1$  and  $M_2$ . Each model  $M_1$  requires 4 hours of grinding and 2 hours of polishing, whereas each model  $M_2$  requires 2 hours of grinding and 6 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for a maximum of 80 hours a week and each polisher works for a maximum of 180 hours a week. Profit on a  $M_1$  model is ₹ 3 and on a  $M_2$  model is ₹ 4. Whatever is produced in a week is sold in the market. Formulate L.P.P.

**Solution :**

Let 'x' be the number of  $M_1$  models and 'y' be the number of  $M_2$  models. The weekly profit of the manufacturer is given by

$Z = 3x + 4y$  which is the objective function. This is on the assumption that he has enough grinders and polishers capacity to produce these number of models. Now in order to produce these number of models, the total number of grinding hours needed per week is given by  $4x + 2y$  and the total number of polishing hours needed per week given by  $2x + 6y$ . Since the manufacturer does not have more than 80 hours of grinding nor does he have 180 hours of polishing, we must have

$$4x + 2y \leq 80$$

$$2x + 6y \leq 180$$

Also it is not possible for the manufacturer to produce a negative number of models, it is obvious that we must also have  $x \geq 0$ ,  $y \geq 0$ . Hence the manufacturer's allocation problem can be put in the following mathematical form

To maximize the profit

$$z = 3x + 4y$$

Subject to the constraints

$$4x + 2y \leq 80$$
$$2x + 6y \leq 180$$
$$x \geq 0, y \geq 0$$

The above mathematical expression is known as mathematical formulation. From the above in equations we find out the values of x and y. The values of x and y are substituted in the objective function z. The maximum value of the objective function is known as optimal value.

**Example 2 :**

A resourceful home decorator Ananthu manufactures two types of lamps say A and B. Both lamps go through two technicians, first a cutter and second a finisher. Lamp 'A' requires 2 hours of cutter's time and 1 hour of finisher's time. Lamp 'B' requires 1 hour of cutter's time and 2 hours of the finisher's time. The cutter has 104 hours and the finisher has 76 hours of available time. Profit on a lamp 'A' is ₹ 6 and profit on a lamp 'B' is ₹ 11. assuming that she can sell all that she produces, formulate L.P.P.

**Solution :**

Let the manufacturer makes  $x$  and  $y$  lamps of type A and B respectively. Then the total profit which is the objective function  $z$  is given by  $z = 6x + 11y$ . Total time of the cutter used in preparing ' $x$ ' lamps of type A and ' $y$ ' lamps of type B is  $2x + y$ . Since the cutter has only 104 hours, we have  $2x + y \leq 104$ . Similarly,

Total time of the finisher in preparing  $x$  lamps of type A and  $y$  lamps of type B is  $x + 2y \leq 76$

Hence the decorator's problem is to find  $x$  and  $y$   
which maximizes  $Z = 6x + 11y$  subject to the constraints

$$2x + y \leq 104$$

$$x + 2y \leq 76$$

$$x \geq 0, y \geq 0$$

**Example 3:**

**(Diet problem)**

Pratheek wants to decide the constituents of diet which will fulfill his daily requirements of proteins, fats and carbohydrates at minimum cost. The combination is made among 2 food products A and B whose contents are indicated below:

Food	A	B	Minimum requirements
Proteins (mg)	5	2	800
Fats (mg)	6	5	200
Carbohydrates (mg)	4	3	700
Cost / Unit in ₹	70	50	

**Formulate L.P.P**

**Solution:**

Let the number of units of food A be  $x$  and that of food B be  $y$  respectively.

Therefore objective function is given by

$$\text{Minimize } z = 70x + 50y$$

Subject to constraints:

$$5x + 2y \geq 800$$

$$6x + 5y \geq 200$$

$$4x + 3y \geq 700$$

$$\text{where } x, y \geq 0$$

**Example 4:**

Arvind, a chemist provides his customers at the least cost, the minimum daily requirements of 2 vitamins A and B by using a mixture of 2 products M and N. The amount of each vitamin in 1 gm of each product, the cost per gram of each product and the minimum daily requirements are given below:

Products	No. of units of each vitamin in one gram of each product			Cost/gm of each product
	Vitamin A	Vitamin B	Minimum requirement of each vitamin	
M	6	2	12	20
N	2	2	8	16

**Formulate L.P.P**

**Solution :**

Let the amount of product M be  $x$  and the amount of product N be  $y$ . Looking at the table, we can formulate the L.P.P. problem as follows:

Objective function is to minimize  $z = 20x + 16y$

Subject to the constraints  $6x + 2y \geq 12$

$$2x + 2y \geq 8$$

$$x, y \geq 0$$

**Example 5 :**

A company owned by Swathi can produce two types of high quality shoes. Each shoe of the first kind requires thrice as much as time as the second kind. If all shoes are of the second kind only, the company can produce a total of 600 pairs a day. Only a maximum of 150 pairs of the first kind and 400 of the second kind can be sold a day. If the profit per pair of the first kind is ₹400 and per pair of the second kind is ₹150 then formulate L.P.P.

**Solution :**

Let the number of pair of shoes produced per day of the first kind requires 3 times as of the

second kind and the maximum possible production of the second kind is 600 pairs, we obtain the constraint :  $3x + y \leq 600$

Next we have the sales constraint. Since only a maximum of 150 pairs of the first kind and 400 pairs of the second kind can be sold, we get  $x \leq 150, y \leq 400$

Since neither production can be negative, we get  $x \geq 0, y \geq 0$

Since profits on the two kinds per pair are respectively ₹400 and ₹150, the objective of the maximizing profit requires to maximize  $z = 400x + 150y$ .

Summarising we get the following L.P.P.

Maximize  $z = 400x + 150y$  subject to the constraints

$$3x + y \leq 600$$

$$x \leq 150, y \leq 400$$

$$x \geq 0, y \geq 0$$

#### Example 6:

**A company owned by Vaishnavi wants to advertise its products in two magazines I and II. Magazine I reaches 2000 potential customers and magazine II reaches 3000 potential customers. The two magazines charge ₹400 and ₹600 per page. The company can spend not more than ₹6000 per month on advertisement. The total reach for the income group under ₹20,000 p.a. should not exceed 4000 potential customers. The reach for this income group in the two magazine I and II is 400 and 200. How many days should the advertisement be placed in the two magazines to reach the maximum.**

**Formulate L.P.P.**

#### Solution :

Let  $x$  and  $y$  be the number of days on which the advertisement is placed in the two magazines I and II respectively.

Then the required LPP will be:

$$\text{Maximise } z = 2000x + 3000y$$

Subject to the constraints:

$$400x + 600y \leq 6000$$

$$400x + 200y \leq 20000$$

$$x, y \geq 0$$

#### Example 7 :

**A producer named Samarth has 30 and 17 units of labour and capital respectively which he can use to produce two types of goods A and B. To produce one unit of A,**

2 units of labour and 3 units of capital are required. Similarly 3 units of labour and 1 unit of capital is required to produce 1 unit of B. If A and B are priced at ₹100 and ₹120 per unit respectively how should the producer use his resources to maximize the total revenue.

Form a LPP to maximize his revenue.

**Solution:**

The above information is given in the following table:

Goods	Labour	Capital	Cost (in ₹)
A	2	3	100
B	3	1	120
Available units	30	17	

Let  $x$  be the number of units of A and  $y$  be the number of units of B produced.

Mathematical formulation of given problem is as follows:

$$\text{Maximise } z = 100x + 120y$$

Subject to the constraints

$$2x + 3y \leq 30$$

$$3x + y \leq 17$$

$$x, y \geq 0$$

**Example 8 :**

A soft drink firm owned by Ektha has two bottling plants one located at P and the other located at Q. Each plant produces three different soft drinks A, B and C. The capacities of two plants in number of bottles per day are as follows:

Product	Plant P	Plant Q
A	3000	1000
B	2000	6000
C	1000	1000

A market survey indicates that during the month of April, there will be a demand for 24000 bottles of A, 48000 bottles of B, 16000 bottles of C. The cost of running the two plants P and Q are respectively ₹6000 and ₹8000 per day.

Formulate L.P.P.

**Solution :**

Let us suppose that the plants P and Q should run for  $x$  days and  $y$  days respectively, then the total cost of running the two plants is ₹  $6000x + 8000y$ . The following table helps us in formulating the problem.

Plant	No. of working days	Production			Cost of running
		A	B	C	
P	$x$	$3000x$	$2000x$	$1000x$	$6000x$
Q	$y$	$1000y$	$6000y$	$1000y$	$8000y$

Our problem is to find  $x$  and  $y$ , so that the cost  $z = 6000x + 8000y$  is minimum subject to constraints

$$x \geq 0, y \geq 0$$

$$3000x + 1000y \geq 24000$$

$$2000x + 6000y \geq 48000$$

$$1000x + 1000y \geq 16000$$

### Example 9:

**Yatharth runs a company which manufactures two products A and B. Product A is sold at ₹200 per unit and takes 30 minutes to make. Product B is sold at ₹300 per unit and takes 1 hour to make. There is a permanent order of 14 units of product A and 16 units of product B. The working week for the production of both A and B consists of 40 hours and the weekly turnover must not be less than ₹10,000. If the profit on each unit of product A is ₹20 and on product B is ₹30, then formulate L.P.P.**

### Solution :

Let  $x$  units of product A and  $y$  units of product B be manufactured.

The problem is to maximise  $P = 20x + 30y$

Subject to constraints

$$x \geq 0, y \geq 0$$

$$x \geq 14, y \geq 16$$

$$200x + 300y \geq 10000$$

$$\frac{1}{2}x + 1y \leq 40$$

### 11.3.1 General Linear programming problem

A problem in linear programming is formulated mathematically by first identifying a set of decision variables say  $x_1, x_2, x_3, \dots, x_n$  which are subjected to certain linear constraints written in the form of inequalities

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n (\leq, \geq) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n (\leq, \geq) b_2$$

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$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n (\leq, \geq) b_m$$

where the coefficients  $a_{ij}$  and  $b_1, b_2, b_3, \dots, b_m$  are constants and the non-negative constraints are given by  $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \dots, x_n \geq 0$

The aim of this mathematical formulation is to optimize a linear expression

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n \text{ for a given constraints } c_1, c_2, \dots, c_n$$

This is called as the **objective function**. Thus the optimisation (maximization or minimization) of the objective function is called the General Linear Programming problem.

### 11.3.2 Structure of Linear Programming Model:

The general structure of Linear Programming model consists of three basic components or elements.

- 1) **Objective function:** This is the function which is to be maximised or minimised in the problem. This function is always a linear function consisting of variables. It is denoted by 'Z'.
- 2) **Decision variables :** These are the variables which affect the value of the objective function. These variables are denoted by  $x_1, x_2, x_3, \dots, x_n$ .
- 3) **Constraints :** These are the conditions existing with regard to the decision variables. These constraints will be in the form of linear inequalities.

Also there exists some other terminologies which are as follows:

- (i) **Feasible region :** while solving the linear programming problem by graphical method, every constraint is satisfied by a part of the co-ordinate plane. The region of the cartesian plane which is the intersection of all these parts, thus representing the region which satisfies all the constraints is called the feasible region.
- (ii) **Feasible solution :** A solution which is a set of real values which satisfies all the constraints is called Feasible solution.
- (iii) **Optimal solution :** The feasible solution which achieves the objective on hand that is either maximizes or minimizes the objective function as the case may be is the optimal solution. It optimizes the objective. In the graphical method this solution occurs at one of the corners of the feasible region.

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**EXERCISE 11.1**

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**Four mark questions:**

- 1) Smaran being a manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts while it takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹2.50 per package on nuts and ₹ 1 per package on bolts. Form a linear programming problem to maximize his profit, if he operates each machine for at most 12 hours a day.
  
- 2) Vishal consumes two types of food, A and B everyday to obtain minimum 8 units of protein, 12 units carbohydrates and 9 units of fat which is his daily requirements. 1 kg of food A contains 2, 6, 1 units of protein, carbohydrates and fat respectively. 1 kg of food B contains 1, 1 and 3 units of protein, carbohydrates and fat respectively. Food A cost ₹8 per kg and food B cost ₹ 5 per kg. Form an L.P.P. to find how many kilogram of each food should he buy daily to minimize his cost of food and still meet minimal nutritional requirements.
  
- 3) Archana, a dietician wishes to mix two types of foods F1 and F2 in such a way that the vitamin contents of the mixture contains at least 6 units of vitamin A and 8 units of vitamin B. Food F1 contains 2 units/kg of vitamin A and 3 units/kg of vitamin B while food F2 contains 3 units/kg of vitamin A and 4 units/kg of vitamin B. Food F1 costs ₹50 per kg and food F2 costs ₹75 per kg. Formulate the problem as L.P.P. to minimize the cost of the mixture.
  
- 4) A furniture maker Jatin has 6 units of wood and 28 hours of free time in which he will make decorative screens. Two models have sold well in the past, so he will restrict himself to those two. He estimates that model 1 requires 2 units of wood and 7 hours of time. Model 2 requires 1 unit of wood and 8 hours of time. The prices of the models are ₹120 and ₹ 80 respectively. Formulate LPP to determine how many screens of each model should the furniture maker assemble if he wishes to maximize his sales revenue.
  
- 5) A firm owned by Sheshnag manufactures two types of products A and B and sell them at a profit of ₹2 on type A and ₹3 on type B. Each product is processed on machines M1 and M2. Type A requires one minute of processing time on M1 and two minutes on M2. Type B requires one minute of time on M1 and one minute on M2. The machine M1 is available for not more than 6 hours 40 minutes while M2 is available for 10 hours during any working day. Formulate the L.P.P. in order to find how many products of each type should the firm produce each day so that profit is maximum.

- 6) Shreya company produces 2 types of shoes A and B. A is of superior type and B is of ordinary type. Each shoe of the first kind requires thrice as much time as the second kind. If all shoes are of the second kind only, the company can produce a total of 650 pairs a day. Only a maximum of 150 pairs of the first kind and 400 of the second kind can be sold per day. If the profit per pair of the first kind is ₹400 and per pair of the second kind is ₹150. Find the optimal product mix to be produced to maximize the profit by formulating on L.P.P. model.
- 7) Producer Rahul has 50 and 85 units of labour and capital respectively which he can use to produce two types of goods A and B. To produce one unit of A, 1 unit of labour and 2 units of capital are required. Similarly 3 units of labour and 2 units of capital is required to produce 1 unit of B. If A and B are priced at ₹100 and ₹150 per unit respectively, how should the producer use his resources to maximize the total revenue. Formulate the L.P.P. to maximize his total revenue?
- 8) Nikhil pesticide company must produce 200 kg mixture consisting of chemicals A and B daily. A cost ₹3 per kg and B cost ₹8 per kg. Maximum 80 kg of chemical A and atleast 60 kg. of chemical B should be used. Formulate L.P.P. model to minimize the cost.

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**ANSWERS 11.1**

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1. Maximize,  $z = 2.5x + y$   
subject to the constraints;

$$\begin{aligned}x + 3y &\leq 12 \\ 3x + y &\leq 12 \\ x, y &\geq 0\end{aligned}$$

2. Minimize,  $z = 8x + 5y$   
subject to the constraints;

$$\begin{aligned}2x + y &\geq 8 \\ 6x + y &\geq 12 \\ x + 3y &\geq 9 \\ x, y &\geq 0\end{aligned}$$

3. Minimize,  $z = 50x + 75y$   
subject to constraints;  $2x + 3y \geq 6$   
 $3x + 4y \geq 8$   
 $x, y \geq 0$

4. Maximize,  $z = 120x + 80y$   
subject to constraints;  $2x + y \leq 6$   
 $7x + 8y \leq 28$   
 $x, y \geq 0$

5. Maximize,  $z = 2x + 3y$   
subject to constraints,  $x + y \leq 400$   
 $2x + y \leq 600$   
 $x, y \geq 0$

6. Maximize,  $z = 400x + 150y$   
subject to constraints,  $3x + y \leq 650$   
 $x \leq 150$   
 $y \leq 400$   
 $x, y \geq 0$

7. Maximize,  $z = 100x + 150y$

subject to constraints;  $x + y \leq 50$   
 $2x + 2y \leq 85$   
 $x, y \geq 0$

8. Minimize,  $z = 3x + 8y$

subject to constraints;  $x + y \geq 200$   
 $x \leq 80$   
 $y \geq 60$   
 $x, y \geq 0$

### 11.4 Graphical solutions for Linear Programming problems:

The graphical method of solving a Linear Programming problem consists of the following steps.

**Step 1:** Represent the given problem in mathematical form i.e. formulate linear programming for the given problem.

**Step 2:** Represent the given constraints as equalities on  $x, y$  (or  $x_1, x_2 \dots$ ) i.e. consider each inequality constraint as equation. Plot each equation on the graph as each will geometrically represent a straight line.

**Step 3:** Mark the region. If the inequality constraint corresponding to the line is  $\leq$ , then the region below the line lying in the first quadrant (due to the non-negativity of the variables) is shaded. This region includes the origin. For the inequality constraint  $\geq$  sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

**Step 4 :** Measure the values of the two decision variables  $x$  and  $y$  graphically at each of the vertices of the feasible region. The values of  $x$  and  $y$  at one of the vertices represents optimal solution.

**Step 5:** Calculate the value of the objective function of the corner points on the feasible region and select the one which gives the optimal solution i.e. in the case of maximization problem, optimal point corresponds to the corner point at which the objective function has a maximum value and in the case of minimization, the corner point which gives the objective function the minimum value is the optimal solution.

**Note:**

- 1) Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through atleast one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through atleast one corner of the feasible region.

The following illustrations demonstrates how L.P.P. solutions can be obtained by the graphical method.

**Example 1:**

A company owned by shree group produces two products P and Q. Each P requires 4 hours of grinding and 2 hours of polishing and each Q requires 2 hours of grinding and 5 hours of polishing. The total available hours for grinding is 20 hours and for polishing is 24 hours. Profit per unit of P is ₹6 and that of Q is ₹8. Formulate L.P.P. and solve graphically.

**Solution :**

Let 'x' be the units of P manufactured and 'y' be the units of Q manufactured. Consider the table formed as per the given details.

Products	Grinding hours	Polishing hours	Profit per unit
P	4	2	6
Q	2	5	8
Availability of time	20	24	

Formulation of L.P.P. can be done as follows

$$\text{Maximise, } z = 6x + 8y.$$

$$\text{Subject to constraints } 4x + 2y \leq 20$$

$$2x + 5y \leq 24$$

$$x, y \geq 0$$

The given L.P.P. can be solved graphically by following the steps as given below.

**Step 1:** The given constraints in the form of inequalities to be converted in the equations format and then draw the graph of the obtained lines equations.

Referring to the given details, we get the equations  $4x + 2y = 20$  and  $2x + 5y = 24$  from the constraints.

Now to draw the graph of the above lines first construct the table.

$$4x + 2y = 20$$

$$2x + 5y = 24$$

x	0	5
y	10	0

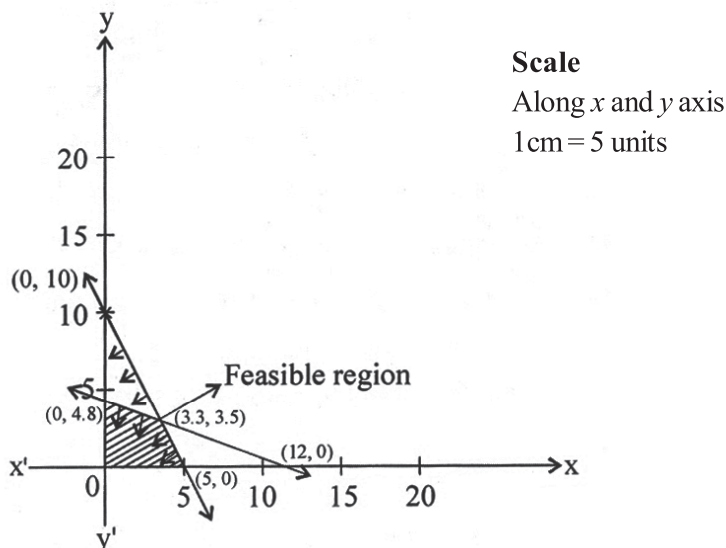
(0,10) (5, 0)

x	0	12
y	4.8	0

(0,48) (12, 0)

Using the obtained points we can draw the graph of the lines separately.

**Step 2:** Since both the given constraints are "less than or equal to" type the shading of the graph to be done in such a way that origin is to be included i.e. the part below the obtained lines to be shaded. Where overlapping of the shaded region occurs, we get feasible region step (1) and (2) are together shown in the graph which is drawn below.



**Note:** Shading direction of the given lines is shown by the arrow marks. The region OABC is the feasible region where we find the overlapping of the shades (arrow marks) and this region gives the corner points O(0,0), A(0,4.8), B(3.3, 3.5) and C(5, 0)

**Step 3:** Optimal solution is obtained by substituting the values of  $x$  and  $y$  of the corner points in the objective function.

$$\text{At } O(0,0), Z = 6(0) + 8(0) = 0$$

$$\text{At } A(0, 4.8), Z = 6(0) + 8(4.8) = 38.4$$

$$\text{At } B(3.3, 3.5), Z = 6(3.3) + 8(3.5) = 47.8$$

$$\text{At } C(5, 0), Z = 6(5) + 8(0) = 30$$

**Step 4:** Thus the optimal solution of the graphical problem is obtained as

When  $x = 3.3$  and  $y = 3.5$  the maximum value of  $Z = 47.8$

### Example 2:

A diet for Sahana must contain at least 4000 units of vitamins, 50 units of minerals and 1400 calories. Two foods A and B available at a cost of ₹5 and ₹4 per unit respectively. If one unit of A contains 200 units of vitamin, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamins, 2 unit of minerals and 40 calories, find by graphical method what combination of foods be used to have least (minimum) cost.

**Solution:**

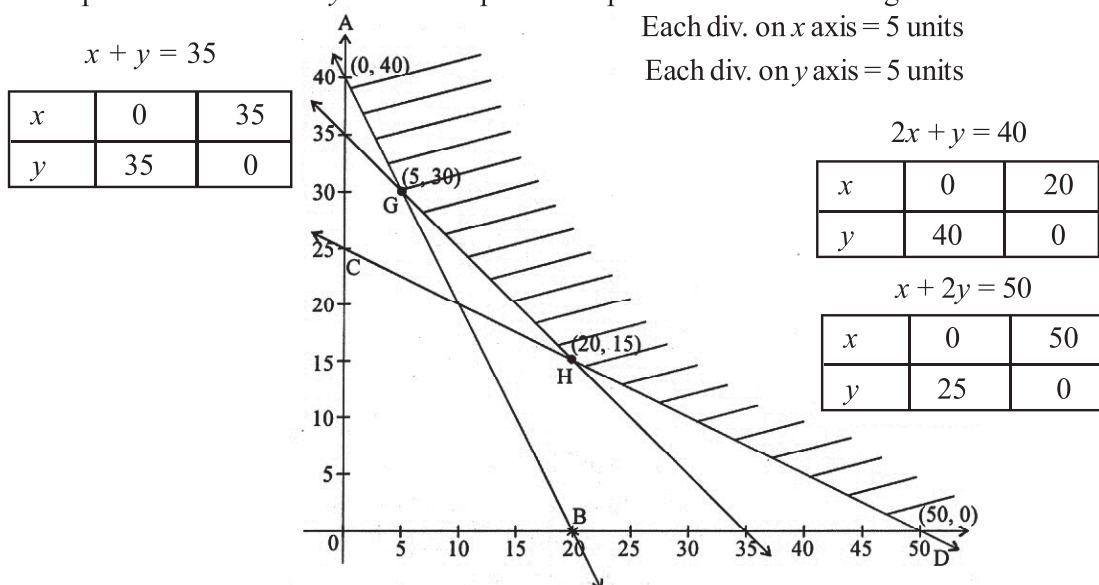
Variables	Food	Unit content of			Cost per unit (₹)
		Vitamins	Minerals	Calories	
$x$	A	200	1	40	5
$y$	B	100	2	40	4
Minimum requirement		400	50	1400	

$x$  = number of units of food A,  $y$  = number of units of food B.

We shall formulate a Linear Programming model minimize  $z = 5x + 4y$ , subject to the constraints  $200x + 100y \geq 4000$ ;  $x + 2y \geq 50$ ;  $40x + 40y \geq 1400$   $x, y \geq 0$

The above constraints can now be written as  $2x + y \geq 40$ ;  $x + 2y \geq 50$ ;  $x + y \geq 35$  and  $x \geq 0, y \geq 0$  indicates that the region is to lie in the first quadrant only.

As already stated in example 1 in equalities are to be graphed taking them as equalities. To draw the graph of  $2x + y = 40$ , put  $x = 0$ , we get i.e. A(0,40) as a point. Similarly put  $y = 0$ , we get  $x = 20$  i.e. B(20,0) is another point joining the point A(0, 40) and B(20,0) by a line, we obtain the line graph  $2x + y = 40$ . Any point  $(x, y)$  lying one or above the line  $2x + y = 40$  satisfies the constraint  $2x + y \geq 40$ . Similarly to draw the graph of  $x + 2y = 50$ ,  $y = 25$  i.e. C(0, 25) is a point and put  $y = 0, x = 50$  i.e. D(50,0) is another point. Joining these two points by a line, we obtain the graph of  $x + 2y = 50$ . Finally we have  $x + y = 35$ , put  $x = 0, y = 35$  and then put  $y = 0, x = 35$ . So E(0,35) and F(35, 0) are the points on the line  $x + y = 35$ . Now plot all the points as shown in the figure.



The set of points  $\{(x, y) : x \geq 0, y \geq 0\}$  satisfying all the constraints constitute a feasible region which is shown as shaded region in the figure.

Solving  $2x + y = 40, x + y = 35$  we get  $x = 5, y = 30 \therefore G = (5, 30)$ , Also by solving  $x + 2y = 50$  and  $x + y = 35$ , we get  $x = 20, y = 15 \therefore H = (20, 15)$ .

We observe that the feasible region is unbounded on the upper side. But since the objective function  $z = 5x + 4y$  is to be minimized, the optimum solution to the problem shall be located only at one of the extreme points A, G, H, D.

We shall evaluate the objective function  $z = 5x + 4y$  at these extreme points.

$$\text{At A } (0, 40) \quad z = 5x + 4y = 5(0) + 4(40) = 160$$

$$\text{At G } (5, 30) \quad z = 5x + 4y = 5(5) + 4(30) = 145$$

$$\text{At H } (20, 15) \quad z = 5x + 4y = 5(20) + 4(15) = 160$$

$$\text{At D } (50, 0) \quad z = 5x + 4y = 5(50) + 4(0) = 250$$

Since the minimum value of the objective function  $z = 145$  occurs at the extreme point  $G(5, 30)$  we conclude that the optimum solution to the given LPP is  $x = 5, y = 30$  and minimum  $z = 145$ . Hence to have least cost, the diet should contain 5 units of food A and 30 units of food B.

Note : Cross check: Solving equations  $2x + y = 40, x + 2y = 50$  and  $x + y = 35$  by pair we get  $G = (5, 30)$  and  $H = (20, 15)$  Graphically also this must be correct.

### Example 3:

**Solve the following LPP graphically maximize  $z = 6x + 8y$ , subject to the constraints  $4x + 2y \leq 20$**

$$2x + 5y \leq 24 \text{ and } x \geq 0, y \geq 0$$

**Mark the feasible region.**

### Solution :

Consider the first constraint  $4x + 2y \leq 20$ , taking it as equality we get  $4x + 2y = 20$

$$\text{put } x = 0, \text{ we get } 2y = 20 \rightarrow y = \frac{20}{2} = 10$$

$$\text{put } y = 0, \text{ we get } 4x = 20 \rightarrow x = \frac{20}{4} = 5$$

$\therefore 4x + 2y = 20$  passes through the points A(0, 10) and B(5, 0)

Similarly for the 2nd constraint  $2x + 5y = 24$

$$\text{put } x = 0, \text{ we get } 5y = 24 \rightarrow y = \frac{24}{5} = 4.8$$

put  $y = 0$ , we get  $2x = 24 \rightarrow x = \frac{24}{2} = 12$

$\therefore 2x + 5y = 24$  passes through the points  $C(0, 4.8)$  and  $D(12, 0)$

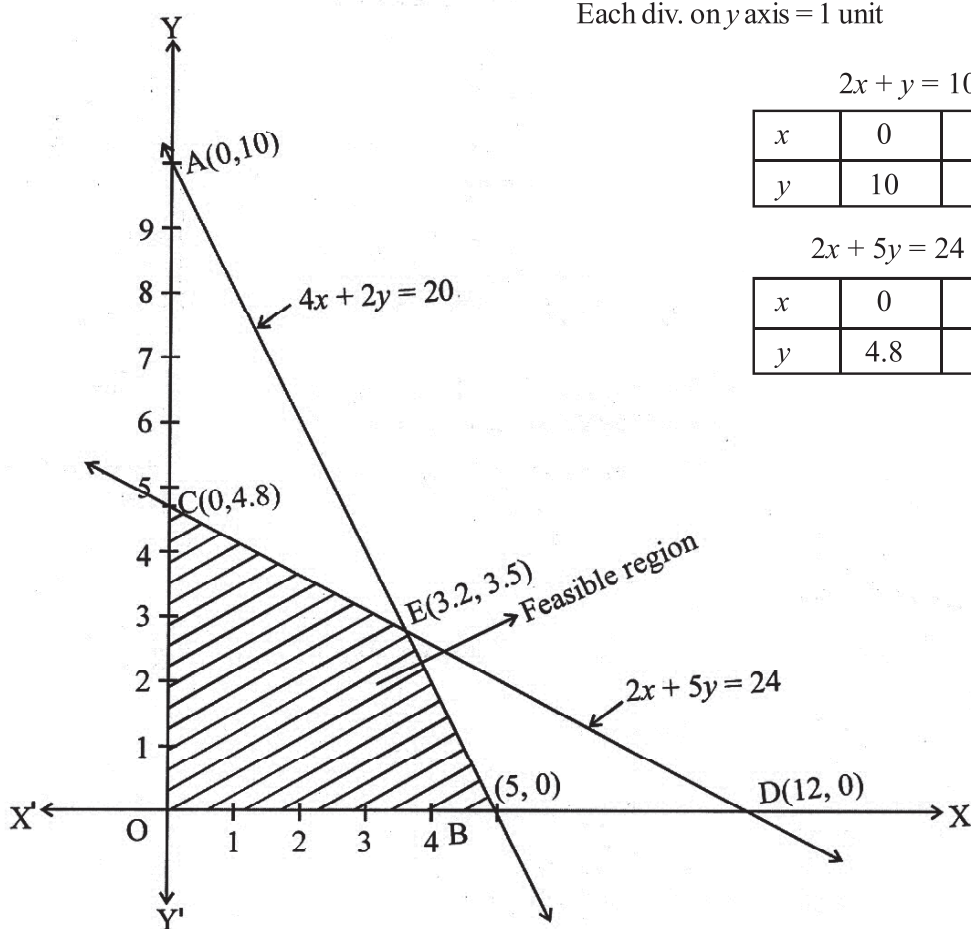
Plot each equation on the graph. The shaded portion in the figure represent the feasible region which is bounded by the two axes and the two lines,

$$4x + 2y = 20 \text{ and } 2x + 5y = 24$$

The shaded area OCEB represents the set of all feasible solutions.

Each div. on  $x$  axis = 1 unit

Each div. on  $y$  axis = 1 unit



Solving  $4x + 2y = 20$  and

$$2x + 5y = 24$$

we get  $y = 3.5$  and  $x = 3.2$

thus  $E = (3.2, 3.5)$

The four extreme points of the feasible region are  $O(0,0)$ ,  $C(0,4.8)$ ,  $E(3.2, 3.5)$  and  $B(5,0)$

Now we have to evaluate the objective function  $z = 6x + 8y$  at these extreme points.

$$\text{At } O(0,0) \quad ; z = 6(0) + 8(0) = 0$$

$$\text{At } C(0, 4.8) \quad ; z = 6(0) + 8(4.8) = 38.4$$

$$\text{At } E(3.2, 3.5) \quad ; z = 6(3.2) + 8(3.5) = 19.2 + 28 = 47.2$$

$$\text{At } B(5, 0) \quad ; z = 6(5) + 8(0) = 30$$

Thus the optimum solution to the given problem occurs at  $E(3.2, 3.5)$  ie  $x = 3.2$ ,  $y = 3.5$  with the objective function value 47.2

$\therefore$  Optimal solution is obtained when  $x = 3.2$  and  $y = 3.5$

#### Example 4 :

**Maximize  $z = -x + 2y$  subject to the constraints:**

$$x \geq 3, x + y \geq 5, x + 2y \geq 6, y \geq 0, x \geq 0$$

#### Solution :

Our problem is to maximize

$$z = -x + 2y$$

subject to the constraints

$$x \geq 3, x + y \geq 5, x + 2y \geq 6$$

$$x \geq 0, y \geq 0$$

First consider the lines  $x = 3$ , which passes through  $A(3,0)$  and is parallel to  $y$  axis. Then  $x + y = 5$ , which passes through the points  $B(5,0)$  and  $C(0,5)$  and the line  $x + 2y = 6$  if drawn is found to be passing through  $D(6,0)$  and  $E(0,3)$ .

Lines  $BC$  and  $DE$  meet at  $R(4,1)$ ;  $BC$  meets  $AT$  in  $Q(3,2)$  and  $DE$  meets  $AT$  in  $P\left(3, \frac{3}{2}\right)$ .

Feasible region is shown shaded in the figure. Note that the point  $D(6,0)$  lies in this region

but  $O(0,0)$  and  $P\left(3, \frac{3}{2}\right)$  do not lie in this region.

The corner points which are to be examined for optimum solution are D(6,0) R(4,1) and Q(3,2).

We find from the graph drawn below:

$$\text{At D(6,0)} \Rightarrow z = -6 + 2 \times 0 = -6$$

$$\text{At R(4,1)} \Rightarrow z = -4 + 2 \times 1 = -2$$

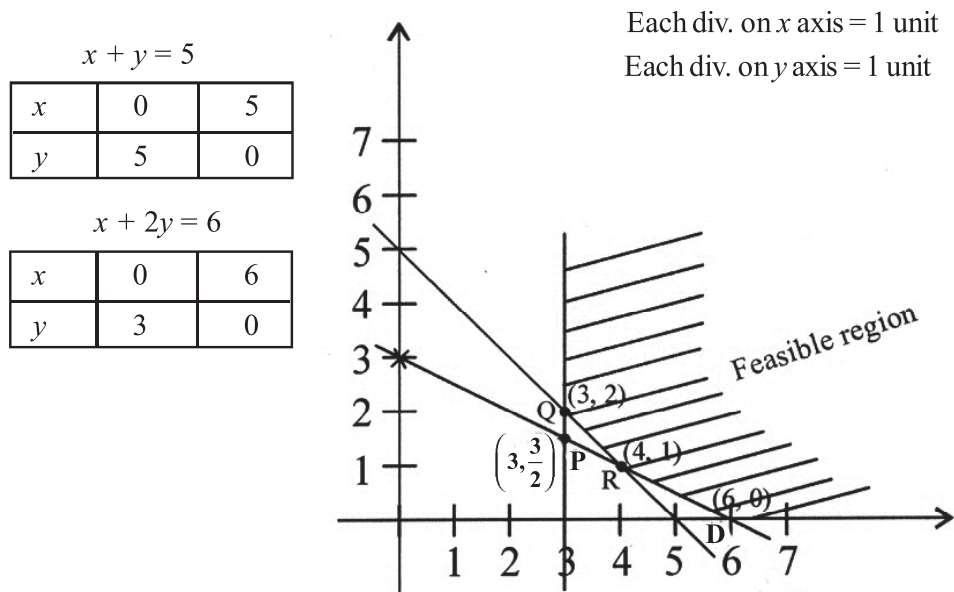
$$\text{At Q(3,2)} \quad z = -3 + 2 \times 2 = 1$$

Hence it appears that  $\max z = 1$  at (3, 2)

Since the feasible region is unbounded

we find the graph of the inequality  $-x + 2y > 1$

As the half plane represented by this inequality has points common with the feasible region, therefore 1 is not the maximum value of  $z$ . Thus  $z$  has no maximum value.



### Example 5 : Solve graphically

Maximize  $z = x + y$ , subject to the constraints;  $2x - y + 1 \geq 0$ ,  $x + y \leq 3$ ,  $x \leq 2$  and  $x, y \geq 0$ .

**Solution :**  $Z = 3$  B(2,1) C( $\frac{3}{2}, \frac{3}{2}$ ) O(0,0)

Consider the lines  $2x - y + 1 = 0$ ,  $x + y = 3$  and  $x = 2$ . Lines  $2x - y + 1 = 0$  passes through

$(-\frac{1}{2}, 0)$ , (0, 1) and  $(\frac{3}{2}, \frac{3}{2})$  whereas the line  $x + y = 3$  passes through (0,3),  $(\frac{3}{2}, \frac{3}{2})$ , (2,1)

and (3,0). Line  $x = 2$  is parallel to  $y$  axis and passes through (2,1). We find lines  $2x - y + 1 = 0$  and  $x + y = 3$  meet at  $B\left(\frac{3}{2}, \frac{3}{2}\right)$  and the lines  $x + y = 3$  and  $x = 2$  meet at  $C(2,1)$  whereas lines  $x = 2$  meets the  $x$  axis at (2,0) and the line  $2x - y + 1 = 0$  meets the  $y$  axis at  $A(0,1)$  and  $O(0,0)$ . Thus the corner points  $A(0,1)$ ,  $B\left(\frac{3}{2}, \frac{3}{2}\right)$ ,  $C(2,1)$ ,  $D(2,0)$  determines the feasible region as shaded below.

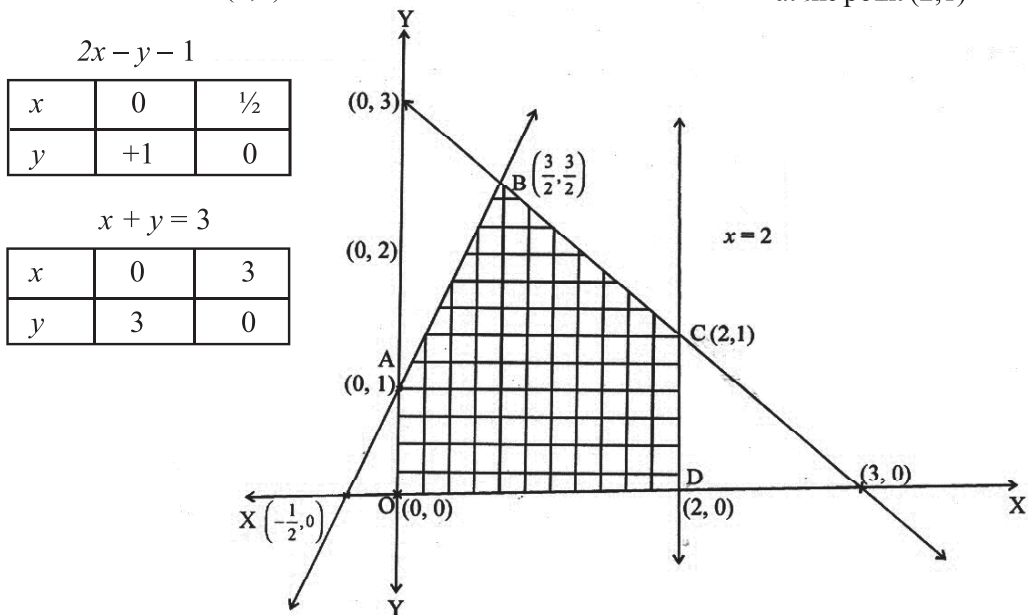
To find the optimal solution we find

at  $A(0,1)$   $z = 1$ , at  $C(2,1)$   $z = 3$

at  $B\left(\frac{3}{2}, \frac{3}{2}\right)$   $z = 3$ , at  $D(2,0)$   $z = 2$

at  $O(0,0)$   $z = 0$

$Z$ , the objective function is having the maximum value 3 at the point (2,1)



#### Example 6 :

Varsha electric appliance company produces two items, Refrigerator and ranges production takes place in two separate departments. Refrigerator are produced in department I and ranges in department II. The company's two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 refrigerators and 35 ranges in department II, because of limited available facilities in these two departments. The company regularly employs a total of 60 workers in the two



we evaluate the value of the objective function  $z$  at each of the corners of this region.

At  $O(0,0)$   $z = 0$

At  $A(25,0)$   $z = 1500$

At  $B(25,10)$   $z = 1900$

At  $C(12.5, 35)$   $z = 2150$

At  $D(0, 35)$   $z = 1400$

Clearly  $z$  is maximum at  $C(12.5, 35)$

Since product of either product cannot be in fractional value, 12 refrigerators and 35 ranges should be produced to earn maximum profit.

**Example 7:**

**A firm owned by Vaidhyanathan plans to purchase atleast 200 quintals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain atleast 100 quintals of X and not more than 35 quintals of Y. The firm can purchase the scrap from two supplies A and B in unlimited quantities. The percentage of X and Y in terms of weight is the scraps supplied by A and B is given below.**

Metals	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

**The price of A's scrap is ₹200 per quintal and that of B's is ₹400 per quintal. Formulate this problem as a L.P.P. and solve to determine the quantities to be bought from the two suppliers to minimize the total purchase cost.**

**Solution :**

Let 'x' and 'y' be the number of quintals of scrap bought by the suppliers A and B respectively. Since the firm is to minimise the cost of purchase and each quintal from supplier A cost ₹ 2000 and that from B costs ₹400, the objective is to minimise  $C = 200x + 400y$

Since it must buy atleast 200 quintals

$$x + y \geq 200$$

Also, the minimum of metal X in the scrap bought is to be 100 quintals. Since scrap from supplier A contains 25% and that from B contains 75% of metal X, we must have

$$0.25x + 0.75y \geq 100$$

Similarly, the scrap must not contain more than 35 quintals of metal y. Since the scrap from supplier A contains 10% of y and that from B contains 20% of y, we get

$$0.10x + 0.20y \leq 35$$

Also since the purchases cannot be negative  $x, y \geq 0$

Consolidating, we arrive at the following L.P.P. minimise  $C = 200x + 400y$  subject to

$$x + y \geq 200$$

$$0.25x + 0.75y \geq 100$$

$$.10x + 0.20y \leq 35$$

$$x, y \geq 0$$

Drawing the graph of each of the inequality constraints treating them as equalities we find the corner points as G(100, 100) H(250, 50) and I(50, 150). Graph is shown below with the proper markings of the points.

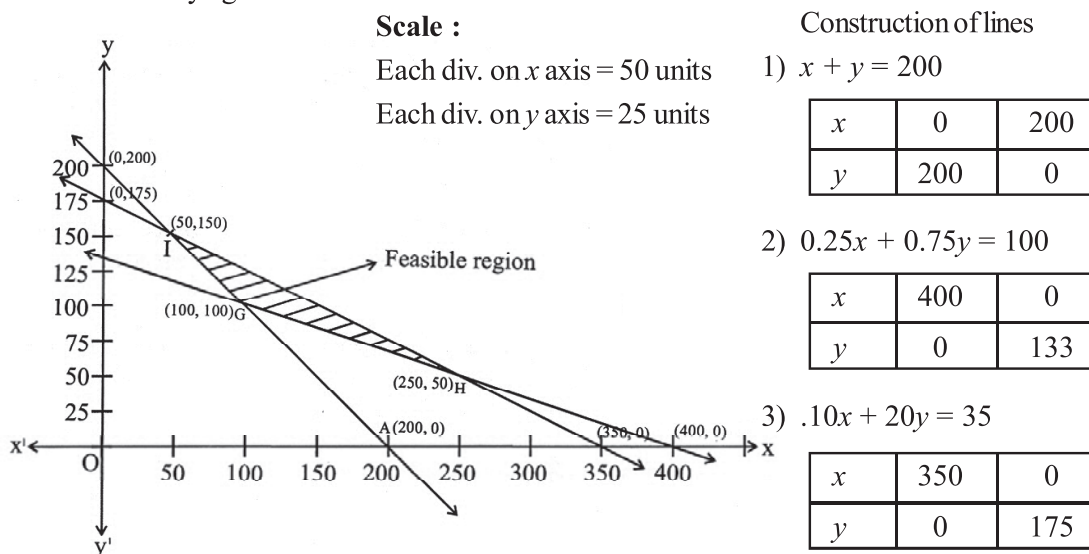
We then evaluate the objective function C at these points.

$$\text{At G (100, 100) } C = 200(100) + 400(100) = 60,000$$

$$\text{At H (250, 50) } C = 200(250) + 400(50) = 70,000$$

$$\begin{aligned} \text{At I (50, 150) } C &= 200(50) + 400(150) \\ &= 70,000 \end{aligned}$$

Since cost is minimum at G(100, 100) that is the optimal point and hence it is recommended that 100 quintals of scrap should be bought by each of the suppliers A and B to minimise the cost of buying.



**Example 8 :**

A co-operative society of farmers presided by Lalith has 50 hectare of land to grow two crops X and Y. The profits from crop X and Y per hectare are estimated as ₹10,500 and ₹9000 respectively. To control weeds a liquid herbicide has to be used for crops X and Y at the rate of 20 litres and 10 litres per hectare. Further not more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects, drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society? Also find the maximum profit.

**Solution :**

Let 'x' hectare of land be allocated to crop X and Y hectare of land to crop Y.

Obviously  $x \geq 0, y \geq 0$

profit per hectare on crop  $x = ₹10,500$

profit per hectare on crop  $y = ₹9000$

Therefore total profit  $= 10500x + 9000y$

Formulation of the problem is as follows

Maximise  $z = 10500x + 9000y$

Subject to the constraints

$$x + y \leq 50 \quad (1)$$

$$20x + 10y \leq 800 \text{ which can be otherwise}$$

written as  $2x + y \leq 80 \quad (2)$

$$x \geq 0, y \geq 0 \quad (3)$$

Let us draw the graph of the system of inequalities (1) to (3). The feasible region OABC is shown in the figure.

Now  $z = 10500x + 9000y$

At  $O(0,0)$ ,  $z = 0$

At  $A(40, 0)$ ,  $z = 420000$

At  $B(30, 20)$ ,  $z = 495000$

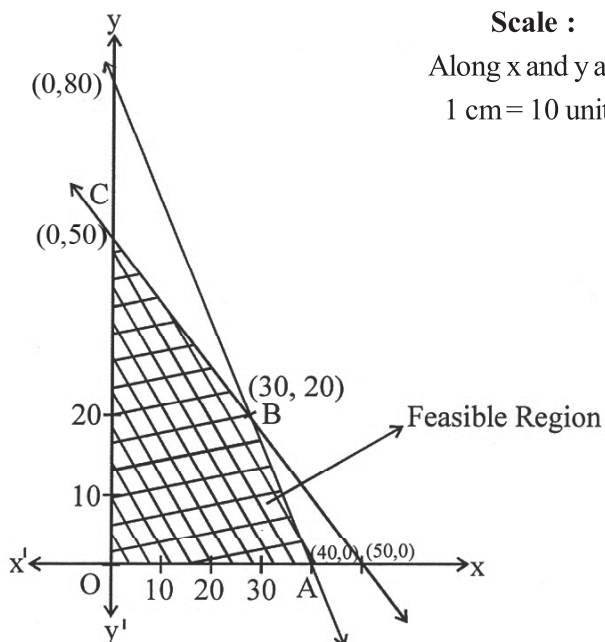
At  $C(0, 50)$ ,  $z = 450000$

$\therefore z$  is maximum at  $B(30, 20)$

$\therefore$  land allocated to crop X = 30 hectare

land allocated to crop Y = 20 hectare

$\therefore$  maximum profit = ₹450000



Construction of lines

$$x + y = 50$$

x	0	50
y	50	0

$$x + y = 50$$

x	0	40
y	80	0

**Example 9 :**

Arjun wants to invest at most ₹12,000 in Bonds A and B. According to the rule, he has to invest atleast ₹2000 in Bond A and atleast ₹4000 in Bond B. If the rates of interest on Bonds A and B respectively are 8% and 10% per annum formulate the problem as L.P.P. and solve it graphically for maximum interest.

**Solution :**

Let Arjun invest in bond A = ₹  $x$

Arjun invests in bond B = ₹  $y$

Then interest in bond A = 8% of  $x$

$$= \frac{8}{100} \times x = \frac{8x}{100} = 0.08x$$

Interest in bond B = 10% of  $y$

$$= \frac{10}{100} \times y = 0.10y$$

Let  $Z$  be the yearly income of Arjun

Then L.P.P. is to maximize,  $z = 0.08x + 0.10y$

subject to  $x + y \leq 12000$

$$x \geq 2000$$

$$y \geq 4000$$

$$x, y \geq 0$$

Draw the lines  $x = 2000$ ,  $y = 4000$  and  $x + y = 12000$

The shaded region ABC represents the feasible region.

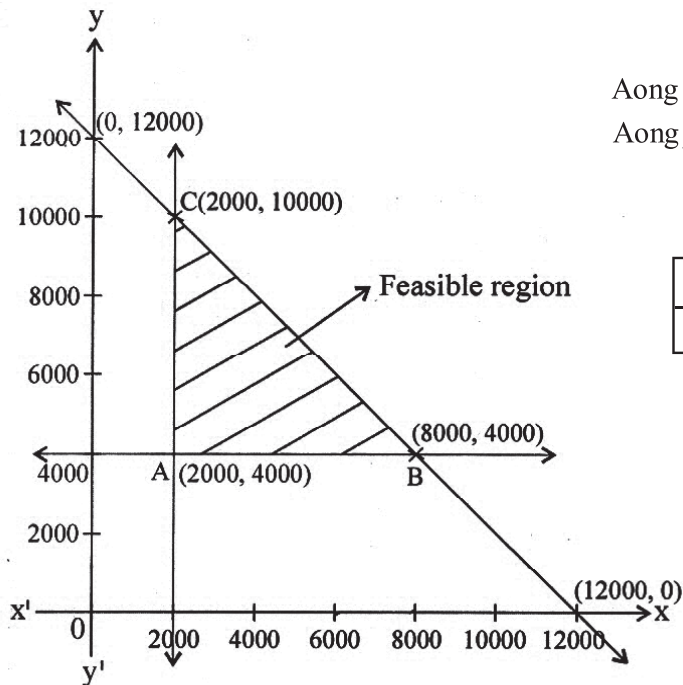
$$I_A = \cdot 08(2000) + \cdot 10(4000) = ₹ 560$$

$$I_B = \cdot 08(8000) + \cdot 10(4000) = ₹ 1040$$

$$I_C = \cdot 08(2000) + \cdot 10(10000) = ₹ 1160$$

The interest is maximum at C(2000, 10000) and the maximum interest is ₹1160

Construction of lines using table



**Scale**

Aong x-axis 1 cm = 2000 units

Aong y-axis 1 cm = 2000 units

$$x + y = 12000$$

$x$	0	12000
$y$	12000	0

(0, 12000) (12000, 0)

### EXERCISE 11.2

**Five marks questions:**

**I. Solve the following linear programming problems using graphical method.**

1) Maximise  $z = 60x + 15y$  subject to

$$x + y \leq 50$$

$$3x + y \leq 90$$

$$x, y \geq 0$$

2) Maximise  $z = 5x + 3y$  subject to

$$3x + 5y \leq 15$$

$$5x + 2y \leq 10$$

$$x \geq 0, y \geq 0$$

3) Minimise  $z = 3x + 5y$  subject to

$$x + 3y \geq 3$$

$$x + y \geq 2$$

$$x, y \geq 0$$

4) Minimise  $z = x - 7y + 190$  subject to

$$x + y \leq 8$$

$$x \leq 5, y \leq 5, x + y \geq 4, x \geq 0, y \geq 0$$

- II.5)** Kellogg is a new cereal formed from a mixture of bran and rice that contains atleast 88gm of protein and atleast 36 milligram of iron per kg. knowing that bran contains 80 gm of protein and 40 milligram of iron per kilogram and that rice contains 100gm of protein and 30 milligram of iron per kilogram, find the minimum cost of producing this new cereal if bran cost ₹5 per kilogram and rice cost ₹4 per kilogram.
- 6) A firm owned by Abhirami has to transport 1200 packages using large vans which can carry 200 packages each and small vans which can take 80 packages each. The cost for engaging each large van is ₹400 and each small van is ₹200. Not more than ₹3000 is to be spent on the job and the number of large vans cannot exceed the number of small vans. Solve this L.P.P. graphically to find the minimum cost.
- 7) There are two types of fertilisers  $F_1$  and  $F_2$ .  $F_1$  consists of 10% nitrogen and 6% phosphoric acid and  $F_2$  consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer Vaidhya finds that he needs atleast 14kg of nitrogen and 14kg of phosphoric acid for the crop. If  $F_1$  costs ₹6 per kg and  $F_2$  costs ₹5 per kg, determine how much of each type of fertilizer should be used so that nutrient requirements are met at a minimum cost. Solve graphically.
- 8) A Television company owned by Priyanka and Bhavana operates two assembly lines, line I and line II. Each of line is used to assemble the components of three types of television: colour standard and economy. The expected daily production on each line is as follows:

TV model	Line I	Line II
Colour	3	1
Standard	1	1
Economy	2	6

The daily running costs for two lines average ₹6000 for line I and ₹4000 for line II. It is given that company must produce at least 24 colour, 16 standard and 48 economy TV sets for which an order is pending. Formulate L.P.P. and solve graphically determining the number of days the two lines should be run to meet the requirements.

- 9) Old hens can be bought at ₹2 each and young ones at ₹5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen cost ₹1 per week to feed. Deepthi has only ₹80 to spend for hens. How many hens of each kind should Deepthi buy to give a profit of more than ₹6 per week assuming that Deepthi cannot house more than 20 hens. Solve graphically.
- 10) A company owned by Viswa Narayana concentrates on two grades of paper A and B, produced on a paper machine. Because of raw material restrictions, not more than 400 tonnes of grade A and 300 tonnes of grade B can be produced in a week. There are 160 production hours in a week. It requires 0.2 hour and 0.4 hour to produce one tonne of products A and B respectively with corresponding profits of ₹20 and ₹50 per tonne. Find the optimum product mix using the graphical method.
- 11) A company owned by Navya manufactures two types of cloth, using three different colours of wool. One yard length of type A cloth required 4 oz (ounce) of red wool, 5 oz of green wool, 3 oz (ounce) of yellow wool. One yard length of type B cloth requires 5 oz red wool, 2 oz of green wool and 8 oz of yellow wool. The wool available for manufacture is 1000 oz of red wool 1000 oz of green wool and 1200 oz of yellow wool. The manufactures can make a profit of ₹5 on one yard of type A cloth and ₹3 on one yard of type B cloth. Find the best combination of the quantities of type A and Type B cloth which gives him maximum profit by solving the L.P.P. by graphical method.

### ANSWERS 11.2

I. 1)  $Z_{\max} = 1800$  when  $x = 30$  and  $y = 0$

2)  $Z_{\max} = \frac{235}{19}$  when  $x = \frac{20}{19}$  and  $y = \frac{45}{19}$

3)  $Z_{\min} = 7$  at  $\left(\frac{3}{2}, \frac{1}{2}\right)$

4) Minimum value of  $z = 155$  when  $x = 0$  and  $y = 5$

5) Minimum cost is ₹4.60 when 0.6kg of bran and 0.4 kg of rice are mixed

6) Min  $z = 400x + 200y$  subject to  $5x + 2y \geq 30$ ,  $2x + y \leq 15$ ,  $x \leq y$ ,  $x \geq 0$ ,  $y \geq 0$

Minimum value of  $z = ₹2600$  when  $x = 4$ ,  $y = 5$

7) Minimum value of  $z = 1000$  when  $x = 100$  and  $y = 80$

8)  $Z_{\min} = 72000$  when  $x = 4$  and  $y = 12$

9)  $Z_{\max} = ₹8$  when  $x = 0$  and  $y = 16$

10)  $Z_{\max} = ₹19,000$  when  $x = 200$  and  $y = 300$

11)  $Z_{\max} = 1000$  when  $x = 200$  and  $y = 0$

\*\*\*\*

**12.1 Introduction :**

The developmental activities done by the government to the public by collecting revenue from various sources from the public in the form of various type of taxes. Among them the sales tax and value added tax plays a vital role in revenue to the government. At each stage of the transaction of the goods from the manufactures, distributors, whole saler and retailer till to customer, **sales tax (ST)** and the **value added tax (VAT)** Exist. In this chapter we discuss and solve the problem in detail.

**12.2 Sales Tax :**

**Definition :** On the purchase of some items, we have to pay certain amount at a specified rate. This amount is called 'Sales Tax'.

- The rate of sales tax depends upon the nature of goods purchased. Some items of daily use are completely exempted from sales tax eg : Match boxes, salt etc.
- Sales Tax is one of the indirect taxes as it affects all individual indirectly. Other examples of indirect taxes are excise duty, custom duty etc. State Government levies tax on sales of goods within the state.
- Calculation of sales tax involves the use of the concept of percentage. The rate of sales tax is different for different items. Sales tax collected by the state government is used for the development of the state as well as to meet the expenses like payment to its employees, health, education etc.

**Important formulae**

1. Profit = Selling price - Cost price = SP – CP
2. Loss = Cost price - Selling price = CP – SP
3. Profit% =  $\frac{\text{Profit}}{\text{CP}} \times 100$
4. Loss% =  $\frac{\text{Loss}}{\text{CP}} \times 100$
5. SP =  $\frac{100 + \text{Gain}\%}{100} \times \text{CP}$
6. SP =  $\frac{100 - \text{Loss}\%}{100} \times \text{CP}$
7. Total amount to be paid = marked price + sales tax on marked price = MP + ST% on MP
8. Discount = discount % of MP
9. ST% =  $\frac{\text{ST}}{\text{MP}} \times 100$

**Note :**

- (i) **Marked price = Market price = Printed price = List price = Sale price = MP**
- (ii) **Sales Tax is always found on MP**
- (iii) **Discount is Calculated on MP**
- (iv) **ST% is found on new marked price (after discount)**

**WORKED EXAMPLES**

**Example 1 :**

**Sharath paid ₹ 40 sales tax on a pair of shoes worth ₹ 500. Find the rate of sales tax?**

$$\begin{aligned}\text{Solution : S.T\%} &= \frac{\text{ST}}{\text{MP}} \times 100 \\ &= \frac{40}{500} \times 100 = 8\%\end{aligned}$$

**Example 2 :**

**The price of a washing machine inclusive of sales tax is ₹ 13,530. If the sales tax is 10%. Find the basic price.**

**Solution :** Suppose the basic price (MP) of the washing machine = ₹  $x$

$\therefore$  total amount paid = MP + ST% of MP

$$13530 = x + 10\% \text{ of } x$$

$$13530 = x + \frac{10x}{100}$$

$$13530 = x + \frac{x}{10}$$

$$13530 = \frac{10x + x}{10} = \frac{11x}{10}$$

$$x = \frac{13530 \times 10}{11} \quad \therefore x = ₹ 12,300$$

**Example 3 :**

**Gopal purchased a scooter costing ₹ 32,450. If the rate of sales tax is 9%. Calculate the total amount payable by him?**

**Solution :** Total amount paid = MP + ST% of MP

$$= 32,450 + 9\% \text{ of } 32,450$$

$$= 32,450 + \frac{9}{100} \times 32450$$

$$= 32450 + 9 \times 32450$$

$$\text{Total amount paid} = ₹ 35,370.50$$

**Example 4 :**

**A colour T.V is marked for sale for ₹17,600 which include sales tax at 10%. Calculate the sales tax in ₹?**

**Solution :** SP = ₹ 17,600, ST% = 10%, ST = ?, MP = x

$$\text{Total amount to be paid (SP)} = \text{MP} + \text{ST\% of MP}$$

$$17600 = x + 10\% \text{ of } x$$

$$17600 = x + \frac{10}{100}x$$

$$17600 = x + \frac{x}{10}$$

$$17600 = \frac{10x + x}{10} = \frac{11x}{10} \quad \therefore x = \frac{17600 \times 10}{11}$$

$$x = ₹ 16,000$$

$$\therefore \text{S.T} = \text{SP} - \text{MP}$$

$$= 17600 - 16000 = ₹ 1,600$$

**Example 5 :**

**Sanju goes to a shop to buy a Bicycle quoted at ₹ 2,000. The rate of sales tax is 12% on it. He asks the shopkeeper for a rebate on the price of the bicycle to such an extent that he has to pay ₹ 2,016 inclusive of sales tax. Find the rebate percentage on the price of the bicycle.**

**Solution :**

$$\text{Let SP} = ₹ 2016$$

$$\text{ST} = 12\%$$

$$\text{MP} = ₹ x \quad \text{Rebate \%} = ?$$

$$\text{Amount paid} = \text{SP} = \text{MP} + 12\% \text{ of MP}$$

$$2016 = x + \frac{12}{100}x$$

$$2016 = \frac{100x + 12x}{100}$$

$$2016 \times 100 = 112x$$

$$\therefore x = \frac{2016 \times 100}{112} = ₹ 1,800$$

$$\begin{aligned}\therefore \text{Rebate} &= 2000 - 1800 \\ &= ₹ 200\end{aligned}$$

$$\begin{aligned}\therefore \text{Rebate} &= \frac{\text{Rebate}}{\text{Quoted price}} \times 100 = \frac{200}{2000} \times 100 \\ &= 10\%\end{aligned}$$

**Example 6 :**

**A shopkeeper announces a discount of 10% on a T.V set. The marked price of the T.V is ₹ 22,000. How much will a customer have to pay for buying the T.V set if the rate of sales tax is 8%.**

**Solution :**

$$\text{MP} = ₹ 22,000$$

$$\text{ST} = 8\%$$

$$\text{Discount} = 10\% \text{ of M.P}$$

$$\begin{aligned}\text{Total amount paid} &= ? \\ &= \frac{10}{100} \times 22000 = ₹ 2,200\end{aligned}$$

$$\begin{aligned}\therefore \text{New MP} &= 22,000 - 2,200 \\ &= ₹ 19,800\end{aligned}$$

$$\begin{aligned}\therefore \text{Amount to be paid} &= \text{MP} + \text{ST}\% \text{ on MP} \\ &= 19800 + 8\% \text{ of } 19800 \\ &= 19800 + \frac{8}{100} \times 19800 \\ &= 19800 + (8 \times 198) = ₹ 21,384\end{aligned}$$

**Example 7 :**

**Mr. Govind buys a tape recorder for ₹ 10,260 including sales tax. If the list price of the tape recorder is ₹ 9,500. Find the rate of sales tax charged?**

**Solution:**

$$\text{SP} = 10,260 \text{ selling price} = \text{list price (MP)} + \text{ST}\% \text{ on MP}$$

$$\text{MP} = ₹ 9500, \text{ ST} = x\%$$

$$10,260 = 9,500 + x\% \text{ of } 9,500$$

$$10,260 - 9500 = x\% \text{ of } 9500$$

$$760 = \frac{x \times 9500}{100}$$

$$760 = 95x$$

$$\therefore x = \frac{760}{95}$$

$$\therefore \text{ST} = 8\%$$

**Example 8 :**

**Sohail goes to purchase a motorcycle which is priced at 35,640 including 10% as sales tax. How was the actual rate of sales tax at the time of purchase is 7%. Find the extra profit made by the shopkeeper if he still charges the original list price?**

**Solution:**

$$\text{Purchase price} = \text{S.P} = ₹ 35,640$$

$$\text{M.P} = ₹ x$$

$$\text{S.T} = 10\%$$

$$\text{S.T at the time of purchase} = 7\%$$

$$\text{Project} = ?$$

$$\text{SP} = \text{MP} + \text{ST}\% \text{ of MP}$$

$$35,640 = x + 10\% \text{ of } x$$

$$35,640 = x + \frac{10x}{100}$$

$$35,640 = x + \frac{x}{10} = \frac{10x + x}{10} = \frac{11x}{10}$$

$$x = \frac{35640 \times 10}{11}$$

$$x = ₹ 32,400$$

$$\text{Amount to be paid} = 32,400 + 7\% \text{ of } 32400$$

$$= 32,400 + \frac{7 \times 32400}{100}$$

$$= 32400 + 2268$$

$$= ₹ 34,668$$

$$\therefore \text{profit made by the shopkeeper} = 35640 - 34668 = ₹ 972$$

**Example 9 :**

**A shopkeeper announces a discount of 10% on a washing machine set. The marked price of the washing machine is ₹ 12000. How much will a customer have to pay for buying the washing machine set if the rate of sales tax is 8%?**

**Solution:** Discount = 10%

$$\text{M.P} = ₹ 12000$$

$$\text{S.T} = 8\%$$

$$\text{Amount paid} = \text{SP} = ?$$

$$\text{Discount of 10\% on MP} = 10\% \text{ of } 12000$$

$$= \frac{10}{100} \times 12000 = ₹ 1200$$

$$\text{New M.P} = 12000 - 1200 = ₹ 10,800$$

$$\therefore \text{Total amount to be paid} = 10,800 + 8\% \text{ of } 10,800$$

$$= 10,800 + \frac{8}{100} \times 10,800$$

$$= 10,800 + 864 = ₹ 11,664$$

**Example 10 :**

**A shopkeeper purchases an Audio system for ₹ 3,000 and sells it off at a gain of 15%. He also charges a sales tax of 10% on the selling price. Calculate the amount that the buyer will pay to the shopkeeper.**

**Solution:**

$$\text{Cost price} = ₹ 3,000$$

$$\text{Gain (profit)} = 15\% \text{ of cost price}$$

$$= \frac{15 \times 3000}{100}$$

$$= ₹ 450$$

$$\therefore \text{S.P} = \text{C.P} + \text{Gain}$$

$$= 3000 + 450 = ₹ 3450$$

$$\therefore \text{Total amount to be paid} = \text{S.P} + \text{S.T}\% \text{ of SP}$$

$$= 3450 + 10\% \text{ of } 3450$$

$$= 3450 + \frac{10 \times 3450}{100}$$

$$= 3450 + 345 = ₹ 3,795$$

**Example 11:**

**When the rate of sales tax is decreased from 9% to 7%. For a Radio, Rahul has to pay ₹632 less for it. What is the listed price of the radio?**

**Solution:** Assume the M.P = ₹  $x$

Total amount paid when S.T is 9% =  $x + \text{S.T}\% \text{ on } x$

$$= x + \frac{9x}{100}$$

Again total amount paid when ST is 7% =  $x + \text{S.T}\% \text{ on } x = x + \frac{7x}{100}$

∴ Total amount paid when ST is 9% = Total amount paid when ST is 7% + 632

$$x + \frac{9x}{100} = \left( x + \frac{7x}{100} \right) + 632$$

$$\frac{100x + 9x}{100} = \left( \frac{100x + 7x}{100} \right) + 632$$

$$\frac{109x}{100} = \frac{107x}{100} + \frac{632}{1} \quad (\text{Take LCM})$$

$$= \frac{107x + 632(100)}{100}$$

$$\frac{109x}{100} = \frac{107x + 63200}{100}$$

$$109x - 107x = 63200$$

$$2x = 63200$$

$$\therefore x = \frac{63200}{2}$$

$$\boxed{\text{MP} = ₹31,600}$$

**Example 12:**

**If the rate of sales tax is 5% Sushma has to pay ₹ 7,140 for the steel cupboard. What amount she has to pay if the sales tax is increased by 2%.**

**Solution:** Let the MP of cupboard = ₹  $x$

S.T = 5%

Amount paid by Sushma at 5% ST = ₹ 7140

$$\therefore 7140 = \text{M.P} + \text{S.T}\% \text{ on MP}$$

$$7140 = x + 5\% \text{ of } x$$

$$7140 = x + \frac{5x}{100}$$

$$7140 = \frac{100x + 5x}{100}$$

$$714000 = 105x$$

$$x = \frac{714000}{105} = ₹ 6,800$$

$$\therefore \boxed{\text{MP} = ₹ 6,800}$$

$$\text{Sales tax charges} = 5 + 2 = 7\%$$

$$\text{Total amount to be paid by Sushma} = 6800 + 7\% \text{ of } 6800$$

$$= 6800 + \frac{7 \times 6800}{100} = 6800 + 476 = ₹ 7276$$

**Example 13:**

**Chandana purchases an article for ₹ 5,400 which include 10% rebate on the marked price and 20% sales tax on the remaining price. Find the marked price of the article?**

**Solution:**

Let the marked price of the article = ₹  $x$

$$\text{Rebate} = \frac{10}{100}x = \boxed{\frac{x}{10}}$$

$$\therefore \text{Remaining price} = x - \frac{x}{10} = \frac{10x - x}{10} = \frac{9x}{10}$$

$$\therefore \text{Total amount to be paid} = \text{Remaining price} + \text{ST}\% \text{ on remaining price} \\ (\text{purchase price})$$

$$5400 = \frac{9x}{10} + 20\% \text{ of } \frac{9x}{10}$$

$$= \frac{9x}{10} + \left( \frac{20}{100} \times \frac{9x}{10} \right)$$

$$= \frac{9x}{10} + \frac{9x}{50}$$

$$5400 = \frac{45x + 9x}{50} = \frac{54x}{50}$$

$$54x = 5400 \times 50$$

$$\therefore x = \frac{\overset{100}{\cancel{5400}} \times 50}{\cancel{54}}$$

$$\therefore \text{M.P} = \boxed{x = ₹5,000}$$

#### Example 14 :

Sharath goes to a departmental store and purchase the following articles

- (i) A raincoat for ₹ 300 S.T @ 10%
- (ii) A pair of shoes for ₹ 460 S.T@ 9%
- (iii) Food article for ₹ 450 S.T@5%
- (iv) Cloth for ₹ 800 ST@1%

Calculate total amount of the bill.

$$\begin{aligned} \text{Solution : Purchase price of Raincoat} &= 300 + \text{S.T}\% 300 \\ &= 300 + 10\% \text{ of } 300 \\ &= 300 + \frac{10}{100} \times \overset{3}{\cancel{300}} = 300 + 30 = ₹ 330 \end{aligned}$$

$$\begin{aligned} \text{Similary, purchase price of shoe} &= 460 + \left( \frac{9}{100} \times 460 \right) \\ &= 460 + 41.40 = ₹ 501.40 \end{aligned}$$

$$\text{Purchase price of food article} = 450 + 22.50 = ₹ 472.50$$

$$\begin{aligned} \text{Purchase price of clothes} &= 800 + \frac{1 \times \overset{8}{\cancel{800}}}{100} \\ &= 800 + 8 = ₹ 808 \end{aligned}$$

$$\begin{aligned} \therefore \text{Total amount of the bill} &= ₹ (330 + 501.40 + 472.50 + 808) \\ &= ₹ \boxed{2,111.90} \end{aligned}$$

**Example 15 :**

**Following purchases were made from a store**

Item	List price (₹)	No. of item	Rate of ST
Tape recorder	10,900	2	10%
Suitcase	4,200	1	5%
Raincoat	400	3	7%
Mixer	2,500	$x$	2%

**Total amount paid by Raju is ₹ 34,774. Find the number of mixers bought.**

**Solution:**

Amount to be paid for 2 tape recorder

$$= 2[10,900 + 10\% 10,900]$$

$$= 2 \left[ 10,900 + \frac{10}{100} \times 10900 \right]$$

$$= 2 \times [10,900 + 1090]$$

$$= 2 \times 11,990 = ₹ 23,980$$

Similarly, amount to be paid for 1 suitcase

$$= 4200 + 5\% (4200)$$

$$= 4200 + \frac{5}{100} \times 4200 = 4200 + 210 = 4410$$

Amount to be paid for 3 Raincoats

$$= 3[400 + 7\% (400)]$$

$$= 3 \left[ 400 + \frac{7 \times 400}{100} \right]$$

$$= 3 \times 428 = ₹ 1284$$

Amount to be paid for ' $x$ ' mixers

$$= x[2500 + 2\% (2500)]$$

$$= x \left( 2500 + \frac{2 \times 2500}{100} \right)$$

$$= x (2500 + 50) = 2550 x$$

$\therefore$  Total amount paid = total amount of all the articles

$$34,774 = 23980 + 4410 + 1284 + 2550 x$$

$$\begin{aligned} 34,774 &= 29,674 + 2550x \\ \therefore 2550x &= 34,774 - 29,674 \\ 2550x &= 5100 \end{aligned}$$

$$\therefore x = \frac{5100}{2550} = 2$$

$\therefore$  number of mixers bought = 2

### EXERCISE 12.1

#### I. 2 Marks question:

1. Abhishek purchase a bicycle costing ₹12,000. If the rate of sales tax is 9%. Calculate the total amount payable by him.
2. Ramu paid ₹ 60 as sales tax on a Titan Raga watch worth ₹1200. Find the rate of sales tax.
3. Refrigerator is marked for sale for ₹ 17,000 which include sales tax at 10% Calculate the sales tax in ₹

#### II. 3 Marks question:

1. Bharath bought a shirt for ₹ 336, including 12% sales tax and a necktie for ₹110 including 10% sales tax. Find the printed price of shirt and necktie together.
2. A furniture dealers sold furniture for ₹ 21,000 and added 5% S.T to the quoted price. The customer agrees to buy it for ₹ 21,000 including S.T. find the discount he received.
3. A shopkeeper sells an item at the price of ₹ 810 including ST of 8% what should a customer pay for the same item if the ST is reduced to 6%.
4. The price of T.V set inclusive of sales tax of 9% is ₹ 13.407. Find its marked price. If the S.T is increased to 13% how much more does the customer Sneha pay for the T.V.

### ANSWERS 12.1

- |     |                   |          |             |
|-----|-------------------|----------|-------------|
| I.  | 1. ₹13,080        | 2. 5%    | 3. ₹1545.45 |
| II. | 1. ₹ 400          | 2. ₹1000 | 3. ₹ 795    |
|     | 4. ₹12,300, ₹ 492 |          |             |

### 12.3 Value Added Tax (VAT)

**It is neither a new tax nor in addition to the existing sales tax. It is the replacement of sales tax.**

**Note:**

- (i) VAT is a form of sales tax only. The only difference in that it is collected in stages rather than at one point from the sales of goods.
- (ii) In VAT, the firm seller pays the first point of tax to the government and sub sequent

seller pays tax to the government on the value added by them, leading to a total tax exactly to the last point tax paid by the customer.

- (iii) Advantage of VAT : (i) It reduces the scope of under evaluation  
(ii) It provide a broad base tax system.
- (iv) VAT paid by the shopkeeper = VAT% of (selling price - Shopkeeper purchase price)
- (v) VAT = VAT% (selling price - cost price)

**Example 1:** Let the rate of sales tax be 10% suppose a trader Siddharth

**Step 1 :** Buys a Radio from a factory owner Venkatesh for ₹1000 then the tax paid by the trader to the factory owner.

$= 10\% \text{ of } 1000 = \frac{10}{100} \times 1000 = ₹100$ . So the factory owner Venkatesh pay ₹100 to the government as a tax.

**Step 2 :** It is trader sell the Radio to a consumer for ₹ 1500 then the tax paid by the

consumer to the trader  $= 10\% \text{ of } 1500 = \frac{10}{100} \times 1500 = ₹150$

Hence the trader has added ₹500 in value of the radio so he will pay tax to the government only on the added value

i.e.,  $10\% \text{ of } ₹ 500 = \frac{10 \times 500}{100} = 50$

Total tax paid to the government = ₹100 by factory owner + ₹ 50 by trader  
= ₹150 tax paid by the consumer

**Example 2 :**

Let rate of sales tax be 5%

	Producer	trader 1	trader 2	consumer
Sale price	₹1000	₹1200	₹1600	
Tax charged	by producer $= \frac{5 \times 1000}{100} = 50$	by trader 1 $= \frac{5 \times 1200}{100} = 60$	by trade 2 $= \frac{5 \times 1600}{100} = ₹80$	Tax paid
Tax paid to Government	by producer ₹50	by trader ₹10 = (60 - 50)	by trader 2 ₹20 (80 - 60)	by the consumer ₹80

Total tax paid by the consumer = ₹80

Total tax collected by the govt = ₹ (50 + 10 + 20) = ₹80

**Note :** The tax is collected at different stages

**Input tax :** Tax paid by a dealer on the purchase of good for resale (subject to tax under VAT) during any fixed tax period is called input tax.

**Output tax :** The tax charged by dealers on the sales of good during any fixed period is called output tax.

**Computation of VAT :** Net tax to be paid by the dealer = (output tax) – (Input tax)

### WORKED EXAMPLES

**Example 1 :**

**A shopkeeper purchase an article for 7,000 and sell it to a customer for 8,200. If the VAT rate is 6%. Find the VAT paid by the shopkeeper?**

**Solution:** VAT paid by the shopkeeper = VAT% of (customer price - shopkeeper price)  

$$= \text{VAT\% of } (SP - CP)$$

$$= 6\% (8200 - 7000)$$

$$= \frac{6 \times (1200)}{100} = ₹ 72$$

**Example 2 :**

**Mr. Arya purchase an article for ₹ 3,100 and sell it to Mr. Aravind for ₹ 4,250. Mr. Aravind in turn sells it to Mr. Anil for ₹ 5,000. If the VAT levied 10%. Find the VAT levied on Arya and Aravind?**

**Solution:**

VAT paid by Arya = 10% of ₹ (4,250 – 3,100)  

$$= \frac{10 \times 1150}{100} = ₹ 115$$
VAT paid by Aravind = 10% of ₹ (5,000 – 4,250)  

$$= \frac{10 \times 750}{100} = ₹ 75$$

**Example 3 :**

**A shopkeeper purchased an item of ₹100 at 8% VAT and sell it at ₹ 120 to a customer and the customer also pay 8% VAT to the shopkeeper. How much amount did the shopkeeper deposit to the government as VAT?**

**Solution:**

	Shopkeeper	Customer
C.P	₹100	₹120
VAT = 8%, VAT	$\frac{8 \times 100}{100} = ₹8$	$\frac{8 \times 120}{100} = 9.60$
Tax charged : VAT	= ₹8	₹(9.60 - 8) = 1.60

Total tax paid to government = (8 + 1.60) = ₹ 9.60

**Example 4 :**

**A shopkeeper buys a mobile set at a discount rate of 20% from the wholesaler, the printed price of the mobile set being ₹1,600 and the rate of sales tax is 6%. The shopkeeper sells it to the buyer at the printed price and charges tax at the same rate.**

- Find (i) the price at which the mobile set can be bought from whole saler  
(ii) the VAT paid by the shopkeeper.

**Solution:** (i) Price at which the mobile net can be bought

$$= 80\% \text{ of } 1,600 + 6\% \text{ VAT of } 1280$$

$$= \left( \frac{80 \times 1600}{100} \right) + \left( \frac{6 \times 1280}{100} \right)$$

$$= 1,280 + 76.8$$

$$= ₹ 1356.80$$

- (ii) for whole saler's VAT = 6% of 1,280

$$= \frac{6 \times 1280}{100}$$

$$= ₹ 76.80$$

for buyer VAT = 6% of 1600

$$= \frac{6 \times 1600}{100}$$

$$= ₹ 96$$

VAT paid by shopkeeper = for buyer VAT for whole sale VAT

$$= 96 - 76.80$$

$$= ₹19.20$$

**Example 5 :**

Assume that the profit of 20 at each stage of the selling chain. The manufacture 'A' the sale distributor B, the whole saler C and the retailer D live in the same state where the rate of VAT is 8%. The cost price of 'A' in commodity is 200. Find the amount of VAT if A sells to B. B sells to C and C sells to D. Also find the selling price of D.

**Solution:**

	Manufacture	Distributor	Whole saler	Retailer
	A	B	C	D
cost price	₹200	₹220	₹240	₹200
selling price	₹220	₹240	₹260	₹280
$\frac{\text{VAT} = 8\%}{\text{Tax charged} \rightarrow}$	$\frac{8 \times 220}{100} = 17.60$	₹19.20	₹20.80	₹22.40
VAT	₹17.60	(19.20-17.60) ₹1.60	₹1.60	₹1.60

$$\therefore \text{Total VAT} = (17.60 + 1.60 + 1.60 + 1.60) = ₹ 22.40$$

$$\text{Actual selling price of D} = ₹260 + ₹20 + 22.40 = ₹302.40$$

**EXERCISE 12.2**

**I. 2 and 3 mark question**

1. A shopkeeper purchase an article for ₹ 9,000 and sell it to a customer for 10,500. If the VAT rate is 4%. Find the VAT paid by the shopkeeper?
2. A shopkeeper purchased an electric iron of ₹1000 at 8% VAT from the whole saler and sell it to the customer of ₹1400 at 8% VAT.

Find

- (a) amount paid by the customer
  - (b) the VAT to be paid by the shopkeeper
3. A shopkeeper bought a TV at a discount of 30% of the listed price of ₹24,000. The shopkeeper offer a discount of 10% of the listed price to the customer. If the VAT is 10%. Find
    - (i) the amount paid by the customer.
    - (ii) the VAT to be paid by the shopkeeper.

4. Mohan a owner of a departmental store purchased an article of ₹ 1,500 at 4% VAT and sell it at ₹ 1,700 to the customer at 4% VAT. How much amount did the shopkeeper deposit to the government as VAT.
5. Sanju a owner of jewellery shop purchased a ear ring of ₹ 2,000 at 12% VAT and sells it at 2,300 to the customer Radhika. If Radhika also pays 12% VAT to the shopkepr how much did the shopkeeper deposit to the government as VAT?
6. 'A' is manufacture of electric iron. The cost price of each electric iron in ₹1,600. He sells to 'B' and 'B' sells to 'C' and 'C' sells to 'D' the retailer. The tax rate is 12.5% and the profit is ₹ 150 at each stage of the selling chain. Find the
  - (i) the total amount of VAT and
  - (ii) the amount that the purchased will have to pay.

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**ANSWERS 12.2**

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- |        |                  |                             |
|--------|------------------|-----------------------------|
| 1. ₹60 | 2. ₹1512 and ₹32 | 3. ₹23,760, ₹480            |
| 4. ₹ 8 | 5. ₹36           | 6. (i) ₹256.25 (ii) ₹ 2,475 |

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## UNIT III - TRIGONOMETRY

Chapter	Title	No. of Teaching hrs.
13.	HEIGHTS AND DISTANCES	04 hrs
14.	COMPOUND ANGLES, MULTIPLE ANGLES, SUBMULTIPLE ANGLES AND TRANSFORMATION FORMULAE	08 hrs
	<b>TOTAL TEACHING HOURS</b>	<b>12 hrs</b>



## 13.1 Introduction :

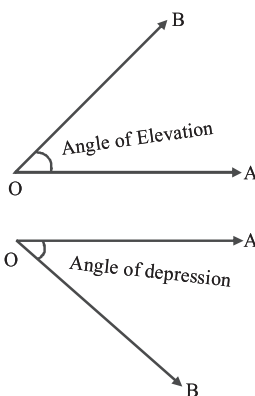
One of the important practical applications of Trigonometry is to find the heights and distances of visible inaccessible objects such as the top of a Tower, the summit of a hill, two ships at sea and so on, from the position of the observer without actually measuring it. To solve these, we often use the terms angle of elevation and angle of depression, which are defined below.

## 13.2 Angle of Elevation and Angle of Depression :

**Angle of Elevation** : The angle between the horizontal line drawn through the observer's eye and the line joining the eye of the observer to any object is called the Angle of Elevation of the object, when the object is at a higher level than the eye.

**Angle of depression** : The angle between the horizontal line drawn through the observer's eye and the line joining the eye of the observer to any object is called the Angle of depression of the object when the object is at a lower level than the eye.

**Note** : In all the problems, objects such as mountains, towers, trees etc. are considered to be linear and unless, otherwise mentioned the height of the observer is neglected.



## WORKED EXAMPLES

## Example 1:

The angle of elevation of the top of a tower at a distance of 200 metres from its foot is  $60^\circ$ , find the height of the tower.

## Solution:

Let O be the position of the observer and AB denote the Tower. Let the heights of the Tower be h.

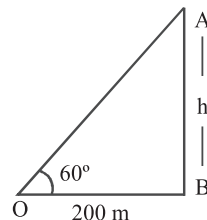
We have  $\angle AOB = 60^\circ$ ,  $OB = 200\text{m}$

From the right angled triangle ABO, we get

$$\tan 60^\circ = \frac{h}{200} \Rightarrow \sqrt{3} = \frac{h}{200}$$

$$\Rightarrow h = 200\sqrt{3} \text{ metres}$$

Hence the height of the Tower is  $200\sqrt{3}$  metres



**Example 2:**

**Find the angle of elevation of the sun's rays from a point on the ground at a distance of 1 metre from the foot of tower  $\sqrt{3}$  m height.**

**Solution:**

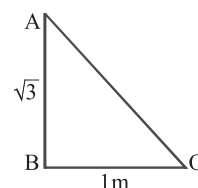
Let AB be the height of the tower &  $AB = \sqrt{3} \text{ m}$

$OB = 1 \text{ m}$  &  $\angle AOB = ?$

From the right angled triangle AOB we have  $\tan \theta = \frac{\sqrt{3}}{1}$

$$\tan \theta = \sqrt{3} \Rightarrow \theta = 60^\circ$$

$\therefore$  The angle of elevation is  $60^\circ$



**Example 3:**

**Find the angle of elevation of the Sun when the length of the shadow of a pole is  $\sqrt{3}$  times the height of the pole.**

**Solution:**

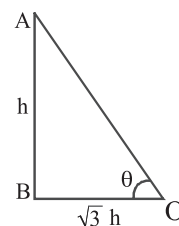
Let AB be the height of the pole

Let  $AB = h$ ,  $OB = \sqrt{3} h$  is the length of the shadow

From the right angled triangle

$$\tan \theta = \frac{h}{\sqrt{3}h} = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^\circ$$

$\therefore$  The angle of elevation is  $30^\circ$



**Example 4:**

**From a ship's mast head 50 metres high, the angle of depression of a boat is observed to be  $30^\circ$ . Find its distance from the ship.**

**Solution:**

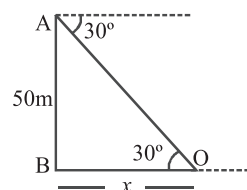
Let AB is mast head of a ship  $AB = 50 \text{ metres}$

O be the position of the boat

we have  $\angle AOB = 30^\circ$

From the right angled triangle

ABO we have  $\tan 30^\circ = \frac{50}{x}$



$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{50}{x} \Rightarrow x = 50\sqrt{3} \text{ metres}$$

$\therefore$  The distance from the ship =  $50\sqrt{3}$  metres

**Example 5:**

**A Ladder leaning against a wall makes an angle of  $60^\circ$  with the ground, the foot of the ladder 36m away from the wall. Find the length of the ladder.**

**Solution:**

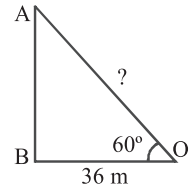
Let AO be the ladder

By data, BO = 36m,  $\angle AOB = 60^\circ$

From a right angled triangle  $\cos 60^\circ = \frac{36}{AO}$

$$\frac{1}{2} = \frac{36}{AO} \Rightarrow AO = 72m$$

$\therefore$  The length of the ladder = 72 metres.



**Example 6:**

**A kite flying at a height of h is tied to a thread which is 50m long. Assuming that there is no kink in the thread and it makes an angle of  $30^\circ$  with the ground. Find the height of the kite.**

**Solution:**

Let AB = h is the height of the kite

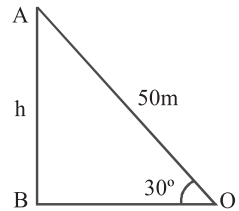
Give AO = 50m,

From the right angled triangle ABO

$$\sin 30^\circ = \frac{h}{50}$$

$$\frac{1}{2} = \frac{h}{50} \Rightarrow h = \frac{50}{2} = 25m$$

$\therefore$  The height of the kite is 25m



**Example 7:**

**A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is  $60^\circ$ . When he returns 40 metres from the bank, he finds the angle to be  $30^\circ$ . Find the height of the tree and the breadth of the river.**

**Solution:**

Let AB = h be the height of the tree,

BC = x be the breadth of the river.

By data  $\angle AOB = 30^\circ$ ,  $\angle ACB = 60^\circ$

OC = 40m

From right angled triangle ACB we get

$$\tan 60^\circ = \frac{h}{x} \Rightarrow \sqrt{3} = \frac{h}{x}$$

$$\Rightarrow x = \frac{h}{\sqrt{3}} \quad \dots\dots\dots(1)$$

Again from right angled triangle AOB

$$\tan 30^\circ = \frac{h}{40+x} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{40+x}$$

$$\Rightarrow h\sqrt{3} = 40+x$$

$$\Rightarrow x = h\sqrt{3} - 40 \quad \dots\dots\dots(2)$$

From equations 1 and 2 we get

$$\frac{h}{\sqrt{3}} = \frac{h\sqrt{3} - 40}{1}$$

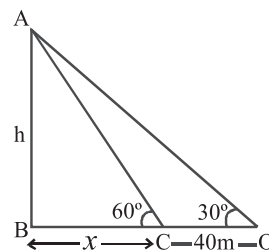
$$h = \sqrt{3}(h\sqrt{3} - 40)$$

$$h = 3h - 40\sqrt{3}$$

$$40\sqrt{3} = 2h \Rightarrow h = 20\sqrt{3} \text{ metres}$$

$$\text{But } x = \frac{h}{\sqrt{3}} = \frac{20\sqrt{3}}{\sqrt{3}} = 20 \text{ metres}$$

$\therefore$  The height of the tree is  $20\sqrt{3}$  metres and the breadth of the river is 20 metres.



### Example 8:

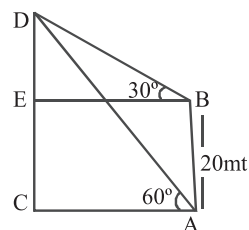
The angles of elevation of the top of a tower from the base and the top of a building are  $60^\circ$  and  $30^\circ$ . The building is 20 metres high. Find the height of the tower.

#### Solution:

Let AB be building and CD represent the tower

Let BE be the horizontal drawn through the point B

By data  $\angle EBD = 30^\circ$ ,  $\angle CAD = 60^\circ$  and AB = 20 metres



From the right angled triangle ADC. We have

$$\tan 60^\circ = \frac{DC}{AC} \Rightarrow \sqrt{3} = \frac{DC}{AC}$$

$$\Rightarrow AC = \frac{DC}{\sqrt{3}} \quad \dots\dots\dots(1)$$

Again from the right angled triangle DEB we have,  $\tan 30^\circ = \frac{DE}{EB} = \frac{DE}{AC}$  ( $\therefore EB = AC$ )

$$\frac{1}{\sqrt{3}} = \frac{DE}{AC} \Rightarrow AC = DE\sqrt{3} \quad \dots\dots\dots(2)$$

[But,  $DE = DC - EC$  ( $EC = AB = 20$ )

$$DE = DC - 20]$$

From equations 1 and 2 we get

$$\frac{DC}{\sqrt{3}} = DE\sqrt{3}$$

$$\frac{DC}{\sqrt{3}} = (DC - 20)\sqrt{3}$$

$$\Rightarrow DC = 3(DC - 20)$$

$$DC = 3DC - 60$$

$$2DC = 60$$

$$\Rightarrow DC = 30 \text{ metres}$$

$\therefore$  The height of the tower = 30 metres.

### Example 9:

The angles of elevation of the Summit of a hill from the top and the bottom of a tower are  $30^\circ$  and  $60^\circ$  respectively. If the height of the tower is  $h$ , show that the

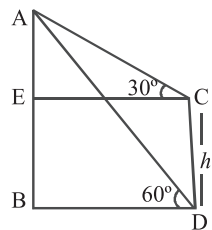
height of the hill is  $\frac{3h}{2}$

### Solution:

Let AB represents the height of the hill, CD represents the height of the tower

By data,  $\angle ADB = 60^\circ$ ,  $\angle ACE = 30^\circ$

$$DC = h$$



From the right angled triangle ABD we have

$$\tan 60^\circ = \frac{AB}{BD} \Rightarrow \sqrt{3} = \frac{AB}{BD} \Rightarrow BD = \frac{AB}{\sqrt{3}} \quad \dots\dots\dots (1)$$

Again from the right angled triangle AEC

$$\tan 30^\circ = \frac{AE}{EC} = \frac{AE}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB - EB}{BD} \Rightarrow \frac{1}{\sqrt{3}} = \frac{AB - h}{BD}$$

$$BD = \sqrt{3}(AB - h)$$

$$\frac{AB}{\sqrt{3}} = \sqrt{3}(AB - h)$$

$$AB = 3(AB - h)$$

$$AB = 3AB - 3h$$

$$3h - 2AB \Rightarrow AB = \frac{3h}{2}$$

$$\therefore \text{The height of the hill} = \frac{3h}{2}$$

### Example 10:

**A person is at the top of a tower 75 feet high, from there he observes a vertical pole and finds the angles of depressions of the top and the bottom of the pole which are  $30^\circ$  and  $60^\circ$  respectively. Find the height of the pole.**

### Solution:

Let AB represent the tower and CD represents the vertical pole. Let AX be the horizontal line drawn through A.

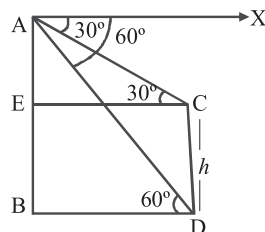
By data  $\angle XAD = 60^\circ$

$$\angle XAC = 30^\circ$$

Also AB = 75 ft. Let CD = h = ?

Since AX is parallel to CD we have  $\angle ACE = \angle XAC$

Again as AX is parallel to BD, we have  $\angle ADB = \angle XAD = 60^\circ$



$$\text{From the right angled triangle ADB } \tan 60^\circ = \frac{AB}{BD} \Rightarrow \sqrt{3} = \frac{75}{BD}$$

$$\Rightarrow BD = \frac{75}{\sqrt{3}} \quad \dots\dots\dots (1)$$

Again from the right angled  $\triangle AEC$ , we have

$$\tan 30^\circ = \frac{AE}{EC} = \frac{AE}{BD} \quad (\because EC = BD)$$

$$\frac{1}{\sqrt{3}} = \frac{AE}{BD} = \frac{AB - EB}{BD} = \frac{75 - CD}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{75 - CD}{BD} \Rightarrow BD = \sqrt{3} (75 - CD) \quad \dots\dots\dots (2)$$

From Eq<sup>ns</sup> 1 and 2 we get

$$\frac{75}{\sqrt{3}} = \sqrt{3}(75 - CD)$$

$$75 = 3 (75 - h) \Rightarrow 3h = 75 \times 2$$

$$h = \frac{75 \times 2}{2} \Rightarrow h = 50 \text{ ft}$$

$\therefore$  height of the vertical pole = 50ft

**Example 11:**

**Two towers of height 14m and 25m stand on level ground. The angles of elevation of their tops from a point on the line joining their feet are  $45^\circ$  and  $60^\circ$  respectively. Find the distance between the towers.**

**Solution:**

Let  $AB = 25\text{m}$  &  $DE = 14\text{m}$  are two towers

By data  $\angle ACB = 60^\circ$ ,  $\angle DCE = 45^\circ$

Let  $BC = x$  &  $CE = y$ ,  $x + y = ?$

from right  $\triangle ABC$

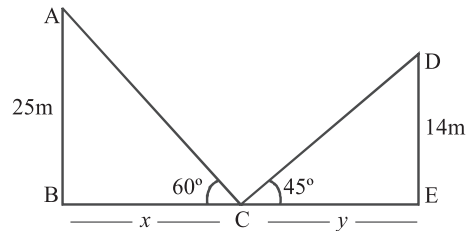
$$\tan 60^\circ = \frac{25}{x} \Rightarrow \sqrt{3} = \frac{25}{x}$$

$$\Rightarrow x = \frac{25}{\sqrt{3}} \quad \dots\dots(1)$$

Again from right angled  $\triangle DCE$

$$\tan 45^\circ = \frac{14}{y} \Rightarrow 1 = \frac{14}{y}$$

$$\Rightarrow y = 14 \quad \dots\dots(2)$$



From Eq<sup>n</sup> (1) and Eq<sup>n</sup>(2) we get

$$x + y = \frac{25}{\sqrt{3}} + \frac{14}{1} = \frac{25 + 14\sqrt{3}}{\sqrt{3}}$$

Hence the distance between the towers =  $\frac{25 + 14\sqrt{3}}{\sqrt{3}}$  metres.

**Example 12:**

**The angles of depressions of two boats as observed from the mast head of a ship 50m high are 45° and 30°. What is the distance between the boats if they are on the same side of the mast head in line with it?**

**Solution:**

Let AB be the mast head, C and D denote the positions of the boats

Given  $AB = 50m$   $\angle XAC = 45^\circ$

$\angle XAD = 30^\circ$

Also  $\angle BCA = 45^\circ$  ( $\therefore BD \parallel AX$ )

$\angle BDA = 30^\circ$  ( $\therefore BD \parallel AX$ )

From the right angled triangle ABC

we get  $\tan 45^\circ = \frac{AB}{BC} \Rightarrow BC = 50$  .....(1)

Again, from a right angled triangle ABD

$$\tan 30^\circ = \frac{AB}{BD} = \frac{50}{BC + x} \quad (\because CD = x)$$

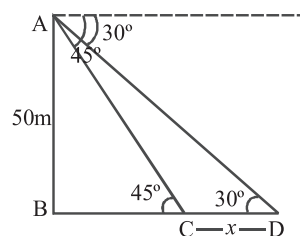
$$\frac{1}{\sqrt{3}} = \frac{50}{BC + x} \Rightarrow BC + x = 50\sqrt{3}$$

$$\Rightarrow BC = 50\sqrt{3} - x \quad \text{.....(2)}$$

From equations 1 & 2 we get  $50 = 50\sqrt{3} - x$

$$x = 50(\sqrt{3} - 1) \text{ metres}$$

Hence, the distance between the boats is  $50(\sqrt{3} - 1)$  metres



**EXERCISE 13.1**

**Two Mark Questions:**

- I 1) The angle of elevation of the top of a tower at a distance 500 metres from its foot is  $30^\circ$ . Find the height of the tower.
- 2) The angle of elevation of the top of a chimney at a distance of 100 metres from a foot is  $30^\circ$ . Find its height.
- 3) From a ship a mast head 40 metres high the angle of depression of a boat is observed to be  $45^\circ$ . Find its distance from the ship.
- 4) What is the angle of elevation of the sun when the length of the shadow of a pole is  $\frac{1}{\sqrt{3}}$  times the height of the pole?
- 5) Find the angle of elevation of the sun when the shadow of a tower 75 metres high is  $25\sqrt{3}$  metres long.
- 6) A kite flying at a height of  $h$  is tied to a thread which is 500m long. Assuming that there is no kink in the thread and  $H$  makes an angle of  $30^\circ$  with the ground. Find the height of the kite.
- 7) A ladder leaning against a wall makes an angle of  $60^\circ$  with the ground. The foot of the ladder is 6m away from the wall. Find the length of the ladder.
- 8) Find the angle of elevation of the Sun's rays from a point on the ground at a distance of  $3\sqrt{3}$  m, from the foot of tower 3m high.

**Four or Five marks questions:**

- 9) The angles of elevation of the top of a tower from the base and the top of a building are  $60^\circ$  and  $45^\circ$ . The building is 20 metres high. Find the height of the tower.
- 10) The shadow of a tower standing on a level plane is found to be 50 metres longer when Sun's altitude is  $30^\circ$ . Than when it is  $60^\circ$ . Find the height of the tower.
- 11) An aeroplane when flying at a heights of 2000 metres passes vertically above another plane at an instant when their angles of elevation from the same point of observation are  $60^\circ$  and  $45^\circ$  respectively. Find the distance between the aeroplanes.
- 12) From a point on the line joining the feet of two poles of equal heights, the angles of elevation of the tops of the poles are observed to be  $30^\circ$  and  $60^\circ$ . If the distance between the poles is a Find (i) the height of the poles (ii) the position of the point of observation.
- 13) The angles of elevation of the top of a tower from two points distant  $a$  and  $b$  ( $a < b$ ) from its foot and the same straight line from it are  $30^\circ$  and  $60^\circ$ . Show that the height of the tower is  $\sqrt{ab}$ .

- 14) A flag staff stands upon the top of a building. At a distance of 20 metres the angles of elevation of the top of the flag staff and building are  $60^\circ$  &  $45^\circ$  respectively. Find the height of the flag staff.
- 15) From the top of a cliff, the angles of depression of two boats in the same vertical plane as the observer are  $30^\circ$  and  $45^\circ$ . If the distance between the boats is 100 metres, find the height of the cliff.
- 16) From a point A due north of the tower, the elevation of the top of the tower is  $60^\circ$ . From a point B due south, the elevation is  $45^\circ$ , if  $AB = 100$  metres. Show that the height of the tower is  $50\sqrt{3}(\sqrt{3} - 1)$  metres.
- 17) A person at the top of a hill observes that the angles of depression of two consecutive kilometres stones on a road leading to the foot of the hill and in the same vertical plane containing the position of the observer are  $30^\circ$  and  $60^\circ$ . Find the height of the hill.
- 18) The angle of elevation of a tower from a point on the ground is  $30^\circ$ . At a point on the horizontal line passing through the foot of the tower and 100 metres nearer it, the angle of elevation is found to be  $60^\circ$ . Find the height of the tower and the distance of the first point from the tower.
- 20) A person is at the top of a tower 75 feet high from there he observes a vertical pole and finds the angles of depressions of the top and the bottom of the pole which are  $30^\circ$  and  $60^\circ$  respectively. Find the height of the pole.

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**ANSWERS 13.1**

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- 1)  $\frac{500\sqrt{3}}{3}$  mts      2)  $\frac{100}{\sqrt{3}}$  mts      3) 40 mts      4)  $60^\circ$       5)  $60^\circ$       6) 250 mts
- 7) 12 mts      8)  $30^\circ$       9)  $10\sqrt{3} + 1$  mts      10) 43.3 mts      11)  $\frac{2000(\sqrt{3} - 1)}{\sqrt{3}}$  metres
- 12) (i)  $\frac{a\sqrt{3}}{4}$       (ii)  $\frac{3a}{4}$       13)  $\sqrt{ab}$       14)  $\frac{20\sqrt{3}}{3}$  mts      15)  $+50(1 + \sqrt{3})$
- 16)  $50(\sqrt{3} + 1)$  mts      17)  $\frac{\sqrt{3}}{2}$  kms      18)  $32(\sqrt{3} + 1)$  mts,  $32\sqrt{3}$  mts
- 19)  $50\sqrt{3}$  mts, 150 mts      20) 50 feet

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### 14.1 Introduction:

The algebraic sum of two or more angles is called compound angles. Thus  $A + B$ ,  $A - B$ ,  $A + B + C$ ,  $A + B - C$  ..... etc are all Compound Angles.

$A$  is a given angle, then the angles  $2A$ ,  $3A$ ,  $4A$  ..... etc are called Multiple Angles.

$A$  is a given angle then  $\frac{A}{2}$ ,  $\frac{A}{3}$ ,  $\frac{A}{4}$  ..... etc are called Sub Multiple angles.

The transformation formulae which we derive here transform the product of two trigonometric functions into a sum or difference of two trigonometric functions and vice versa.

### 14.2 Trigonometrical ratios of Compound Angles

If  $A$  and  $B$  are the two angles then  $(A + B)$  and  $(A - B)$  are called compound angles. Let us now find the trigonometric ratios of  $(A + B)$  and  $(A - B)$  and use them to find the T-ratios of other compound angles.

#### Trigonometric ratios of $(A + B)$

(without proof)

1.  $\sin(A + B) = \sin A \cos B + \cos A \sin B$
2.  $\cos(A + B) = \cos A \cos B - \sin A \sin B$
3.  $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

#### Trigonometric - Ratios of $(A - B)$

4.  $\sin(A - B) = \sin A \cos B - \cos A \sin B$
5.  $\cos(A - B) = \cos A \cos B + \sin A \sin B$
6.  $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

**Note :** (i) We can find the T-ratios of  $(A - B)$  by replacing  $B$  by  $-B$  in T-Ratio's of  $(A + B)$  using

The results  $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = +\cos\theta$

$\tan(-\theta) = -\tan\theta$

(ii)  $\sin(A + B) \neq \sin A + \sin B$

But  $\sin(A + B) = \sin A \cos B + \cos A \sin B$

**WORKED EXAMPLES**

**Example 1:**

Prove that  $\cot(A + B) = \frac{\cot A \cdot \cot B - 1}{\cot B + \cot A}$

**Solution:**

$$\begin{aligned} \text{L.H.S} = \cot(A + B) &= \frac{1}{\tan(A + B)} \\ &= \frac{1}{\left[ \frac{\tan A + \tan B}{1 - \tan A \tan B} \right]} \\ &= \frac{1 - \tan A \cdot \tan B}{\tan A + \tan B} \end{aligned}$$

Divide both Numerator and Denominator by  $\tan A \cdot \tan B$ .

$$= \frac{\cot A \cdot \cot B - 1}{\cot B + \cot A} = \text{RHS}$$

$$\boxed{\cot(A + B) = \frac{\cot A \cdot \cot B - 1}{\cot B + \cot A}}$$

Similarly we can prove

$$\cot(A - B) = \frac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$$

**Example 2:**

Find the values of  $\sin 75^\circ$ ,  $\cos 75^\circ$  and  $\tan 75^\circ$

**Solution:**

$$\begin{aligned} \text{(i) } \sin(75^\circ) &= \sin(45^\circ + 30^\circ) \\ &= \sin 45^\circ \cdot \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}} \\ \therefore \sin 75^\circ &= \frac{\sqrt{3} + 1}{2\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{(ii) } \cos 75^\circ &= \cos (45^\circ + 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \end{aligned}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\text{(iii) } \tan 75^\circ = \tan (45^\circ + 30^\circ)$$

$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{\sqrt{3}}}$$

$$= \frac{\frac{\sqrt{3}+1}{\cancel{\sqrt{3}}}}{\frac{\sqrt{3}-1}{\cancel{\sqrt{3}}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{(3+1)^2}{(\sqrt{3})^2 - (1)^2} = \frac{3+1+2\sqrt{3}}{3-1}$$

$$= \frac{4+2\sqrt{3}}{2} = 2 + \sqrt{3}$$

$$\therefore \tan 75^\circ = 2 + \sqrt{3}$$

**Example 3:**

**Find the value of  $\sin 15^\circ$ ,  $\cos 15^\circ$  and  $\tan 15^\circ$ .**

**Solution:**

$$\begin{aligned} \text{(i) } \sin 15^\circ &= \sin (45^\circ - 30^\circ) \\ &= \sin 45^\circ \cdot \cos 30^\circ - \cos 45^\circ \cdot \sin 30^\circ \end{aligned}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\therefore \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\begin{aligned} \text{(ii) } \cos 15^\circ &= \cos (45^\circ - 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \end{aligned}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\therefore \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\text{(iii) } \tan 15^\circ = \tan (45^\circ - 30^\circ)$$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \cdot \tan 30^\circ}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2 - (1)^2} = \frac{3+1-2\sqrt{3}}{3-1}$$

$$= \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

$$\therefore \tan 15^\circ = 2-\sqrt{3}$$

**Example 4:**

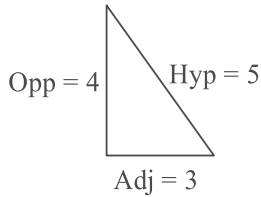
If  $\sin A = \frac{4}{5}$  and  $\sin B = \frac{5}{13}$  where A and B are acute angles. Find  $\sin (A + B)$ ,  $\sin (A - B)$ ,  $\cos (A + B)$ ,  $\cos (A - B)$ ,  $\tan (A + B)$ ,  $\tan (A - B)$

**Solution:**

$$\text{Given: } \sin A = \frac{4}{5} \quad \therefore \cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

$$\text{Also given } \sin B = \frac{5}{13} \quad \therefore \cos B = \sqrt{1 - \sin^2 B} = \sqrt{1 - \frac{25}{169}} = \frac{12}{13}$$

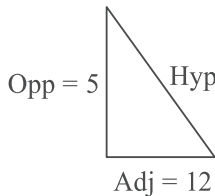
OR



$$\text{Hyp}^2 = \text{Opp}^2 + \text{Adj}^2$$

$$\therefore \text{Adj} = \sqrt{\text{Hyp}^2 - \text{Opp}^2} = \sqrt{5^2 - 4^2} = 3$$

$$\text{||}^{\text{ly}} \text{adj} = \sqrt{13^2 - 5^2} = 12$$



$$\text{Opp} = 5 \quad \text{Hyp} = 13 \quad \therefore \cos A = \frac{\text{Adj}}{\text{Hyp}} = \frac{3}{5}$$

$$\tan A = \frac{\text{Opp}}{\text{Adj}} = \frac{4}{3}, \quad \cos B = \frac{12}{13}, \quad \tan B = \frac{5}{12}$$

$$(i) \sin (A + B) = \sin A \cos B + \cos A \sin B$$

$$= \frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13} = \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$$

$$\therefore \sin (A + B) = \frac{63}{65}$$

$$(ii) \sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B$$

$$= \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

$$\sin(A - B) = \frac{33}{65}$$

$$(iii) \cos(A + B) = \cos A \cos B - \sin A \cdot \sin B$$

$$= \frac{3}{5} \cdot \frac{12}{13} - \frac{4}{5} \cdot \frac{5}{13}$$

$$= \frac{36 - 20}{65} = \frac{16}{25}$$

$$\therefore \cos(A + B) = \frac{16}{25}$$

$$(iv) \cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

$$= \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{13}$$

$$= \frac{36}{65} + \frac{20}{65}$$

$$= \frac{36 + 20}{65} = \frac{56}{65}$$

$$\therefore \cos(A - B) = \frac{56}{65}$$

$$(v) \tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{\frac{63}{65}}{\frac{16}{65}} = \frac{63}{16}$$

$$\therefore \tan(A + B) = \frac{63}{16}$$

$$(vi) \tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)}$$

$$= \frac{\frac{33}{\cancel{65}}}{\frac{\cancel{56}}{\cancel{65}}} = \frac{33}{56}$$

$$\therefore \tan(A - B) = \frac{33}{56}$$

**Example 5:**

If  $\tan A = \frac{1}{2}$  and  $\tan(A - B) = \frac{2}{7}$  find  $\tan B$  and  $\tan(A + B)$ .

**Solution:**

$$\text{w.k.t. } \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

$$\frac{2}{7} = \frac{\frac{1}{2} - \tan B}{1 + \frac{1}{2} \tan B}$$

$$\frac{2}{7} = \frac{1 - 2 \tan B}{2 + \tan B}$$

$$4 + 2 \tan B = 7 - 14 \tan B$$

$$2 \tan B + 14 \tan B = 7 - 4$$

$$16 \tan B = 3$$

$$\therefore \tan B = \frac{3}{16}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = \frac{\frac{1}{2} + \frac{3}{16}}{1 - \frac{1}{2} \cdot \frac{3}{16}} = \frac{16 + 6}{32 - 3}$$

$$\therefore \tan(A + B) = \frac{22}{29}$$

**Example 6:**

If  $A + B = 45^\circ$  S.T  $(1 + \tan A)(1 + \tan B) = 2$  and hence deduce  $\tan A = \sqrt{2} - 1$ .  
If  $A = B$

**Solution:**

Given:  $(A + B) = 45^\circ$

$$\therefore \tan(A + B) = \tan 45^\circ$$

$$\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = 1$$

$$\tan A + \tan B = 1 - \tan A \cdot \tan B$$

$$\tan A + \tan B + \tan A \cdot \tan B = 1$$

Add 1 to both sides we get

$$1 + \tan A + \tan B + \tan A \cdot \tan B = 1 + 1$$

$$(1 + \tan A) + \tan B (1 + \tan A) = 2$$

$$(1 + \tan A)(1 + \tan B) = 2$$

Given  $A = B$

$$\therefore (1 + \tan A)(1 + \tan A) = 2$$

$$(1 + \tan A)^2 = 2$$

$$1 + \tan A = \pm\sqrt{2}$$

But  $A$  is an acute angle and therefore  $\tan A$  is positive

$$1 + \tan A = \sqrt{2}$$

$$\tan A = \sqrt{2} - 1$$

**Example 7:**

Show that  $\frac{\cos 2A}{\sec A} - \frac{\sin 2A}{\operatorname{cosec} A} = \cos 3A$

**Solution:**

$$\text{L.H.S.} = \cos 2A \cdot \frac{1}{\sec A} - \sin 2A \cdot \frac{1}{\operatorname{cosec} A}$$

$$= \cos 2A \cdot \cos A - \sin 2A \cdot \sin A \quad [\text{using } \cos(A + B) \text{ formula}]$$

$$= \cos(2A + A)$$

$$= \cos 3A$$

$$= \text{RHS}$$

**Example 8:**

$$\text{S.T } \cos(A - B) \cos(A + B) - \sin(A - B) \sin(A + B) = \cos 2A$$

**Solution:**

Take  $A - B = X$ ,  $A + B = Y$  in the LHS

$$\begin{aligned} \therefore \text{LHS} &= \cos X \cos Y - \sin X \sin Y \\ &= \cos(X + Y) \\ &= \cos(A - \cancel{B} + A + \cancel{B}) \\ &= \cos 2A = \text{R.H.S} \end{aligned}$$

**Example 9:**

$$\text{S.T } \cos\left[\frac{\pi}{3} + A\right] \cdot \cos\left[\frac{\pi}{3} - A\right] - \sin\left(\frac{\pi}{3} + A\right) \cdot \sin\left[\frac{\pi}{3} - A\right] = \frac{-1}{2}$$

**Solution:**

Take  $\frac{\pi}{3} + A = X$  &  $\frac{\pi}{3} - A = Y$

$$\begin{aligned} \therefore \text{LHS} &= \cos X \cos Y - \sin X \sin Y \\ &= \cos(X + Y) \\ &= \cos\left[\frac{\pi}{3} + \cancel{A} + \frac{\pi}{3} - \cancel{A}\right] \\ &= \cos\frac{2\pi}{3} = \cos 120^\circ = \cos[180 - 60] = -\cos 60^\circ \\ &= \frac{-1}{2} = \text{RHS} \end{aligned}$$

**Example 10:**

$$\text{S.T } \sin 2A \cdot \cos A + \cos 2A \cdot \sin A = \sin 4A \cdot \cos A - \cos 4A \cdot \sin A$$

**Solution:**

LHS =  $\sin 2A \cdot \cos A + \cos 2A \cdot \sin A$  [Using Compound Angle Formula]

$$= \sin(2A + A) = \sin 3A \quad \text{———— (1)}$$

RHS =  $\sin 4A \cdot \cos A - \cos 4A \cdot \sin A$

$$= \sin(4A - A) = \sin 3A \quad \text{———— (2)}$$

From (1) and (2) we get

$$\text{LHS} = \text{RHS}$$

**Example 11:**

$$\text{S.T } \cot 2\theta + \tan \theta = \operatorname{cosec} 2\theta$$

**Solution:**

$$\begin{aligned}\text{L.H.S.} &= \cot 2\theta + \tan \theta \\&= \frac{\cos 2\theta}{\sin 2\theta} + \frac{\sin \theta}{\cos \theta} \\&= \frac{\cos 2\theta \cdot \cos \theta + \sin 2\theta \sin \theta}{\sin 2\theta \cdot \cos \theta} \\&= \frac{\cos(2\theta - \theta)}{\sin 2\theta \cdot \cos \theta} \\&= \frac{\cancel{\cos \theta}}{\sin 2\theta \cdot \cancel{\cos \theta}} \\&= \frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta = \text{R.H.S.}\end{aligned}$$

**Example 12:**

If  $\tan A = \frac{1}{3}$ ,  $\tan B = \frac{2}{7}$ , then find  $\cot(A - B)$ .

**Solution:**

$$\begin{aligned}\text{Given: } \tan A &= \frac{1}{3} \Rightarrow \cot A = 3 \\ \tan B &= \frac{2}{7} \Rightarrow \cot B = \frac{7}{2} \\ \text{w.k.t } \cot(A - B) &= \frac{\cot B \cot A + 1}{\cot B - \cot A} \\&= \frac{\frac{7}{2} \cdot 3 + 1}{\frac{7}{2} - 3} = \frac{\frac{21}{2} + 1}{\frac{7 - 6}{2}} \\&= \frac{21 + 2}{7 - 6} = \frac{23}{1} \\ \therefore \cot(A - B) &= 23\end{aligned}$$

**Example 13:**

**Prove that (i)  $\tan(45^\circ + A) = \frac{1 + \tan A}{1 - \tan A}$       (ii)  $\tan(45^\circ - A) = \frac{1 - \tan A}{1 + \tan A}$**

**Solution:**

$$\begin{aligned} \text{(i) LHS} &= \tan(45^\circ + A) \\ &= \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \cdot \tan A} \\ &= \frac{1 + \tan A}{1 - \tan A} = \text{RHS} \end{aligned}$$

(ii) Try yourself

**Example 14:**

**If  $A + B + C = 180^\circ$  and  $\tan A = 1$ ,  $\tan B = 2$ , show that  $\tan C = 3$ .**

**Solution:**

$$\begin{aligned} \text{Given: } A + B + C &= 180^\circ \\ \Rightarrow A + B &= 180^\circ - C \\ \therefore \tan(A + B) &= \tan(180^\circ - C) \\ \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} &= -\tan C \\ \frac{1 + 2}{1 - 2} &= -\tan C \\ \frac{3}{-1} &= -\tan C \Rightarrow \tan C = 3 \end{aligned}$$

**Example 15:**

**P.T  $\tan 2A \cdot \tan 3A \cdot \tan 5A = \tan 5A - \tan 3A - \tan 2A$**

**Solution:**

$$\begin{aligned} \text{Consider, } \tan 5A &= \tan(2A + 3A) \\ \tan 5A &= \frac{\tan 2A + \tan 3A}{1 - \tan 2A \cdot \tan 3A} \\ \tan 2A + \tan 3A &= \tan 5A (1 - \tan 2A \cdot \tan 3A) \\ \tan 2A + \tan 3A &= \tan 5A - \tan 2A \cdot \tan 3A \cdot \tan 5A \\ \tan 2A \cdot \tan 3A \cdot \tan 5A &= \tan 5A - \tan 3A - \tan 2A \end{aligned}$$

**Example 16:**

If  $\tan A = \frac{5}{6}$ ,  $\tan B = \frac{1}{11}$  S.T.  $A + B = \frac{\pi}{4}$

**Solution:**

$$\text{Consider, } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{5}{6} + \frac{1}{11}}{1 - \frac{5}{6} \cdot \frac{1}{11}} = \frac{\frac{55+6}{66}}{\frac{66-5}{66}}$$

$$\tan(A+B) = \frac{61}{61} = 1$$

$$\Rightarrow A+B = \pi/4 \left( \because \tan \frac{\pi}{4} = 1 \right)$$

17. If  $\sin A = \frac{5}{13}$ ,  $\cos B = \frac{-4}{5}$ ,  $\frac{\pi}{2} < A < \pi$  and  $\pi < B < \frac{3\pi}{2}$

Find (i)  $\sin(A+B)$  (ii)  $\cos(A+B)$  (iii)  $\sin(A-B)$  (iv)  $\cos(A-B)$   
(v)  $\tan(A+B)$  (vi)  $\tan(A-B)$

**Solution:**

By the given data angle A lies in II quadrant and angle B lies in III quadrant

Given:  $\sin A = \frac{5}{13} = \frac{\text{Opp}}{\text{Hyp}}$ ,  $\therefore \text{adj} = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$

$$\therefore \cos A = \frac{-12}{13} \text{ (-ve sign is taken) } (\because \cos A \text{ is negative in II quadrant})$$

Also,  $\cos B = \frac{-4}{5}$  opp =  $\sqrt{5^2 - 4^2} = \sqrt{9} = 3$

$$\sin B = \frac{-3}{5} \text{ (sin B is negative in III quadrant)}$$

(i)  $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$$= \frac{5}{13} \cdot \frac{-4}{5} + \frac{-12}{13} \cdot \frac{-3}{5}$$

$$= \frac{-20}{65} + \frac{36}{65} = \frac{16}{65}$$

$$(ii) \cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$= \frac{-12}{13} \cdot \frac{-4}{5} - \frac{5}{13} \cdot \frac{-3}{5}$$

$$= \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$$

$$(iii) \sin(A-B) = \sin A \cdot \cos B - \cos A \sin B$$

$$= \frac{-12}{13} \cdot \frac{-4}{5} + \frac{5}{13} \cdot \frac{-3}{5}$$

$$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

$$(v) \tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\cancel{16}/\cancel{65}}{\cancel{63}/\cancel{65}} = \frac{16}{63}$$

$$(vi) \tan(A-B) = \frac{\sin(A-B)}{\cos A - B} = \frac{\cancel{-56}/\cancel{65}}{\cancel{33}/\cancel{65}} = \frac{-56}{33}$$

**Example 18:**

$$P.T \cos A + \cos(120^\circ + A) + \cos(120^\circ - A) = 0$$

**Solution:**

$$LHS = \cos A + \cos(120^\circ + A) + \cos(120^\circ - A)$$

$$= \cos A + \cos 120^\circ \cdot \cos A - \sin 120^\circ \sin A + \cos 120^\circ \cdot \cos A + \cancel{\sin 120^\circ} \sin A$$

$$= \cos A + 2 \cdot \cos 120^\circ \cdot \cos A$$

$$= \cos A + 2 \cdot \cos(180^\circ - 60^\circ) \cdot \cos A$$

$$= \cos A + 2 \cdot (-\cos 60^\circ) \cos A$$

$$= \cos A - \cancel{2} \cdot \frac{1}{\cancel{2}} \cdot \cos A = \cancel{\cos A} - \cancel{\cos A} = 0 = RHS$$

**Example 19:**

$$S.T \tan(45^\circ + A) \tan(45^\circ - A) = 1$$

**Solution:**

$$LHS = \left( \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \cdot \tan A} \right) \left( \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \cdot \tan A} \right)$$

$$= 1 = RHS \quad (\because \tan 45^\circ = 1)$$

**Example 20:**

$$\text{P.T } \sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B$$

**Solution:**

$$\begin{aligned} \text{LHS} &= (\sin A \cdot \cos B + \cos A \cdot \sin B) (\sin A \cdot \cos B - \cos A \cdot \sin B) \\ &= \sin^2 A \cdot \cos^2 B - \cos^2 A \cdot \sin^2 B \quad [\text{using } (a+b)(a-b) = a^2 - b^2] \\ &= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B \\ &= \sin^2 A - \sin^2 A \cdot \cancel{\sin^2 B} - \sin^2 B + \sin^2 A \cdot \cancel{\sin^2 B} \\ &= \sin^2 A - \sin^2 B = \text{RHS} \end{aligned}$$

**Note :** It is always useful to remember the following

$$\text{Results : (i) } \sin 15^\circ = \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{(ii) } \cos 15^\circ = \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\text{(iii) } \cos 105^\circ = \frac{1-\sqrt{3}}{2\sqrt{2}} \quad \text{(iv) } \sin 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

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**EXERCISE 14.1**

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**Two marks questions:**

**I. 1.** Obtain the values of the trigonometric functions of  $75^\circ$ ,  $15^\circ$  and  $105^\circ$  and prove that

$$\text{(i) } \tan 75^\circ + \cot 75^\circ = 4$$

$$\text{(ii) } \sin 105^\circ + \cos 105^\circ = \frac{1}{\sqrt{2}}$$

$$\text{(iii) } \sec 15^\circ + \operatorname{cosec} 15^\circ = 2\sqrt{6}$$

**2.** If  $\sin A = \frac{3}{5}$ ,  $\cos B = \frac{4}{5}$  find  $\sin(A+B)$  and  $\cos(A-B)$ . Where A and B are acute angles.

**3.** If  $\cos A = \frac{5}{13}$ ,  $\cos B = \frac{24}{25}$  find  $\cos(A+B)$  and  $\sin(A-B)$ . A and B are acute angles.

**4.** If  $\sec A = \frac{17}{8}$ ,  $\operatorname{cosec} B = \frac{5}{4}$  find  $\sec(A+B)$ ,  $\operatorname{cosec}(A-B)$ , A and B are acute angles.

5. If  $\sin A = \frac{3}{5}$ ,  $\cos B = \frac{-8}{17}$ ,  $\frac{\pi}{2} < A < \pi$  and  $\frac{\pi}{2} < B < \pi$

Find the values of  $\sin(A + B)$  and  $\cos(A - B)$

6. If  $\sin A = \frac{7}{25}$ ,  $\cos B = \frac{-12}{13}$  where  $\frac{\pi}{2} < A < \pi$  and  $\pi < B < \frac{3\pi}{2}$

Find the values of:

(i)  $\sin(A + B)$

(ii)  $\cos(A + B)$

(iii)  $\sin(A - B)$

(iv)  $\cos(A - B)$

(v)  $\tan(A + B)$

(vi)  $\tan(A - B)$

7. If  $\tan A = \frac{1}{2}$ ,  $\tan B = \frac{1}{3}$ , Find  $\tan(A + B)$ ,  $\tan(A - B)$

8.  $\tan(A - B) = \frac{1}{7}$ ,  $\tan A = \frac{1}{2}$  show that  $A + B = 45^\circ$

9. If  $\tan A = \frac{1}{3}$ ,  $\tan(A + B) = \frac{1}{7}$  find  $\tan B$ .

10.  $\tan A = \frac{3}{4}$  and  $\tan B = \frac{1}{7}$  S.T  $\tan(A + B) = 1$

11. If  $\tan A = \frac{5}{6}$  and  $\tan(A + B) = 1$  S.T  $\tan B = \frac{1}{11}$

12. If  $\tan \alpha = \frac{n}{n+1}$  and  $\tan \beta = \frac{1}{2n+1}$  S.T  $\alpha + \beta = \frac{\pi}{4}$

13. P.T  $\frac{\cos 2A}{\sec A} + \frac{\sin 2A}{\cos \sec A} = \cos A$ .

14. P.T  $\sin(45^\circ + A) + \cos(45^\circ + A) = \sqrt{2} \cos A$

15. P.T  $\cos\left(A + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\cos A - \sin A)$

16. P.T  $\cos\left(\frac{\pi}{6} + A\right) \cdot \cos\left(\frac{\pi}{6} - A\right) - \sin\left(\frac{\pi}{6} + A\right) \cdot \sin\left(\frac{\pi}{6} - A\right) = \frac{1}{2}$

**Five marks questions:****II. Prove the following**

$$17. \frac{\sin(A+B)}{\cos A \cdot \cos B} = \tan A + \tan B$$

$$18. \frac{\sin(A-B)}{\sin A \sin B} = \cot B - \cot A$$

$$19. S.T \cos(A+B) \cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

$$20. \sin(A+B) \sin(A-B) = \cos^2 B - \cos^2 A.$$

$$21. \cos\left(\frac{\pi}{4} - A\right) - \sin\left(\frac{\pi}{4} + A\right) = 0$$

$$22. \sin\left(\frac{\pi}{3} - A\right) \cdot \cos\left(\frac{\pi}{6} + A\right) + \cos\left(\frac{\pi}{3} - A\right) \cdot \sin\left(\frac{\pi}{6} + A\right) = 1$$

$$23. \tan 15^\circ + \cot 15^\circ = 4$$

$$24. \sin 105^\circ + \cos 105^\circ = \cos 45^\circ$$

$$25. \cot 2A + \tan A = \operatorname{cosec} 2A$$

$$26. P.T \tan A \tan 3A \cdot \tan 4A = \tan 4A - \tan 3A - \tan A.$$

$$27. P.T \cos(45^\circ - A) \cdot \cos(45^\circ - B) - \sin(45^\circ - A) \sin(45^\circ - B) = \sin(A+B)$$

$$28. S.T \sum \frac{\sin(A-B)}{\cos A \cdot \cos B} = 0$$

$$29. \cos(120^\circ + A) + \cos(120^\circ - A) = -\cos A$$

$$30. \frac{\sin(A+B)}{\sin(A-B)} = \frac{\tan A + \tan B}{\tan A - \tan B}$$

**ANSWERS 14.1**

$$I. 1. \frac{\sqrt{3}+1}{2\sqrt{2}}, \frac{\sqrt{3}-1}{2\sqrt{2}}, 2+\sqrt{3}, \frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}, 2-\sqrt{3}, \frac{\sqrt{3}+1}{2\sqrt{2}}, \frac{1-\sqrt{3}}{2\sqrt{2}}, \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$2. \frac{24}{25}, 1$$

$$3. \frac{36}{325}, \frac{253}{325}$$

$$4. \frac{-85}{36}, \frac{85}{13}$$

$$5. \frac{-84}{85}, \frac{77}{85}$$

$$6. \frac{36}{325} \cdot \frac{323}{325}, \frac{-204}{325}, \frac{253}{325}, \frac{36}{323}, \frac{-204}{253}$$

$$7. 1, \frac{1}{7}$$

$$8. A+B=45^\circ$$

$$9. -\frac{2}{11}$$

### 14.3 Multiple angles :

If  $A$  is the given angle then the angles  $2A$ ,  $3A$ ,  $4A$  - etc are called multiple angles.

#### I. Trigonometric Ratios of $2A$ :

Using compound angle formulae, we get multiple angles formula as follows

(i) w.k.t.  $\sin(A + B) = \sin A \cdot \cos B + \cos A \sin B$

$$\text{put } B = A$$

$$\sin(A + A) = \sin A \cdot \cos A + \cos A \sin A$$

$$\sin 2A = 2 \sin A \cdot \cos A$$

$$\boxed{\sin 2A = 2 \sin A \cdot \cos A}$$

(ii) w.k.t.  $\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

$$\text{put } B = A$$

$$\cos(A + A) = \cos A \cdot \cos A - \sin A \cdot \sin A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\boxed{\cos 2A = \cos^2 A - \sin^2 A} \quad \text{———— (1)}$$

we can express  $\cos 2A$  in to other useful forms using the identity  $\sin^2 A + \cos^2 A = 1$

From eq<sup>n</sup> (1)  $\cos 2A = \cos^2 A - \sin^2 A$

$$= \cos^2 A - (1 - \cos^2 A)$$

$$= \cos^2 A - 1 + \cos^2 A$$

$$= 2 \cos^2 A - 1$$

$$\therefore \boxed{\cos 2A = 2 \cos^2 A - 1}$$

Again from (1)  $\cos 2A = \cos^2 A - \sin^2 A$

$$= (1 - \sin^2 A) - \sin^2 A$$

$$= 1 - 2 \sin^2 A$$

$$\boxed{\cos 2A = 1 - 2 \sin^2 A}$$

Hence  $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$

$$(iii) \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

put  $B = A$

$$\tan(A+A) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan A}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\therefore \boxed{\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}}$$

## II. Trigonometric rotios of 3A

(i) Prove that  $\sin 3A = 3 \sin A - 4 \sin^3 A$

$$\begin{aligned} \text{LHS} = \sin 3A &= \sin(A+2A) \\ &= \sin A \cdot \cos 2A + \sin 2A \cdot \cos A \\ &= \sin A (1 - 2 \sin^2 A) + (2 \sin A \cdot \cos A) \cdot \cos A \\ &= \sin A - 2 \sin^3 A + 2 \sin A (\cos^2 A) \\ &= \sin A - 2 \sin^3 A + 2 \sin A (1 - \sin^2 A) \\ &= \sin A - 2 \sin^3 A + 2 \sin A - 2 \sin^3 A \\ &= 3 \sin A - 4 \sin^3 A \end{aligned}$$

$$\therefore \boxed{\sin 3A = 3 \sin A - 4 \sin^3 A}$$

(ii) Prove that  $\cos 3A = 4 \cos^3 A - 3 \cos A$

$$\begin{aligned} \text{LHS} = \cos 3A &= \cos(A+2A) \\ &= \cos A \cdot \cos^2 A - \sin A \cdot \sin 2A \\ &= \cos A (2 \cos^2 A - 1) - \sin A (2 \sin A \cdot \cos A) \\ &= 2 \cos^3 A - \cos A - 2 \sin^2 A \cdot \cos A \\ &= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A) \\ &= 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A \\ &= 4 \cos^3 A - 3 \cos A \end{aligned}$$

$$\therefore \boxed{\cos 3A = 4 \cos^3 A - 3 \cos A}$$

(iii) Prove that  $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

$$\text{LHS} = \tan 3A = \tan (A + 2A)$$

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \frac{2 \tan A}{1 - \tan^2 A}}{1 - \tan A \cdot \frac{2 \tan A}{1 - \tan^2 A}}$$

$$= \frac{\tan A(1 - \tan^2 A) + 2 \tan A}{1 - \tan^2 A - 2 \tan^2 A}$$

$$= \frac{\tan A - \tan^3 A + 2 \tan A}{1 - 3 \tan^2 A}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

$$\therefore \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

### III. To express $\sin 2A$ and $\cos 2A$ in terms of $\tan A$

Prove that  $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$

$$\text{LHS} = \sin 2A = 2 \sin A \cos A$$

$$= \frac{2 \sin A}{\cos A} \cdot \cos^2 A$$

$$= \frac{2 \tan A}{1 + \tan^2 A} \quad \left[ \begin{array}{l} \because \sec^2 A = 1 + \tan^2 A \\ \& \frac{\sin A}{\cos A} = \tan A \end{array} \right]$$

$$\therefore \sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

(ii) P.T  $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

$$\text{LHS} = \cos 2A = \cos^2 A - \sin^2 A = \cos^2 A \left( 1 - \frac{\sin^2 A}{\cos^2 A} \right)$$

$$= \cos^2 A (1 - \tan^2 A)$$

$$= \frac{1 - \tan^2 A}{\sec^2 A}$$

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\therefore \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

**List of Multiple angles formulae**

$$(1) \sin 2A = \begin{cases} (i) 2 \sin A \cos A \\ (ii) \frac{2 \tan A}{1 + \tan^2 A} \end{cases}$$

$$(2) \cos 2A = \begin{cases} (i) \cos^2 A - \sin^2 A \\ (ii) 2 \cos^2 A - 1 \\ (iii) 1 - 2 \sin^2 A \\ (iv) \frac{1 - \tan^2 A}{1 + \tan^2 A} \end{cases}$$

$$(3) \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$(4) \sin 3A = 3 \sin A - 4 \sin^3 A$$

$$(5) \cos 3A = 4 \cos^3 A - 3 \cos A$$

$$(6) \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

### 14.4 Sub multiple angles :

If 'A' is an angle then  $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}, \dots$  are called submultiple angle

In multiple angle formulae, replace 2A by A and A by  $\frac{A}{2}$ , we get sub multiple angle formulae as follows:

$$\therefore \sin 2A = 2 \sin A \cdot \cos A$$

$$\therefore \sin A = 2 \sin\left(\frac{A}{2}\right) \cdot \cos\left(\frac{A}{2}\right)$$

Hence all the multiple angle formula of '2A' can be changed as follows

#### List of Half Angle Formulae or Submultiple Angles Formulae (Without Proof & Problem)

$$(1) \sin A = \begin{cases} (i) 2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right) \\ (ii) \frac{2 \tan\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)} \end{cases}$$

$$(2) \cos A = \begin{cases} (i) \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right) \\ (ii) 2 \cos^2\left(\frac{A}{2}\right) - 1 \\ (iii) 1 - 2 \sin^2\left(\frac{A}{2}\right) \\ (iv) \frac{1 - \tan^2\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)} \end{cases}$$

$$(3) \tan A = \frac{2 \tan\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)}$$

**Note :** The above all formulae are very useful in the problems of differentiation and integration

**Note :** If  $\sin 2A = 2 \sin A \cos A$

then for  $\sin 4A$  multiply 'A' by 2 on both sides

$$\sin 4A = 2 \sin 2A \cos 2A$$

|||ly  $\sin 6A = 2 \sin 3A \cos 3A$  Multiply 'A' by 3 on both sides

Similarly we can find  $\cos 4A$ ,  $\cos 6A$ ,  $\tan 4A$  ..... etc by multiplying 'A' by respective positive number on both in all the above formula

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**WORKED EXAMPLES**

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**Example 1:**

P. T  $\frac{\cos 2A}{1 - \sin 2A} = \tan(45^\circ + A)$

**Solution:**

$$\begin{aligned} \text{LHS} &= \frac{\cos 2A}{1 - \sin 2A} \\ &= \frac{\cos^2 A - \sin^2 A}{1 - 2 \sin A \cos A} \\ &= \frac{(\cos A - \sin A)(\cos A + \sin A)}{(\cos A - \sin A)} \quad \text{by } \cos A \text{ to both NR and DNR} \\ &= \frac{1 + \tan A}{1 - \tan A} \\ &= \tan(45^\circ + A) = \text{RHS} \end{aligned}$$

**Example 2:**

P. T  $\frac{\sin 3\theta}{1 + 2 \cos 2\theta} = \sin \theta$ . Hence deduce the value of  $\sin 15^\circ$

**Solution:**

$$\begin{aligned} \text{LHS} &= \frac{\sin 3\theta}{1 + 2 \cos 2\theta} \\ &= \frac{3 \sin \theta - 4 \sin^3 \theta}{1 + 2(1 - 2 \sin^2 \theta)} \end{aligned}$$

$$= \frac{3\sin\theta - 4\sin^3\theta}{3 - 4\sin^2\theta}$$

$$= \sin\theta \frac{(3 - 4\sin^2\theta)}{(3 - 4\sin^2\theta)}$$

$$= \sin\theta = \text{RHS}$$

Put  $\theta = 15^\circ$

$$\text{Then } \sin 3\theta = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\cos 2\theta = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\therefore \sin\theta = \frac{\sin 3\theta}{1 + 2\cos 2\theta}$$

$$\sin 15^\circ = \frac{\sin 45^\circ}{1 + 2\cos 30^\circ}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1 + \frac{\cancel{2} \cdot \sqrt{3}}{\cancel{2}}} = \frac{1}{\sqrt{2}(\sqrt{3} + 1)} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \quad (\text{Multiply and divide by } (\sqrt{3} - 1))$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3} - 1}{(3 - 1)} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

### Example 3 :

$$\text{P. T } \cos^4\theta - \sin^4\theta = \cos 2\theta$$

**Solution:**

$$\text{LHS} = \cos^4\theta - \sin^4\theta$$

$$= (\cos^2\theta)^2 - (\sin^2\theta)^2$$

$$= (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$$

$$= \cos 2\theta \times 1$$

$$= \cos 2\theta = \text{RHS}$$

**Example 4 :**

$$\text{P. T } \frac{1 - \cos 2A}{\sin 2A} = \tan A$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \frac{1 - \cos^2 A}{\sin^2 A} = \frac{1 - (1 - 2\sin^2 A)}{2\sin A \cos A} \\ &= \frac{\cancel{1} - \cancel{1} + 2\sin^2 A}{2\sin A \cdot \cos A} \\ &= \frac{\cancel{2} \sin^{\cancel{2}} A}{\cancel{2} \sin^{\cancel{1}} A \cdot \cos A} \\ &= \frac{\sin A}{\cos A} = \tan A = \text{RHS} \end{aligned}$$

**Example 5 :**

$$\text{P. T } \cos^6 A + \sin^6 A = 1 - \frac{3}{4} \sin^2 (2A)$$

**Solution:**

$$\begin{aligned} \text{LHS} &= (\cos^2 A)^3 + (\sin^2 A)^3 \\ &= (\cos^2 A + \sin^2 A)^3 - 3\cos^2 A \cdot \sin^2 A (\cos^2 A + \sin^2 A) \\ &= 1^3 - 3\cos^2 A \cdot \sin^2 A (1) \\ &= 1 - 3 \frac{(2\cos A \sin A)^2}{4} \quad (\text{Divide \& Multiply by 4}) \\ &= 1 - \frac{3}{4} \sin^2 2A = \text{RHS} \end{aligned}$$

**Example 6:**

$$\text{P. T } \frac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} = \tan A$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \frac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} \\ &= \frac{2\sin^2 A + 2\sin A \cdot \cos A}{2\cos^2 A + 2\sin A \cdot \cos A} \end{aligned}$$

$$= \frac{2 \sin A (\cancel{\sin A + \cos A})}{2 \cos A (\cancel{\cos A + \sin A})}$$

$$= \frac{\cancel{2} \sin A}{\cancel{2} \cos A}$$

$$= \tan A = \text{RHS}$$

**Example 7:**

**P. T**  $(\sin A + \cos A)^2 = 1 + \sin 2A$

**Solution:**

$$\begin{aligned} \text{LHS} &= (\sin A + \cos A)^2 \\ &= \sin^2 A + \cos^2 A + 2 \sin A \cos A \\ &= 1 + \sin 2A = \text{RHS} \end{aligned}$$

**Example 8:**

**P. T**  $\frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$

**Solution:**

$$\text{LHS} = \frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \frac{\sin A + 2 \sin A \cos A}{\cancel{1} + \cos A + 2 \cos^2 A - \cancel{1}}$$

$$= \frac{\sin A (\cancel{1 + 2 \cos A})}{\cos A (\cancel{1 + 2 \cos A})}$$

$$\frac{\sin A}{\cos A} = \tan A = \text{RHS}$$

**Example 9:**

**P. T**  $\frac{\cot A}{\cot A - \cot 3A} + \frac{\tan A}{\tan A - \tan 3A} = 1$

**Solution:**

$$\text{LHS} = \frac{\cot A}{\frac{1}{\tan A} - \frac{1}{\tan 3A}} + \frac{\tan A}{\tan A - \tan 3A}$$

$$= \frac{(\cot A \cdot \tan A) \tan 3A}{\tan 3A - \tan A} + \frac{\tan A}{(\tan 3A - \tan A)}$$

$$\begin{aligned} &= \frac{1 \times \tan 3A}{\tan 3A - \tan A} - \frac{\tan A}{\tan 3A - \tan A} \\ &= \frac{\cancel{\tan 3A} - \cancel{\tan 3A}}{\cancel{\tan 3A} - \tan A} \\ &= 1 = \text{RHS} \end{aligned}$$

**Example 10:**

$$\text{P.T } \sec(45^\circ + A) \cdot \sec(45^\circ - A) = 2 \sec 2A$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \sec(45^\circ + A) \cdot \sec(90^\circ - (45^\circ + A)) \\ &= \frac{1}{\cos(45^\circ + A) \cdot \sin(45^\circ + A)} \\ &= \frac{2}{2 \cdot \sin(45^\circ + A) \cdot \cos(45^\circ + A)} = \frac{2}{\sin 2(45^\circ + A)} \\ &= \frac{2}{\sin(90^\circ + 2A)} = \frac{2}{\cos 2A} = 2 \sec 2A = \text{RHS} \end{aligned}$$

**Example 11:**

$$\text{Find the value of } 4 \cos^3 10^\circ - 3 \cos 10^\circ$$

**Solution:**

$$\text{Consider } \cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\text{Put } A = 10^\circ \quad \therefore 4 \cos^3 10^\circ - 3 \cos 10^\circ = \cos 3 \cdot 10^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

**Example 12:**

$$\text{P.T } \tan\left(\frac{\pi}{4} + A\right) - \tan\left(\frac{\pi}{4} - A\right) = 2 \tan 2A$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \tan\left(\frac{\pi}{4} + A\right) - \tan\left(\frac{\pi}{4} - A\right) \\ &= \frac{\tan \frac{\pi}{4} + \tan A}{1 - \tan \frac{\pi}{4} \cdot \tan A} - \frac{\tan \frac{\pi}{4} - \tan A}{1 + \tan \frac{\pi}{4} \cdot \tan A} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1 + \tan A}{1 - \tan A} - \frac{1 - \tan A}{1 + \tan A} = \frac{(1 + \tan A)^2 - (1 - \tan A)^2}{1^2 - \tan^2 A} \\
 &= \frac{1 + \tan^2 A + 2 \tan A - 1 - \tan^2 A + 2 \tan A}{1 - \tan^2 A} \\
 &= \frac{4 \tan A}{1 - \tan^2 A} = 2 \cdot \frac{2 \tan A}{1 - \tan^2 A} = 2 \cdot \tan 2A = \text{RHS}
 \end{aligned}$$

**Example 13:**

**S.T**  $\frac{1 + \tan^2 (45^\circ - \theta)}{1 - \tan^2 (45^\circ - \theta)} = \text{cosec} 2\theta$

**Solution:**

$$\begin{aligned}
 \text{LHS} &= \frac{1 + \frac{\sin^2 (45^\circ - \theta)}{\cos^2 (45^\circ - \theta)}}{1 - \frac{\sin^2 (45^\circ - \theta)}{\cos^2 (45^\circ - \theta)}} \\
 &= \frac{\cos^2 (45^\circ - \theta) + \sin^2 (45^\circ - \theta)}{\cos^2 (45^\circ - \theta) - \sin^2 (45^\circ - \theta)} \\
 &= \frac{1}{\cos 2(45^\circ - \theta)} \\
 &= \frac{1}{\cos (90^\circ - 2\theta)} \\
 &= \frac{1}{\sin 2\theta} \\
 &= \text{cosec} 2\theta = \text{RHS}
 \end{aligned}$$

**Example 14:**

**P.T**  $\tan 2A = \frac{\tan(A + B) + \tan(A - B)}{1 - \tan(A + B) \cdot \tan(A - B)}$

**Solution:**

$$\begin{aligned}
 \text{LHS} &= \tan 2A = \tan [(A + B) + (A - B)] \\
 &= \frac{\tan(A + B) + \tan(A - B)}{1 - \tan(A + B) \cdot \tan(A - B)} = \text{RHS}
 \end{aligned}$$

**Example 15:**

$$\text{If } 2 \cos \theta = \left(x + \frac{1}{x}\right) \text{ S.T } 2 \cos 2\theta = \left(x^2 + \frac{1}{x^2}\right)$$

**Solution:**

$$\begin{aligned} \text{LHS} &= 2 \cos 2\theta = 2(2 \cos^2 \theta - 1) \\ &= 4 \cos^2 \theta - 2 = (2 \cos \theta)^2 - 2 \\ &= \left(x + \frac{1}{x}\right)^2 - 2 = x^2 + \frac{1}{x^2} + 2 \cdot \cancel{x} \cdot \cancel{\frac{1}{x}} - 2 \\ &= x^2 + \frac{1}{x^2} = \text{RHS} \end{aligned}$$

**Example 16 :**

$$\text{P. T } \frac{\sin 3A}{1 + 2 \cos 2A} = \sin A \text{ and hence find } \sin 30^\circ$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \frac{\sin 3A}{1 + 2 \cos 2A} = \frac{3 \sin A - 4 \sin^3 A}{1 + 2(1 - 2 \sin^2 A)} \\ &= \frac{\sin A (3 - 4 \sin^2 A)}{(3 - 4 \sin^2 A)} = \sin A = \text{RHS} \end{aligned}$$

$$\begin{aligned} \sin 30^\circ &= \frac{\sin 3 \cdot (30^\circ)}{1 + 2 \cdot \cos 2 \cdot (30^\circ)} = \frac{\sin 90^\circ}{1 + 2 \cos 60^\circ} \\ &= \frac{1}{1 + \cancel{2} \cdot \cancel{\frac{1}{2}}} = \frac{1}{2} \end{aligned}$$

**Example 17 :**

$$\text{P. T } \frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\tan 8A}{\tan 2A}$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\frac{1 - \cos 8A}{\cos 8A}}{\frac{1 - \cos 4A}{\cos 4A}} = \frac{1 - \cos 8A}{\cos 8A} \times \frac{\cos 4A}{1 - \cos 4A} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2 \sin^2 4A}{\cos 8A} \cdot \frac{\cos 4A}{2 \sin^2 2A} \\
 &= \frac{(2 \sin 4A \cdot \cos 4A)}{\cos 8A} \cdot \frac{\sin 4A}{2 \sin 2A \sin 2A} \\
 &= \frac{\sin 8A}{\cos 8A} \times \frac{\cancel{2 \sin 2A} \cos 2A}{\cancel{2 \sin 2A} \cdot \sin 2A} \\
 &= \tan 8A \cdot \cot 2A \\
 &= \frac{\tan 8A}{\tan 2A} = \text{RHS}
 \end{aligned}$$

**Example 18 :**

**P. T**  $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$

**Solution:**

$$\begin{aligned}
 \text{LHS} &= \frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} \\
 &= \left( \frac{3 \sin A - 4 \sin^3 A}{\sin A} \right) - \left( \frac{4 \cos^3 A - 3 \cos A}{\cos A} \right) \\
 &= 3 - 4 \sin^2 A - 4 \cos^2 A + 3 \\
 &= 6 - 4 (\cos^2 A + \sin^2 A) \\
 &= 6 - 4 = 2 = \text{RHS}
 \end{aligned}$$

**Example 19:**

**Express  $\cos 4A$  in terms of  $\sin A$**

**Solution:**

$$\begin{aligned}
 \cos 4A &= \cos 2(2A) = 1 - 2 \sin^2(2A) \\
 &= 1 - 2 (2 \sin A \cos A)^2 \\
 &= 1 - 8 \sin^2 A \cos^2 A \\
 \cos 4A &= 1 - 8 \sin^2 A (1 - \sin^2 A) \\
 \therefore \cos 4A &= 1 - 8 \sin^2 A + 8 \sin^4 A
 \end{aligned}$$

**Example 20 :**

**If  $\sin \theta = \frac{12}{13}$  and  $\theta$  is acute find  $\sin 2\theta$ ,  $\cos 2\theta$ ,  $\tan 2\theta$**

**Find (i)  $\sin 2\theta$ , (ii)  $\cos 2\theta$ , (iii)  $\tan 2\theta$**

**Solution:**

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{12}{13} \cdot \frac{5}{13} = \frac{120}{169}$$

$$\cos 2\theta = 1 - 2 \sin^2 \theta = 1 - 2\left(\frac{144}{169}\right) = \frac{-119}{169}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{120}{-119}$$

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**EXERCISE 14.2**

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**Two marks questions:**

1. If  $\sin A = \frac{1}{2}$  find  $\sin 2A$
2. If  $\cos A = \frac{\sqrt{3}}{2}$  find  $\cos 2A$
3. If  $\tan A = \frac{1}{\sqrt{3}}$  find  $\tan 2A$
4. If  $\sin A = \frac{3}{5}$  find  $\sin 3A$
5. If  $\cos A = \frac{4}{5}$  find  $\cos 3A$
6. If  $\tan A = \frac{3}{4}$  find  $\tan 3A$
7. Find the value of  $3 \sin 10^\circ - 4 \sin^3 10^\circ$
8. If  $\cot A = \frac{12}{5}$  and  $A$  is acute find  $\sin 3A$  and  $\cos 3A$ .

$$9. \text{ S.T } \tan A = \frac{\tan(A - B) + \tan B}{1 - \tan(A - B)\tan B}$$

$$10. \text{ P.T } \frac{\cos 2A}{1 + \sin 2A} = \frac{\cos A - \sin A}{\cos A + \sin A}$$

$$11. \text{ P.T } \frac{1 + \sin 2\theta}{\cos 2\theta} = \frac{1 + \tan \theta}{1 - \tan \theta}$$

$$12. \text{ P.T } \frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$$

$$13. \text{ P.T } \frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta$$

$$14. \text{ P.T } (\sin A - \cos A)^2 = 1 - \sin 2A$$

$$15. \text{ P.T } \cos^4 \theta - \sin^4 \theta = 2\cos^2 \theta - 1$$

$$16. \text{ P.T } \frac{\cos^3 A - \sin^3 A}{\cos A - \sin A} = 1 + \frac{1}{2} \sin 2A$$

**Five marks questions:**

Prove the following

$$1. \frac{1 + \cos 2A + \sin 2A}{1 - \cos 2A + \sin 2A} = \cot A$$

$$2. \frac{\cos 3A}{2\cos 2A - 1} = \cos A \text{ and hence find } \cos 15^\circ$$

$$3. \frac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} = \tan A$$

$$4. \cos^6 A + \sin^6 A = 1 - \frac{3}{4} \sin^2(2A)$$

$$5. \text{ P.T } \frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = 2$$

$$6. \sec(45^\circ + A) \cdot \sec(45^\circ - A) = 2 \sec 2A$$

$$7. \frac{\cot A}{\cot A - \cot 3A} + \frac{\tan A}{\tan A - \tan 3A} = 1$$

$$8. \text{ P.T } \frac{\cos 2A}{1 + \sin 2A} = \tan(45^\circ - A)$$

9. If  $\tan \alpha = \frac{1}{3}$ ,  $\tan \beta = \frac{1}{7}$  P.T  $\tan (2\alpha + \beta) = 45^\circ$

10. If  $\tan^2 (45^\circ + \theta) = \frac{a}{b}$  P.T  $\frac{b-a}{b+a} = -\sin 2\theta$

11. P.T  $\tan 2\theta - \tan \theta = \tan \theta \cdot \sec 2\theta$

12. P.T  $\cos 2\alpha - \tan \alpha = \frac{\cos 3\alpha}{\cos \alpha \cdot \sin 2\alpha}$

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**ANSWERS 14.2**

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I. 1.  $\frac{\sqrt{3}}{2}$

2.  $\frac{1}{2}$

3.  $\sqrt{3}$

4.  $\frac{117}{125}$

5.  $\frac{-44}{125}$

6.  $\frac{117}{-44}$

7.  $\frac{1}{2}$

8.  $\frac{-828}{2197}, \frac{2035}{2197}$

**14.5 Transformation Formulae:**

Transformation formulae are the formulae used to convert the product of two trigonometric functions into sum or of difference. And sum of difference into product of two function as follows.

w.k.t  $\sin (A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$  — (1)

$\sin (A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B$  — (2)

$\cos (A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$  — (3)

$\cos (A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$  — (4)

Adding (1) and (2) we get

$\sin (A + B) + \sin (A - B) = 2 \sin A \cdot \cos B$  — (5)

$\therefore \sin A \cdot \cos B = \frac{1}{2} [\sin (A + B) + \sin (A - B)]$

Subtracting (2) from (1), we get

$2 \cos A \cdot \sin B = \sin (A + B) - \sin (A - B)$  — (6)

$\therefore \cos A \cdot \sin B = \frac{1}{2} [\sin (A + B) - \sin (A - B)]$

Similarly adding (3) and (4), we get:

$2 \cos A \cdot \cos B = \cos (A + B) + \cos (A - B)$  — (7)

$\cos A \cdot \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$

subtracting (4) from (3), we get

$$-2 \sin A \cdot \sin B = \cos(A + B) - \cos(A - B) \text{ ——— (8)}$$

$$\therefore \sin A \cdot \sin B = -\frac{1}{2} [\cos(A + B) - \cos(A - B)]$$

Hence the transformation formulae from product into sum or difference are

$$1. \sin A \cdot \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

$$2. \cos A \cdot \sin B = \frac{1}{2} [\sin(A + B) - \sin(A - B)]$$

$$3. \cos A \cdot \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

$$4. \sin A \cdot \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

In the above formulae 5, 6, 7 and 8. Let us take  $A + B = C$  and  $A - B = D$ . So we get,

$A = \frac{C + D}{2}$  and  $B = \frac{C - D}{2}$  then we get the sum or difference of trigonometric functions into product of two functions as follows:

$$5. \sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right) \cdot \cos\left(\frac{C - D}{2}\right)$$

$$6. \sin C - \sin D = 2 \cos\left(\frac{C + D}{2}\right) \cdot \sin\left(\frac{C - D}{2}\right)$$

$$7. \cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right) \cdot \cos\left(\frac{C - D}{2}\right)$$

$$8. \cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right) \cdot \sin\left(\frac{C - D}{2}\right)$$

**Trigonometrical identities in a triangle:**

If A, B and C are the three angles of a triangle. Then  $A + B + C = 180^\circ$  and we have the following useful relation.

I  $A + B + C = 180^\circ$

$$\frac{A}{2} = \frac{B}{2} + \frac{C}{2} = 90^\circ$$

$$\frac{A}{2} + \frac{B}{2} = 90^\circ - \frac{C}{2}$$

$$A + B = 180^\circ - C$$

$$\sin\left(\frac{A+B}{2}\right) = \sin\left(90^\circ - \frac{C}{2}\right)$$

$$\sin(A+B) = \sin(180^\circ - C)$$

$$\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$$

$$\sin(A+B) = \sin C$$

$$\text{|||}^y \cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$$

Also  $\sin(B+C) = \sin A$

$$\sin(C+A) = \sin B$$

II  $A + B + C = 180^\circ$

$$A + B = 180^\circ - C$$

$$\cos(A+B) = \cos(180^\circ - C) \quad (\cos \text{ is } 2^{\text{nd}} \text{ quadrant is negative})$$

$$\cos(A+B) = -\cos C$$

Also  $\cos(B+C) = -\cos A$

$$\cos(C+A) = -\cos B$$

III  $A + B + C = 180^\circ$

$$A + B = 180^\circ - C$$

$$\tan(A+B) = \tan(180^\circ - C) \quad (\because \tan \theta \text{ in } 2^{\text{nd}} \text{ quadrant is negative})$$

$$\tan(A+B) = -\tan C$$

Also  $\tan(B+C) = -\tan A$

$$\tan(C+A) = -\tan B$$

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**WORKED EXAMPLES**


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**Example 1:**

Express the following as sum or difference of two trigonometric functions.

(i)  $\sin 4A \cdot \cos 2A$

(ii)  $\cos 65^\circ \cos 15^\circ$

(iii)  $\cos \frac{7\theta}{2} \sin \frac{3\theta}{2}$

(iv)  $\sin \frac{\pi}{5} \cdot \sin \frac{3\pi}{5}$

**Solution:**

$$(i) \sin 4A \cdot \cos 2A = \frac{1}{2} [\sin(4A + 2A) + \sin(4A - 2A)]$$

$$= \frac{1}{2} [\sin 6A + \sin 2A]$$

$$(ii) \cos 65^\circ \cdot \cos 15^\circ = \frac{1}{2} [\cos(65^\circ + 15^\circ) + \cos(65^\circ - 15^\circ)]$$

$$= \frac{1}{2} [\cos 80^\circ + \cos 50^\circ]$$

$$(iii) \cos\left(\frac{7\theta}{2}\right) \cdot \sin\left(\frac{3\theta}{2}\right) = \frac{1}{2} \left[ \sin\left(\frac{7\theta}{2} + \frac{3\theta}{2}\right) - \sin\left(\frac{7\theta}{2} - \frac{3\theta}{2}\right) \right]$$

$$= \frac{1}{2} [\sin 5\theta - \sin 2\theta]$$

$$(iv) \sin\left(\frac{\pi}{5}\right) \cdot \sin\left(\frac{3\pi}{5}\right) = \frac{1}{2} \left[ \cos\left(\frac{\pi}{5} - \frac{3\pi}{5}\right) - \cos\left(\frac{\pi}{5} + \frac{3\pi}{5}\right) \right]$$

$$= \frac{1}{2} \left[ \cos\left(-\frac{2\pi}{5}\right) - \cos\left(\frac{4\pi}{5}\right) \right]$$

$$= \frac{1}{2} \left[ \cos\left(\frac{2\pi}{5}\right) - \cos\left(\frac{4\pi}{5}\right) \right]$$

**Example 2:**

**Express the following as product of two functions.**

(i)  $\sin 5\theta + \sin \theta$       (ii)  $\sin 4\theta - \sin 3\theta$

(iii)  $\cos 4A + \cos 2A$     (iv)  $\cos 10^\circ - \cos 50^\circ$

**Solution:**

$$\begin{aligned} \text{(i) } \sin 5\theta + \sin \theta &= 2\sin\left(\frac{5\theta + \theta}{2}\right) \cdot \cos\left(\frac{5\theta - \theta}{2}\right) \\ &= 2\sin 3\theta \cdot \cos 2\theta \end{aligned}$$

$$\begin{aligned} \text{(ii) } \sin 4\theta - \sin 3\theta &= 2\cos\left(\frac{4\theta + 3\theta}{2}\right) \cdot \sin\left(\frac{4\theta - 3\theta}{2}\right) \\ &= 2\cos\frac{7\theta}{2} \cdot \sin\frac{\theta}{2} \end{aligned}$$

$$\begin{aligned} \text{(iii) } \cos 4A + \cos 2A &= 2\cos\left(\frac{4A + 2A}{2}\right) \cos\left(\frac{4A - 2A}{2}\right) \\ &= 2\cos 3A \cdot \cos A \end{aligned}$$

$$\begin{aligned} \text{(iv) } \cos 10^\circ - \cos 50^\circ &= -2\sin\left(\frac{10^\circ + 50^\circ}{2}\right) \cdot \sin\left(\frac{10^\circ - 50^\circ}{2}\right) \\ &= -2 \cdot \sin 30^\circ \cdot \sin(-20^\circ) \\ &= -2\left(\frac{1}{2}\right)(-\sin 20^\circ) \\ &= \sin 20^\circ \end{aligned}$$

**Example 3:**

**Prove the following**

$$\frac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A} = \tan 3A \cdot \cot A$$

**Solution:**

$$\text{LHS} = \frac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A}$$

$$\begin{aligned}
 &= \frac{\cancel{2} \sin\left(\frac{4A+2A}{2}\right) \cdot \cos\left(\frac{4A-2A}{2}\right)}{\cancel{2} \cos\left(\frac{4A+2A}{2}\right) \cdot \sin\left(\frac{4A-2A}{2}\right)} \\
 &= \frac{\sin 3A \cdot \cos A}{\cos 3A \cdot \sin A} = \tan 3A \cdot \cot A = \text{RHS}
 \end{aligned}$$

**Example 4 :**

**P.T**  $\frac{\cos 2y - \cos 2x}{\sin 2x + \sin 2y} = \tan(x - y)$

**Solution:**

$$\begin{aligned}
 \text{LHS} &= \frac{\cos 2y - \cos 2x}{\sin 2x + \sin 2y} \\
 &= \frac{\cancel{2} \sin\left(\frac{\cancel{2}y + 2x}{2}\right) \cdot \sin\left(\frac{2x - 2y}{2}\right)}{\cancel{2} \sin\left(\frac{2x + \cancel{2}y}{2}\right) \cos\left(\frac{2x - 2y}{2}\right)} \\
 &= \frac{\sin(x - y)}{\cos(x - y)} = \tan(x - y) = \text{RHS}
 \end{aligned}$$

**Example 5:**

$$\frac{\cos 75^\circ + \cos 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \sqrt{3}$$

**Solution:**

$$\begin{aligned}
 \text{LHS} &= \frac{\cos 75^\circ + \cos 15^\circ}{\sin 75^\circ - \sin 15^\circ} \\
 &= \frac{\cancel{2} \cos\left(\frac{75+\cancel{15}}{2}\right) \cdot \cos\left(\frac{75-15}{2}\right)}{\cancel{2} \cos\left(\frac{75+\cancel{15}}{2}\right) \cdot \sin\left(\frac{75-15}{2}\right)} \\
 &= \frac{\cos 30}{\sin 30} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} = \text{RHS}
 \end{aligned}$$

**Example 6:**

$$\cos\left(\frac{\pi}{4} - \alpha\right) - \cos\left(\frac{\pi}{4} + \alpha\right) = \sqrt{2} \sin \alpha$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \cos\left(\frac{\pi}{4} - \alpha\right) - \cos\left(\frac{\pi}{4} + \alpha\right) \\ &= -2 \sin\left[\frac{\left(\frac{\pi}{4} - \alpha\right) + \left(\frac{\pi}{4} + \alpha\right)}{2}\right] \sin\left[\frac{\left(\frac{\pi}{4} - \alpha\right) - \left(\frac{\pi}{4} + \alpha\right)}{2}\right] \\ &= -2 \sin \frac{\pi}{4} \cdot \sin(-\alpha) = 2 \cdot \frac{1}{\sqrt{2}} \sin \alpha = \sqrt{2} \sin \alpha = \text{RHS} \end{aligned}$$

**Example 7:**

$$\cos 130^\circ + \cos 110^\circ + \cos 10^\circ = 0$$

**Solution:**

$$\begin{aligned} \text{LHS} &= (\cos 130^\circ + \cos 10^\circ) + \cos 110^\circ \\ &= 2 \cos\left(\frac{130^\circ + 10^\circ}{2}\right) \cdot \cos\left(\frac{130^\circ - 10^\circ}{2}\right) + \cos 110^\circ \\ &= 2 \cdot \cos 70^\circ \cdot \cos 60^\circ + \cos 110^\circ \\ &= \cancel{2} \cdot \cos 70^\circ \cdot \frac{1}{\cancel{2}} + \cos 110^\circ \\ &= \cos (180^\circ - 110^\circ) + \cos 110^\circ \\ &= -\cancel{\cos 110^\circ} + \cancel{\cos 110^\circ} = 0 = \text{RHS} \end{aligned}$$

**Example 8:**

$$4 \sin A \cdot \sin (60^\circ + A) \cdot \sin (60^\circ - A) = \sin 3A$$

**Solution.**

$$\begin{aligned} \text{LHS} &= 4 \sin A [\sin^2 60^\circ - \sin^2 A] \\ &= 4 \sin A \left[ \left(\frac{\sqrt{3}}{2}\right)^2 - \sin^2 A \right] \quad \left[ \because \sin(A+B) \cdot \sin(A-B) \right. \\ &\quad \left. = \sin^2 A - \sin^2 B \right] \\ &= 4 \sin A \left[ \frac{3}{4} - \sin^2 A \right] \\ &= 3 \sin A - 4 \sin^3 A = \sin 3A = \text{RHS} \end{aligned}$$

**Example 9:**

$$\text{P.T } \frac{\sin 6A + \sin 2A + 2\sin 4A}{\sin 7A + \sin 3A + 2\sin 5A} = \frac{\sin 4A}{\sin 5A}$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \frac{(\sin 6A + \sin 2A) + 2\sin 4A}{(\sin 7A + \sin 3A) + 2\sin 5A} \\ &= \frac{2\sin\left(\frac{6A+2A}{2}\right) \cdot \cos\left(\frac{6A-2A}{2}\right) + 2\sin 4A}{2\sin\left(\frac{7A+3A}{2}\right) \cdot \cos\left(\frac{7A-3A}{2}\right) + 2\sin 5A} \\ &= \frac{2 \cdot \sin 4A \cdot \cos 2A + 2\sin 4A}{2\sin 5A \cdot \cos 2A + 2\sin 5A} \\ &= \frac{\cancel{2} \sin 4A (\cancel{\cos 2A} + 1)}{\cancel{2} \sin 5A (\cancel{\cos 2A} + 1)} \\ &= \frac{\sin 4A}{\sin 5A} = \text{RHS} \end{aligned}$$

**Example 10:**

$$\text{If } A + B + C = \pi \quad \text{P.T } \tan\left(\frac{A}{2}\right) \tan\left(\frac{B}{2}\right) + \tan\left(\frac{B}{2}\right) \tan\left(\frac{C}{2}\right) + \tan\left(\frac{C}{2}\right) \tan\left(\frac{A}{2}\right) = 1$$

**Solution.**

$$\text{Given: } A + B + C = \pi$$

$$\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$$

$$\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$$

$$\tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\Rightarrow \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \cdot \tan \frac{B}{2}} = \cot \frac{C}{2} = \frac{1}{\tan \frac{C}{2}}$$

$$\Rightarrow \tan \frac{C}{2} \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right) = 1 - \tan \frac{A}{2} \tan \frac{B}{2}$$

$$\Rightarrow \tan \frac{A}{2} \cdot \tan \frac{C}{2} + \tan \frac{B}{2} \cdot \tan \frac{C}{2} = 1 - \tan \frac{A}{2} \cdot \tan \frac{B}{2}$$

$$\Rightarrow \tan \frac{A}{2} \cdot \tan \frac{B}{2} + \tan \frac{B}{2} \cdot \tan \frac{C}{2} - \tan \frac{A}{2} \cdot \tan \frac{C}{2} = 1$$

**Example 11:**

$$\frac{\sin 5A + \sin 4A + \sin 2A + \sin A}{\cos 5A + \cos 4A + \cos 2A + \cos A} = \tan 3A$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \frac{(\sin 5A + \sin A) + (\sin 4A + \sin 2A)}{(\cos 5A + \cos A) + (\cos 4A + \cos 2A)} \\ &= \frac{2 \sin 3A \cdot \cos 2A + 2 \sin 3A \cdot \cos A}{2 \cos 3A \cdot \cos 2A + 2 \cos 3A \cdot \cos A} \quad (\text{By using transformation formula}) \\ &= \frac{\cancel{2} \sin 3A (\cancel{\cos 2A} + \cos A)}{\cancel{2} \cos 3A (\cancel{\cos 2A} + \cos A)} \\ &= \tan 3A = \text{RHS} \end{aligned}$$

**Example 12:**

$$\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ = \frac{3}{16}$$

**Solution:**

$$\begin{aligned} \text{LHS} &= (\sin 40^\circ \cdot \sin 20^\circ) \cdot \frac{\sqrt{3}}{2} \cdot \sin 80^\circ \\ &= \frac{\sqrt{3}}{2} \left[ -\frac{1}{2} (\cos 60^\circ - \cos 20^\circ) \right] \sin 80^\circ \\ &= \frac{-\sqrt{3}}{4} [\cos 60^\circ \cdot \sin 80^\circ] + \frac{\sqrt{3}}{4} [\cos 20^\circ \cdot \sin 80^\circ] \\ &= \frac{-\sqrt{3}}{4} \cdot \frac{1}{2} \sin 80^\circ + \frac{\sqrt{3}}{4} \cdot \frac{1}{2} [\sin(80^\circ + 20^\circ) + \sin(80^\circ - 20^\circ)] \end{aligned}$$

$$\begin{aligned}
 &= \frac{-\sqrt{3}}{8} \cdot \sin 80^\circ + \frac{\sqrt{3}}{8} [\sin 100^\circ + \sin 60^\circ] \\
 &= \frac{-\sqrt{3}}{8} \cdot \sin 80^\circ + \frac{\sqrt{3}}{8} \cdot \sin 100^\circ + \frac{\sqrt{3}}{8} \cdot \sin 60^\circ \\
 &= \frac{-\sqrt{3}}{8} \cdot \sin 80^\circ + \frac{\sqrt{3}}{8} \cdot \sin(180^\circ - 80^\circ) + \frac{\sqrt{3}}{8} \cdot \frac{\sqrt{3}}{2} \\
 &= \frac{-\sqrt{3}}{8} \cdot \cancel{\sin 80^\circ} + \frac{\sqrt{3}}{8} \cdot \cancel{\sin 80^\circ} + \frac{3}{16} \\
 &= \frac{3}{16} = \text{RHS}
 \end{aligned}$$

**Example 13:**

$$\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \frac{1}{8}$$

**Solution:**

$$\begin{aligned}
 \text{LHS} &= (\cos 40^\circ \cdot \cos 20^\circ) \cos 80^\circ \\
 &= \frac{1}{2} [\cos 60^\circ + \cos 20^\circ] \cdot \cos 80^\circ \\
 &= \frac{1}{2} \left[ \frac{1}{2} + \cos 20^\circ \right] \cos 80^\circ \\
 &= \frac{1}{4} \cdot \cos 80^\circ + \frac{1}{2} \cdot \cos 80^\circ \cdot \cos 20^\circ \\
 &= \frac{1}{4} \cdot \cos 80^\circ + \frac{1}{2} \cdot \frac{1}{2} [\cos 100^\circ + \cos 60^\circ] \\
 &= \frac{1}{4} \cdot \cos 80^\circ + \frac{1}{4} \cdot \cos 100^\circ + \frac{1}{4} \cdot \cos 60^\circ \\
 &= \frac{1}{4} \cdot \cancel{\cos 80^\circ} - \frac{1}{4} \cdot \cancel{\cos 80^\circ} + \frac{1}{4} \cdot \frac{1}{2} \quad \left[ \begin{array}{l} \because \cos 100^\circ = \cos(180^\circ - 80^\circ) \\ = -\cos 80^\circ \end{array} \right] \\
 &= \frac{1}{8} = \text{RHS}
 \end{aligned}$$

**Example 14:**

$$\text{S.T } \cos^2 \theta + \cos^2 (60^\circ + \theta) + \cos^2 (60^\circ - \theta) = \frac{3}{2}$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \cos^2 \theta + \cos^2 (60^\circ + \theta) + \cos^2 (60^\circ - \theta) \\ &= \frac{1 + \cos 2\theta}{2} + \frac{1 + \cos 2(60^\circ + \theta)}{2} + \frac{1 + \cos 2(60^\circ - \theta)}{2} \\ &= \frac{1}{2} [1 + \cos 2\theta + 1 + \cos(120^\circ + 2\theta) + 1 + \cos(120^\circ - 2\theta)] \\ &= \frac{1}{2} [3 + \cos 2\theta + \cos(120^\circ + 2\theta) + \cos(120^\circ - 2\theta)] \\ &= \frac{1}{2} [3 + \cos 2\theta + 2 \cos 120^\circ \cdot \cos 2\theta] \\ &= \frac{1}{2} \left[ 3 + \cos 2\theta + 2 \cdot \frac{-1}{2} \cos 2\theta \right] \\ &= \frac{1}{2} [3 + \cancel{\cos 2\theta} - \cancel{\cos 2\theta}] = \frac{3}{2} = \text{RHS} \end{aligned}$$

**Example 15:**

$$\text{In any } \triangle ABC \text{ P.T } \sin 2A + \sin 2B - \sin 2C = 4 \cos A \cdot \cos B \cdot \sin C$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \sin 2A + \sin 2B - \sin 2C \\ &= 2 \sin(A + B) \cdot \cos(A - B) - \sin 2C \quad [\because \sin 2C = 2 \sin C \cdot \cos C \text{ and } \sin(A + B) = \sin C] \\ &= 2 \sin C \cos(A - B) - 2 \sin C \cdot \cos C \\ &= 2 \sin C [\cos(A - B) - \cos C] \\ &= 2 \sin C [\cos(A - B) + \cos(A + B)] \quad [\because \cos(A + B) = -\cos C] \\ &= 2 \sin C \cdot 2 \cos A \cdot \cos B \\ &= 4 \cos A \cdot \cos B \cdot \sin C \\ &= \text{RHS} \end{aligned}$$

**Example 16:**

**If  $A + B + C = 180^\circ$**

**$P.T \cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$**

**Solution:**

$$\begin{aligned}
 LHS &= \cos 2A + \cos 2B + \cos 2C \\
 &= 2 \cos \left( \frac{2A+2B}{2} \right) \cos \left( \frac{2A-2B}{2} \right) + \cos 2C \\
 &= 2 \cos(A+B) \cos(A-B) + \cos 2C \\
 &= -2 \cos C \cos(A-B) + 2 \cos^2 C - 1 \\
 &= -1 - 2 \cos C \cos(A-B) + 2 \cos^2 C \\
 &= -1 - 2 \cos C [\cos(A-B) - \cos C] \\
 &\quad = -1 - 2 \cos C [\cos(A-B) + \cos(A+B)] \\
 &\quad = -1 - 2 \cos C [2 \cos A \cos B] \\
 &\quad = -1 - 4 \cos A \cos B \cos C \\
 &= RHS
 \end{aligned}$$

**Example 17:**

**$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$**

**Solution:**

$$\begin{aligned}
 LHS &= \sin^2 A + \sin^2 B + \sin^2 C \\
 &= \frac{1 - \cos 2A}{2} + \frac{1 - \cos 2B}{2} + \frac{1 - \cos 2C}{2} \\
 &= \frac{1}{2} [3 - (\cos 2A + \cos 2B + \cos 2C)]
 \end{aligned}$$

(Using the result of example 16)

$$\begin{aligned}
 &= \frac{1}{2} [3 - (-1 - 4 \cos A \cos B \cos C)] \\
 &= \frac{1}{2} [3 + 1 + 4 \cos A \cos B \cos C] \\
 &= \frac{1}{2} [4 + 4 \cos A \cos B \cos C] \\
 &= 2 + 2 \cos A \cos B \cos C = RHS
 \end{aligned}$$

**Example 18:**

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$$

**Solution:**

$$\begin{aligned} \text{LHS} &= \cos^2 A + \cos^2 B + \cos^2 C \\ &= \frac{1 + \cos 2A}{2} + \frac{1 + \cos 2B}{2} + \frac{1 + \cos 2C}{2} \\ &= \frac{1}{2} [3 + \cos 2A + \cos 2B + \cos 2C] \quad \left[ \begin{array}{l} \text{since} \\ \cos 2A + \cos 2B + \cos 2C \\ = -1 - 4 \cos A \cdot \cos B \cdot \cos C \end{array} \right] \\ &= \frac{1}{2} [3 - 1 - 4 \cos A \cdot \cos B \cdot \cos C] \\ &= \frac{1}{2} [2 - 4 \cos A \cos B \cos C] \\ &= 1 - 2 \cos A \cos B \cos C \\ &= \text{RHS} \end{aligned}$$

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**EXERCISE 14.3**

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**One marks questions:**

I. 1. Express each of the following as sum or difference of two trigonometric functions.

i.  $\sin 5A \cdot \cos 3A$

ii.  $\cos 4A \sin 2A$

iii.  $\cos\left(\frac{5\theta}{2}\right) \cdot \sin\left(\frac{\theta}{2}\right)$

iv.  $2 \cos 70^\circ \cos 10^\circ$

2. Express each of the following as the product of two trigonometric functions.

i.  $\sin 12x + \sin 4x$

ii.  $\sin 7A - \sin 3A$

iii.  $\cos 2\theta + \cos 6\theta$

iv.  $\sin 80^\circ - \sin 40^\circ$

**Two marks questions :**

**II. Prove the following:**

1.  $\frac{\cos 2A - \cos 12A}{\sin 12A - \sin 2A} = \tan 7A$

2.  $\frac{\sin x - \sin y}{\sin x + \sin y} = \tan\left(\frac{x-y}{2}\right) \cdot \cot\left(\frac{x+y}{2}\right)$

$$3. \frac{\sin 2\alpha + \sin 3\alpha}{\cos 2\alpha - \cos 3\alpha} = \cot \left( \frac{\alpha}{2} \right)$$

$$4. \sin \left( \frac{\pi}{3} + A \right) - \sin \left( \frac{\pi}{3} - A \right) = \sin A$$

$$5. \cos A + \cos (120 - A) + \cos (120 + A) = 0$$

$$6. \sin 65^\circ + \cos 65^\circ = \sqrt{2} \cos 20^\circ$$

$$7. \text{ If } A + B + C = \pi \text{ P.T. } \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$$

$$8. \text{ P.T. } \frac{\sin 2A + \sin 5A - \sin A}{\cos 2A + \cos 5A + \cos A} = \tan 2A$$

$$9. \text{ If } A + B + C = 180^\circ, \text{ P.T. } \cot B \cdot \cot C + \cot C \cdot \cot A + \cot A \times \cot B = 1.$$

$$10. \text{ IF } A + B + C = \frac{\pi}{2}, \text{ P.T. } \tan A \cdot \tan B + \tan B \cdot \tan C + \tan C \cdot \tan A = 1$$

### Five marks questions:

#### III. Prove the following

$$1. \frac{\cos 7x + \cos 3x - \cos 5x - \cos x}{\sin 7x - \sin 3x - \sin 5x + \sin x} = \cot 2x$$

$$2. \cos 10^\circ \cdot \cos 30^\circ \cdot \cos 50^\circ \cdot \cos 70^\circ = \frac{3}{16}$$

$$3. \cos 20^\circ \cdot \cos 40^\circ \cdot \cos 60^\circ \cdot \cos 80^\circ = \frac{1}{16}$$

**If  $A + B + C = 180^\circ$ , prove that**

$$4. \sin 2A + \sin 2B + \sin 2C = 4 \sin A \cdot \sin B \cdot \sin C$$

$$5. \tan 2A + \tan 2B + \tan 2C = \tan 2A \cdot \tan 2B \cdot \tan 2C$$

$$6. \sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \cdot \sin 2B \cdot \sin 2C$$

$$7. \cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \cdot \sin B \cdot \sin C$$

$$8. \frac{\sin 2A + \sin 2B + \sin 2C}{\sin 2A + \sin 2B - \sin 2C} = \tan A \cdot \tan B$$

9.  $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$ .

10.  $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \cdot \sin B \cdot \cos C$

11. If  $A + B + C = \pi$ . Prove that:

$$\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \cdot \sin B \cdot \cos C$$

12. If  $A + B + C = 180^\circ$ . Prove that:

$$\sin 2A - \sin 2B + \sin 2C = 4 \cos A \cdot \sin B \cdot \cos C$$

13.  $\cos 2A - \cos 2B + \cos 2C = 1 - 4 \sin A \cdot \cos B \cdot \sin C$

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**ANSWERS 14.3**

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I. 1.  $\frac{1}{2}(\sin 8A + \sin 2A)$

2.  $\frac{1}{2}(\sin 6A - \sin 2A)$

3.  $\frac{1}{2}[\sin 3\theta - \sin 2\theta]$

4.  $\cos 80^\circ + \frac{1}{2}$

II. 1.  $2 \sin 8x \cdot \cos 4x$

2.  $2 \cos 5A \cdot \sin 2A$

3.  $2 \cos 4\theta \cdot \cos 2\theta$

4.  $\sin 20^\circ$

\* \* \* \* \*

## UNIT IV - ANALYTICAL GEOMETRY

Chapter	Title	No. of Teaching hrs.
15.	CIRCLES	06 hrs
16.	PARABOLA	04 hrs
	<b>TOTAL TEACHING HOURS</b>	<b>10 hrs</b>



**15.1 Introduction :**

Of all the curves, the circle is the simplest one and occurs most frequently in nature, architecture, science and industry. We see applications of circles everywhere. Various objects round (circular) in shape are used by us in our daily life. Few examples are coins, wheels, compact disc etc. Therefore it is necessary to pay attention to the study of circles.

**15.2 Definition :**

A circle is the locus of a point which moves in a plane so that its distance from a fixed point always remain constant. This fixed point is called the **centre** of the circle and the constant distance is called the **radius** of the circle.

**Equation of the circle in different forms:****(a) Equation of a circle whose centre is the origin and the radius being 'r'**

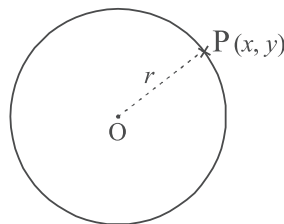
Consider a circle with centre at the origin  $O(0,0)$  and radius ' $r$ '. Let  $P(x,y)$  be a point on the circle. Join  $OP$ , by given data  $OP = r$

Using distance formula

$$OP = \sqrt{(x-0)^2 + (y-0)^2}$$

$$r = \sqrt{x^2 + y^2}$$

$$\boxed{r^2 = x^2 + y^2} \text{ which is the required equation.}$$



**Example 1 :** The equation of the circle with centre at the origin and radius 5 unit is given by

$$x^2 + y^2 = 5^2 \text{ i.e. } x^2 + y^2 = 25$$

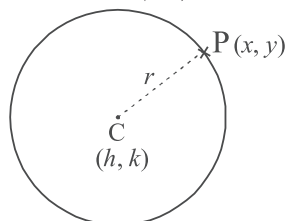
**(b) Equation of the circle whose centre is (h,k) and radius being 'r'**

Consider a point  $p(x,y)$  on the circle. Let the centre of the circle be  $c(h,k)$  Join  $CP$  which is by definition equal to ' $r$ '.

By distance formula

$$CP = r = \sqrt{(x-h)^2 + (y-k)^2}$$

$$\text{i.e. } \boxed{r^2 = (x-h)^2 + (y-k)^2}$$



This is the equation of the circle in standard form with centre  $(h,k)$  and radius  $r$

**Note :** In particular if  $h = 0$  and  $k = 0$  then the equation of the circle becomes  $x^2 + y^2 = r^2$

**WORKED EXAMPLES**

**Example 1:**

**Find the equation of the circle with centre  $(-2, -1)$  and the radius is 2.**

**Solution:**

Given  $(h, k) = (-2, -1)$  and  $r = 2$

we have the equation of the circle given by

$$r^2 = (x - h)^2 + (y - k)^2$$

$$2^2 = (x - (-2))^2 + (y - (-1))^2$$

$$4 = (x + 2)^2 + (y + 1)^2$$

Simplifying the above equation we get the equation of the circle as

$$x^2 + y^2 + 4x + 2y + 1 = 0$$

**Example 2:**

**Find the equation of the circle with the centre as  $(3, -4)$  and radius is 5 units. Also show that the circle passes through origin.**

**Solution:**

Given  $(h, k) = (3, -4)$  and  $r = 5$

we have the equation of the circle given by

$$r^2 = (x - h)^2 + (y - k)^2$$

$$5^2 = (x - 3)^2 + (y - (-4))^2$$

$$25 = (x - 3)^2 + (y + 4)^2$$

Simplifying the above equation we get the equation of the circle as

$$x^2 + y^2 - 6x + 8y = 0$$

The co-ordinates of the origin i.e.,  $x = 0$  and  $y = 0$  satisfies the above equation and so the circle passes through the origin.

**Example 3:**

**Find the equation of the circle whose centre is  $(2, -3)$  and passes through the point of intersection of the lines  $3x - 2y = 1$  and  $4x + y = 27$ .**

**Solution:**

$$3x - 2y = 1 \quad \dots(1)$$

$$4x + y = 27 \quad \dots(2)$$

Solving (1) and (2) we get  $x = 5$  and  $y = 7$

point of intersection is  $(x, y) = (5, 7)$ . If  $C(2, -3)$  and  $P(x, y)$

Using distance formula

$$\therefore CP = \sqrt{(5-2)^2 + (7+3)^2} = \sqrt{109} = \text{Radius}$$

Hence the equation of the circle is

$$(x-h)^2 + (y-k) = r^2$$

$$(x-2)^2 + (y+3)^2 = (\sqrt{109})^2$$

$$x^2 + y^2 - 4x + 6y - 96 = 0$$

**Example 4:**

**Find the equation of the circle with two of whose diameters  $x + y = 6$  and  $x + 2y = 4$  having radius  $2\sqrt{5}$  units.**

**Solution:**

Solving  $x + y = 6$  and  $x + 2y = 4$  we get  $y = -2$  and  $x = 8$ .

The centre of the circle is the meeting point of the two diameters which is given by  $(8, -2)$

$$\text{Given } r = 2\sqrt{5}$$

$$\text{Equation of the circle is } (x-h)^2 + (y-k)^2 = r^2$$

$$\text{i.e. } (x-8)^2 + (y+2)^2 = (2\sqrt{5})^2$$

$$x^2 + y^2 - 16x + 4y + 48 = 0$$

**Example 5:**

**If the lines  $2x + 3y + 1 = 0$  and  $3x - y - 4 = 0$  lie along the diameters of a circle of circumference  $10\pi$ , find the equation of the circle**

**Solution :**

Solving  $2x + 3y + 1 = 0$  and  $3x - y - 4 = 0$  we get  $x = 1, y = -1 \therefore$  centre  $(1, -1)$

Circumference of the circle  $= 2\pi r = 10\pi$

$$\therefore r = 5$$

Thus the equation of the circle is given by

$$(x-1)^2 + (y+1)^2 = 5^2$$

i.e.  $x^2 + y^2 - 2x + 2y - 23 = 0$  is the required equation.

**(c) Equation of the circle with  $A(x_1, y_1)$  and  $B(x_2, y_2)$  as the co-ordinates of the end points of a diameter**

Let  $p(x, y)$  be a point on the circle. Join PA and PB.

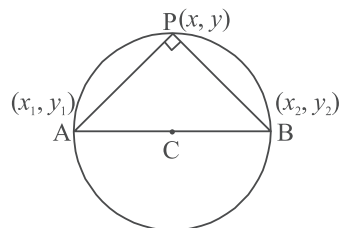
We find  $\angle APB = 90^\circ$  (Angle in a semi circle)

$\therefore$  slope of AP  $\times$  slope PB =  $-1$  ( $\because AP \perp BP$ )

$$\text{i.e.} \left( \frac{y - y_1}{x - x_1} \right) \left( \frac{y - y_2}{x - x_2} \right) = -1$$

$$(y - y_1)(y - y_2) = -1(x - x_1)(x - x_2)$$

i.e.  $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$  is the required equation of the circle.



**Example 6:**

**Find the equation of the circle with  $(-4, 3)$  and  $(12, -1)$  as the extremities of a diameter.**

**Solution :**

The equation of the circle is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$(x + 4)(x - 12) + (y - 3)(y + 1) = 0$$

$$\text{i.e. } x^2 + y^2 - 8x - 2y - 51 = 0$$

**(d) Equation of the point circle**

The circle whose **radius is zero** is called as the **point circle**.

**Note:** Consider the equation  $r^2 = (x - h)^2 + (y - k)^2$  where centre is  $(h, k)$  and radius is  $r$ .

- (i) If  $r > 0$ , we say it is a real circle.
- (ii) If  $r < 0$ , then the circle is called as an **imaginary circle or virtual circle**.
- (iii) If  $r = 0$  then the circle is said to be a **point circle**
- (iv) If  $r = 1$ , then the circle is said to be a **unit circle**

**Example 7:**

**Write the equation of the point circle with centre at  $(3, -5)$**

**Solution :**

Centre  $(h, k) = (3, -5)$

Since it is a point circle, radius  $r = 0$

The equation of the point circle is given by

$$r^2 = (x - h)^2 + (y - k)^2$$

$$0 = (x - 3)^2 + (y + 5)^2$$

$$\text{i.e., } x^2 + y^2 - 6x + 10y + 34 = 0$$

**EXERCISE 15.1****One mark questions:**

- I. Find the equation of the circle with centre 'C' and radius 'r' in each of the following:
- a)  $c(-3, 2)$  and  $r = 5$  units                      b)  $c(0, 0)$  and  $r = 4$  units
- c)  $c(-1, -2)$  and diameter  $r = 25$  units                      d)  $c(1, 1)$  and  $r = \sqrt{2}$  units
- e)  $c(a \cos \theta, a \sin \theta)$  and  $r = a$  units
- II. Find the centre of the circle, two of the diameters are
- a)  $x + y = 2$ ,  $x - y = 0$                       b)  $2x - 3y = 1$ ,  $3x - 2y = 2$
- c)  $y = 0$  and  $y = x - 5$
- III. Write the equation of the point circle with centre at (a)  $(4, -5)$     (b)  $(-3, 2)$     (c)  $(1, 0)$

**Two marks questions:**

- IV. Find the equation of the circle.
- a) two of the diameters are  $x + y = 6$  and  $x + 2y = 4$  and its radius is 10 units.
- b) two of the diameters  $x + y = 4$  and  $x - y = 2$  and passing through the point  $(2, -1)$
- c) two of the diameters are  $2x - 3y = 5$  and  $3x - 4y = 7$  are the diameters of a circle of area 154 sq.cm
- V. a) Find the equation of the circle with centre at  $(-2, 1)$  and passing through the origin.
- b) Find the equation of the circle with centre at  $(2, 1)$  and passing through  $(0, -1)$
- VI. Find the equation of the circle described on the line joining the points A and B as diameter where
- a)  $A(-5, 1)$  and  $B(1, 3)$                       b)  $A(2, 0)$  and  $B(0, 2)$
- c)  $A(3, 4)$  and  $B(1, -2)$

**ANSWERS 15.1****One mark questions:**

- I. a)  $x^2 + y^2 + 6x - 4y = 12$                       b)  $x^2 + y^2 = 16$
- c)  $4x^2 + 4y^2 + 8x + 16y = 605$                       d)  $x^2 + y^2 - 2x - 2y = 0$
- e)  $x^2 + y^2 - (2a \cos \theta)x - (2a \sin \theta)y = 0$
- II. a) centre  $(1, 1)$                       b) centre  $\left(\frac{4}{5}, \frac{1}{5}\right)$                       c) centre  $(5, 0)$
- III. a)  $x^2 + y^2 - 8x + 10y + 41 = 0$                       b)  $x^2 + y^2 + 6x - 4y + 13 = 0$
- c)  $x^2 + y^2 - 2x + 1 = 0$

**Two marks questions:**

IV. a)  $x^2 + y^2 - 16x + 4y - 32 = 0$

b)  $x^2 + y^2 - 6x - 2y + 5 = 0$

c)  $x^2 + y^2 - 2x + 2y - 47 = 0$

V. a)  $x^2 + y^2 + 4x - 2y = 0$

b)  $x^2 + y^2 - 4x - 2y = 3$

VI. a)  $x^2 + y^2 + 4x - 4y = 2$

b)  $x^2 + y^2 - 2x - 2y = 0$

c)  $x^2 + y^2 - 4x - 2y = 5$

**15.3 General equation of the circle**

**Show that the second degree equation  $x^2 + y^2 + 2gx + 2fy + c = 0$  always represents a circle**

**Proof:** Given  $x^2 + y^2 + 2gx + 2fy + c = 0$  ....(1)

i.e.  $x^2 + y^2 + 2gx + 2fy = -c$

Adding  $g^2 + f^2$  on both sides of the above equation we get

$$x^2 + y^2 + 2gx + 2fy + g^2 + f^2 = g^2 + f^2 - c$$

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$$

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

$$\Rightarrow (x - (-g))^2 + (y - (-f))^2 = (\sqrt{g^2 + f^2 - c})^2$$

which is of the form  $(x - h)^2 + (y - k)^2 = r^2$

Hence the given equation (1) always represent a circle where centre  $c(h, k) = (-g, -f)$  and

radius  $r = \sqrt{g^2 + f^2 - c}$

**Definition :** The equation of the circle in the form  $x^2 + y^2 + 2gx + 2fy + c = 0$  is called the standard form of the equation of the circle.

Consider the general second degree equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ . The equation represents a circle if  $a = b$  and  $h = 0$  i.e. the equation  $ax^2 + ay^2 + 2gx + 2fy + c = 0$  always represent a circle. This equation is called the general equation of the circle. In particular if  $a = b = 1$ , we get the standard form of the equation of the circle. Which is given by  $x^2 + y^2 + 2gx + 2fy + c = 0$ , the centre of the circle is  $(-g, -f)$  and radius

$$r = \sqrt{g^2 + f^2 - c}$$

**Note:**

$$* \quad x \text{ co-ordinate of the centre of the circle} = \frac{-(\text{coefficient of } x)}{2}$$

$$* \quad y \text{ coordinate of the centre of the circle} = \frac{-(\text{coefficient of } y)}{2}$$

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**WORKED EXAMPLES**

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**Example 1:**

**Find the centre and radius of the following circles.**

**(a)**  $x^2 + y^2 - 2x + 6y - 20 = 0$

**(b)**  $3x^2 + 3y^2 - 6x + 4y + 1 = 0$

**(c)**  $2x^2 + 2y^2 + 6x - 10y + 9 = 0$

**Solution :**

(a) The equation of the circle is given by

$$x^2 + y^2 - 2x + 6y - 20 = 0$$

Here  $2g = -2$ ,  $2f = 6$  and  $c = -20$

$$\Rightarrow g = -1, f = 3 \text{ and } c = -20$$

$$c = (-g, -f) = (1, -3), \text{radius} = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 9 + 20} = \sqrt{30}$$

(b) The equation of the circle is given by  $3x^2 + 3y^2 - 6x + 4y + 1 = 0$

In this equation, the coefficient of  $x^2$  and  $y^2$  are not 1. Thus, we have to divide, throughout by 3.

we get,  $x^2 + y^2 - 2x + \frac{4}{3}y + \frac{1}{3} = 0$

Here  $2g = -2$ ,  $2f = \frac{4}{3}$ ,  $c = \frac{1}{3}$

$$\Rightarrow g = -1, f = \frac{2}{3}, c = \frac{1}{3}$$

centre  $C(-g, -f) = \left(1, -\frac{2}{3}\right)$

$$\text{radius} = \sqrt{g^2 + f^2 - c} = \sqrt{1 + \frac{4}{9} - \frac{1}{3}} = \frac{\sqrt{10}}{3}$$

(c) The equation of the given circle is

$$2x^2 + 2y^2 + 6x - 10y + 9 = 0$$

Here also we have to make the coefficient of  $x^2$  and  $y^2$  as 1. To this, we shall divide throughout by 2. We get

$$x^2 + y^2 + 3x - 5y + \frac{9}{2} = 0$$

$$\text{Here } 2g = 3, 2f = -5, c = \frac{9}{2}$$

$$\Rightarrow g = \frac{3}{2}, f = -\frac{5}{2}, c = \frac{9}{2}$$

$$\therefore \text{centre } c = (-g, -f) = \left(-\frac{3}{2}, \frac{5}{2}\right)$$

$$\text{and radius } r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{9}{4} + \frac{25}{4} - \frac{9}{2}} = 2 \text{ units}$$

**Example 2:**

If the radius of the circle  $x^2 + y^2 - 2x + 3y + k = 0$  is  $\frac{5}{2}$ , find  $k$ .

**Solution:**

The given equation is  $x^2 + y^2 - 2x + 3y + k = 0$

$$\text{By data } r = \frac{5}{2} = \sqrt{g^2 + f^2 - c}, \quad 2g = -2 \quad \therefore g = -1$$

$$\text{i.e. } \left(\frac{5}{2}\right)^2 = (+1)^2 + \left(\frac{-3}{2}\right)^2 - (+k) \quad 2f = 3 \quad \therefore f = \frac{3}{2}, \quad c = k$$

$$\frac{25}{4} = 1 + \frac{9}{4} - k$$

simplifying,  $k = -3$

**Example 3:**

If one end of the diameter of the circle  $x^2 + y^2 + 2x + 6y - 22 = 0$  is  $(3, -7)$ , find the co-ordinate of the other end.

**Solution:**

The equation of the circle is  $x^2 + y^2 + 2x + 6y - 22 = 0$

Now centre  $C = (-g, -f) = C(-1, -3)$

One end of the diameter is A (3, -7)

Let the other end of the diameter be B(x,y)

$$C = \text{Midpoint of } AB \Rightarrow (-1, -3) = \left( \frac{3+x}{2}, \frac{-7+y}{2} \right)$$

$$-1 = \frac{3+x}{2}, \quad -3 = \frac{-7+y}{2} \Rightarrow -2 = 3+x, \quad -6 = -7+y$$

$$x = -5 \quad \text{and} \quad y = 1$$

$\therefore$  the other end of the diameter is (-5,1)

#### Example 4:

**If one end of the diameter of the circle  $x^2 + y^2 - 4x - 2y - 5 = 0$  is (3,4), find co-ordinate of the other end.**

#### Solution:

The equation of the circle is  $x^2 + y^2 - 4x - 2y - 5 = 0$

Now, centre  $C = (-g, -f) = (2, 1)$ . One end of the diameter is  $A = (3, 4)$ . Let the other end be  $B = (h, k)$

$$\therefore C = \text{midpoint of } AB \Rightarrow (2, 1) = \left( \frac{3+h}{2}, \frac{4+k}{2} \right) \text{ by equating we get}$$

$$\Rightarrow 2 = \frac{3+h}{2}, \quad 1 = \frac{4+k}{2}$$

$$\Rightarrow 4 = 3+h, \quad 2 = 4+k \Rightarrow h = 1, \quad k = -2$$

$\therefore$  the other end of the diameter is (1, -2)

#### Example 5:

**If  $x^2 + y^2 - 14x + by + 53 = 0$  represents a point circle, find 'b'**

#### Solution:

By data  $x^2 + y^2 - 14x + by + 53 = 0$  represents a point circle. Thus, the radius = 0

$$\begin{aligned} \Rightarrow \sqrt{g^2 + f^2 - c} &= 0 \rightarrow \sqrt{(-7)^2 + \left(\frac{b}{2}\right)^2} - 53 = 0 & \begin{cases} 2g = -14 \therefore g = -7 \\ 2f = b \therefore f = \frac{b}{2} \\ c = 53 \end{cases} \\ \Rightarrow 49 + \frac{b^2}{4} - 53 &= 0 \\ \Rightarrow \frac{b^2}{4} &= 4 \rightarrow b^2 = 16 \rightarrow b = \pm 4 \end{aligned}$$

**Note :** Consider the equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \dots\dots\dots(1)$$

Any circle concentric with this circle, will have the same centre. Thus, in its equation the coefficients of  $x^2$  and  $y^2$  will be same as that of (1). Hence we have the following:

The equation of the circle concentric with the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  is given by  $x^2 + y^2 + 2gx + 2fy + k = 0$ .

For different values of  $k$ , we get different circles concentric with the given circle.

**Example 6:**

**Find the equation of the circle concentric with the circle  $x^2 + y^2 - 6x - 2y + 4 = 0$  and passing through the point (3,2).**

**Solution:**

The equation of the circle concentric with

$$x^2 + y^2 - 6x - 2y + 4 = 0 \text{ is of the form}$$

$$x^2 + y^2 - 6x - 2y + k = 0$$

By data this must pass through (3,2). Thus the equation (1) must satisfy

$$x = 3, y = 2 \rightarrow 9 + 4 - 18 - 4 + k = 0 \rightarrow k = 9$$

Hence, the equation of the required circle is  $x^2 + y^2 - 6x - 2y + 9 = 0$

**Example 7:**

**Find the equation of the circle whose centre is (-3,1) and passing through the centre of the circle  $x^2 + y^2 + 2x - 4y + 4 = 0$ .**

**Solution:**

Let  $C = (-3,1)$ . The centre of the given circle  $x^2 + y^2 + 2x - 4y + 4 = 0$  is  $A = (-1,2)$

$$(\because 2g = 2, 2f = -4)$$

The required circle passes through A. Thus,

$$\text{radius} = CA \rightarrow \sqrt{(-3+1)^2 + (1-2)^2} = \sqrt{5}$$

Now, the equation of the circle with centre  $(-3, 1)$  and radius  $\sqrt{5}$  units is given by

$$(x+3)^2 + (y-1)^2 = (\sqrt{5})^2 \Rightarrow x^2 + y^2 + 6x - 2y + 5 = 0$$

### 2nd method

By data the centre of the required circle is  $(-3, 1)$ ,  $\therefore -g = -3$  and  $-f = 1 \rightarrow g = 3, f = -1$

Thus, the required circle is of the form  $x^2 + y^2 + 6x - 2y + c = 0$

This must pass through the centre of  $x^2 + y^2 + 2x - 4y + 4 = 0$ , i.e., it must pass through  $(-1, 2) \rightarrow 1 + 4 - 6 - 4 + c = 0 \rightarrow c = 5$ . Putting  $c = 5$ , we get the required equation

$$x^2 + y^2 + 6x - 2y + 5 = 0$$

**Note:** The standard form of the equation of the circle has three parameters  $g, f$  and  $c$ . Thus to find the equation of a circle, we have to find corresponding values of  $g, f$ , and  $c$ . Hence, we require at least three conditions to determine the equation of a circle.

### **Example 8:**

**Find the equation of the circle passing through the points  $(0,0)$  and  $(1,1)$  and has its centre on  $x$ -axis.**

### **Solution:**

Let the equation of the required circle be  $x^2 + y^2 + 2gx + 2fy + c = 0$  .....(1)

Using the conditions in the problem, we shall determine the values of  $g, f$  and  $c$

By data the circle passes through the point  $(0,0)$ . Thus the equation (1) must satisfy  $x = 0$  and  $y = 0$

$$\Rightarrow 0 + 0 + 2g(0) + 2f(0) + c = 0 \Rightarrow c = 0$$

Again by data the circle passes through the point  $(1,1)$ . Thus the equation (1) must satisfy  $x = 1, y = 1$

$$1 + 1 + 2g + 2f + c = 0 \Rightarrow 2g + 2f + 2 = 0 \quad (\because c = 0)$$

$$\Rightarrow g + f + 1 = 0 \text{ .....(2)}$$

Again by data, the centre  $(-g, -f)$  of (1) lies on  $x$ -axis

$$\Rightarrow y\text{-coordinate of the centre} = 0 \Rightarrow f = 0$$

putting  $f = 0$  in (2), we get  $g + 1 = 0 \Rightarrow g = -1$

putting  $g = -1, f = 0, c = 0$  in (1) we get the equation of the circle  $x^2 + y^2 - 2x = 0$

**Note:** When the circle passes through the origin  $O(0,0)$  then the standard form of the equation of the circle is given by  $x^2 + y^2 + 2gx + 2fy = 0$

**Example 9:**

Find the equation of the circle passing through the points  $(-1, 2)$  and  $(3, -2)$ , and has its centre on  $x = 2y$ .

**Solution:**

Let the equation of the required circle be  $x^2 + y^2 + 2gx + 2fy + c = 0$  .....(1)

By data this circle passes through the point  $(-1, 2)$  and  $(3, -2)$

$$\Rightarrow -2g + 4f + c + 5 = 0 \text{ .....(2)}$$

$$\text{and } 6g - 4f + c + 13 = 0 \text{ .....(3)}$$

Again by data the centre  $(-g, -f)$  lies on  $x = 2y$

$$\Rightarrow -g = -2f \Rightarrow g = +2f$$

Solving the equations (1), (2) and (3) we get  $g, f$  and  $c$

consider (2) - (3) we get  $-8g + 8f - 8 = 0 \Rightarrow g - f + 1 = 0$

putting  $g = 2f$ , we get  $2f - f + 1 = 0 \Rightarrow f = -1$

$$\text{Now, } g = 2f \Rightarrow g = -2$$

$$\text{From (2) we have, } c = 2g - 4f - 5 \Rightarrow c = 4 + 4 - 5 \Rightarrow c = -5$$

Thus the equation of the required circle is  $x^2 + y^2 - 4x - 2y - 5 = 0$

**Example 10:**

Find the equation of the circle passing through the points  $(5, 3)$ ,  $(1, 5)$  and  $(3, -1)$ .

**Solution:**

Let the equation of the circle be  $x^2 + y^2 + 2gx + 2fy + c = 0$  ....(1)

By data this passes through the point

$(5, 3)$ ,  $(1, 5)$  and  $(3, -1)$  Thus we have

$$10g + 6f + c + 34 = 0 \text{ .....(2)}$$

$$2g + 10f + c + 26 = 0 \text{ .....(3)}$$

$$6g - 2f + c + 10 = 0 \text{ .....(4)}$$

Solving these three equations we get  $f, g$  and  $c$ . To this end consider,

$$(2) - (3) \rightarrow 8g - 4f + 8 = 0 \rightarrow 2g - f + 2 = 0 \text{ .....(5)}$$

$$(3) - (4) \rightarrow -4g + 12f + 16 = 0 \rightarrow g - 3f - 4 = 0 \text{ .....(6)}$$

Solving (5) and (6), we get  $g = -2, f = -2$

$$\text{Now, from (1) } \Rightarrow c = -10g - 6f - 34 \Rightarrow c = 20 + 12 - 34 \Rightarrow c = -2$$

Thus, the equation of the required circle is  $x^2 + y^2 - 4x - 4y - 2 = 0$

**Example 11:**

**Show that the following points (2,0), (-1, 3), (-2, 0) and ( 1,-1) are concyclic.**

**Solution:**

We shall find the equation of the circle passing through the points (2, 0), (-1, 3) and (-2, 0) and then we shall show that the fourth point (1, -1) lies on it.

Let the equation of the circle passing through (2, 0), (-1, 3) and (-2, 0) be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \dots\dots\dots(1)$$

Since it passes through (2,0), (-1,3) and (-2, 0)

$$\text{we have } 4g + c + 4 = 0 \dots\dots\dots(2)$$

$$-2g + 6f + c + 10 = 0 \dots\dots\dots(3)$$

$$-4g + c + 4 = 0 \dots\dots\dots(4)$$

Solving these equations we get  $f, g$  and  $c$ .

consider, (2) - (4) we get,  $8g = 8g = 0 \rightarrow g = 0$

putting  $g = 0$  in (2) we get,  $c = -4$

Now (3)  $\Rightarrow 6f = 2g - c - 10 \rightarrow 6f = +4 - 10 \Rightarrow f = -1$

Thus, the equation of the circle is  $x^2 + y^2 - 2y - 4 = 0$

Now, putting  $x = 1$  and  $y = -1$  in LHS of (1)

we get,  $1 + 1 + 2 - 4 = 4 - 4 = 0 = \text{RHS}$

Thus, the point (1, -1) lies on the circle.

Hence, the given four points are concyclic

**Example 12:**

**Find the equation of the diameter of the circle  $x^2 + y^2 + 2x + 4y = 4$  which when produced passes through the point (-6, 2).**

**Solution :** Equation of the circle is  $x^2 + y^2 + 2x + 4y = 4$

centre  $C = (-1, -2)$

Required diameter is the line joining the points  $C(-1, -2)$  and the point  $(-6, 2)$

Its equation is  $\frac{y+2}{x+1} = \frac{2+2}{-6+1}$

$$\frac{y+2}{x+1} = \frac{4}{-5} \Rightarrow -5y - 10 = 4x + 4$$

$$\Rightarrow 4x + 5y + 14 = 0$$

**Example 13:**

Show that the line  $2x + y = 4$  passes through the centre of the circle

$$x^2 + y^2 - 3x - 2y + 2 = 0$$

**Solution:**

The centre of the circle  $x^2 + y^2 - 3x - 2y + 2 = 0$  is  $C = \left(\frac{3}{2}, 1\right)$ . Put  $x = \frac{3}{2}$ ,  $y = 1$  in the left

hand side of  $2x + y = 4$ , we get  $2\left(\frac{3}{2}\right) + 1 = 3 + 1 = 4 = \text{RHS}$

Thus  $\left(\frac{3}{2}, 1\right)$  lies on  $2x + y = 4$

**15.4 Length of the chord of the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  intercepted by the co-ordinate axes.**

**(a) To find the length of the chord of the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  intercepted by the  $x$  axis**

**Solution:**

The equation of the circle is  $x^2 + y^2 + 2gx + 2fy + c = 0$ . Let the circle cut the  $x$  axis at A and B.

Draw perpendicular CM from the centre onto AB. Clearly M is the midpoint of AB.

Now  $x$  intercept  $2AM$

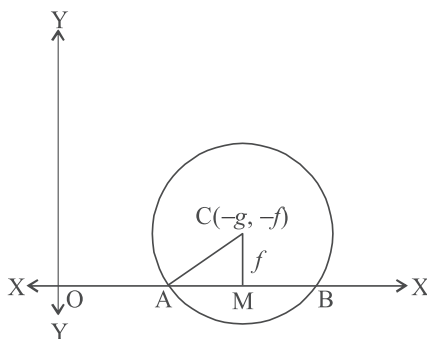
$$x \text{ intercept} = 2\sqrt{AC^2 - CM^2}$$

[ $\because$  AMC is a right triangle]

$$x \text{ intercept} = 2\sqrt{g^2 + f^2 - c - f^2}$$

[ $\because$  AC = radius and CM =  $-f$ ]

$$\boxed{x \text{ intercept} = 2\sqrt{g^2 - c}}$$



**(b) To find the length of the chord of the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  intercepted by the  $y$  axis**

**Solution:**

Let the circle cut the  $y$  axis at D and E. Draw perpendicular CN from the centre on DE clearly N is the midpoint of DE

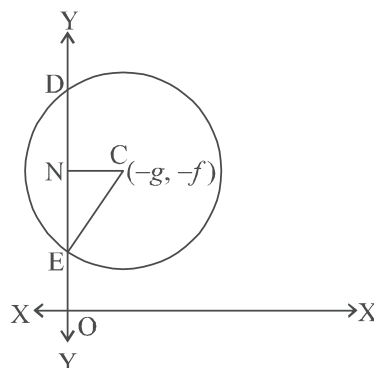
Now  $y$  intercept  $= 2 \text{ EN}$

$$y \text{ intercept} = 2\sqrt{\text{CE}^2 - \text{CN}^2}$$

$$y \text{ intercept} = 2\sqrt{g^2 + f^2 - c - g^2}$$

( $\because \text{CE} = \text{radius}, \text{CN} = g$ )

$$y \text{ intercept } 2\sqrt{f^2 - c}$$



**Example 14:**

**Find the length of the chord of the circle  $x^2 + y^2 - 6x + 4y + 5 = 0$  intercepted by  $x$  axis.**

**Solution:**

The equation of the circle  $x^2 + y^2 - 6x + 4y + 5 = 0$

$$x \text{ intercept} = 2\sqrt{g^2 - c} = 2\sqrt{9 - 5} = 2\sqrt{4} = 4$$

**Example 15:**

**Find the length of the chord of the circle  $x^2 + y^2 + 3x - 2 = 0$  intercepted by  $y$  axis**

**Solution:**

The equation of the circle is

$$x^2 + y^2 + 3x - 2 = 0$$

$$y \text{ intercept} = 2\sqrt{f^2 - c} = 2\sqrt{0 + 2} = 2\sqrt{2}$$

**Example 16:**

**Find the equation of the circle passing through the origin and making positive intercepts 3 and 5 on the co-ordinate axes.**

**Solution:**

Let the equation of the circle be  $x^2 + y^2 + 2gx + 2fy + c = 0$

The circle passes through the origin  $\Rightarrow C = (0, 0)$

By data  $x$  intercept  $= 3$

$$\text{i.e. } 2\sqrt{g^2 - c} = 3$$

$$2\sqrt{g^2} = 3 \Rightarrow -2g = 3 \Rightarrow g = \frac{-3}{2}$$

we take negative sign, as the centre must lie in the first quadrant.

$$2\sqrt{f^2 - c} = 5$$

$$2\sqrt{f^2} = 5$$

$$-2f = 5 \Rightarrow f = \frac{-5}{2}$$

we take negative sign, as the centre must lie in the first quadrant.

Thus the equation of the circle is  $x^2 + y^2 - 3x - 5y = 0$

**Note:** To find the length of the chord intercepted by a line with the given circle whose radius is ' $r$ ' and the length of the perpendicular from the centre on to the line.

**Solution:**

Let the given line cut the circle at A and B. Clearly the length of the chord is AB

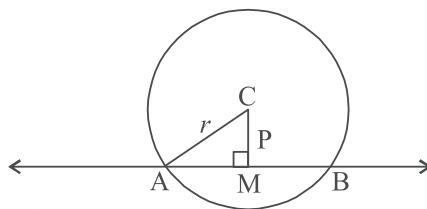
Let C be the centre of the circle.

Draw CM perpendicular to the line AB

Now  $AB = 2 AM$

$$= 2\sqrt{CA^2 - CM^2}$$

$$AB = 2\sqrt{r^2 - p^2} \quad (\because CA = r, CM = p)$$



Thus length of the chord  $= 2\sqrt{r^2 - p^2}$

**Example 17 :** Find the length of the chord intercepted by the circle,  $x^2 + y^2 + 4x + 6y = 12$  and  $x + 4y = 6$

**Solution:** The equation of the circle is  $x^2 + y^2 + 4x + 6y - 12 = 0$

Equation of the line is  $x + 4y - 6 = 0$

Now centre C and radius  $r$  of the circle are  $C(-2, -3)$   $r = \sqrt{4 + 9 + 12} = 5$

$p$  = length of perpendicular from C onto the line

$$p = \left| \frac{-2 + 4(-3) - 6}{\sqrt{1 + 16}} \right| = \frac{20}{\sqrt{17}}$$

Length of the chord  $= 2\sqrt{r^2 - p^2}$

$$= 2\sqrt{25 - \frac{400}{17}} = \frac{10}{\sqrt{17}}$$

**Example 18:**

**Find the equation of the chord of the circle  $x^2 + y^2 - 2x + 4y - 17 = 0$  bisected at  $(-1, 2)$**

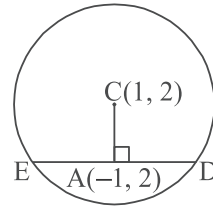
**Solution:**

The equation of the circle is  $x^2 + y^2 - 2x + 4y - 17 = 0$  centre =  $(1, -2)$ . The mid point of the chord A =  $(-1, 2)$ . The chord is perpendicular to AC

$\therefore$  (slope of the chord) (slope of AC) =  $-1$

$$\text{slope of the AC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 + 2}{-1 - 1}$$

$$\text{slope of the chord} = \frac{-1}{\left(\frac{2+2}{-1-1}\right)} = \frac{1}{2}$$



The chord ED passes through  $(-1, 2)$ . Thus the equation of the chord ED is

$$y - 2 = \frac{1}{2}(x + 1) \Rightarrow 2y - 4 = x + 1$$

$$\Rightarrow x - 2y + 5 = 0$$

**EXERCISE 15.2**

**One mark questions:**

- Find the centre and the radius of the circle.
  - $x^2 + y^2 - 4x - y - 5 = 0$
  - $3x^2 + 3y^2 - 6x - 12y - 2 = 0$
  - $(x - 2)(x - 4) + (y - 1)(y - 3) = 0$
  - $x^2 + y^2 - 2x \cos \alpha - 2y \sin \alpha = 1$
- If the radius of the circle  $x^2 + y^2 + 4x - 2y - k = 0$  is 4 units find  $k$ .
- Find the other end of the diameter, if one end of the diameter of the circle
  - $x^2 + y^2 = 25$  is  $(5, 0)$
  - $x^2 + y^2 + 4x - 6y - 12 = 0$  is  $(-5, -1)$
  - $x^2 + y^2 - 6x + 2y = 31$  is  $(7, 4)$
- If  $(a, b)$  and  $(-5, 1)$  are the two end points of diameter of the circle  $x^2 + y^2 + 4x - 4y = 2$ . Find the values of  $a$  and  $b$ .
- If  $x^2 + y^2 - 4x - 8y + k = 0$  represents a point circle find  $k$ .

6. If  $x^2 + y^2 + ax + 2y + 5 = 0$  represent a point circle find  $a$ .
7. If  $x^2 + y^2 + ax + by = 3$  represents a circle with centre at  $(1, -3)$ , find  $a$  and  $b$ .
8. Find the unit circle concentric with the circle  $x^2 + y^2 - 8x + 4y = 8$ .

**Two marks questions:**

1. Find the equation of the circle whose centre is same as the centre of the circle  $x^2 + y^2 + 6x + 2y + 1 = 0$ , and passing through the point  $(-2, -1)$ .
2. Find the equation of the circle whose centre is same as the centre of the circle  $x^2 + y^2 - 6x + 4y + 9 = 0$  and passing through the point  $(-2, 3)$ .
3. Find the equation of the circle whose centre is  $(-2, 3)$  and passing through the centre of the circle  $x^2 + y^2 - 6x + 4y + 9 = 0$ .
4. Find the equation of the circle passing through the centre of the circle  $x^2 + y^2 - 2x - 4y - 20 = 0$  and centre at  $(4, -2)$ .
5. Find the equation of the circle two of whose diameters are  $x + y = 3$  and  $2x + y = 2$  and passing through the centre of the circle  $x^2 + y^2 - 4x + 2y - 1 = 0$ .
6. Find the equation of the circle concentric with the centre of the circle  $x^2 + y^2 - 2x + 2y - 1 = 0$  and having double its area.
7. Find the equation of the circle concentric with the centre of the circle  $3x^2 + 3y^2 - 6x + 9y - 2 = 0$  and having  $\frac{2}{3}$  of its area.
8. Find the equation of the diameter of the circle  $x^2 + y^2 + 6x - 2y = 6$  which when produced passes through the point  $(1, -2)$ .
9. Find the equation of the diameter of the circle  $2x^2 + 2y^2 + 3x - 5y - 1 = 0$ , which when produced passes through the point  $(-1, 2)$ .
10. Show that the line  $4x - y = 17$  passes through the centre of the circle  $x^2 + y^2 - 8x + 2y = 0$ .
11. Find the length of the chord of the circle  $x^2 + y^2 - 6x + 15y - 16 = 0$  intercepted by the  $x$ -axis.
12. Find the length of the chord of the circle  $x^2 + y^2 + 3x - y - 6 = 0$  intercepted by the  $y$ -axis.
13. Find the length of the chord of the circle  $x^2 + y^2 - 6x - 4y - 12 = 0$  on the coordinate axes.

**Five marks questions:**

1. Find the equation of the circle.
  - a) passing through the origin, having its centre on the  $x$  - axis and radius 2 units.
  - b) passing through (2,3) having its centre on the  $x$  - axis and radius 5 units.
  - c) passing through the points (5,1), (3,4) and has its centre on the  $x$  - axis.
  - d) passing through the points (1,2) and (2,1) and has its centre on the  $y$  - axis.
  - e) passing through the points (0,5) and (6,1) and has its centre on the line  $12x + 5y = 25$
  - f) passing through the points (1,-4) and (5,2) and has its centre on the line  $x - 2y + 9 = 0$ .
  - g) passing through the points (0,-3) and (0,5) and whose centre lies on  $x - 2y + 5 = 0$
  - h) passing through (1,1) and (2,2) and having radius 1.
2. Find the equation of the circle passing through the points.
 

a) (0,2), (3,0), (3,2)	b) (1,1), (-2,2), (-6,0)
c) (1,1), (5,-5), (6,-4)	d) (1,0), (3,0), (0,2)
e) (5,7), (6,6), (2,-2)	f) (p,q), (p,0), (0,q)
g) (0,1), (2,3), (-2,5)	h) (0,0), (a,0), (0,b)
3. Find the equations of the circles whose radius is 5 and which passes through the points on  $x$  - axis at distances 3 from the origin.
4. A circle has radius 3 units and its centre lies on the line  $y = x - 1$ . Find the equation of the circle if it passes through (7,3).
5. Find the equation of the circle which cuts intercepts of the lengths ' $a$ ' and ' $b$ ' on axes and passing through the origin.
6. Find the equation of the circle passing through the origin and making positive  $x$  - intercept ' $a$ ' units and negative  $y$  - intercepts ' $b$ ' units.
7. Find the length of the chord intercepted by the
  - a) circle  $x^2 + y^2 - 8x - 6y = 0$  and the line  $x - 7y - 8 = 0$
  - b) circle  $x^2 + y^2 = 9$  and the line  $x + 2y = 3$ .
  - c) circle  $x^2 + y^2 - 6x - 2y + 5 = 0$  and the line  $x - y + 1 = 0$
8. Find the points of intersection of the circle
  - a)  $x^2 + y^2 = 9$  and the line  $x + 2y = 3$ .
  - b)  $x^2 + y^2 - 6x - 2y + 5 = 0$  and the line  $x - y + 1 = 0$ .
  - c)  $x^2 + y^2 + 4x + 6y - 12 = 0$  and the line  $x + 4y - 6 = 0$

Also find the length of the chord.

**Six marks questions:**

3. Show that the following points are concyclic

- a)  $(0,0), (1,1), (5,-5), (6,-4)$       b)  $(2,-4), (3,-1), (3,-3), (0,0)$   
c)  $(1,0), (2,-7), (8,1), (9,-6)$

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**ANSWERS 15.2**

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- I. 1. a)  $\left(2, \frac{1}{2}\right), \frac{\sqrt{37}}{2}$       b)  $(1,2)\sqrt{\frac{17}{3}}$   
c)  $(3,-1)\sqrt{5}$       d)  $(\cos \alpha, \sin \alpha), \sqrt{2}$   
2.  $k = 11$   
3. a)  $(-5,0)$       b)  $(1,7)$       c)  $(-1,-6)$   
4.  $a = 1, b = 3$       5.  $k = 20$       6.  $a = \pm 4$       7.  $a = -2, b = 6$   
8.  $x^2 + y^2 - 8x + 4y + 19 = 0$   
II. 1.  $x^2 + y^2 + 6x + 2y + 9 = 0$       2.  $x^2 + y^2 - 6x + 4y - 37 = 0$   
3.  $x^2 + y^2 + 4x - 6y - 37 = 0$       4.  $x^2 + y^2 - 8x + 4y - 5 = 0$   
5.  $x^2 + y^2 + 2x - 8y - 17 = 0$       6.  $x^2 + y^2 - 2x + 2y - 4 = 0$   
7.  $36x^2 + 36y^2 - 72x + 108y + 23 = 0$       8.  $3x + 4y + 5 = 0$   
9.  $3x + y + 1 = 0$       11. 10 units  
12. 5 units      13.  $2\sqrt{21}, 8$  units  
III. 1. a)  $x^2 + y^2 \pm 4x = 0$       b)  $\begin{matrix} x^2 + y^2 - 12x + 11 = 0 \\ x^2 + y^2 + 4x - 21 = 0 \end{matrix}$   
c)  $2x^2 + 2y^2 - x - 47 = 0$       d)  $x^2 + y^2 - 5 = 0$   
e)  $3x^2 + 3y^2 - 10x - 6y - 45 = 0$       f)  $x^2 + y^2 + 6x - 6y - 47 = 0$   
g)  $x^2 + y^2 + 6x - 2y - 15 = 0$       h)  $\begin{matrix} x^2 + y^2 - 2x - 4y + 4 = 0 \\ x^2 + y^2 - 4x - 2y + 4 = 0 \end{matrix}$   
2. a)  $x^2 + y^2 - 3x - 2y = 0$       b)  $x^2 + y^2 + 4x + 6y - 12 = 0$   
c)  $x^2 + y^2 - 6x + 4y = 0$       d)  $2x^2 + 2y^2 - 8x - 7y + 6 = 0$

- e)  $x^2 + y^2 - 4x - 6y - 12 = 0$       f)  $x^2 + y^2 - px - qy = 0$   
 g)  $3x^2 + 3y^2 + 2x - 20y + 17 = 0$       h)  $x^2 + y^2 - ax - by = 0$   
 3.  $x^2 + y^2 - 8y - 9 = 0, x^2 + y^2 + 8y - 9 = 0$   
 4.  $x^2 + y^2 - 14x - 12y + 76 = 0, x^2 + y^2 - 8x - 6y + 16 = 0$   
 5.  $x^2 + y^2 \pm ax \pm by = 0$   
 6.  $x^2 + y^2 - ax + by = 0$   
 7. a)  $5\sqrt{2}$       b)  $\frac{12\sqrt{5}}{5}$       c)  $\sqrt{2}$   
 8. a)  $(3,0), \left(\frac{-9}{5}, \frac{12}{5}\right), \frac{12}{\sqrt{5}}$       b)  $(2,3), (1,2), \frac{8}{\sqrt{5}}$   
 c)  $(-2,2), \left(\frac{6}{17}, \frac{24}{17}\right), \frac{10}{\sqrt{17}}$

### 15.5 Point of Intersection of a line and a circle

With a line and a circle, we have the following three cases, namely

**Case (i) :** The line may cut the circle at two points A and B as shown in fig (i)

In this case the length of the perpendicular from the centre is less than the radius of the circle. AB is length of the chord of the circle.

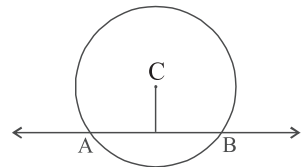


Fig. (i)

**Case (ii) :** The line may be tangent to the circle i.e. the line and the circle have only one point in common A as shown in fig(ii)

In this case the length of the perpendicular from the centre on to the line is equal to the radius of the circle.

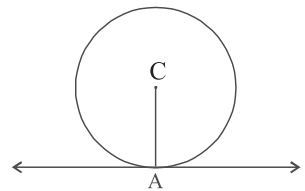


Fig. (ii)

**Case (iii):** The line and the circle have no points in common as shown in fig(iii)

In this case the length of the perpendicular from the centre on the line is greater than the radius of the circle.

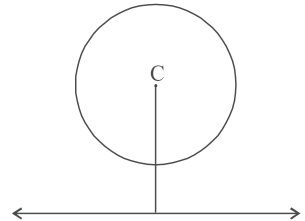


Fig. (iii)

**Important Observation :** A line will be a tangent to the given circle if and only if the length of the perpendicular from the centre on to the line is equal to radius of the circle.

Let  $y = mx + c$  be the equation of a line and  $x^2 + y^2 + 2gx + 2fy + k = 0$  be the equation of a circle. Substituting the value of 'y' in the equation of the circle, we get

$$x^2 + (mx + c)^2 + 2gx + 2f(mx + c) + k = 0$$

$$\text{i.e. } x^2 + m^2x^2 + c^2 + 2mxc + 2gx + 2fmx + 2fc + k = 0$$

$$\Rightarrow (1 + m)^2 x^2 + (2mc + 2g + 2fm)x + (c^2 + 2fc + k) = 0 \dots\dots(1)$$

which is a quadratic equation in  $x$ .

- 1) The equation (1) has two real and distinct roots if the discriminant of the equation is greater than zero and in this case the line cuts the circle in two distinct points say A and B as shown in fig(i).

The length AB is called the length of the chord.

- 2) The roots of the equation (1) are real and coincident if the discriminant of the equation is equal to zero and in this case the line touches the circle at only one point as shown in fig(ii). Thus the line will be a tangent to the circle.
- 3) The roots of the equation (i) are imaginary if the discriminant of the equation is less than zero and in this case, there will be no common points between the line and the circles, as shown in fig (iii).

**Note :**

- 1) Length of the chord of the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  intercepted by  $x$  - axis is  $2\sqrt{g^2 - c}$
- 2) Length of the chord of the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  intercepted by  $y$  - axis is  $2\sqrt{f^2 - c}$
- 3) Length of the chord intercepted by the line with the given circle is  $2\sqrt{r^2 - p^2}$
- 4) Condition for the circle to touch  $x$  - axis is  $g^2 = c$
- 5) Condition for the circle to touch  $y$  - axis is  $f^2 = c$
- 6) Condition for the circle to touch both the coordinate axes is  $g^2 = f^2 = c$

**Example 19:**

**Show that the line  $3x - 4y + 6 = 0$  touches the circle  $x^2 + y^2 - 6x + 10y - 15 = 0$ .**

**Solution:**

We have  $x^2 + y^2 - 6x + 10y - 15 = 0$ .

$$2g = -6 \Rightarrow g = -3, \quad 2f = 10 \Rightarrow f = 5$$

$$\therefore \text{centre} = C = (-g, -f) = (3, -5)$$

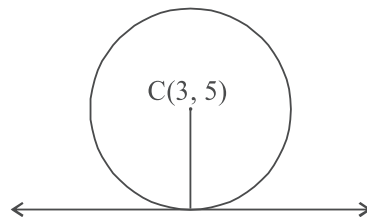
$$\text{radius} = r = \sqrt{g^2 + f^2 - c} = \sqrt{9 + 25 + 15} = \sqrt{49} = 7$$

we know that a line touches the circle if and only if the length of the perpendicular from the centre on to the line is equal to radius of the circle.

$$\text{i.e., } \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right| = r \quad \text{i.e., } \frac{3(3) - 4(-5) + 6}{\sqrt{9 + 16}} = 7$$

$$\Rightarrow \frac{35}{5} = 7$$

$\therefore$  The line  $3x - 4y + 6 = 0$  touches the circle.



**Example 20:**

**Show that the line  $2x + y + 2 = 0$  is a tangent to the circle  $x^2 + y^2 + 6x + 2y + 5 = 0$ .**

**Also find the point of contact.**

**Solution :**

We have  $x^2 + y^2 + 6x + 2y + 5 = 0$

$$2g = 6 \Rightarrow g = 3, \quad 2f = 2 \Rightarrow f = 1 \quad c = 5$$

$$\text{centre} = C = (-g, -f) = (-3, -1);$$

$$\text{radius} = r = \sqrt{g^2 + f^2 - c} = \sqrt{9 + 1 - 5} = \sqrt{5}$$

Radius = length of the perpendicular from centre to the line

$$\text{i.e., } r = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \rightarrow \sqrt{5} = \left| \frac{2(-3) + 1(-1) + 2}{\sqrt{4 + 1}} \right|$$

Thus we find  $r = \sqrt{5}$  and length of perpendicular from centre to the line  $= \sqrt{5}$

Therefore the line  $2x + y + 2 = 0$  is a tangent to the given circle.

**To find the contact :** We shall find the equation of the line through the centre  $(-3, -1)$  and perpendicular  $2x + y + 2 = 0$ . We know any line perpendicular to  $2x + y + 2 = 0$  is of the form  $x - 2y + k = 0$ . This line passes through  $(-3, -1)$ . Therefore put  $x = -3, y = -1$  we get  $-3 + 2 + k = 0 \rightarrow k = 1$

So equation becomes  $x - 2y + 1 = 0$ . Now solving  $2x + y + 2 = 0$  and  $x - 2y + 1 = 0$  we get  $x = -1, y = 0$

Hence the point of contact is  $(-1, 0)$

**Example 21:**

Find  $k$ , such that the straight line  $x - 2y + r = 0$  may be tangent to the circle  $x^2 + y^2 + 3x - 2y + 2 = 0$

**Solution:**

We have  $x^2 + y^2 + 3x - 2y + 2 = 0$

$$2g = 3 \rightarrow g = \frac{3}{2}; 2f = -2 \rightarrow f = -1, c = 2$$

$$\text{centre} = C = (-g, -f) = \left(-\frac{3}{2}, 1\right);$$

$$\text{radius} = r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{9}{4} + 1 - 2} = \sqrt{\frac{9}{4} - 1} = \frac{\sqrt{5}}{2}$$

radius = length of the perpendicular from the centre to the line  $x - 2y + r = 0$

$$\text{ie } \frac{\sqrt{5}}{2} = \left| \frac{1\left(-\frac{3}{2}\right) - 2(1) + r}{\sqrt{1+4}} \right|$$

$$\frac{\sqrt{5}}{2} = \left| \frac{\frac{-7}{2} + \lambda}{\sqrt{5}} \right| \rightarrow \pm \frac{5}{2} = \frac{-7}{2} + \lambda$$

$$\rightarrow \lambda = \frac{5}{2} + \frac{7}{2} = \frac{12}{2} = 6$$

$$\text{Also } \lambda = \frac{-5}{2} + \frac{7}{2} = \frac{2}{2} = 1$$

Thus  $\lambda = 1$  or  $6$

**Example 22:**

Find the equation of the tangents to the circles  $x^2 + y^2 + 2x + 4y - 4 = 0$  which are parallel to the line  $5x + 12y + 6 = 0$

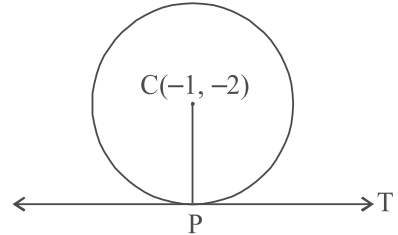
**Solution :**

Given circle is

$$x^2 + y^2 + 2x + 4y - 4 = 0$$

$$2g = 2, g = 1, 2f = 4, f = 2, c = -4$$

$$\text{centre} = (-g, -f) = (-1, -2)$$



$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 4 + 4} = 3$$

Equation of the tangent parallel to the given line can be taken as  $5x + 12y + k = 0$  .....(1)

Length of perpendicular from  $(-1, -2)$  to the line (1)

$$CP = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right| = \left| \frac{5(-1) + 12(-2) + k}{\sqrt{25 + 144}} \right| = \left| \frac{-5 - 24 + k}{13} \right|$$

$$\text{i.e., } 3 = \left| \frac{-29 + k}{13} \right| \Rightarrow \frac{k - 29}{13} = \pm 3$$

$$\text{i.e., } k - 29 = \pm 39 \Rightarrow k = 68, k = -10$$

$\therefore$  Equations of the tangents parallel to the given line are  $5x + 12y + 68 = 0$  and  $5x + 12y - 10 = 0$

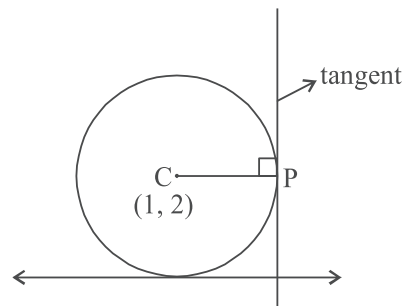
**Example 23:**

Find the equation of the tangents to the circle  $x^2 + y^2 - 2x - 4y + 1 = 0$  which are perpendicular to the line  $3x - 4y + 7 = 0$

**Solution:**

The equation of a tangent  $\perp$  to the given line can be taken as  $4x + 3y + k = 0$  .....(1)

Given circle  $x^2 + y^2 - 2x - 4y + 1 = 0$ ,  $g = -1, f = -2$ ,  $c = 1$  centre  $(1, 2)$ , radius = 2



Length of  $\perp$  from the centre (1,2) to the line (1) is

$$CP = \left| \frac{4(1) + 3(2) + k}{\sqrt{16+9}} \right| = \left| \frac{10+k}{5} \right|$$

$$\text{i.e., } 2 = \left| \frac{10+k}{5} \right| \Rightarrow \frac{10+k}{5} = \pm 2$$

$$10 \pm k = +10$$

$$\therefore k = 0, -20$$

$\therefore$  The equation of the tangents  $\perp$  to the given line are  $4x + 3y = 0$  and  $4x + 3y - 20 = 0$

**Example 24:**

**Find the value of  $k$  for which the line  $x + ky - 5 = 0$  may touch the circle  $x^2 + y^2 - 2x - 6y - 6 = 0$**

**Solution:**

$$g = -1, f = -3, c = -6; \text{ centre} = (1, 3)$$

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 9 + 6} = 4$$

The line  $x + ky - 5 = 0$  touches the circle if the length of the perpendicular from the centre (1,3) to the line is equal to the radius of the circle.

$$\text{i.e., } \left| \frac{1 + 3k - 5}{\sqrt{1 + k^2}} \right| = 4 \quad \text{i.e., } |3k - 4| = 4\sqrt{1 + k^2}$$

$$\text{or } 9k^2 + 16 - 24k = 16 + 16k^2$$

$$\text{or } 7k^2 + 24k = 0 \quad \text{i.e., } k(7k + 24) = 0$$

$$\Rightarrow k = 0, k = -\frac{24}{7}$$

**Example 25:**

**Show that the circle  $x^2 + y^2 + 4x - 3y + 4 = 0$  touches  $x$  - axis**

**Solution:**

$$2g = 4 \Rightarrow g = 2, c = 4$$

Condition for the circle to touch

$$x - \text{axis is } g^2 = c$$

i.e.,  $4 = 4 \therefore$  given circle touches  $x$  - axis.

**Example 26:**

Show that the circle  $x^2 + y^2 - 3x + 8y + 16 = 0$  touches  $y$  - axis.

**Solution:**

$2f = 8 \rightarrow f = 4, c = 16$ , Condition for the circle to touch  $y$  - axis is  $f^2 = c$

Given circle touches  $y$  - axis, since  $4^2 = 16$

**Example 27:**

Show that the circle  $x^2 + y^2 - 2x + 2y + 1 = 0$  touches both the co-ordinate axes.

**Solution :**

Condition for the circle to touch both the axes is  $g^2 = f^2 = c$ . Here  $g = 1, f = -1$  and  $c = 1$ .

Since  $(1)^2 = (-1)^2 = 1$ , we find the given circle touches both the axes.

**EXERCISE 15.3**

**One mark questions:**

**I. Show that the circle**

1.  $x^2 + y^2 - 8x + 2y + 16 = 0$  touches  $x$  - axis
2.  $x^2 + y^2 - 4x + 4y + 4 = 0$  touches  $x$  - axis
3.  $x^2 + y^2 + 8x + 6y + 9 = 0$  touches  $y$  - axis
4.  $x^2 + y^2 - x + 4y + 4 = 0$  touches  $y$  - axis

**II. 5. Show that the circle  $x^2 + y^2 + 4x - 3y + 4 = 0$  touches  $x$  - axis**

6. Show that the circle  $x^2 + y^2 - 3x + 8y + 16 = 0$  touches  $y$  - axis

**Two marks questions:**

**III. 7. Find the length of the chord of the circle  $x^2 + y^2 - 6x + 4y + 5 = 0$  intercepted by  $x$  - axis**

8. Find the length of the chord of the circle  $x^2 + y^2 - 6x + 15y - 16 = 0$  intercepted by the  $y$  - axis.
9. Find the length of the chord of the circle  $x^2 + y^2 + 3x - 2 = 0$  intercepted by  $y$  - axis.
10. Find the length of the chord of the circle  $x^2 + y^2 + 3x - y - 6 = 0$  intercepted by the  $y$  - axis.
11. Find the length of the chord of the circle  $x^2 + y^2 + 4x + 6y - 12 = 0$  and  $x + 4y - 6 = 0$

12. Find the length of the chord intercepted by the circle  $x^2 + y^2 - 8x - 6y - 6 = 0$  and the line  $x - 7y - 8 = 0$
13. Find the length of the chord intercepted by the circle  $x^2 + y^2 - 6x - 2y + 5 = 0$  and the line  $x - y + 1 = 0$
14. Find the equation of the chord of the circle  $x^2 + y^2 - 2x + 4y - 17 = 0$  bisected at  $(-1, 2)$
15. Find the equation of the chord of the circle  $x^2 + y^2 - 4x - 2y - 20 = 0$  bisected at  $(2, 3)$
16. Prove that the length of the intercept made by the circle  $x^2 + y^2 + 10x - 6y + 9 = 0$  on the  $x$  - axis is 8 units.
17. Prove that the length of the chord  $x + 2y = 5$  of the circle  $x^2 + y^2 = 9$  is 4 units.

**Five marks questions:**

18. Show that the line  $3x + 4y + 7 = 0$  touches the circle  $x^2 + y^2 - 4x - 6y - 12 = 0$  and find the point of contact.
19. Find  $k$  if the line  $3x + y + k = 0$  touches the circle  $x^2 + y^2 - 2x - 4y - 5 = 0$
20. Find  $k$  if the line  $2x + y + k = 0$  touches the circle  $x^2 + y^2 + 6x + 2y + 5 = 0$
21. Find  $k$  if the line  $4x - y + k = 0$  touches the circle  $x^2 + y^2 + 4x - 8y + 3 = 0$
22. Find  $k$  such that the line  $x - 2y + k = 0$  touches the circle  $x^2 + y^2 + 3x - 2y + 2 = 0$

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**ANSWERS 15.3**

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- |                            |                 |                  |                      |
|----------------------------|-----------------|------------------|----------------------|
| III. 7) 4                  | 8) 10           | 9) $2\sqrt{2}$   | 10) 5                |
| 11) $\frac{10}{\sqrt{17}}$ | 12) $5\sqrt{2}$ | 13) $\sqrt{2}$   | 14) $x - 2y + 5 = 0$ |
| 15) $y = 3$                | 18) $(-1, -1)$  | 19) $k = 5, -15$ | 20) $k = 2$          |
| 21) $k = -5$               | 22) $k = 6, 1$  |                  |                      |

\* \* \* \* \*

**16.1 Introduction to Conic Section - Parabola:**

Definition : A conic section is the locus of the point which moves such that the ratio of its distance from the fixed point  $S$  to its distance from the fixed line  $l$ , is always a fixed positive constant.

The fixed point  $S$  is called the focus of the conic section.

The fixed line  $l$  is called the directrix of the conic section.

If  $P$  is any point on the locus (conic section) then by

definition we have  $\frac{PS}{PM} = a$  fixed constant

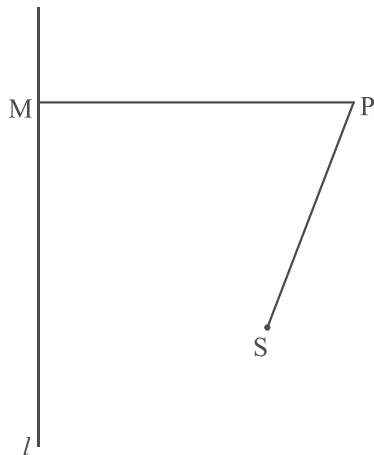
This constant ratio  $\frac{PS}{PM}$  is called the eccentricity of the conic section and it is denoted by  $e$ .

If the eccentricity  $e = 1$ , the conic section is called a parabola.

If the eccentricity  $e < 1$ , the conic section is called an ellipse.

If the eccentricity  $e > 1$ , the conic section is called a hyperbola.

Now we shall see the conic section parabola and study few properties and results associated to them Parabola.

**16.2 Definition of Parabola and other forms of Parabola:**

Parabola is a conic section whose eccentricity is 1. That is, Parabola is the locus of the point which moves such that its distance from the **fixed point (called the focus)** is equal to its distance from the **fixed line (called the directrix)**

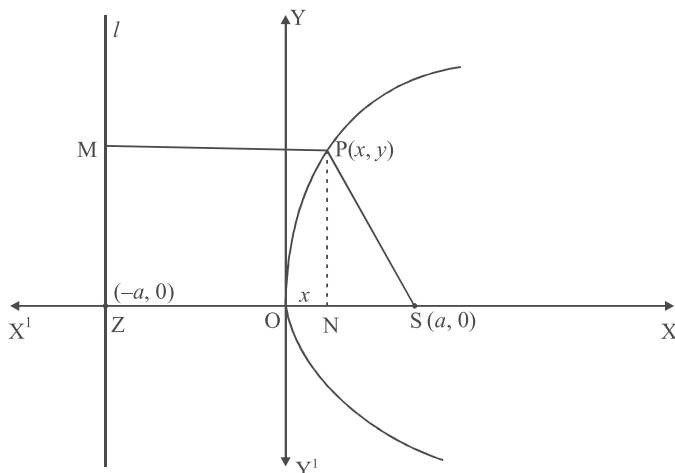
We shall derive the equation of the parabola in its standard form.

**Theorem: The equation of a Parabola with proper choice of co-ordinates axes is  $y^2 = 4ax$**

**Proof :** Let  $S$  be the focus and the line  $l$  be the directrix. Draw  $SZ$  through  $S$  and perpendicular to the directrix. Let  $O$  be the midpoint of the line segment  $SZ$ .

Let  $SZ = 2a$  ( $a > 0$ )

$$\Rightarrow OZ = OS = a$$



We shall choose O as the origin and the line ZOSX as  $x$  axis. The line YOY<sup>1</sup> through O, perpendicular to the  $x$  axis will be  $y$  axis.

With this choice of co-ordinate axes we have  $S(a, 0)$  and  $Z(-a, 0)$

The equation of the directrix is  $l$  is  $x = -a$

Let  $P(x, y)$  be any point on the Parabola. Then by the definition of the parabola we have  
Distance of P from S = Distance of P from the line ' $l$ '

$$|SP| = |PM|$$

$$SP^2 = ZN^2 \quad \text{since } PM = ZN$$

$$SP^2 = (OZ + ON)^2$$

$$(x - a)^2 + y^2 = (a + x)^2 \quad (\because |OZ| = a, |ON| = x)$$

$$x^2 - 2ax + a^2 + y^2 = a^2 + 2ax + x^2$$

$$\boxed{y^2 = 4ax} \text{ which is the equation of the parabola.}$$

**Note :** The equation  $y^2 = 4ax$  is also called standard form of the equation of the parabola.

### Shape of the Parabola $y^2 = 4ax$

We shall note few observations from the equation  $y^2 = 4ax$ , which will help us to trace the curve parabola.

1. If  $y$  is replaced by  $-y$  in the equation remains same, i.e.,  $(-y)^2 = 4ax \rightarrow y^2 = 4ax$ . This shows that if  $(x, y)$  is any point on the curve  $y^2 = 4ax$ , then  $(x, -y)$  is also a point on the

curve. Thus, the curve is symmetric about the  $x$  - axis, ie the shape of the curve above the  $x$  axis is the mirror image (about the  $x$  axis) of the shape of the curve below the  $x$  - axis.

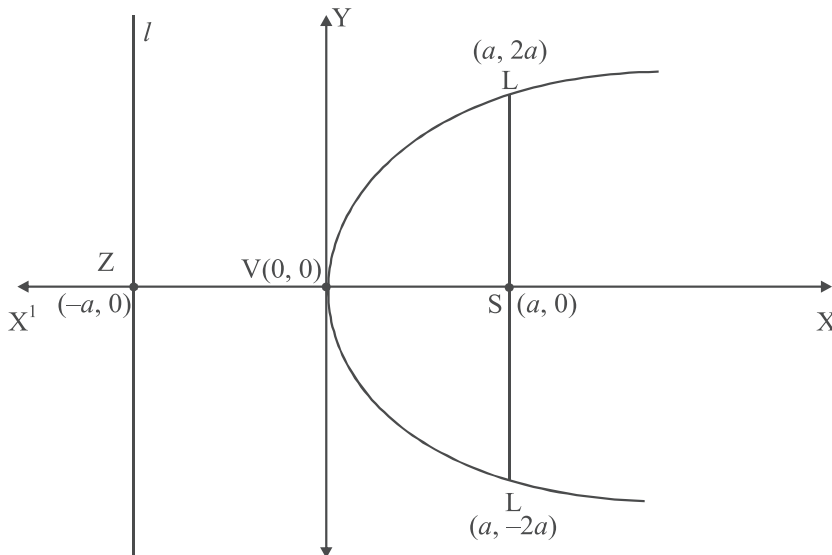
2. If  $x < 0$ , then  $y^2$  will be a negative quantity (note that  $a > 0$ ) and therefore  $y^2 = 4ax$  will have no real solution for  $y$ . This shows that no part of the curve lies to the left side of the  $x$  - axis.
3. If  $y = 0$ , the only value of  $x$  we get is zero. Thus the curve cuts the  $x$  - axis at the origin  $(0, 0)$
4. If  $x = 0$ , we get  $y^2 = 0$ , which gives  $y = 0$ . Thus the curve cuts the  $y$  - axis at the origin and further the  $y$  - axis meets the curve only at the origin. That is,  $y$  - axis is a tangent to the curve at the origin.
5. For any point  $P(x, y)$  on the parabola we have

$$y^2 = 4ax \Rightarrow y = \pm 2\sqrt{ax}$$

$$y = 2\sqrt{ax} \quad \text{and} \quad y = -2\sqrt{ax}$$

This shows that, as  $x$  increases from 0 to  $\infty$ ,  $y$  also increases from 0 to  $\infty$  or  $y$  decreases from 0 to  $-\infty$ . Thus, the two branches of the parabola, laying on opposite sides of the  $x$  - axis, will extend to infinity towards the positive directions of the  $x$  - axis.

From the above discussion and by plotting few points, whose coordinates satisfy  $y^2 = 4ax$ , it is found the shape of the parabola is as shown in the following figure.



The origin 0 is called the **Vertex** of the parabola,  $y^2 = 4ax$ , it is also denoted by  $V(0, 0)$ .

The line  $ZSX$  is called the **axis of the parabola** and its equation is  $y = 0$

The focus  $S = (a, 0)$  and the equation of the directrix is  $x = -a$  i.e.,  $x + a = 0$

The  $y$  - axis is called the **tangent at the vertex** - its equation is  $x = 0$ .

**Note:** The distance between the vertex and the focus is the distance between the vertex and the directrix is equal to  $a$  and the distance between the directrix and the focus is  $2a$ .

**Definition :** The chord passing through the focus and perpendicular to the axis of the parabola is called the latus rectum of the parabola.

In the figure  $LSL^1$  is the latus rectum.

The points  $L$  and  $L^1$  on the parabola are called end points of the latus rectum, the length  $LL^1$  is called the length of the latus rectum.

Clearly, the  $x$  - coordinates of  $L$  and  $L^1$  is  $a$  because  $OS = a$ . To find the corresponding  $y$  - coordinates, we shall put  $x = a$  in  $y^2 = 4ax$ , we get  $y = \pm 2a$ . Thus,  $y$  - coordinates of  $L$  and  $L^1$  respectively.

$$L = (a, 2a) \text{ and } L^1 = (a, -2a)$$

$$\therefore \text{ consider } |LL^1| = \sqrt{(a-a)^2 + (2a+2a)^2} = \sqrt{(4a)^2} = 4a$$

Thus, the length of the latus rectum,  $LL^1 = 4a$

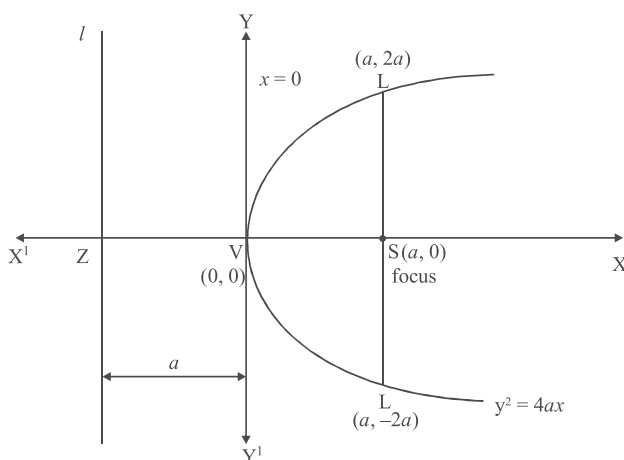
**Note:** Any chord passing through the focus is called a **focal chord**. Focal chord need not be perpendicular to the axis of the parabola.

### 16.3 Four Standard forms of Parabola and their Graphs

There are four standard forms of parabola viz  $y^2 = 4ax$ ,  $y^2 = -4ax$ ,  $x^2 = 4ay$  and  $x^2 = -4ay$ , depending upon the choice of the axes along either of coordinate axes and vertex at the origin.

#### I. Right handed parabola

ie., parabola opening to right is of the form  $y^2 = 4ax$ ,  $a > 0$



### Characteristics of Right Handed Parabola

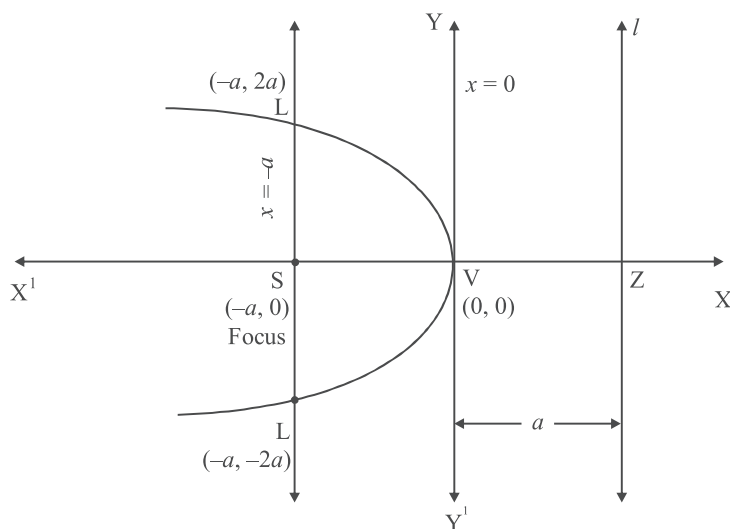
- 1) Vertex is V (0,0)
- 2) Focus is S (a,0)
- 3) Equation of directrix  $l$  is  $x = -a$  or  $x + a = 0$
- 4) Equation of the axis is  $y = 0$  ie  $x$  - axis
- 5) Equation of the tangent at the vertex is  $x = 0$  ie  $y$  - axis
- 6) Length of latus rectum =  $LL^1 = 4a$
- 7) Equation of LR is  $x - a = 0$  or  $x = a$
- 8) Coordinates of ends of latus rectum are  $L(-a, 2a)$  and  $L^1(-a, -2a)$
- 9) Parametric coordinates is  $(at^2, 2at)$
- 10) It is symmetric about  $x$  - axis

### II. Left handed parabola

ie., parabola opening to left is of the form  $y^2 = -4ax$ ,  $a > 0$

### Characteristics of Left Handed Parabola

- 1) Vertex is V (0,0)
- 2) Focus is S (-a, 0)
- 3) Equation of directrix MZ is  $x = a$  or  $x - a = 0$



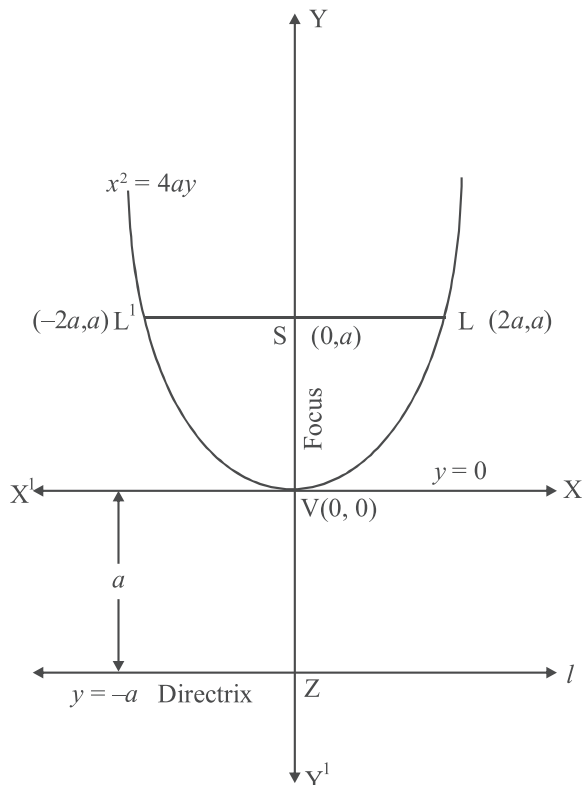
- 4) Equation of axis  $y = 0$  is  $x$  - axis
- 5) Equation of the tangent at the vertex is  $x = 0$  ie.,  $y$  - axis
- 6) Length of latus rectum  $LL^1 = 4a$
- 7) Equation of latus rectum =  $x = -a$  or  $x + a = 0$
- 8) Co-ordinates of ends of latus rectum is  $L(-a, 2a)$  and  $L^1(-a, -2a)$

9) Parametric coordinates is  $(-at^2, 2at)$

10) It is symmetric about  $x$  - axis

### III. Upward parabola

ie., parabola opening upwards is of the form  $x^2 = 4ay, (a > 0)$

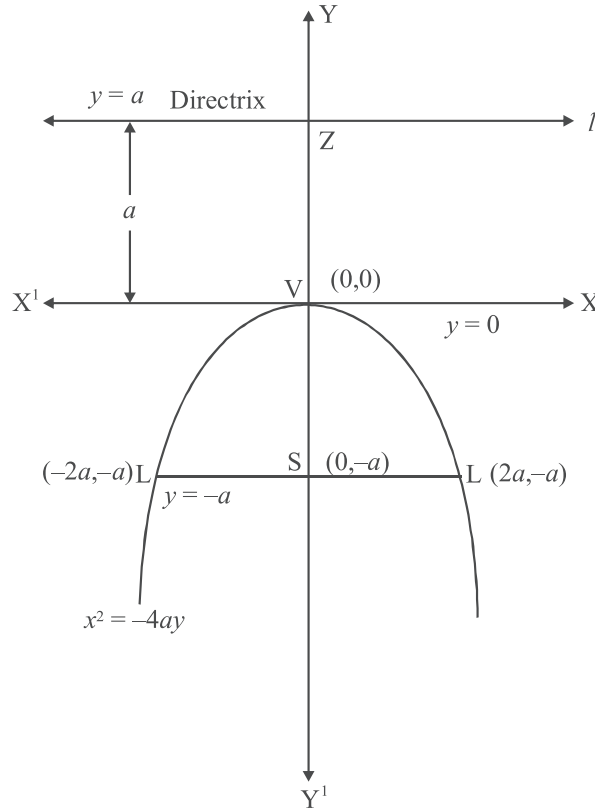


#### Characteristics of Upward Parabola

- 1) Vertex is A(0,0)
- 2) Focus is S (0,a)
- 3) Equation of directrix MZ is  $y = -a$  or  $y + a = 0$
- 4) Equation of axis is  $x = 0$  ie,  $y$  - axis
- 5) Equation of tangent at the vertex is  $y = 0$  ie,  $x$  - axis
- 6) Length of latus rectum  $LL' = 4a$
- 7) Equation of latus rectum  $y = a$  or  $y - a = 0$
- 8) Co-ordinate of the ends of latus rectum are  $L(2a, a)$  and  $L'(-2a, a)$
- 9) Parametric coordinates is  $(2at, at^2)$
- 10) It is symmetric about the  $y$  - axis

#### IV Downward parabola

ie, parabola opening downwards is of the form  $x^2 = -4ay$ , ( $a > 0$ )



#### Characteristics of Downward Parabola

- 1) Vertex is  $V(0, 0)$
- 2) Focus is  $S(0, -a)$
- 3) Equation of directrix  $MZ$  is  $y = a$  or  $y - a = 0$
- 4) Equation of axis is  $x = 0$  ie,  $y$  - axis
- 5) Equation of tangent at the vertex is  $y = 0$  ie,  $x$  - axis
- 6) Length of latus rectum  $LL' = 4a$
- 7) Equation of latus rectum  $y = -a$  or  $y + a = 0$
- 8) Coordinate of ends of latus rectum are  $L(2a, -a)$  and  $L'(-2a, -a)$
- 9) Parametric coordinates are  $(2at, -at^2)$
- 10) It is symmetric about the  $y$  - axis

Equation	$y^2 = +4ax$	$y^2 = -4ax$	$x^2 = +4ay$	$x^2 = -4ay$
Directrix	$x = -a$	$x = +a$	$y = -a$	$y = +a$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Axis	$x$ - axis ( $y = 0$ )	$x$ - axis ( $y = 0$ )	$y$ - axis ( $x = 0$ )	$y$ - axis ( $x = 0$ )
Equation of tangent vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Equation and length of latus rectum	$x = a$ and $4a$	$x = -a$ and $4a$	$y = a$ and $4a$	$y = -a$ and $4a$
Coordinates of ends of LL	$(a, 2a)$ and $(a, -2a)$	$(-a, 2a)$ and $(-a, -2a)$	$(2a, a)$ and $(-2a, a)$	$(-2a, -a)$ and $(-2a, a)$

### WORKED EXAMPLES

**Type I :** Finding the focus, directrix, latus rectum, Axis etc. for a given Parabola in one of the standard forms.

**Find the characteristics of the following parabolas**

- |                 |                 |
|-----------------|-----------------|
| 1) $y^2 = 8x$   | 2) $x^2 = 6y$   |
| 3) $y^2 = -12x$ | 4) $x^2 = -16y$ |

**Solution :**

- 1)  $y^2 = 8x$  It is in the form of  $y^2 = 4ax$ , where  $4a = 8 \rightarrow a = 2$

$\therefore$  the characteristics are given by

Directrix	$x = -a \Rightarrow x = -2$
Focus	$(a, 0) \Rightarrow (2, 0)$
Vertex	$(0, 0) \Rightarrow (0, 0)$
Axis	$x$ - axis ( $y = 0$ ) $\Rightarrow y = 0$
Equation of tangent at vertex	$y$ - axis i.e., $x = 0$
Equation of L.R.	$x = a \Rightarrow x = 2$
Length of L.R.	$4a \Rightarrow 4 \times 2 = 8$
Co-ordinates of ends of L.R.	$(a, 2a) = (2, 4)$ $(a, -2a) = (2, -4)$

2)  $x^2 = 6y$

It is in the form of  $x^2 = 4ay$  where  $4a = 6 \Rightarrow a = \frac{3}{2}$  the characteristics are

Directrix	$y = -a \Rightarrow y = -\frac{3}{2}$
Focus	$(0, a) \Rightarrow \left(0, \frac{3}{2}\right)$
Vertex	$(0, 0) \Rightarrow (0, 0)$
Axis	$y - \text{axis} \Rightarrow x = 0$
Equation of tangent at vertex	$x - \text{axis} \Rightarrow y = 0$
Equation of L.R.	$y = a \Rightarrow y = \frac{3}{2}$
Length of L.R.	$4a \Rightarrow 4 \times \frac{3}{2} = 6$
Co-ordinates of ends of L.R.	$(2a, a) = \left(2 \times \frac{3}{2}, \frac{3}{2}\right) = \left(3, \frac{3}{2}\right)$ and $(-2a, a) = \left(-2 \times \frac{3}{2}, \frac{3}{2}\right) = \left(-3, \frac{3}{2}\right)$

3) It is the form of  $y^2 = -4ax$  where  $4a = 12 \Rightarrow a = 3$  the characteristics are

Directrix	$x = a \Rightarrow x = 3$
Focus	$(-a, 0) \Rightarrow (-3, 0)$
Vertex	$(0, 0)$
Axis	$x - \text{axis} \Rightarrow y = 0$
Equation of tangent at vertex	$y - \text{axis} \Rightarrow x = 0$
Equation of L.R.	$x = -a \Rightarrow x = -3$ or $x + 3 = 0$
Length of L.R.	$4a \Rightarrow 4 \times 3 = 12$
Co-ordinates of ends of L.R.	$(-a, 2a) = (-3, 6)$ and $(-a, -2a) = (-3, -6)$

4)  $x^2 = -16y$

It is in the form of  $x^2 = -4ay$  where  $4a = 16 \Rightarrow a = 4$ .

Characteristics are given by

Directrix	$y = a \Rightarrow y = 4$
Focus	$(0, -a) \Rightarrow (0, -4)$
Vertex	$(0, 0)$
Axis	$y - axis \Rightarrow x = 0$
Equation of tangent at vertex	$x - axis \Rightarrow y = 0$
Equation of L.R.	$y = -a \Rightarrow y = -4$
Length of L.R.	$4a \Rightarrow 4 \times 4 = 16$
Co-ordinates of ends of L.R.	$(2a, -a) = (8, -4)$ and $(-2a, -a) = (-8, -4)$

**Type II :** Find the standard equation of parabola when vertex and focus are given.

**Find the equation of parabola given that**

- |                                       |                                       |
|---------------------------------------|---------------------------------------|
| a) Vertex is (0, 0) and focus (3, 0)  | b) Vertex is (0, 0) and focus (4, 0)  |
| c) Vertex is (0, 0) and focus (-6, 0) | d) Vertex is (0, 0) and focus (-4, 0) |
| e) Vertex is (0, 0) and focus (0, 3)  | f) Vertex is (0, 0) and focus (0, 4)  |
| g) Vertex is (0, 0) and focus (0, -8) | h) Vertex is (0, 0) and focus (0, 16) |

**Solution :**

- a) Given vertex A = (0, 0) and focus S = (3, 0) clearly focus is to the right side of vertex.

$\therefore$  required parabola is in the form  $y^2 = 4ax \Rightarrow a = 3$

Thus equation given by  $y^2 = 4 \times 3x \Rightarrow y^2 = 12x$

- b) Given vertex is (0, 0) and focus (4, 0). Clearly focus is to the right side of vertex

$\therefore$  required parabola is  $y^2 = 4ax \Rightarrow a = 4$

Thus equation is given by  $y^2 = 4 \times 4x \Rightarrow y^2 = 16x$

- c) Given vertex is (0, 0) and focus (-6, 0) clearly focus is left side of vertex

$\therefore$  required parabola is in the focus

$$y^2 = -4ax \Rightarrow -a = -6 \Rightarrow a = 6$$

Thus equation is given by  $y^2 = -4(6)x \Rightarrow y^2 = -24x$

- d) Vertex (0,0) and focus (-4, 0) clearly focus is to the left side of the vertex

∴ required parabola is in the form  $y^2 = -4ax$

$$\Rightarrow -a = -4 \Rightarrow a = 4$$

Thus equation is given by  $y^2 = -4ax \Rightarrow y^2 = -4 \times 4 \times x \Rightarrow y^2 = -16x$

- e) Vertex (0,0) and focus (0,3) clearly focus is to the left side of the vertex

∴ required parabola is in the form of  $x^2 = 4ay \Rightarrow a = 3$

Thus equation is given by  $x^2 = 4ay \Rightarrow x^2 = 4 \times 3y \Rightarrow x^2 = 12y$

- f) Vertex (0,0) and focus (0,4) clearly focus is above the vertex

∴ required parabola is of the focus  $x^2 = 4ay \Rightarrow a = 4$

Thus equation is given by  $x^2 = 4ay \Rightarrow x^2 = 4 \times 4 \times y \Rightarrow x^2 = 16y$

- g) Vertex (0,0) and focus (0,-8) clearly focus is below the vertex.

∴ required parabola is in the form

$$x^2 = -4ay \Rightarrow -a = -8 \Rightarrow a = 8$$

Thus equation is given by

$$x^2 = -4ay \Rightarrow x^2 = -4 \times 8y \Rightarrow x^2 = -32y$$

### Type III : Finding the equation of parabola when vertex and directrix are given.

**Find the equation of parabola given that**

- Vertex is (0, 0) and directrix  $y = -4$
- Vertex is (0, 0) and directrix  $y = -2$
- Vertex is (0, 0) and directrix  $y = 6$
- Vertex is (0, 0) and directrix  $y = 8$
- Vertex is (0, 0) and directrix  $x = -3$
- Vertex is (0, 0) and directrix  $x = -5$
- Vertex is (0, 0) and directrix  $x = 4$
- Vertex is (0, 0) and directrix  $x = 7$

### Solution :

- a) Let the vertex of the parabola be A (0,0) and directrix is  $y = -4$

Thus, the directrix is a line parallel to the  $x$  - axis at distance of 4 units below the  $x$  - axis.

So the focus of the parabola lies on the  $y$ -axis consequently the focus is S(0,4). Hence the equation of the parabola is  $x^2 = 4ay$  where  $a = 4 \Rightarrow x^2 = 16y$

- b) Let the vertex of the parabola be a (0,0) and directrix is  $y = -2$ .

Thus directrix is a line parallel to  $x$  - axis at 'a' distance of 2 units below the axis. So the focus of parabola lies on  $y$  - axis consequently the focus is S(0,2). Hence the equation of the parabola is  $x^2 = 4ay$  where  $a = 2 \Rightarrow x^2 = 8y$ .

- c) Let vertex is (0,0) and directrix is  $y = 6$ .

Thus directrix is a line parallel to the  $x$ -axis at a distance of 6 units above the vertex. So the focus of the parabola lie on  $y$ -axis consequently the focus is (0, -6)  $\Rightarrow a = 6$ . Hence the equation the parabola is  $x^2 = 4ay$  where  $a = 6$  then  $x^2 = 4 \times 6y \Rightarrow x^2 = -24y$

- d) Let the vertex of the parabola be (0,0) and directrix is  $y = 8$ .

Thus directrix is a line parallel to axis at a distance of 8 units above the vertex. So the focus of the parabola lies on  $y$  - axis consequently the focus is (0, -8). Hence the equation of the parabola is  $x^2 = -4ay$  where  $a = 8 \Rightarrow x^2 = -32y$

- e) Let the vertex of the parabola be (0,0) and directrix  $x = -3$ .

Thus directrix is a line parallel to  $y$  axis at a distance of 3 units to the left side of the vertex. So the focus is (3, 0). Hence the equation of the parabola is  $y^2 = 4ax$  where  $a = 3 \Rightarrow y^2 = 12x$

- f) Let the vertex of the parabola be A(0,0) and directrix is  $x = -5$ .

Thus the directrix is a line parallel to  $y$  - axis at a distance of 5 units to the left side of the vertex so the focus of the parabola lies on the  $x$  - axis. Consequently the focus is S(5,0). Hence the equation of the parabola is  $y^2 = 4ax$  where  $a = 5 \Rightarrow y^2 = 20x$

- g) Let the vertex of the parabola be A (0,0) and directrics is  $x = 4$ .

Thus the directrix is line parallel to  $y$  - axis at distance of 4 units to the right side of the vertex. So the focus of the parabola lies on the  $x$  - axis. Consequently the focus is S(-4, 0). Hence the equation of parabola is  $y^2 = 4ax$  where  $a = 4 \Rightarrow y^2 = -16x$

- h) Let the vertex of the parabola be A(0,0) and directrix is  $x = 7$ .

Thus the directrix is a line parallel to the  $y$  axis at a distance of 7 units to the right side of the vertex.

So the focus of the parabola lies on the  $x$  - axis, consequently the focus is S(-7, 0). Hence the equation of a parabola is  $y^2 = 4ax$  where  $a = 7 \Rightarrow y^2 = -28x$

**Type IV : Find the equation of parabola when focus and directrix are given.**

**Find the equation of parabola given that**

- a) Focus (3,0) and directrix is  $x = -3$

- b) Focus (4,0) and directrix is  $x = -4$

- |  |  |
|--|--|
| c) Focus $(-3,0)$ and directrix is $x = 3$ | d) Focus $(-5,0)$ and directrix is $x = 5$ |
| e) Focus $(0,2)$ and directrix is $y = -2$ | f) Focus $(0,3)$ and directrix is $y = -3$ |
| g) Focus $(0,-4)$ and directrix is $y = 4$ | h) Focus $(0,-5)$ and directrix is $y = 5$ |

**Solution :**

- a) Since focus lies on the  $x$ -axis,  $\therefore x$ -axis is the axis of the parabola.

Focus  $S(3,0)$  lies right side of the origin. So it is a right hand parabola.

$\therefore$  required equation of parabola is  $y^2 = 4ax$

Since focus is  $S(a,0) = (3,0) \Rightarrow a = 3$

Hence the required equation is  $y^2 = 4 \times 3x \Rightarrow y^2 = 12x$

or

Let  $S = (3,0)$  and  $P(x,y)$  be any point on the parabola

$\therefore$  According definition of parabola

$PS = \perp^r$  distance, from point  $P(x,y)$  to the directrix  $y + 3 = 0$  is given by

$$\sqrt{(x-3)^2 + (y-0)^2} = \frac{|x+3|}{\sqrt{1+0}}$$

$$\Rightarrow x^2 + 9 - 6x + y^2 = x^2 + 9 + 6x \Rightarrow y^2 = 12x$$

- b) Since focus lies on the  $x$ -axis  $\therefore x$ -axis is the axis of the parabola

Focus  $S(4,0)$  lies rightside of the origin, so it is a right hand parabola.

$\therefore$  required equation of parabola is  $y^2 = 4ax$

Since focus  $S(a,0) = (4,0) \Rightarrow a = 4$

Hence the required equation is  $y^2 = 4 \times 4x \Rightarrow y^2 = 16x$

or

Let  $S(4,0)$  and  $P(x,y)$  be any point of parabola

$\therefore$  According to definition of parabola

$PS = \perp^r$  distance from point  $P(x,y)$  to the directrix  $y + 4 = 0$  is given by

$$\sqrt{(x-4)^2 + (y-0)^2} = \frac{|x+4|}{\sqrt{1+0}}$$

$$\Rightarrow x^2 + y^2 - 8x + 16 = (x+4)^2$$

$$x^2 + y^2 - 8x + 16 = x^2 + 8x + 16 \Rightarrow y^2 = 16x$$

- c) Since Focus lies on the x-axis, x-axis is the axis of the parabola. Focus S(-3,0) lies to the left side of the origin.

So it is a left handed parabola

∴ required equation of parabola is  $y^2 = 4ax$

Since focus is  $(-a, 0) = (-3, 0) \Rightarrow a = 3$

Hence the required equation is  $y^2 = -4ax \Rightarrow y^2 = -12x$

or

Let S = (-3, 0) and P(x,y) be any point on the parabola

∴ According to definition of parabola

PS =  $\perp^r$  distance from P(x,y) to the directrix  $x - 3 = 0$

$$\Rightarrow \sqrt{(x - (-3))^2 + (y - 0)^2} = \frac{|x - 3|}{\sqrt{1 + 0}}$$

$$\Rightarrow (x + 3)^2 + (y - 0)^2 = (x - 3)^2 \Rightarrow x^2 + y^2 + 6x + 9 = x^2 - 6x + 9 \Rightarrow y^2 = -12x$$

- d) Since focus lies on the x - axis; x - axis is the axis of the parabola

Focus S(-5, 0) lies to the left of the origin.

So it is a left hand parabola

∴ required equation of parabola is  $y^2 = -4ax$

Since focus is  $(-a, 0) = (-5, 0) \Rightarrow a = 5$

Hence the required equation is  $y^2 = -4 \times 5x \Rightarrow y^2 = -20x$

- e) Since focus lies on the y - axis

∴ y axis is the axis of parabola. Focus S(0, 2) lies above the origin.

∴ required equation of parabola is  $x^2 = 4ay$

since focus is S(0,a) = (0, 2)  $\Rightarrow a = 2$

Hence the required equation is  $x^2 = 4ay \Rightarrow x^2 = 4 \times 2y \Rightarrow x^2 = 8y$

- f) Since focus lies on y axis

∴ y axis is the axis of parabola. Focus S(0,3) lies above the origin.

∴ required equation of parabola is  $x^2 = 4ay$

since focus is S(0,a) = (0,3)  $\Rightarrow a = 3$

Hence the required equation is  $x^2 = 4ay \Rightarrow x^2 = 4 \times 3y \Rightarrow x^2 = 12y$

g) Since focus lies on  $y$  - axis

$\therefore y$  axis is the axis of parabola. Focus  $S(0, -4)$  lies below the origin.

$\therefore$  required equation of parabola is  $x^2 = -4ay$

Since focus is  $S(0, -a) \Rightarrow (0, -4) \Rightarrow a = 4$

Hence the required equation is  $x^2 = -4ay = 4 \times 4y \Rightarrow x^2 = -16y$

h) Since focus lies on  $y$  axis

$\therefore y$  axis is the axis of parabola. Focus  $S(0, -5)$  lies below the origin.

$\therefore$  required equation of parabola is  $x^2 = -7ay$

since focus is  $S(0, -a) \Rightarrow (0, -5) \Rightarrow a = 5$

Hence the required equation is  $x^2 = -4ay \Rightarrow x^2 = -4 \times 5y \Rightarrow x^2 = -20y$

**Type V : Find the equation of parabola if the vertex and axis of parabola are given.**

**Find the equation of parabola given that**

- a) Vertex is the origin and passing through the point  $P(3, -4)$  and symmetric about the  $y$  - axis.
- b) Vertex at the origin and axis along  $x$  - axis and passing through the point  $P(2, 3)$ .
- c) Vertex at the origin and passing through the point  $P(5, 2)$  and symmetric with respect to the  $y$  - axis.
- d) Vertex at the origin and passing through the point  $P(2, -3)$  and which is symmetric about the  $y$  - axis.
- e) Vertex at the origin and symmetric about  $y$  - axis and passing through  $(-1, -3)$
- f) Vertex at the origin, axis is  $y$  - axis and passes through  $\left(\frac{1}{2}, 2\right)$

**Solution :**

- a) It is given that the vertex of the parabola is  $V(0,0)$  and symmetric about the  $y$ -axis so, its equation is  $x^2 = 4ay$  or  $x^2 = -4ay$ , since the parabola passes through the point  $P(3, -4)$ . So, it lies in the 4th quadrant.  
 $\therefore$  It is a downward parabola

Let its equation be  $x^2 = -4ay$ , since it passes through the point  $P(3, -4)$ ,

$$\text{we have } 3^2 = -4 \times a \times (-4) \Rightarrow a = \frac{+9}{16}$$

$$\text{So its equation is } x^2 = -4\left(\frac{9}{16}\right)y \Rightarrow x^2 = -\frac{9}{4}y \Rightarrow 4x^2 + 9y = 0$$

- b) It is given that vertex of the parabola is  $V(0,0)$  and symmetric about the  $x$  - axis so its equation is  $y^2 = 4ax$  or  $y^2 = -4ax$  , since the parabola passes through the point  $P(2,3)$ , so it lies in the 1st quadrant.

$\therefore$  It is a right handed parabola. Let its equation be  $y^2 = 4ax$  . Since it passes through the point  $P(2,3)$ , we have  $3^2 = 4 \times a(2) \Rightarrow a = \frac{9}{8}$  . So its equation is

$$y^2 = 4 \times \frac{9}{8} x \Rightarrow y^2 = \frac{9}{2} x \Rightarrow 2y^2 - 9x = 0$$

- c) It is given that vertex of the parabola is  $V(0,0)$  and symmetric about the  $y$  - axis, so its equation is  $x^2 = 4ay$  or  $x^2 = -4ay$  , since the parabola passes through the point  $P(5,2)$ . So it lies in the 1st quadrant

$\therefore$  It is a upward parabola. Let its equation be  $x^2 = 4ay$  , since it passes through the point  $P(5,2)$  we have  $5^2 = 4 \times a \times 2 \Rightarrow a = \frac{25}{8}$  so its equation is given by

$$x^2 = 4ay \Rightarrow x^2 = 4 \times \frac{25}{8} y \Rightarrow 2x^2 - 25y = 0$$

- d) It is given that the vertex of the parabola is  $V(0,0)$  and symmetric about the  $y$ -axis so its equation is  $x^2 = 4ay$  or  $x^2 = -4ay$  since the parabola passes through the point  $P(2, -3)$  so it lies in the 4th quadrant.

$\therefore$  It is a downward parabola. Let its equation be  $x^2 = -4ay$  , since it passes through the point  $P(2, -3)$  we have  $2^2 = -4 \times a(-3) \Rightarrow a = \frac{1}{3}$  so its equation is

$$x^2 = -4 \times \frac{1}{3} y \Rightarrow x^2 = -\frac{4}{3} y \Rightarrow 3x^2 + 4y = 0$$

- e) It is given that vertex of the parabola is  $V(0,0)$  and symmetric about the  $y$  - axis so its equation is  $x^2 = -4ay$  or  $x^2 = 4ay$  since parabola passes through the point  $P(-1, -3)$ . So it lies in the 3rd quadrant.

$\therefore$  It is a downward parabola, let its equation be  $x^2 = -4ay$  , since it passes through the point  $P(-1,-3)$ , we have  $(-1)^2 = -4 \times a \times (-3) \Rightarrow a = \frac{1}{12}$  . So its equation is

$$x^2 = -4 \left( \frac{1}{12} \right) y \Rightarrow x^2 = -\frac{y}{3} \Rightarrow 3x^2 + y = 0$$

f) It is given that vertex of the parabola is  $V(0,0)$  and symmetric about  $y$ -axis so, its equation is  $x^2 = 4ay$  or  $x^2 = -4ay$ , since the parabola passes through the point  $P\left(\frac{1}{2}, 2\right)$  so it lies in the 1st quadrant.

$\therefore$  It is a upward parabola. Let its equation be  $x^2 = 4ay$

Since it passes through the point  $P\left(\frac{1}{2}, 2\right)$  we have  $\left(\frac{1}{2}\right)^2 = 4 \times a \times 2 \Rightarrow a = \frac{1}{32}$

so its equation is  $x^2 = 4\left(\frac{1}{32}\right)y \Rightarrow x^2 = \frac{1}{8}y$  or  $8x^2 - y = 0$

### EXERCISE 16.1

#### One mark questions:

I.1. Write the characteristics of the following parabolas

- |                    |                    |                   |
|--------------------|--------------------|-------------------|
| a) $y^2 = 16x$     | b) $y^2 = -8x$     | c) $3x^2 = -8y$   |
| d) $x^2 = 8y$      | e) $y^2 = 8x$      | f) $y^2 + 4x = 0$ |
| g) $3x^2 + 4y = 0$ | h) $x^2 + 16y = 0$ |                   |

2. If the length of the latus rectum of  $y^2 = 8kx$  is 4 find k.

3. If the length of the latus rectum of  $x^2 = 4ky$  is 8, find the value of k.

#### Two marks questions:

II. 1. Find the equation of the parabola given that its

- a) vertex is  $(0,0)$  and focus is  $(4,0)$
- b) vertex is  $(0,0)$  and directrix is  $y = -3$
- c) focus is  $(1,0)$  and directrix is  $x = -1$
- d) focus is  $(-4,0)$  and directrix is  $x = 4$
- e) focus is  $(0,-3)$  and directrix is  $y = 3$
- f) focus is  $(0,6)$  and vertex is  $(0,0)$
- g) vertex is  $(0,0)$ , axis is  $y$  axis and passes through  $\left(\frac{1}{2}, 2\right)$
- h) vertex is  $(0,0)$ , axis is  $y$  axis and passes through  $(-1, -3)$

### ANSWERS 16.1

	Vertex	Focus	Directrix	Axis	Tangent	L and L' Co-ordinates	LR length	Equation of LR
I. a)	(0,0)	(4,0)	$x = -4$	$y = 0$	$x = 0$	(4,8), (4, -8)	16	$x = 4$
b)	(0,0)	(-2,0)	$x = 2$	$y = 0$	$x = 0$	(-2,4), (-2,-4)	8	$x = -2$
c)	(0,0)	$\left(0, -\frac{2}{3}\right)$	$y = \frac{2}{3}$	$x = 0$	$y = 0$	$\left(\frac{4}{3}, -\frac{2}{3}\right)\left(-\frac{4}{3}, -\frac{2}{3}\right)$	$\frac{8}{3}$	$y = \frac{2}{3}$
d)	(0,0)	(0,2)	$y = -2$	$x = 0$	$y = 0$	(4,2), (-4,2)	8	$y = 2$
e)	(0,0)	(0,2)	$x = -2$	$y = 0$	$x = 0$	(2,4), (2,-4)	8	$x = 2$
f)	(0,0)	(-1,0)	$x = 1$	$y = 0$	$x = 0$	(-1,2), (-1,-2)	4	$x = 1$
g)	(0,0)	$\left(0, -\frac{1}{3}\right)$	$y = \frac{1}{3}$	$x = 0$	$y = 0$	$\left(\frac{2}{3}, -\frac{1}{3}\right)\left(-\frac{2}{3}, -\frac{1}{3}\right)$	$\frac{4}{3}$	$y = -\frac{1}{3}$
h)	(0,0)	(0,-4)	$y = 4$	$x = 0$	$y = 0$	(8,-4)(-8,-4)	16	$y = -4$

2. a)  $k = \frac{1}{2}$

3.  $k = 2$

II. 1. a)  $y^2 = 16x$

b)  $x^2 = 12y$

c)  $y^2 = 4x$

d)  $y^2 = -16x$

e)  $x^2 = -12y$

f)  $x^2 = 24y$

g)  $8x^2 - y = 0$

h)  $3x^2 + y = 0$

\* \* \* \* \*