UNIT V - CALCULUS

Chapter	Title	No. of Teaching hrs.
17.	LIMITS AND CONTINUITY OF A FUNCTION	8 hrs
18.	DIFFERENTIAL CALCULUS	10 hrs
19.	APPLICATION OF DERIVATIVES	08 hrs
20.	INDEFINITE INTEGRALS	08 hrs
21.	DEFINITE INTEGRALS AND APPLICATION TO AREAS	08 hrs
	TOTAL TEACHING HOURS	**42 hrs**

Chapter 17

LIMIT AND CONTINUITY OF A FUNCTION

17.1 Introduction :

Calculus is a latin word which means pebble or a small stone used for calculating. The word calculation is also derived from the same latin word. Sir Issac Newton (1642-1727 A.D) and the German mathematician G.W. Leibnitz (1646 - 1716 A.D) invented and developed the subject independently and almost simultaneously. In this chapter we shall study about different types of functions, definition of limit, standard limits and continuity.

17.2 Variables and Constants:

A quantity which changes in its values is called a **variable.**

If a variable takes real values only then the variable is called a real variable.

For example : *x, y z*.... etc are variables.

A quantity which remains same in its value is called a **constant.**

If the value of a constant is a real number then it is called real constants.

For example: 5,11, e .... are constants.

17.3 Definition of a function

If to each value of real varible *'x'* a unique real number *'y'* is associated by means of rule *'f'* then we say the variable y is a real valued function of the real variable x. This is denoted by $y = f(x)$.

The variable x is called **independent** variable

The variable y is called **dependent** variable

17.4 Types of functions

1. **Polynomial function:**

   A function of the form

   $f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$

   where $a_0 \neq 0$, n is a non-negative integer and $a_0, a_1, a_2 .... a_n$ are real constants.

   Ex : $f(x) = x^4 + 5x^2 + 2x + 1$

2. **Modulus Function**

   The function which associates to each real number x, the number $|x|$ is called the modulus function.

   $$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

   Ex: $|3| = 3, |-3| = 3$

3. Exponential function

A function of the form $f(x) = e^x$ where e > 0 amd e ≠ 1 is called an exponential function. The value of e = 2.71828.... and is an irrational number.

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots\ldots\ldots + \frac{1}{\lfloor n} + \ldots\ldots\ldots$$

4. Logarithmic function

A function of the form $f(x) = \log_a x$ where $a > 0$ and $a \neq 1$ is called a logarithmic function with base 'a'.

For $a > 0,\ a \neq 1,\ \log_a x = y \Rightarrow a^y = x$

Logarithms with base e are called **natural logarithms.**

5. Rational function

The function defined by $y = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions is called a rational function.

$$Ex:\ y = \frac{2x - 4}{3x^2 + 2x + 1}$$

17.5 Limit of a function

Consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ Clearly, $f(1) = \dfrac{0}{0}$, which is meaningless.

Thus $f(x)$ is not defined at $x = 1$

Now, $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{x - 1} = (x+1)$, only when $x \neq 1$

If we give to x, a value not exactly 1 but slightly more than 1 then clearly, the value of $f(x)$ is slightly more than 2. Now if we go on decreasing this value and take it nearer to 1 then clearly the value of $f(x)$ will come nearer to 2, as shown below.

If $x = 1.1$ then $f(x) = 2.1$

$x = 1.01$ then $f(x) = 2.01$

$x = 1.001$ then $f(x) = 2.001$

....

Thus as the value of x approaches 1, the value of $f(x)$ approaches 2

i.e. As $x \to 1,\ f(x) \to 2$

(The symbol '$\to$' stands for 'approaches to' or tends to)

Similarly if we give to x, a value slightly less than 1 then the value of $f(x)$ is slightly less than 2. Now, if we go on increasing this value and take it nearer to 1, the value of $f(x)$ will come nearer to 2, as shown below.

If	$x = 0.9$	then $f(x) = 1.9$
	$x = 0.99$	then $f(x) = 1.99$
	$x = 0.999$	then $f(x) = 1.999$
	....	....

Thus as the value of x approaches 1, the value of $f(x)$ approaches 2.

i.e. As $x \to 1, f(x) \to 2$

We express this fact as $\lim_{x \to 1}\left[\frac{x^2-1}{x-1}\right] = 2$

In general a number 'l' is called the limit of a function $f(x)$ as x tends to 'a' if the value of $f(x)$ is very nearly equal to 'l', whenever the value of x is very nearly equal to 'a'. We write this as $\lim_{x \to a}[f(x)] = l$

Definition of Limit

A function $y = f(x)$ is said to tend to limit 'l' as x tends to 'a' iff the numerical difference **$|f(x) - l|$ can be made as small as we like by taking x, nearer and nearer to 'a'.**

17.6 Algebra of Limits

(properties of limits without proof)

1. The limit of a constant function is the same constant.

   i.e. If $f(x) = k$, a constant function then $\lim_{x \to a}[f(x)] = k$

2. The limit of a scalar product is equal to the scalar product of the limit.

   i.e. $\lim_{x \to a}[k.f(x)] = k \, . \, \lim_{x \to a}[f(x)]$

3. The limit of a sum is equal to the sum of the limits.

   i.e $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a}[f(x)] + \lim_{x \to a}[g(x)]$

4. The limit of a difference is equal to the difference of the limits.

   i.e $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a}[f(x)] - \lim_{x \to a}[g(x)]$

5. The limit of a product is equal to the product of the limits.

   i.e. $\lim_{x \to a}[f(x)\, g(x)] = \lim_{x \to a}[f(x)] \lim_{x \to a}[g(x)]$

6. The limit of a quotient is equal to the quotient of the limits provided the limit of the denominator is non–zero

i.e. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a}[f(x)]}{\lim_{x\to a}[g(x)]}$, provided $\lim_{x\to a}[g(x)]\neq 0$

7. The limit of the n[th] root is equal to the n[th] root of the limit provided that the n[th] root of the limit is a real number.

i.e. $\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}f(x)}$, provided $\sqrt[n]{\lim_{x\to a}f(x)}$ is a real number.

Note:

1. Limits of polynomials can be found by substitution

$$\lim_{x\to a}\left[p_n x^n+p_{n-1}x^{n-1}+........+p_1x+p_0\right]$$

$$=\ p_n a^n+p_{n-1}a^{n-1}+.........+p_1a+p_0$$

2. Indeterminate forms:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 0^o,\ \infty^o,\ 1^\infty$$

3. Meaningful forms:

(i) $\frac{a}{0}=\infty(a\neq 0)$ (ii) $\frac{0}{a}=0(a\neq 0)$ (iii) $\infty\times a=\infty$

(iv) $\infty\times\infty=\infty$ (v) $0\times 0=0$ (vi) $\infty-b=\infty$

(vii) $a-\infty=-\infty$ (viii) $\infty+\infty=\infty$

17.7 Evaluation of limits:

Type A: Direct substitution method

Put $x=a$ in the given function. If *f(a)* is a definite value then $\lim_{x\to a}[f(x)]=f(a)$

Example 1:

Evaluate the following limits

1. $\lim_{x\to 1}\left[x^2+6x+4\right]=\lim_{x\to 1}x^2+\lim_{x\to 1}6x+\lim_{x\to 1}4$

$$=1^2+6(1)+4$$

$$=11$$

2. $\lim_{x\to 4}\left[\dfrac{x^3+4}{1-x}\right] = \dfrac{\lim_{x\to 4}(x^3+4)}{\lim_{x\to 4}(1-x)}$

$$= \frac{4^3+4}{1-4} = -\frac{68}{3}$$

3. $\lim_{x\to 2}\sqrt{36-x^2} = \sqrt{\lim_{x\to 2}(36-x^2)}$

$$= \sqrt{36-\lim_{x\to 2}x^2} = \sqrt{36-2^2} = \sqrt{36-4} = \sqrt{32}$$

Type B : Method of factors:

In this method, *f(x)* and *g(x)* are factorised and the quotient $\dfrac{f(x)}{g(x)}$ is simplified by cancelling the common factors. If the limit of the resulting denominator as $x \to a$ is non-zero, then the quotient rule of limits is applied to get the required limit.

Example 2 :

Evaluate the following limits

1. $\lim_{x\to 2}\dfrac{x^2-3x+2}{x^2-x-2}$

Solution : At $x = 2$, $\dfrac{x^2-3x+2}{x^2-x-2} = \dfrac{0}{0}$ form

$$\therefore \lim_{x\to 2}\frac{x^2-3x+2}{x^2-x-2} = \lim_{x\to 2}\frac{(x-2)(x-1)}{(x-2)(x+1)}$$

$$= \lim_{x\to 2}\frac{x-1}{x+1} = \frac{2-1}{2+1} = \frac{1}{3}$$

2. $\lim_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right)$

Solution: At $x = 3$, $\dfrac{x^2-9}{x-3} = \dfrac{0}{0}$ form

$$\therefore \lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to 3}\left[\frac{(x+3)(x-3)}{x-3}\right] = \lim_{x\to 3}(x+3)$$

$$= 3 + 3 = 6$$

3. $\lim\limits_{x\to -1}\left[\dfrac{x^3+1}{2x^2+5x+3}\right]$

Solution: At $x=-1$, $\dfrac{x^3+1}{2x^2+5x+3}=\dfrac{0}{0}$ form

$$\therefore \lim_{x\to -1}\left[\frac{x^3+1}{2x^2+5x+3}\right] = \therefore \lim_{x\to -1}\left[\frac{(x+1)(x^2-x+1)}{(x+1)(2x+3)}\right]$$

$$= \therefore \lim_{x\to -1}\left[\frac{x^2-x+1}{2x+3}\right] = \frac{(-1)^2-(-1)+1}{2(-1)+3}$$

$$= \frac{3}{1} = 3$$

4. $\lim\limits_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}\right]$

Solution: At $x=2$, $\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}=\infty-\infty$ form

$$\therefore \lim_{x\to 2}\left[\frac{1}{x-2}-\frac{4}{x^3-2x^2}\right] = \lim_{x\to 2}\left[\frac{1}{x-2}-\frac{4}{x^2(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-4}{x^2(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{(x-2)(x+2)}{x^2(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x+2}{x^2}\right]$$

$$= \frac{4}{4} = 1$$

Type C: Method of Rationalisation

In this method the expressions involving square roots are rationalised. After the process of rationalisation, if the limit of the resulting denominator as $x \to a$ is non-zero, then the quotient rule of limits is applied to get the required limit.

Example 3:

Evaluate the following limits

1. $\lim\limits_{x\to 0}\dfrac{\sqrt{2-x}-\sqrt{2+x}}{x}$

Solution:

At $x=0$, $\dfrac{\sqrt{2-x}-\sqrt{2+x}}{x}=\dfrac{0}{0}$ form Rationalise in the Numerator

$$\therefore \lim_{x\to 0}\frac{\sqrt{2-x}-\sqrt{2+x}}{x} = \lim_{x\to 0}\left[\frac{\sqrt{2-x}-\sqrt{2+x}}{x}\times\frac{\sqrt{2-x}+\sqrt{2+x}}{\sqrt{2-x}+\sqrt{2+x}}\right]$$

$$= \lim_{x\to 0}\left[\frac{(2-x)-(2+x)}{x\left(\sqrt{2-x}+\sqrt{2+x}\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{-2x}{x\left(\sqrt{2-x}+\sqrt{2+x}\right)}\right]$$

$$= \lim_{x\to 0}\frac{-2}{\sqrt{2-0}+\sqrt{2+0}}$$

$$= \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

2. $\lim\limits_{x\to 3}\left[\dfrac{x^2-9}{\sqrt{3x+7}-\sqrt{5x+1}}\right]$

Solution: At $x=3$, $\dfrac{x^2-9}{\sqrt{3x+7}-\sqrt{5x+1}}=\dfrac{0}{0}$ form

$$\therefore \lim_{x\to 3}\left[\frac{x^2-9}{\sqrt{3x+7}-\sqrt{5x+1}}\right] = \lim_{x\to 3}\left[\frac{x^2-9}{\sqrt{3x+7}-\sqrt{5x+1}}\times\frac{\sqrt{3x+7}+\sqrt{5x+1}}{\sqrt{3x+7}+\sqrt{5x+1}}\right]$$

$$= \lim_{x\to 3}\left[\frac{(x^2-9)\left(\sqrt{3x+7}+\sqrt{5x+1}\right)}{(3x+7)-(5x+1)}\right]$$

$$= \lim_{x\to 3}\left[\frac{(x^2-9)\left(\sqrt{3x+7}+\sqrt{5x+1}\right)}{-2x+6}\right]$$

$$= \lim_{x\to 3}\left[\frac{(x-3)(x+3)\left(\sqrt{3x+7}+\sqrt{5x+1}\right)}{-2(x-3)}\right]$$

$$= \lim_{x\to 3}\left[\frac{-(x+3)\left(\sqrt{3x+7}+\sqrt{5x+1}\right)}{2}\right]$$

$$= \frac{-(3+3)\left(\sqrt{9+7}+\sqrt{15+1}\right)}{2}$$

$$= \frac{-6\left(\sqrt{16}+\sqrt{16}\right)}{2} = \frac{-6(4+4)}{2}$$

$$= \frac{-6(8)}{2} = -24$$

3. $\lim_{x\to 1}\left[\frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}\right]$

Solution: At $x = 1$, $\frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1} = \frac{0}{0}$ form

$$\therefore \lim_{x\to 1}\frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1} = \lim_{x\to 1}\left[\frac{\left(\sqrt{3+x}-\sqrt{5-x}\right)\left(\sqrt{3+x}+\sqrt{5-x}\right)}{\left(x^2-1\right)\left(\sqrt{3+x}+\sqrt{5-x}\right)}\right]$$

$$= \lim_{x\to 1}\frac{(3+x)-(5-x)}{(x^2-1)\left(\sqrt{3+x}+\sqrt{5-x}\right)}$$

$$= \lim_{x\to 1}\frac{2x-2}{(x^2-1)\left(\sqrt{3+x}+\sqrt{5-x}\right)}$$

$$= \lim_{x\to 1}\frac{2(x-1)}{(x-1)(x+1)\left(\sqrt{3+x}+\sqrt{5-x}\right)}$$

$$= \lim_{x\to 1} \frac{2}{(x+1)\left(\sqrt{3+x}+\sqrt{5-x}\right)}$$

$$= \frac{2}{(1+1)\left(\sqrt{3+1}+\sqrt{5-1}\right)} = \frac{1}{4}$$

17.8 Evaluation of Standard Limits

Theorem 1 : If *n* is a rational number and a is non-zero real number, then

prove that $\lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = n.a^{n-1}$

Proof :

Case 1: Let *n* be a positive integer.

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}.a + x^{n-3}a^2 + ... + a^{n-1})$$

$\div$ by $(x-a)$ and apply lim $x \to a$ on both sides

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = \lim_{x\to a}\left[\frac{(x-a)\left(x^{n-1}+x^{n-2}.a+....+a^{n-1}\right)}{x-a}\right]$$

$$= \lim_{x\to a}\left[x^{n-1}+x^{n-2}.a+....+a^{n-1}\right]$$

$$= a^{n-1}+a^{n-2}.a+........+a^{n-1}$$

$$= a^{n-1}+a^{n-1}+........+a^{n-1}$$

$$= n.\ a^{n-1}$$

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = n.\ a^{n-1}$$

Case 2: Let *n* be a negative integer

put $n = -m, \quad m > 0$

consider $\frac{x^n - a^n}{x-a} = \frac{x^{-m} - a^{-m}}{x-a}$

$$= \left[\frac{\frac{1}{x^m} - \frac{1}{a^m}}{x-a}\right]$$

$$= \frac{a^m - x^m}{x^m a^m (x-a)}$$

$$= \frac{-1}{a^m x^m}\left(\frac{x^m - a^m}{x-a}\right)$$ apply lim $x \to a$ on both sides

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = \lim_{x\to a}\frac{-1}{a^m x^m}\left(\frac{x^m - a^m}{x-a}\right)$$

$$= \frac{-1}{a^m a^m} m.a^{m-1}$$ [By case 1]

$$= (-m)\frac{a^{m-1}}{a^{2m}} = (-m)a^{-m-1}$$

$$= n.\ a^{n-1} \quad [\because -m = n]$$

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = n.a^{n-1}$$

Case 3: Let $n = \frac{p}{q}$ where p and q are integers and $q \neq 0$

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = \lim_{x\to a}\left[\frac{x^{p/q} - a^{p/q}}{x-a}\right]$$

$$= \lim_{x\to a}\frac{\left(x^{1/q}\right)^p - \left(a^{1/q}\right)^p}{x-a}$$

Let $x^{1/q} = y$ and $a^{1/q} = b$

$\Rightarrow x = y^q$ and $a = b^q$

Also $x \to a$ changes to $y \to b$

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = \lim_{y\to b}\left[\frac{y^p - b^p}{y^q - b^q}\right]$$

$$= \lim_{y\to b}\left[\frac{\frac{y^p - b^p}{y-b}}{\frac{y^q - b^q}{y-b}}\right]$$

$$= \frac{p \cdot b^{p-1}}{q \cdot b^{q-1}} \text{ (by case 1)}$$

$$= \frac{p}{q} \cdot b^{p-1-q+1} \quad = \quad \frac{p}{q} \cdot b^{p-q}$$

$$= \frac{p}{q} \cdot \left(a^{1/q}\right)^{p-q} \quad = \quad \frac{p}{q} \cdot a^{p/q-1}$$

$$= n \cdot a^{n-1} \quad [\because p/q = n]$$

$$\because \lim_{x \to a}\left[\frac{x^n - a^n}{n - a}\right] = n \cdot a^{n-1}$$

Hence the result is true for all rational values of n.

Example 4 :

Evaluate the following limits

1. $\lim_{x \to 2}\left[\frac{x^3 - 8}{x - 2}\right] = \lim_{x \to 2}\left[\frac{x^3 - 2^3}{x - 2}\right]$

$$= 3 \cdot 2^{3-1} \quad = \quad 3(2)^2$$

$$= 3(4) \quad = \quad 12$$

2. $\lim_{x \to -2}\left[\frac{x^5 + 32}{x + 2}\right] = \lim_{x \to -2}\left[\frac{x^5 - (-2)^5}{x - (-2)}\right]$

$$= 5(-2)^{5-1} \quad = \quad 5(-2)^4$$

$$= 5(16) \quad = \quad 80$$

3. $\lim_{x \to 2}\frac{x^{10} - 1024}{x^4 - 16} = \lim_{x \to 2}\frac{\frac{x^{10} - 2^{10}}{x - 2}}{\frac{x^4 - 2^4}{x - 2}} = \frac{\lim_{x \to 2}\left[\frac{x^{10} - 2^{10}}{x - 2}\right]}{\lim_{x \to 2}\left[\frac{x^4 - 2^4}{x - 2}\right]}$ Divide both Nr and Dr by $(x-2)$

$$= \frac{10(2)^{10-1}}{4(2)^{4-1}} = \frac{10(2)^9}{4(2)^3} = \frac{10(2)^9}{2^2 \times 2^3}$$

$$= 10(2)^{9-5} = 10(2)^4 = 10(16)$$

$$= 160$$

4. $\lim\limits_{x\to 2}\dfrac{x^{5/2}-2^{5/2}}{x^{3/2}-2^{3/2}} \quad = \quad \lim\limits_{x\to 2}\dfrac{\left(\dfrac{x^{5/2}-2^{5/2}}{x-2}\right)}{\left(\dfrac{x^{3/2}-2^{3/2}}{x-2}\right)}$

$$= \frac{\left(\frac{5}{2}\right) 2^{\frac{5}{2}-1}}{\left(\frac{3}{2}\right) 2^{\frac{3}{2}-1}} = \frac{5}{3}\cdot 2$$

$$= \frac{10}{3}$$

5. $\lim\limits_{x\to 2}\dfrac{x^5\sqrt{x}-32\sqrt{2}}{x^3\sqrt{x}-8\sqrt{2}}$

$$= \lim_{x\to 2}\frac{x^5\cdot x^{1/2}-2^5\cdot 2^{1/2}}{x^3\cdot x^{1/2}-2^3\cdot 2^{1/2}}$$

$$= \lim_{x\to 2}\frac{x^{11/2}-2^{11/2}}{x^{7/2}-2^{7/2}}$$

$$= \lim_{x\to 2}\frac{\dfrac{x^{11/2}-2^{11/2}}{x-2}}{\dfrac{x^{7/2}-2^{7/2}}{x-2}} \qquad \div \text{ both Nr and Dr by } (x-2)$$

$$= \frac{\frac{11}{2}\cdot 2^{\frac{11}{2}-1}}{\frac{7}{2}\cdot 2^{\frac{7}{2}-1}}$$

$$= \frac{11}{7}\cdot 2^{\frac{9}{2}-\frac{5}{2}} \qquad = \frac{11}{7}\cdot 2^2$$

$$= \frac{11}{7}(4) \qquad = \frac{44}{7}$$

6. $\lim\limits_{x\to 0}\dfrac{(x+1)^5-1}{x}$

$= \lim\limits_{x+1\to 1}\dfrac{(x+1)^5-1}{(x+1)-1}$

$= \lim\limits_{y\to 1}\dfrac{y^5-1^5}{y-1}$ where $y = x+1$

$= 5(1)^{5-1} = 5$

7. $\lim\limits_{x\to 3}\dfrac{(x+2)^{5/3}-5^{5/3}}{x-3}$

Put $x+2=y$

As $x\to 3$

$\Rightarrow y\to 5$

$= \lim\limits_{x\to 3}\dfrac{(x+2)^{5/3}-(3+2)^{5/3}}{(x+2)-(3+2)}$

$= \lim\limits_{x\to 5}\dfrac{y^{5/3}-5^{5/3}}{y-5} = \dfrac{5}{3}(5)^{5/3-1}$

$= \dfrac{5}{3}(5)^{2/3}$

$= \dfrac{5^{5/3}}{3}$

8. $\lim\limits_{x\to -2}\dfrac{\frac{1}{x}+\frac{1}{2}}{x+2}$

$= \lim\limits_{x\to -2}\dfrac{x^{-1}-(-2)^{-1}}{x-(-2)}$

$= (-1)\,(-2)^{-1-1} = (-1)\left(\dfrac{1}{4}\right)$

$= \dfrac{-1}{4}$

Exercise 17.1

I. One and two marks questions:

1. Evaluate the following limits

1. $\lim\limits_{x\to 4}\left(\dfrac{4x+3}{x-2}\right)$ 2. $\lim\limits_{x\to 1}\left(\dfrac{ax^2+bx+c}{cx^2+bx+a}\right)$ 3. $\lim\limits_{x\to -3}\left(\dfrac{x^2+9}{x+3}\right)$

4. $\lim\limits_{x\to 3}\left(\dfrac{x^2-4x}{x-2}\right)$ 5. $\lim\limits_{x\to 1}\left(\dfrac{x^3-1}{x-1}\right)$ 6. $\lim\limits_{x\to 1/2}\left(\dfrac{4x^2-1}{2x-1}\right)$

7. $\lim\limits_{x\to 2}\dfrac{x-2}{x^{1/3}-2^{1/3}}$ 8. $\lim\limits_{x\to -1}\dfrac{x^9+1}{x^5+1}$

II. 3 marks questions:

1. $\lim\limits_{x\to 3}\left[\dfrac{x^2-4x+3}{x^2-2x-3}\right]$ 2. $\lim\limits_{x\to 2}\left[\dfrac{2x^2-5x+2}{x^2-3x+2}\right]$ 3. $\lim\limits_{x\to 2}\left[\dfrac{3x^2-x-10}{x^2-4}\right]$

4. $\lim\limits_{x\to 3}\dfrac{x^4-81}{2x^2-5x-3}$ 5. $\lim\limits_{x\to \frac{1}{2}}\dfrac{2x^2+3x-2}{2x^2-3x+1}$ 6. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x\sqrt{x}-2\sqrt{2}}$

7. $\lim\limits_{x\to 0}\left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}\right]$ 8. $\lim\limits_{x\to 1}\dfrac{\sqrt{x+3}-2}{x-1}$ 9. $\lim\limits_{x\to a}\dfrac{x^{3/8}-a^{3/8}}{x^{1/3}-a^{1/3}}$

III. 5 marks questions:

1. $\lim\limits_{x\to 3}\left[\dfrac{1}{x-3}-\dfrac{2}{x^2-4x+3}\right]$ 2. $\lim\limits_{x\to 1}\left[\dfrac{1}{x^2+x-2}-\dfrac{x}{x^3-1}\right]$

3. $\lim\limits_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{2(2x-3)}{x^3-3x^2+2x}\right]$ 4. $\lim\limits_{x\to 2}\left[\dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right]$

Answers 17.1

I. 1) $\dfrac{19}{2}$ 2) 1 3) –6 4) –3 5) 3 6) 2

7) $3(2)^{\frac{2}{3}}$ 8) $\dfrac{9}{5}$

II. 1) $\frac{1}{2}$ 2) 3 3) $\frac{11}{4}$ 4) $\frac{108}{7}$ 5) –5

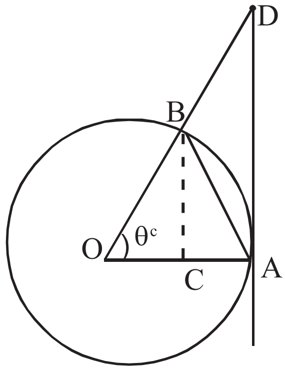
6) $\frac{4\sqrt{2}}{3}$ 7) 1 8) $\frac{1}{4}$ 9) $\frac{9}{8}a^{\frac{1}{24}}$

III. 1) $\frac{1}{2}$ 2) $\frac{-1}{9}$ 3) $\frac{-1}{2}$ 4) –8

Evaluation of Trigonometrical Limits:

Theorem 2 : If angel θ is measured in radians, then prove that $\lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$

Proof:



Let O be the centre of a unit circle. Let us first assume that the angle θ is positive in radian.

Let $\underline{|AOB} = \theta^c$. From A draw AD ⊥ OA, meeting OB produced to D.

Let OB = OA = radian.

From the figure, we have Area of ΔOAB < Area of sector OAB < Area of ΔOAD

$$\Rightarrow \frac{1}{\cancel{2}}\,\cancel{OA}\,.\,BC < \frac{1}{\cancel{2}}\,OA^{\cancel{2}}\,.\,\theta < \frac{1}{\cancel{2}}\,\cancel{OA}\,.\,DA \text{ ....(1)}$$

[∵ Area of a sector = $\frac{1}{2}$ (radius)^2 × angle in radians]

From the Δ^{le} DOA $\tan\theta = \frac{DA}{OA}$ $\therefore \boxed{DA = r\,tan\ \theta}$

From the ΔBOC, $\sin\theta = \frac{BC}{OB}$ $\therefore \boxed{BC = r\,sin\ \theta}$

(1) becomes,

$\Rightarrow$ BC < θ < DA

$\Rightarrow$ sinθ < θ < tanθ (Dividing by sin θ)

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\Rightarrow \cos\theta < \frac{\sin\theta}{\theta} < 1$$

apply $\text{Lt}\ \theta \to 0$

$$\lim_{\theta\to 0} \cos\theta < \lim_{\theta\to 0}\frac{\sin\theta}{\theta} < \lim_{\theta\to 0} 1$$

$$\Rightarrow 1 < \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) < 1$$

$$\Rightarrow \boxed{\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1}$$

$\therefore$ If θ is positive then $\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$ ....(2)

If angle θ is negative, then $-\theta$ is positive and

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = \lim_{\theta\to 0}\frac{-\sin\theta}{-\theta}$$

$$= \lim_{-\theta\to 0}\frac{\sin(-\theta)}{-\theta} = 1$$

[$\because$ $-\theta$ is +ve and so (2) is applicable]

$\therefore$ For any angle θ, $\boxed{\lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\right] = 1}$

Corollary:

1. $\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1$

Proof: $\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = \lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}$

$$= \frac{\lim_{\theta\to 0} 1}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}} = \frac{1}{1} = 1$$

2. $\lim\limits_{\theta\to 0}\left(\dfrac{\tan\theta}{\theta}\right)=1$

Proof:
$$\lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right) = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\times\frac{1}{\cos\theta}\right)$$
$$= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{1}{\cos\theta}\right)$$
$$= (1)\left(\frac{1}{1}\right) = 1$$

Example 5 :

Evaluate the following limits:

1. $\lim\limits_{x\to 0}\dfrac{\sin x^0}{x}=\lim\limits_{x\to 0}\dfrac{\sin\left(\dfrac{x\pi}{180}\right)}{x}$ $\quad\left[\begin{array}{l}180^{o}=\pi^{c}\\ \therefore\quad x^{o}=\left(\dfrac{x\pi}{180}\right)^{c}\end{array}\right]$

Solution:
$$\lim_{x\to 0}\frac{\sin\left(\dfrac{x\pi}{180}\right)}{\dfrac{x\pi}{180}\cdot\dfrac{180}{\pi}} = \frac{\pi}{180}\cdot\lim_{x\to 0}\frac{\sin\left(x\pi/180\right)}{\left(x\pi/180\right)}$$
$$= \frac{\pi}{180}.1 = \frac{\pi}{180}$$

2. $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$, $ab\neq 0$

$ab\neq 0 \Rightarrow a\neq 0$ and $b\neq 0$

Solution: $\therefore \lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin ax}{ax}\cdot ax}{\dfrac{\sin bx}{bx}\cdot bx}$

$$= \frac{a}{b}\cdot\frac{\lim\limits_{x\to 0}\dfrac{\sin ax}{ax}}{\lim\limits_{x\to 0}\dfrac{\sin bx}{bx}} = \frac{a}{b}\cdot\frac{1}{1} = \frac{a}{b}$$

3. $\lim_{x\to 0}\dfrac{ax+x\cos x}{b\sin x}$

$\div$ both Nr and Dr by x

Solution: $\lim_{x\to 0}\dfrac{\frac{1}{x}(ax+x\cos x)}{\frac{b\sin x}{x}}$

$$= \lim_{x\to 0}\frac{a+\cos x}{b\cdot\frac{\sin x}{x}} = \frac{a+\lim_{x\to 0}\cos x}{b\cdot\lim_{x\to 0}\frac{\sin x}{x}} = \frac{a+1}{b}$$

4. $\lim_{x\to 0}\dfrac{\cos^2 x}{1-\sin x}$

Solution: $\lim_{x\to 0}\dfrac{1-\sin^2 x}{1-\sin x} = \dfrac{\lim_{x\to 0}(1+\sin x)\cancel{(1-\sin x)}}{\cancel{(1-\sin x)}} = \lim_{x\to 0}(1+\sin x)$

$= 1+0 = 1$

5. $\lim_{\theta\to 0}\dfrac{1-\cos m\theta}{1-\cos n\theta}$

From sub multiple angle

$$\left(\begin{array}{l}\cos\theta = 1-2\sin^2\theta/2 \\ \therefore 2\sin^2\dfrac{\theta}{2} = 1-\cos\theta\end{array}\right)$$

Solution: $\lim_{\theta\to 0}\dfrac{2\sin^2\left(m\theta/2\right)}{2\sin^2\left(n\theta/2\right)} = \lim_{\theta\to 0}\left[\dfrac{\sin\left(m\theta/2\right)}{\sin\left(n\theta/2\right)}\right]^2$

$$= \lim_{\theta\to 0}\left[\frac{\sin\left(\frac{m\theta}{2}\right)}{\frac{m\theta}{2}}\times\frac{\frac{n\theta}{2}}{\sin\left(\frac{n\theta}{2}\right)}\times\frac{m}{n}\right]^2$$

$$= \frac{m^2}{n^2}\left[\lim_{\theta\to 0}\frac{\sin\left(m\theta/2\right)}{m\theta/2}\times\lim_{\theta\to 0}\frac{n\theta/2}{\sin n\theta/2}\right]^2$$

$$= \frac{m^2}{n^2}(1\times 1)^2 = \frac{m^2}{n^2}$$

6. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\tan 3x}$

Solution: $\lim\limits_{x\to 0}\left[\dfrac{\sin 5x}{5x}\times\dfrac{3x}{\tan 3x}\times\dfrac{5}{3}\right]$

$$= \frac{5}{3}\cdot\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)\cdot\lim_{x\to 0}\left(\frac{3x}{\tan 3x}\right)$$

$$= \frac{5}{3}\times 1\times 1 \quad = \quad \frac{5}{3}$$

7. $\lim\limits_{x\to 0}\dfrac{\cos x}{\pi - x} = \dfrac{\lim\limits_{x\to 0}\cos x}{\lim\limits_{x\to 0}\pi - x}$ $\quad (\because \cos 0^{\circ} = 1)$

$$= \frac{1}{\pi - 0} \quad = \quad \frac{1}{\pi}$$

8. $\lim\limits_{x\to 0}(\operatorname{cosec}x - \cot x) \quad = \quad \lim\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right)$

Solution: $\quad = \quad \lim\limits_{x\to 0}\left(\dfrac{1-\cos x}{\sin x}\right)$

$$= \lim_{x\to 0}\frac{2\sin^2 x/2}{2\sin x/2\,\cos x/2} \qquad \text{[from sub multiple angle]}$$

$$= \lim_{x\to 0}\frac{\sin x/2}{\cos x/2} \quad = \quad \frac{0}{1} \quad = \quad 0$$

Example 6:

Evaluate the following limits:

1. $\lim\limits_{\theta\to\pi/2}\dfrac{\cot\theta}{\pi/2 - \theta}$

Solution: Let $\theta = \dfrac{\pi}{2} + h \quad \therefore\ h \to 0$ as $\theta \to \dfrac{\pi}{2}$

$$\therefore \lim_{\theta \to \pi/2} \frac{\cot\theta}{\pi/2 - \theta} = \lim_{h \to 0} \frac{\cot\left(\pi/2 + h\right)}{\pi/2 - \left(\pi/2 + h\right)}$$

$$= \lim_{h \to 0} \frac{-\tan h}{-h} = \lim_{h \to 0} \frac{\tan h}{h}$$

$$= 1$$

2. $\lim\limits_{x \to \pi} \dfrac{\sin(\pi - x)}{\pi(\pi - x)}$

Solution: Let $x = \pi + h$ $\therefore\ h \to 0$ as $x \to \pi$

$$\therefore \lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \lim_{h \to 0} \frac{\sin[\pi - (\pi + h)]}{\pi[\pi - (\pi + h)]}$$

$$= \lim_{h \to 0} \frac{\sin(-h)}{\pi(-h)} = \lim_{h \to 0} \frac{-\sinh}{-\pi h}$$

$$= \frac{1}{\pi} \cdot \lim_{h \to 0} \frac{\sin h}{h} = \frac{1}{\pi}(1)$$

$$= \frac{1}{\pi}$$

3. $\lim\limits_{x \to \pi/2} \dfrac{\tan 2x}{x - \pi/2}$

Solution : Let $x = \pi/2 + h$ $\therefore\ h \to 0$ as $x \to \pi/2$

$$\therefore \lim_{x \to \pi/2} \frac{\tan 2x}{x - \pi/2} = \lim_{h \to 0} \frac{\tan 2\left(\pi/2 + h\right)}{\pi/2 + h - \pi/2}$$

$$= \lim_{h \to 0} \frac{\tan(\pi + 2h)}{h}$$

$$= \mathop{lt}_{h \to 0} \frac{\tan 2h}{h} \qquad \therefore \tan(180^{o} + \theta) = \tan\theta$$

$$= 2. \mathop{lt}_{h \to 0} \frac{\tan 2h}{2h} = 2(1) = 2$$

4. $\lim_{x\to\pi/4} \dfrac{\sin x - \cos x}{\pi/4 - x}$

Solution: Let $x = \pi/4 + h$ $\quad\therefore\ h \to 0$ as $x \to \dfrac{\pi}{4}$

$$\therefore \lim_{x\to\pi/4} \frac{\sin x - \cos x}{\pi/4 - x} = \lim_{h\to 0} \frac{\sin\left(\pi/4 + h\right) - \cos\left(\pi/4 + h\right)}{\pi/4 - \left(\pi/4 + h\right)}$$

$\because$ from compound angle

$$\begin{bmatrix} \sin(A+B) = \sin A\ \cos B + \cos A \sin B \\ \cos(A+B) = \cos A\ \cos B - \sin A \sin B \end{bmatrix}$$

$$= \lim_{h\to 0} \frac{\left(\frac{1}{\sqrt{2}}\cos h + \frac{1}{\sqrt{2}}\sin h\right) - \left(\frac{1}{\sqrt{2}}\cos h - \frac{1}{\sqrt{2}}\sin h\right)}{-h}$$

$$= -\lim_{h\to 0} \frac{\sqrt{2}\sin h}{h} = -\sqrt{2} \times 1 = -\sqrt{2}$$

5. $\lim_{x\to 0} \dfrac{\tan x - \sin x}{\sin^3 x}$

$$= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \lim_{x\to 0} \frac{\sin x - \sin x \cdot \cos x}{\cos x \cdot \sin^3 x}$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{\cos x \cdot \sin^2 x} = \lim_{x\to 0} \frac{1}{\cos x(1 + \cos x)}$$

$$= \frac{1}{1 \times (1+1)} = \frac{1}{2}$$

EXERCISE 17.2

I. One and Two marks questions:

Evaluate the following limits:

1. $\lim_{x\to 0} \dfrac{\sin ax}{bx}, a, b \neq 0$
2. $\lim_{x\to 0} \dfrac{\sin x^2}{x}$
3. $\lim_{x\to 0} \dfrac{\sin 4x}{\sin 2x}$

4. $\lim_{\theta \to 0} \frac{\sin 3\theta}{\tan 2\theta}$ 5. $\lim_{x \to 0} \frac{\tan 4x}{\tan 3x}$ 6. $\lim_{x \to 0} \frac{x\cos x + \sin x}{x + \tan x}$

7. $\lim_{x \to 0} \frac{\sin 3x + 7x}{4x + \sin 2x}$ 8. $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$ 9. $\lim_{x \to 0} \frac{1 - \cos x}{x}$

10. $\lim_{x \to 0} \frac{x\tan 4x}{1 - \cos 4x}$

II. Three and Five marks questions:

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{1 - \cos 2x}{\sin^2 2x}$ 2. $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$ 3. $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

4. $\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3}$ 5. $\lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x}$ 6. $\lim_{x \to 0} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{x}$

ANSWERS 17.2

I. 1. $\frac{a}{b}$ 2. 0 3. 2 4. $\frac{3}{2}$ 5. $\frac{4}{3}$

6. 1 7. $\frac{5}{3}$ 8. 1 9. 0 10. $\frac{1}{2}$

II. 1. $\frac{1}{2}$ 2. $\frac{4}{9}$ 3. 4 4. 4 5. 4 6. 1

17.9 Statement of some standard limits (without proofs)

1. $\lim_{x \to 0} \left(\frac{e^x - 1}{x} \right) = 1$ **2.** $\lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log_e a$ **5.** $\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = e$

3. $\lim_{x \to 0} (1 + x)^{1/x} = e$ **4.** $\lim_{x \to 0} \frac{1}{x} \log(1 + x) = 1$ **6.** $\lim_{n \to \infty} \left(1 + \frac{x}{n} \right)^n = e^x$

Example 7 :

Evaluate the following limits:

1. $\lim_{x \to 0} \left(\frac{e^{3x} - 1}{x} \right) = \lim_{x \to 0} \left(\frac{e^{3x} - 1}{3x} \times 3 \right)$

$$= \quad 3 \times \lim_{x \to 0} \left(\frac{e^{3x} - 1}{3x} \right)$$

$$= \quad 3(1) \quad = \quad 3$$

2. $$\lim_{x \to 0} \left(\frac{e^{-x} - 1}{x} \right) \quad = \quad \lim_{y \to 0} \left(\frac{e^{y} - 1}{-y} \right) \text{ where } -x = y$$

$$= \quad -\lim_{y \to 0} \left(\frac{e^{y} - 1}{y} \right) \quad \text{or} \quad -\lim_{x \to 0} \left(\frac{e^{-x} - 1}{-x} \right)$$

$$= \quad -1$$

3. $$\lim_{x \to 0} \left(\frac{e^{x} - e^{-x}}{x} \right) \quad = \quad \lim_{x \to 0} \left\{ \frac{(e^{x} - 1) - (e^{-x} - 1)}{x} \right\}$$

$$= \quad \lim_{x \to 0} \left\{ \left(\frac{e^{x} - 1}{x} \right) - \left(\frac{e^{-x} - 1}{x} \right) \right\}$$

$$= \quad \lim_{x \to 0} \left(\frac{e^{x} - 1}{x} \right) - \lim_{x \to 0} \left(\frac{e^{-x} - 1}{x} \right)$$

$$= \quad 1 - \lim_{y \to 0} \left(\frac{e^{y} - 1}{-y} \right), \text{ where } -x = y \left[\because \lim_{x \to 0} \left(\frac{e^{x} - 1}{x} \right) = 1 \right]$$

$$= \quad 1 + \lim_{y \to 0} \left(\frac{e^{y} - 1}{y} \right)$$

$$= \quad 1 + 1 \quad = \quad 2$$

4. $$\lim_{x \to 0} \left[\frac{e^{x} + e^{-x} - 2}{x^2} \right] \quad = \quad \lim_{x \to 0} \left[\frac{e^{2x} + 1 - 2e^{x}}{x^2 e^{x}} \right]$$

$$= \quad \lim_{x \to 0} \left[\left(\frac{e^{x} - 1}{x} \right)^2 \cdot \frac{1}{e^{x}} \right] \quad = \quad \lim_{x \to 0} \left(\frac{e^{x} - 1}{x} \right)^2 \cdot \lim_{x \to 0} \frac{1}{e^{x}}$$

$$= \quad 1^2 \times \frac{1}{e^0} \quad = \quad 1 \times 1 \quad = \quad 1$$

5. $\lim\limits_{x\to 0}\left(\dfrac{3^x-2^x}{x}\right)$

$$= \lim_{x\to 0}\left\{\frac{(3^x-1)-(2^x-1)}{x}\right\} = \lim_{x\to 0}\left(\frac{3^x-1}{x}\right)-\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)$$

$$= \log 3 - \log 2$$

$$= \log\left(\frac{3}{2}\right)$$

6. $\lim\limits_{x\to 0}\left(\dfrac{3^{2x}-1}{2^{3x}-1}\right)$

$$= \mathop{lt}_{x\to 0}\left\{\frac{\dfrac{3^{2x}-1}{2x}\cdot 2x}{\dfrac{2^{3x}-1}{3x}\cdot 3x}\right\}$$

$$= \frac{2}{3}\cdot\frac{\lim\limits_{x\to 0}\left(\dfrac{3^{2x}-1}{2x}\right)}{\lim\limits_{x\to 0}\left(\dfrac{2^{3x}-1}{3x}\right)} = \frac{2}{3}\cdot\frac{\log 3}{\log 2}$$

$$= \frac{\log 3^2}{\log 2^2} = \frac{\log 9}{\log 8}$$

7. $\lim\limits_{x\to 1}\dfrac{\log x}{x-1}$

put $x = y+1$ As $x\to 1$ we have $y\to 0$

$$\therefore \mathop{lt}_{x\to 1}\frac{\log x}{x-1} = \lim_{y\to 0}\frac{\log(1+y)}{y} = 1 \quad \text{(from std Lt)}$$

8. $\lim\limits_{n\to\infty}\left(\dfrac{n+1}{n}\right)^{3n}$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{3} = e^3$$

9. $\lim\limits_{x\to0}\frac{2}{x}\log(1+x)$

$$= 2\cdot\lim_{x\to0}\frac{1}{x}\log(1+x) \quad = \quad 2\cdot(1) \qquad \left(\because \lim_{x\to0}\frac{1}{x}\log(1+x)=1\right)$$

$$= 2$$

10. $\lim\limits_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{x+3}$

$$= \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{x+3+1-1}$$

$$= \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{x-1} \quad \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{4}$$

$$= \lim_{x\to\infty}\left\{\left(1+\frac{4}{x-1}\right)^{\frac{x-1}{4}}\right\}^{4} \times \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{4}$$

$$= e^4\times1 \quad = \quad e^4$$

EXERCISE 17.3

I. One and Two marks questions:

Evaluate the following limits:

1. $\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^{n}$
2. $\lim\limits_{x\to0}(1+3x)^{\frac{1}{x}}$
3. $\underset{n\to0}{lt}\left(\frac{n+3}{3}\right)^{\frac{2}{n}}$
4. $\lim\limits_{x\to0}\frac{e^{-3x}-1}{x}$
5. $\lim\limits_{x\to0}\left(\frac{2^x-1}{3x}\right)$

II. Three marks questions:

Evaluate the following limits:

1. $\lim\limits_{x\to0}\left[\frac{2^x-1}{\sqrt{1+x}-1}\right]$
2. $\lim\limits_{x\to0}\frac{9^x+9^{-x}-2}{x^2}$
3. $\lim\limits_{x\to\infty}\left(\frac{x-3}{x+3}\right)^{x+3}$
4. $\lim\limits_{x\to0}\left(\frac{3^{2x}-2^{3x}}{x}\right)$
5. $\lim\limits_{x\to0}\left(\frac{3^{2+x}-9}{x}\right)$

ANSWERS 17.3

I. 1. e^2 2. e^3 3. $e^{\frac{2}{3}}$ 4. -3 5. $\frac{1}{3}\log_e^2$

II. 1. $2\log 2$ 2. $(\log 9)^2$ 3. $\frac{1}{e^6}$ 4. $\log\left(\frac{9}{8}\right)$ 5. $9\log 3$

17.10 Limit at infinity and infinite limits:

(i) Meaning of $x \to \infty$:

If x is a variable such that it can take any real value howsoever large, then we say that the variable x approaches $+\infty$ and write this as $x \to +\infty$

(ii) Meaning of $x \to \infty$:

If x is a variable such that it can take any real value howsoever small, then we say that the variable x approaches $-\infty$ and write this as $x \to -\infty$

(iii) Infinite limit of a function:

Let *f(x)* be a function of x. If the value of *f(x)* can be made greater than any pre-assigned number by taking x close to a, then we say that the function *f(x)* becomes positively infinite as x approaches a. We write this as $\lim\limits_{x \to a} f(x) = +\infty$

Similarly if the value of *f(x)* can be made less than any pre-assigned number by taking x close to a, then we say that the function *f(x)* becomes negatively infinite as x approaches a.

We write this as $\lim\limits_{x \to a} f(x) = -\infty$.

Corollary:

1. $\lim\limits_{x \to \infty}\left(\frac{1}{x}\right) = 0$

we have $x \to \infty$

$\therefore$ The variable x can be made as large as we like.

$\therefore$ $\frac{1}{x}$ is positive and grows smaller and smaller as x becomes larger and larger.

$$\therefore \quad \frac{1}{x} \to 0 \quad \text{i.e. } \lim_{x \to \infty}\left(\frac{1}{x}\right) = 0$$

2. $\lim\limits_{x \to -\infty}\left(\frac{1}{x}\right) = 0$

we have $x \to -\infty$

$\therefore$ The variable x can be made as small as we like.

$\therefore \left(\frac{1}{x}\right)$ is negative and numerically grows smaller and smaller as x becomes smaller and smaller.

$\therefore \quad \frac{1}{x} \to 0 \qquad$ i.e. $\lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0$

Working rule for evaluating the limit when $x \to \infty$

Rule 1: Write the given expression in the form of rational function i.e. $p(x)/q(x)$ where $p(x), q(x)$ are polynomials and $q(x) \neq 0$.

Rule 2: Divide the numerator and denominator by the highest power of x and simplify and then take the limit.

Example 8 : Evaluate the following limits.

1. $\lim_{x \to \infty} \frac{3x^2 - 4x + 2}{2x^2 + 5x + 7}$

Here the highest power of x in numerator and denominator is 2.

Divide the numerator and denominator by x^2

$$\therefore \quad \lim_{x \to \infty} \left(\frac{3x^2 - 4x + 2}{2x^2 + 5x + 7}\right) = \lim_{x \to \infty} \left(\frac{3 - \frac{4}{x} + \frac{2}{x^2}}{2 + \frac{5}{x} + \frac{7}{x^2}}\right)$$

$$= \frac{3 - 0 + 0}{2 - 0 + 0} = \frac{3}{2}$$

2. $\lim_{x \to \infty} \frac{x^7 + x^3 + 1}{x^8 - 3x^6 + 5}$

Here, Degree of Nr.(7) < Degree of Dr. (8)

$\therefore$ we divide the numerator and the denominator by x^8.

$$\therefore \quad \lim_{x \to \infty} \frac{x^7 + x^3 + 1}{x^8 - 3x^6 + 5} = \lim_{x \to \infty} \frac{\frac{1}{x} + \frac{1}{x^5} + \frac{1}{x^8}}{1 - \frac{3}{x^2} + \frac{5}{x^8}}$$

$$= \frac{0 + 0 + 0}{1 - 0 + 0} = 0$$

3. $\lim_{x \to \infty} \frac{3x^2 + x + 5}{7x - 9}$

Here, Degree of Nr.(2) > Degree of Dr. (1)

$$\therefore \lim_{x\to\infty} \frac{3x^2+x+5}{7x-9} = \lim_{x\to\infty} \frac{3x^2\left(1+\frac{1}{3x}+\frac{5}{3x^2}\right)}{7x\left(1-\frac{9}{x}\right)}$$

$$= \lim_{x\to\infty} \left[\frac{3}{7}x\cdot\frac{1+\frac{1}{3x}+\frac{5}{3x^2}}{1-\frac{9}{x}}\right]$$

$$= \infty \times \frac{1+0+0}{1-0} = \infty$$

4. $\lim\limits_{x\to\infty} \dfrac{(2x-1)^{20}(3x-1)^{30}}{(2x+1)^{50}}$

Here, degree of Nr.(50) = degree of Dr.(50)

$\therefore$ we divide the Nr. and the Dr. by x^{50}

$$\therefore \lim_{x\to\infty} \frac{(2x-1)^{20}(3x-1)^{30}}{(2x+1)^{50}} = \lim_{x\to\infty} \frac{\frac{(2x-1)^{20}}{x^{20}}\cdot\frac{(3x-1)^{30}}{x^{30}}}{\frac{(2x+1)^{50}}{x^{50}}}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{2x-1}{x}\right)^{20}\left(\frac{3x-1}{x}\right)^{30}}{\left(\frac{2x+1}{x}\right)^{50}} = \lim_{x\to\infty} \frac{\left(2-\frac{1}{x}\right)^{20}\left(3-\frac{1}{x}\right)^{30}}{\left(2+\frac{1}{x}\right)^{50}}$$

$$= \frac{(2-0)^{20}(3-0)^{30}}{(2+0)^{50}} = \frac{2^{20}\times 3^{30}}{2^{50}}$$

$$= \frac{3^{30}}{2^{30}} = \left(\frac{3}{2}\right)^{30}$$

5. $\lim\limits_{n\to\infty} \dfrac{1+2+3+........+n}{(n+5)^2}$

$$= \lim_{n\to\infty} \frac{n(n+1)}{2(n+5)^2} \qquad \left[\because \sum n = 1+2+3+....+n = \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1\cdot\left(1+\frac{1}{n}\right)}{2\left(1+\frac{5}{n}\right)^2}$$

$$= \frac{1+0}{2(1+0)} \quad = \quad \frac{1}{2}$$

6. $\lim_{x\to\infty} \frac{1^3+2^3+3^3+\dots\dots+n^3}{(\sum n)\ (2n^2+3n+1)}$

$$= \left(\because\ 1^3+2^3+\dots.+n^3 = \frac{n^2(n+1)^2}{4} = \sum n^3\right)$$

$$= \lim_{n\to\infty} \frac{(n+1)^2\cdot n^2\cdot 2}{4\cdot n\cdot(n+1)(2n^2+3n+1)}$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{2(2n^2+3n+1)} \quad = \quad \lim_{x\to\infty} \frac{1\left(1+\frac{1}{n}\right)}{2\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}$$

$$= \frac{1+0}{2(2+0+0)} \quad = \quad \frac{1}{4}$$

7. $\lim_{x\to\infty}\left(\sqrt{x+2}-\sqrt{x}\right)$

$$= \lim_{x\to\infty}\left[\sqrt{x+2}-\sqrt{x}\times\frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}\right]$$

$$= \lim_{x\to\infty} \frac{(x+2)-x}{\sqrt{x+2}+\sqrt{x}} \quad = \quad \lim_{x\to\infty} \frac{2}{\sqrt{x+2}+\sqrt{x}} \qquad \div \text{ both Nr and Dr by } \sqrt{x}$$

$$= \lim_{x\to\infty} \frac{2/\sqrt{x}}{\sqrt{\frac{x+2}{x}}+\sqrt{\frac{x}{x}}} \quad = \quad \underset{x\to\infty}{lt} \frac{\frac{2}{\sqrt{x}}}{\sqrt{1+\frac{2}{x}}+1}$$

$$= \frac{0}{\sqrt{1+0}+1} \quad = \quad \frac{0}{2} \quad = \quad 0$$

8. $\lim_{x\to\infty}\left[\sqrt{x}\left(\sqrt{x+5}-\sqrt{x}\right)\right]$

$$= \lim_{x\to\infty}\frac{\sqrt{x}\left(\sqrt{x+5}-\sqrt{x}\right)\left(\sqrt{x+5}+\sqrt{x}\right)}{\sqrt{x+5}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{x}(x+5-x)}{\sqrt{x+5}+\sqrt{x}} \quad = \lim_{x\to\infty}\frac{5\sqrt{x}}{\sqrt{x+5}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{5}{\sqrt{1+5/x}+1} \quad = \frac{5}{\sqrt{1+0}+1} \quad = \frac{5}{2}$$

EXERCISE 17.4

I. Two marks questions:

1. $\lim_{x\to\infty}\frac{3x^2-4x+5}{2x^2+5x-1}$
2. $\lim_{x\to\infty}\frac{3x^3-4x+7}{2x^4-3x+6}$
3. $\lim_{x\to\infty}\frac{(3x+4)(4x+3)}{(2x-7)(x+4)}$
4. $\lim_{x\to\infty}\frac{(2x-1)^{30}(3x-1)^{30}}{(2x+4)^{60}}$
5. $\lim_{x\to\infty}\frac{x}{\sqrt{4x^2+1}-1}$
6. $\lim_{n\to\infty}\frac{1+2+3+.....+n}{n^2+3}$
7. $\lim_{n\to\infty}\frac{\sum n^2}{\left(n^2+2n+1\right)(n-1)}$
8. $\lim_{n\to\infty}\frac{\sum n^3}{n^2\cdot\sum n}$

II. Three marks questions:

1. $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$
2. $\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x+3}-\sqrt{x}\right)$
3. $\lim_{x\to\infty}x\left(\sqrt{4+x^2}-x\right)$
4. $\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$
5. $\lim_{x\to\infty}x\left(\sqrt{x^2+1}-x\right)$

ANSWERS 17.4

I. 1) $\frac{3}{2}$ 2) 0   3) 6   4) $\left(\frac{3}{2}\right)^{30}$ 5) $\frac{1}{2}$

6) $\frac{1}{2}$ 7) $\frac{1}{3}$ 8) $\frac{1}{2}$

II. 1) 0   2) $\frac{3}{2}$ 3) 2   4) $\frac{5}{2}$ 5) $\frac{1}{2}$

17.11 Left Hand and Right Hand Limit (one sided limits)

The limit of the function *f(x)*, when *x* tends to *'a'* from the left side of *'a'* is called the left hand limit of *f(x)* as $x \to a$ $\quad \therefore\ LHL = \underset{x \to a-}{lt}\ f(x)$

The limit of the function *f(x)*, when *x* tends to *'a'* from the right side of *'a'* is called the right hand limit of *f(x)* as $x \to a$ $\quad \therefore\ RHL = \lim_{x \to a^+} f(x)$

WORKING RULE

I. To find the left hand limit.

1. Put $x = a - h$ in *f(x)*, where *h* is a small positive quantity.
2. Take the limit of *f(a–h)* as $h \to 0$ i.e. $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

II. To find the right hand limit

1. put $x = a + h$ in *f(x)*, where *h* is a small positive quantity.
2. Take the limit of *f(a+h)* as $h \to 0$ i.e. $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$

17.12 Continuity of a Function

A function $y = f(x)$ is said to be continuous at $x = a$ if

(i) *f(a)* exists (ii) $\lim_{x \to a} f(x) = f(a)$

If a function *f(x)* is not continuous at $x = a$ then it is said to be discontinuous at $x = a$

Note: The discontinuity of a function *f(x)* at a point $x = a$ can arise in any one of the following ways

1. The function *f(x)* is not defined at $x = a$
   i.e. *f(a)* does not exists.
2. $\lim_{x \to a} f(x)$ exists and *f(a)* exists

   but $\lim_{x \to a} f(x) \neq f(a)$

Example 9 :

1. Show that the function

Solution : $f(x) = |x|$ is continuous at $x = 0$

$f(x) = |x|$

$$= \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

LHL $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0$

($\because$ *f(x)* $= -x$ when $x < 0$)

RHL $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x) = 0$

($\because$ *f(x)* $= x$ when $x < 0$)

$\therefore$ LHL $=$ RHL $= f(0) = 0$

$\Rightarrow$ *f(x)* is continuous at $x = 0$

2. Find $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$

Solution : LHL $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1}(x^2 - 1) = 1 - 1 = 0$

$\left(\because f(x) = x^2 - 1,\ \ x < 1\right)$

RHL $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1}(-x^2 - 1)$

$= -1 - 1 = -2$

Thus LHL $\neq$ RHL

$\Rightarrow$ The *f (x)* is discontinuous at $x = 1$ and the limit does not exist at $x = 1$

3. Prove that

$$f(x) = \begin{cases} x^2 + 1 & when \quad x < 2 \\ 5 & when \quad x = 2 \\ 4x - 3 & when \quad x > 2 \end{cases}$$ is continuous at $x = 2$

Solution : LHL : $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^2 + 1)$

$= \lim_{h\to 0}(2 - h)^2 + 1 = 5$

RHL : $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(4x - 3)$

$= \lim_{h\to 0}\{4(2 + h) - 3\} = 5$

LHL $=$ RHL $\therefore \lim_{x\to 2} f(x) = 5$

By data $f(2) = 5$

i.e. $\lim_{x\to 2} f(x) = f(2)$

Hence *f(x)* is continuous at $x = 2$

4. Find k if

$$f(x)=\begin{cases}\dfrac{e^{5x}-1}{2x}, & x\neq 0\\ \dfrac{k+x}{2}, & x=0\end{cases}$$ is continuous at $x=0$

Solution : We have $f(0)=\dfrac{k+0}{2}=\dfrac{k}{2}$

$$\lim_{x\to 0}\frac{e^{5x}-1}{2x}=\frac{1}{2}\cdot\lim_{x\to 0}\frac{e^{5x}-1}{5x}\times 5$$

$$=\frac{5}{2}\cdot\lim_{x\to 0}\left(\frac{e^{5x}-1}{5x}\right)$$

$$=\frac{5}{2}(1)\quad=\quad\frac{5}{2}$$

Since $f(x)$ is continuous at $x=0$

$$\lim_{x\to 0}f(x)=f(0)$$

$$\Rightarrow\ \frac{5}{2}=\frac{k}{2}\quad\Rightarrow\quad k=5$$

5. If $f(x)=\begin{cases}\dfrac{x^4-256}{x-4} & x\neq 4\\ a & x=4\end{cases}$ is continuous at $x=4$, find a

Solution : $\lim\limits_{x\to 4}f(x)=\lim\limits_{x\to 4}\left(\dfrac{x^4-256}{x-4}\right)$

$$=\lim_{x\to 4}\left(\frac{x^4-4^4}{x-4}\right)$$

$$=\quad 4\cdot 4^3\quad=\quad 256$$

Since $f(x)$ is continuous at $x=4$

$$\lim_{x\to 4}f(x)=f(4)\ \Rightarrow\ 256\ =\ a$$

or $a=256$

Exercise 17.5

I. 2 and 3 marks question:

1. Show $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & when \quad x \neq 5 \\ 10 & when \quad x = 5 \end{cases} \quad \text{is continuous at } x = 5$$

2. Show that the function

$$f(x) \begin{cases} \dfrac{x^2 - 9}{x - 3} & when \quad x \neq 3 \\ 4 & when \quad x = 3 \end{cases} \quad \text{is discontinuous at } x = 3$$

3. Define $f(0)$ so that $f(x) = \dfrac{x}{1 - \sqrt{1 - x}}$ becomes continuous at $x = 0$

4. Show that the function $f(x) = \begin{cases} (1 + 3x)^{1/x} & x \neq 0 \\ e^3 & x = 0 \end{cases}$ is continuous at $x = 0$

5. Find k for which

$$f(x) = \begin{cases} k + x, & x = 1 \\ 4x + 3, & x \neq 1 \end{cases} \quad \text{is continuous at } x = 1$$

6. If the function

$$f(x) = \begin{cases} (1 + 2x)^{1/x}, & x \neq 0 \\ k \quad , & x = 0 \end{cases} \quad \text{is continuous at } x = 0 \text{, find } k$$

7. Discuss the continuity of $f(x) = \begin{cases} 3x^2 + 1 & if \quad x < 1 \\ 4 & if \quad x = 1 \\ 2x + 2 & if \quad x > 1 \end{cases}$ at $x = 1$

8. Find the value of k if the function $f(x) = \begin{cases} \dfrac{e^{2x} - 1}{x} & x \neq 0 \\ k & x = 0 \end{cases}$ is continuous at $x = 0$

Answers 17.5

3) 2 5) 6 6) e^2 7) $f(x)$ is continuous at $x = 1$ 8) 2

* * * *

Chapter 18 DIFFERENTIAL CALCULUS

18.1 Introduction:

Calculus is the primary mathematical tool for dealing with change. The concept of 'derivative' is a basic tool in science of calculus. Calculus is essentially concerned with the rate of change of dependent variable with respect to an independent variable.

18.2 Increment:

An increment is any change in a variable

$\therefore$ Increment = Final value – Initial value

Ex: If a variable x changes from

(i) 3.01 to 3.02 then increment = 0.01

(ii) 3.01 to 2.99 then increment = –0.02

Note: 1) The increment of x is denoted by Δx or δx

2) Δx is one symbol and not a product of Δ and x.

3) The ratio $\frac{\Delta y}{\Delta x}$ is called incrementary ratio or average rate of change of $\boldsymbol{y}$ w.r.t. $\boldsymbol{x}$ per unit.

18.3 Derivative of a function:

Let $y = f(x)$ be a given continuous function. Then the value of y depends upon the value of x and it changes with a change in the value of x.

Let Δy be an increment in y corresponding to an increment Δx in x then

$y = f(x)$ and $y + \Delta y = f(x + \Delta x)$ on subtraction,

we get $\Delta y = f(x + \Delta x) - f(x)$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

This $\frac{\Delta y}{\Delta x}$ is called Incremental ratio

$\frac{f(x+\Delta x) - f(x)}{\Delta x}$ is called difference quotient

$$\text{so, } \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

The above limit, if it exists finitely is called the **derivative** or **differential coefficient** of $y = f(x)$ with respect to x and it is denoted by

$$\frac{dy}{dx} \text{ or } \frac{d}{dx}\left[f(x)\right] \text{ or } \left[f^{1}(x)\right] \text{ or } y_1 \text{ or } y^1$$

The process of finding the derivative is known as **differentiation.** The above method is also known as **differentiation from the first principle** or ab initio or by delta method.

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

18.4 Derivative at a point:

The value of $f'(x)$ obtained by putting $x = a$ is called the derivative of $f(x)$ at $x = a$ and is denoted by $f'(a)$ or $\left(\frac{dy}{dx}\right)_{x=a}$ and is defined as

$$\frac{dy}{dx} \text{ or } f'(a) = \lim_{h \to 0}\frac{f(a+h) - f(a)}{h}$$

If $f'(a)$ exists, we say that $f(x)$ is differentiable at $x = a$.

If $f'(x)$ exists for every value of x in the domain of the function, we say that $f(x)$ is differentiable.

Definition (Left hand derivative):

The left hand derivative of a function $y = f(x)$ at $x = a$ is denoted and defined by,

$$L\{f'(a)\} = \lim_{x \to a^-}\frac{f(x) - f(a)}{x - a}, \text{ provided the limit exists.}$$

Definition (Right hand derivative):

The right hand derivative of a function $y = f(x)$ at $x = a$ is denoted and defined by

$$R\{f'(a)\} = \lim_{x \to a^+}\frac{f(x) - f(a)}{x - a}, \text{ provided the limit exists.}$$

18.5 Differentiability:

A function $y = f(x)$ is said to be differentiable at $x = a$ if $L\{f'(a)\}$ and $R\{f'(a)\}$ exist and are equal. This is denoted by $f'(a)$ or $\left(\frac{dy}{dx}\right)_{x=a}$

If for a function $y = f(x)$, any one of $L\{f'(a)\}$ or $R\{f'(a)\}$ (or both) does not exists, or even they exist but not equal then we say $f(x)$ is not differentiable at $x = a$.

18.6 Relation between continuity and differentiability

THEOREM: If a function f is differentiable at a point c, then it is also continuous at that point.

Proof: Since f is differentiable at c, we have

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c)$$

But for $x\neq c$, we have

$$f(x)-f(c)=\frac{f(x)-f(c)}{x-c}\cdot x-c$$

$$\therefore\ \lim_{x\to c}\left[f(x)-f(c)\right]=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\cdot x-c\right]$$

or $\lim_{x\to c}\left[f(x)\right]-\lim_{x\to c}\left[f(c)\right]=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]\cdot\lim_{x\to c}(x-c)$

$$= f'(c)\cdot 0 \quad = \quad 0$$

$$\Rightarrow \lim_{x\to c}f(x)=f(c)$$

Hence f is continuous at $x = c$

Corollary 1 : Every differentiable function is continuous.

We remark that the converse of the above statement is not true. Indeed we have seen that the function defined by $f(x)=|x|$ is a continuous function.

Consider the left hand limit

$$\lim_{h\to 0^-}\frac{f(0-h)-f(0)}{h}=\frac{-h}{h}=-1$$

The right hand limit $\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\frac{h}{h}=1$

Since the above left and right hand limits at 0 are not equal, $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist and hence f is not differentiable at 0. Thus f is not a differentiable function.

Note:

Differentiability implies continuity but continuity does not imply differentiability.

18.7 Differentiation of some important standard function from first principles:

1) To differentiate a constant function from first principles.

Let $y = f(x) = k$ where k is constant

$\therefore \; y + \Delta y = f(x + \Delta x) = k$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{k - k}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0$$

$$\Rightarrow \frac{dy}{dx} = 0$$

Thus the derivative of a constant function is zero.

i.e. $y = f(x) = k \qquad \Rightarrow \qquad \boxed{\frac{d(k)}{dx} = 0}$

Ex : (i) $\frac{d}{dx}(2) = 0$ (ii) $\frac{d}{dx}\left(\frac{2}{\pi}\right) = 0$ (iii) $\frac{d}{dx}\left(e^{\pi}\right) = 0$

2) To differentiate x^n from the first principles

Let $y = f(x) = x^n$

$\therefore \; y + \Delta y = f(x + \Delta x) = (x + \Delta x)^n$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x}\right]$$

put $x + \Delta x = z$

$\Delta x = z - x$

As $\Delta x \to o$

$z \to x$

$$= \lim_{z \to x}\left[\frac{z^n - x^n}{z - x}\right] \qquad \left(\because \underset{x \to a}{\text{lt}} \frac{x^n - a^n}{x - a} = n, a^{n-1}\right)$$

$$\Rightarrow \boxed{\frac{d}{dx}(x^n) = n \cdot x^{n-1}}$$

Ex: (i) $\frac{d}{dx}(x^9) = 9 \cdot x^{9-1} = 9x^8$

(ii) $\frac{d}{dx}\left(\sqrt[3]{x}\right) = \frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3} \cdot x^{1/3 - 1} = \frac{1}{3} \cdot x^{-2/3}$

Note : Consider the function $y = \frac{1}{x^n}$, then

$$y = \frac{1}{x^n} \Rightarrow y = x^{-n} \Rightarrow \frac{dy}{dx} = -n \cdot x^{-n-1} \Rightarrow \frac{dy}{dx} = \frac{-n}{x^{n+1}}$$

Thus if $y = \frac{1}{x^n}$ then $\boxed{\frac{dy}{dx} = \frac{-n}{x^{n+1}}}$

Ex: (i) $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{-3}{x^4}$

(ii) $\frac{d}{dx}\left(\frac{1}{\sqrt[5]{x^4}}\right) = \frac{d}{dx}\left(\frac{1}{x^{4/5}}\right) = \frac{-4}{5x^{9/5}}$

3) To differentiate e^x from the first principes.

Let $y = f(x) = e^x$

$\therefore\ y + \Delta y = f(x + \Delta x) = e^{x + \Delta x}$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{e^{x + \Delta x} - e^x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[e^x\left(\frac{e^{\Delta x} - 1}{\Delta x}\right)\right]$$

$$= e^x \cdot \lim_{\Delta x \to 0}\left(\frac{e^{\Delta x}-1}{\Delta x}\right) = e^x \cdot 1 \qquad \left[\because \lim_{\Delta x \to 0}\frac{e^x-1}{x}=1\right]$$

$$= \boxed{\frac{d}{dx}\left(e^x\right)=e^x}$$

4) To differentiate a^x where ($a > 0$) and $a \neq 1$ from the first principles.

Let $y = f(x) = a^x$

$\therefore\ y + \Delta y = f(x+\Delta x) = a^{x+\Delta x}$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{a^{x+\Delta x}-a^x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[a^x\left(\frac{a^{\Delta x}-1}{\Delta x}\right)\right] \qquad \left[\because \lim_{x \to 0}\frac{a^x-1}{x}=\log a\right]$$

$$= a^x \cdot \lim_{\Delta x \to 0}\left[\frac{a^{\Delta x}-1}{\Delta x}\right]$$

$$\therefore \boxed{\frac{d}{dx}\left(a^x\right)=a^x \log a}$$

5) To differentiate $\log x$ ($x > 0$) from the first principles.

Let $y = f(x) = \log x$

$\therefore\ y + \Delta y = f(x+\Delta x) = \log(x+\Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\log(x+\Delta x)-\log x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x}\cdot\log\left(\frac{x+\Delta x}{x}\right)\right] \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x}\cdot\log\left(1+\frac{\Delta x}{x}\right)\right]$$

put $\dfrac{\Delta x}{x} = h$

$\Delta x = h \cdot x$

As $\Delta x \to 0$

$h \to 0$

$$= \lim_{h \to 0}\left[\frac{1}{h\cdot x}\cdot\log(1+h)\right]$$

$$= \frac{1}{x}\cdot\lim_{h \to 0}\left[\frac{1}{h}\log(1+h)\right] \qquad \left[\because \lim_{x\to 0}\frac{1}{x}\ \log(1+x)=1\right]$$

$$= \frac{1}{x}(1)$$

$$\therefore \boxed{\frac{d}{dx}(\log x) = \frac{1}{x}}$$

6) To differentiate '$\sin x$' **from the first principles.**

Let $y = f(x) = \sin x$

$\therefore\ y + \Delta y = f(x+\Delta x) = \sin(x+\Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[\frac{\sin(x+\Delta x)-\sin x}{\Delta x}\right]$$

$$\left[\text{using } \sin C - \sin D = 2\cos\frac{C+D}{2}\cdot\sin\frac{C-D}{2}\right]$$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\left[\frac{2\cos\left(\frac{x+\Delta x+x}{2}\right)\sin\left(\frac{x+\Delta x-x}{2}\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[2\cos\left(\frac{2x+\Delta x}{2}\right)\frac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\cdot\frac{1}{2}\right] \qquad \left[\because \lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1\right]$$

$$= 2\cos\left(\frac{2x}{2}\right)\cdot 1\cdot\frac{1}{2}$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

7) To differentiate cos *x* from the first principles.

Let $y = f(x) = \cos x$

$\therefore \ y + \Delta y = f(x + \Delta x) = \cos(x + \Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\cos(x+\Delta x) - \cos x}{\Delta x}\right]$$

$$\left[\text{using } \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{-2\sin\left(\frac{x+\Delta x+x}{2}\right)\sin\left(\frac{x+\Delta x-x}{2}\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[-2\sin\left(\frac{2x+\Delta x}{2}\right)\cdot \sin\frac{\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\cdot\frac{1}{2}\right] \qquad \left[\because \ \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1\right]$$

$$= -2\sin\left(\frac{2x}{2}\right)\cdot 1 \cdot \frac{1}{2}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

8. To differentiate 'tan *x*' from the first principles.

Let $y = f(x) = \tan x$

$\therefore \ y + \Delta y = f(x + \Delta x) = \tan(x + \Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\tan(x+\Delta x) - \tan x}{\Delta x}\right]$$

$\left[\text{using} \tan A - \tan B = \tan(A-B)\left[1+\tan A \ \tan B\right]\right]$ $\left[\begin{array}{l}\because A = x+\Delta x \\ \quad B = x\end{array}\right]$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{\tan(x+\Delta x - x)\{1+\tan(x+\Delta x)\tan x\}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\tan \Delta x}{\Delta x}\{1+\tan(x+\Delta x)\cdot \tan x\}\right] \qquad \left[\because \lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1\right]$$

$$= 1\cdot\{1+\tan x \cdot \tan x\} \quad = \quad 1+\tan^2 x$$

$$\therefore \boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

9) To differentiate '$\cot x$' from first principles.

Let $y = f(x) = \cot x$

$\therefore \ y + \Delta y = f(x+\Delta x) = \cot(x+\Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\cot(x+\Delta x) - \cot x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left\{\frac{\dfrac{\cos(x+\Delta x)}{\sin(x+\Delta x)} - \dfrac{\cos x}{\sin x}}{\Delta x}\right\}$$

$$= \lim_{\Delta x \to 0}\frac{\sin x \cdot \cos(x+\Delta x) - \cos x \cdot \sin(x+\Delta x)}{\sin(x+\Delta x)\sin x \cdot \Delta x}$$

$[\because \ \sin A \cos B - \cos A \sin B = \sin(A-B)]$

$$= \frac{-1}{\sin x}\cdot \lim_{\Delta x \to 0}\frac{\sin \Delta x}{\Delta x}\cdot\frac{1}{\lim\limits_{\Delta x \to 0}\sin(x+\Delta x)}$$

$$= \frac{-1}{\sin x}\cdot 1 \cdot \frac{1}{\sin x} \quad = \quad -\operatorname{cosec}^2 x$$

$$\therefore \boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

10. To differentiate 'cosec x' from first principles.

Let $y = f(x) = \operatorname{cosec} x$

$\therefore \; y + \Delta y = f(x + \Delta x) = \operatorname{cosec}(x + \Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\operatorname{cosec}(x+\Delta x) - \operatorname{cosec} x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\frac{\dfrac{1}{\sin(x+\Delta x)} - \dfrac{1}{\sin x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\frac{\sin x - \sin(x+\Delta x)}{\sin(x+\Delta x)\cdot \sin x \cdot \Delta x}$$

$$= \lim_{\Delta x \to 0}\frac{-2\cos\left(x + \dfrac{\Delta x}{2}\right)\sin\left(\dfrac{\Delta x}{2}\right)}{\sin(x+\Delta x)\cdot \sin x \cdot \Delta x}$$

$$\left[\therefore \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \frac{-1}{\sin x}\cdot \lim_{\Delta x \to 0}\cos\left(x + \frac{\Delta x}{2}\right)\cdot \lim_{\Delta x \to 0}\frac{\sin\left(\dfrac{\Delta x}{2}\right)}{\left(\dfrac{\Delta x}{2}\right)}\cdot \lim_{\Delta x \to 0}\frac{1}{\sin(x+\Delta x)}$$

$$= \frac{-1}{\sin x}\times \cos x \times 1 \times \frac{1}{\sin x} \quad = \quad -\operatorname{cosec} x \cdot \cot x \qquad \left(\therefore \frac{\cos x}{\sin x} = \cot x\right)$$

$$\therefore \boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x}$$

11. To differentiate sec x from first principles.

Let $y = f(x) = \sec x$

$\therefore \; y + \Delta y = f(x + \Delta x) = \sec(x + \Delta x)$

By definition,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\sec(x+\Delta x) - \sec x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\frac{\dfrac{1}{\cos(x+\Delta x)} - \dfrac{1}{\cos x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\frac{\cos x - \cos(x+\Delta x)}{\Delta x \cdot \cos(x+\Delta x)\cdot \cos x}$$

$$= \lim_{\Delta x \to 0}\frac{2\sin\left(x+\dfrac{\Delta x}{2}\right)\sin\left(\dfrac{\Delta x}{2}\right)}{\cos(x+\Delta x)\cdot \cos x \cdot \Delta x}$$

$$\left[\because \cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)\right]$$

$$= \frac{1}{\cos x}\cdot \lim_{\Delta x \to 0}\sin\left(x+\frac{\Delta x}{2}\right)\lim_{\Delta x \to 0}\frac{1}{\cos(x+\Delta x)}\cdot \lim_{\Delta x \to 0}\frac{\sin\left(\dfrac{\Delta x}{2}\right)}{\left(\dfrac{\Delta x}{2}\right)}$$

$$= \frac{1}{\cos x}\times \sin x \times \frac{1}{\cos x}\times 1 \quad = \sec x \cdot \tan x \qquad \left(\because \frac{\sin x}{\cos x} = \tan x\right)$$

$$\therefore \boxed{\frac{d}{dx}(\sec x) = \sec x \cdot \tan x}$$

12. To differentiate $\sqrt{x}$ from first principles

Let $y = f(x) = \sqrt{x}$

$\therefore \; y + \Delta y = f(x+\Delta x) = \sqrt{x+\Delta x}$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{1/2} - x^{1/2}}{(x+\Delta x) - x}$$

$$= \frac{1}{2} \cdot x^{\frac{1}{2}-1} \qquad \left[\because \lim_{x \to a}\left(\frac{x^n - a^n}{x-a}\right) = n \cdot a^{n-1}\right]$$

$$\boxed{\therefore \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}}$$

List of formulae of Derivatives of Basic Functions:

$y = f(x)$ (Function)	$\frac{dy}{dx} = f'(x)$ (Derivative)
1. $y = k$	$\frac{dy}{dx} = 0$
2. $y = x^n$	$\frac{dy}{dx} = nx^{n-1}$
3. $y = \frac{1}{x^n}$	$\frac{dy}{dx} = \frac{-n}{x^{n+1}}$
4. $\frac{1}{x}$	$\frac{dy}{dx} = \frac{-1}{x^2}$
5. $\sqrt{x}$	$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$
6. $\sin x$	$\frac{dy}{dx} = \cos x$
7. $\cos x$	$\frac{dy}{dx} = -\sin x$
8. $\tan x$	$\frac{dy}{dx} = \sec^2 x$
9. $\operatorname{cosec} x$	$\frac{dy}{dx} = -\operatorname{cosec} x \cot x$
10. $\sec x$	$\frac{dy}{dx} = \sec x \cdot \tan x$

11. $\cot x$	$\frac{dy}{dx} = -\text{cosec}^2 x$
12. e^x	$\frac{dy}{dx} = e^x$
13. $\log x$	$\frac{dy}{dx} = \frac{1}{x}$
14. a^x	$\frac{dy}{dx} = a^x \log_e a$

18.8 Algebra of Derivative of Functions:

Let f and g be two functions such that their derivatives are defined in a common domain. Then

(i) Derivative of sum of two functions is sum of the derivatives of the functions

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

(ii) Derivative of difference of two functions is difference of the derivative of the functions

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

(iii) Derivative of product of two functions is given by the following product rule.

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)$$

(iv) Derivative of quotient of two functions is given by the following quotient rule (when ever the denominator is non-zero)

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)}{[g(x)]^2}$$

(v) Derivative of scalar multiple of a function is the product of the constant and the derivative of the function

$$\frac{d}{dx}[k.f(x)] = k\cdot\frac{d}{dx}[f(x)]$$

Note 1) If $y = u \pm v \pm w \pm ....$ are function of x

then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm ......$

2) If $y = uvw.....$ are function of x

then $\frac{dy}{dx} = uv \cdot \frac{dw}{dx} + uw \cdot \frac{dv}{dx} + vw \cdot \frac{du}{dx}$

Example 1: Differentiate the following with respect to x :

1. $\mathbf{x^2 + \frac{4}{x^2} - \frac{2}{3}\tan x + 6e}$

Solution: Let $y = x^2 + \frac{4}{x^2} - \frac{2}{3}\tan x + 6e$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) + 4 \cdot \frac{d}{dx}\left(\frac{1}{x^2}\right) - \frac{2}{3}\frac{d}{dx}(\tan x) + 6 \cdot \frac{d}{dx}(e)$$

$$= 2x + 4\left(\frac{-2}{x^3}\right) - \frac{2}{3}\sec^2 x + 6(0)$$

$$= 2x - \frac{8}{x^3} - \frac{2}{3}\sec^2 x$$

2. $\mathbf{x^e + e^x + e^e}$

Solution:

Let $y = x^e + e^x + e^e$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^e\right) + \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^e\right)$$

$$= ex^{e-1} + e^x + 0$$

$$= ex^{e-1} + e^x$$

3. $\mathbf{2^x - x^2 - x + 2 - 3\log x - 4\sqrt{x}}$

Solution:

Let $y = 2^x - x^2 - x + 2 - 3\log x - 4\sqrt{x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(2^x\right) - \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(x) + \frac{d}{dx}(2) - 3\frac{d}{dx}(\log x) - 4\frac{d}{dx}\sqrt{x}$$

$$= 2^x \log_e 2 - 2x - 1 + 0 - 3\left(\frac{1}{x}\right) - 4\left(\frac{1}{2\sqrt{x}}\right)$$

$$= 2^x \log_e 2 - 2x - 1 - \frac{3}{x} - \frac{2}{\sqrt{x}}$$

4. $\dfrac{3x^2+2x+5}{\sqrt{x}}$

Solution:

Let $y=\dfrac{3x^2+2x+5}{\sqrt{x}}$

$$y = 3x^{3/2}+2x^{1/2}+5x^{-1/2}$$

[on dividing each term by $\sqrt{x}$)

$$\frac{dy}{dx}=3\cdot\frac{d}{dx}\left(x^{3/2}\right)+2\cdot\frac{d}{dx}\left(x^{1/2}\right)+5\cdot\frac{d}{dx}\left(x^{-1/2}\right)$$

$$= 3\cdot\frac{3}{2}x^{1/2}+2\cdot\frac{1}{2}x^{-1/2}+5\left(\frac{-1}{2}\right)x^{-3/2}$$

$$= \frac{9}{2}\sqrt{x}+\frac{1}{\sqrt{x}}-\frac{5}{2}x^{-3/2}$$

5. $\dfrac{3}{\sqrt[3]{x}}-\dfrac{5}{\cos x}+\dfrac{6}{\sin x}-\dfrac{2\tan x}{\sec x}+7$

Solution:

Let $y=\dfrac{3}{\sqrt[3]{x}}-\dfrac{5}{\cos x}+\dfrac{6}{\sin x}-\dfrac{2\tan x}{\sec x}+7$ $\qquad\left[\tan x=\dfrac{\sin x}{\cos x}\right]$

$$y = 3x^{-1/3}-5\sec x+6\ \text{cosec}\ x-2\sin x+7$$

$$\frac{dy}{dx}=3\cdot\frac{d}{dx}\left(x^{-1/3}\right)-5\cdot\frac{d}{dx}(\sec x)+6\cdot\frac{d}{dx}(\text{cosec}\ x)-2\cdot\frac{d}{dx}(\sin x)+\frac{d}{dx}(7)$$

$$=3\left(\frac{-1}{3}\right)x^{-4/3}-5\sec x\tan x-6\text{cosec}\ x\cot x-2\cos x+0$$

$$=\frac{-1}{x^{4/3}}-5\sec x\tan x-6\ \text{cosec}\ x\cot x-2\cos x$$

Example 2: **Differentiate the following w.r.t. x [By Product Rule]**

1. $x^9\cdot 9^x$

Solution: Let $y=x^9\cdot 9^x$

$$\frac{dy}{dx} = x^9 \cdot \frac{d}{dx}\left(9^x\right) + 9^x \cdot \frac{d}{dx}\left(x^9\right)$$

$$= x^9 \cdot 9^x \log 9 + 9^x \cdot 9x^8$$

2. $\mathbf{(x^2 - 2x + 1)(e^x + 4)}$

Solution: Let $y = (x^2 - 2x + 1)(e^x + 4)$

$$\frac{dy}{dx} = \left(x^2 - 2x + 1\right)\frac{d}{dx}(e^x + 4) + \left(e^x + 4\right)\frac{d}{dx}\left(x^2 - 2x + 1\right)$$

$$= \left(x^2 - 2x + 1\right)\left(e^x\right) + \left(e^x + 4\right)(2x - 2)$$

3. $\mathbf{\sec x \cdot \tan x}$

Solution:

Let $y = \sec x \tan x$

$$\frac{dy}{dx} = \sec x \cdot \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(\sec x)$$

$$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$$

$$= \sec x\left(\sec^2 x + \tan^2 x\right)$$

4. $\mathbf{\cot x \cdot \left(\sqrt{3} - 4e^x\right)}$

Solution:

Let $y = \cot x \cdot \left(\sqrt{3} - 4e^x\right)$

$$\frac{dy}{dx} = \cot x \cdot \frac{d}{dx}\left(\sqrt{3} - 4e^x\right) + \left(\sqrt{3} - 4e^x\right) \cdot \frac{d}{dx}(\cot x)$$

$$= \cot x\left(0 - 4e^x\right) + \left(\sqrt{3} - 4e^x\right)\left(-\text{cosec}^2 x\right)$$

$$= -4e^x \cot x - \text{cosec}^2 x\left(\sqrt{3} - 4e^x\right)$$

5. $\mathbf{x^2 e^x (\cos x - 4)}$

Solution:

Let $y = x^2 e^x (\cos x - 4)$

$$\frac{dy}{dx} = x^2 e^x \cdot \frac{d}{dx}(\cos x - 4) + x^2(\cos x - 4) \cdot \frac{d}{dx}(e^x) + (\cos x - 4)e^x \cdot \frac{d}{dx}\left(x^2\right)$$

$$= -x^2 e^x \sin x + x^2 e^x (\cos x - 4) + 2xe^x(\cos x - 4)$$

Example 3: **Differentiate the following w.r.t. x (By Quotient Rule)**

1. $\dfrac{a^2 + x^2}{a^2 - x^2}$

Solution:

Let $y = \dfrac{a^2 + x^2}{a^2 - x^2}$

$$\frac{dy}{dx} = \frac{\left(a^2 - x^2\right) \cdot \frac{d}{dx}\left(a^2 + x^2\right) - \left(a^2 + x^2\right)\frac{d}{dx}\left(a^2 - x^2\right)}{\left(a^2 - x^2\right)^2}$$

$$= \frac{\left(a^2 - x^2\right)(2x) - \left(a^2 + x^2\right)(-2x)}{\left(a^2 - x^2\right)^2}$$

$$= \frac{2xa^2 - 2x^3 + 2xa^2 + 2x^3}{(a^2 - x^2)^2}$$

$$= \frac{4xa^2}{(a^2 - x^2)^2}$$

2. $\dfrac{e^x - 1}{e^x + 1}$

Solution:

Let $y = \dfrac{e^x - 1}{e^x + 1}$

$$\frac{dy}{dx} = \frac{(e^x + 1) \cdot \frac{d}{dx}(e^x - 1) - (e^x - 1) \cdot \frac{d}{dx}(e^x)}{(e^x + 1)^2}$$

$$= \frac{(e^x + 1)(e^x) - (e^x - 1)(e^x)}{(e^x + 1)^2}$$

$$= \frac{e^{2x} + e^x - e^{2x} + e^x}{\left(e^x + 1\right)^2}$$

$$= \frac{2e^x}{\left(e^x + 1\right)^2}$$

3. $\dfrac{\cos x+\sin x}{\cos x-\sin x}$

Solution:

Let $y = \dfrac{\cos x+\sin x}{\cos x-\sin x}$

$$\frac{dy}{dx}=\frac{(\cos x-\sin x)\frac{d}{dx}(\cos x+\sin x)-(\cos x+\sin x)\frac{d}{dx}(\cos x-\sin x)}{(\cos x-\sin x)^2}$$

$$=\frac{(\cos x-\sin x)(-\sin x+\cos x)-(\cos x+\sin x)(-\sin x-\cos x)}{(\cos x-\sin x)^2}$$

$$=\frac{(\cos x-\sin x)^2+(\cos x+\sin x)^2}{(\cos x-\sin x)^2}=\frac{2(\cos^2 x+\sin^2 x)}{(\cos x-\sin x)^2} \qquad \left(\because \cos^2 x+\sin^2 x=1\right)$$

$$=\frac{2}{(\cos x-\sin x)^2}$$

4. $4\left(2^{6\log 2^{x}}\right)$

Solution:

Let $y=4\left(2^{\log_2^{x^6}}\right) \qquad \left(\because a^{\log_a^x}=x\right)$

$y=4\cdot x^6$

$$\frac{dy}{dx}=24x^5$$

5. $\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

Solution:

Let $y=\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}} \qquad y=\sqrt{\dfrac{2\sin^2 x}{2\cos^2 x}} \qquad \left[\begin{array}{l}\because \text{From Multiple angle}\\ \cos 2x=2\cos^2 x-1\\ \cos 2x=1-\sin^2 x\end{array}\right]$

$y=\tan x$

$$\frac{dy}{dx}=\sec^2 x$$

6. $\dfrac{e^x(x-1)}{x^2+1}$

Solution:

Let $y=\dfrac{e^x(x-1)}{x^2+1}$

$$\frac{dy}{dx}=\frac{\left(x^2+1\right)\frac{dx}{dx}\left\{e^x(x-1)\right\}-e^x(x-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$=\frac{(x^2+1)\left\{e^x(1)+(x-1)e^x\right\}-e^x(x-1)(2x)}{(x^2+1)^2}$$

EXERCISE 18.1

One mark questions:

I. Differentiate the following w.r.t. x

1. $5e^x-\log x-3\sqrt{x}$
2. $\log e^e$
3. $\dfrac{1}{x^{4/3}}-\dfrac{3}{x^{3/2}}$
4. $\dfrac{4x^2-3x}{x}$
5. $\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2$
6. $\sqrt[3]{x^2}+\dfrac{4}{\sqrt[4]{x^5}}+\dfrac{1}{x^7}+x\sqrt{x}$

Two marks questions:

II. Differentiate the following w.r.t. x

1. $(x-a)(x-b)$
2. $\dfrac{x-a}{x-b}$
3. $\left(5x^2+3x-1\right)(x-1)$
4. $x^{-3}(5+3x)$
5. $x^5(3-6x^{-9})$
6. $\sin^2 x$
7. $\dfrac{x+1}{x}$
8. $\dfrac{1}{ax^2+bx+c}$
9. $\dfrac{\cos x}{1+\sin x}$
10. $(x+\cos x)(x-\tan x)$

III. 1. If $f(x)=x^2-3x+10$. Find $f'(50)$ and $f'(11)$

2. If $f(x)=x^n$ and if $f'(1)=10$. Find the value of n.

3. If $y=x+\dfrac{1}{x}$ show that $x^2\dfrac{dy}{dx}-xy+2=0$

Three marks questions:

IV. Differentiate the following w.r.t. x

1. $\dfrac{x^5-\cos x}{\sin x}$ 2. $\dfrac{x^n-a^n}{x-a}$ 3. $\dfrac{2}{x+1}-\dfrac{x^2}{3x-1}$

4. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$ 5. $\dfrac{2^x\log x}{\sqrt{x}}$

V. 1. Find the derivative of $f(x)=\dfrac{1}{x}$ with respect to x from first principle.

2. If $y=x+\tan x$. Show that $\cos^2 x\cdot\dfrac{dy}{dx}=2-\sin^2 x$

ANSWERS 18.1

One mark questions:

I. 1. $5e^x-\frac{1}{x}-\frac{3}{2}\sqrt{x}$ 2. 0 3. $-\frac{4}{3}x^{7/3}+\dfrac{9}{2x^{5/2}}$

4. 4 5. $1-\dfrac{1}{x^2}$ 6. $\dfrac{2}{3}x^{-1/3}-\dfrac{5}{x^{9/4}}-\dfrac{7}{x^8}+\dfrac{3x^{1/2}}{2}$

Two marks questions:

II. 1. $2x-a-b$ 2. $\dfrac{a-b}{(x-b)^2}$ 3. $15x^2-4x-4$

4. $\dfrac{-3}{x^4}(5+2x)$ 5. $15x^4+\dfrac{24}{x^5}$ 6. $\sin 2x$

7. $\dfrac{-1}{x^2}$ 8. $\dfrac{-(2ax+b)}{(ax^2+bx+c)^2}$ 9. $\dfrac{-1}{1+\sin x}$

10. $(1-\sec^2 x)(x+\cos x)+(x-\tan x)(1-\sin x)$

III. 1. 97, 19 2. 10

Three marks questions:

IV. 1. $\dfrac{-x^5\cos x+5x^4\sin x+1}{(\sin x)^2}$ 2. $\dfrac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$

3. $\dfrac{-2}{(x+1)^2} - \dfrac{x(3x-2)}{(3x-1)^2}$

4. $\dfrac{\sqrt{a}}{\sqrt{x}\left(\sqrt{a}-\sqrt{x}\right)^2}$

5. $\dfrac{2x\left[\dfrac{2^x}{x} + 2^x \log 2 \cdot \log x\right] - 2^x \log x}{2x^{\frac{3}{2}}}$

18.9 Composite functions (function of a function)

If $y = f(u)$ and $u = g(x)$ then $y = f[g(x)]$. This function $y = f[g(x)]$ is called a composite function.

Theorem (Chain Rule)

Let f be a real valued function which is composite of two functions u and v, i.e. $f = vou$.

Suppose $y = f(x)$ and $u = g(x)$ $\dfrac{dy}{dx}$ and $\dfrac{du}{dx}$ exists

we have $\therefore \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Note : The chain rule can be extended to the functions which are composition of more than two function.

If $y = f(u),\ u = g(v),\ v = h(x)$

then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$

Example 1 : Differentiate the following w.r.t. *x*:

1. $\mathbf{\left(3x^2 + 4x + 5\right)^6}$

Solution:

Let $y = \left(3x^2 + 4x + 5\right)^6$

put $u = 3x^2 + 4x + 5$ then $y = u^6$

$\therefore \dfrac{dy}{du} = 6u^5, \dfrac{du}{dx} = 6x + 4$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = 6u^5(6x+4)$

$\Rightarrow \quad \frac{dy}{dx} = 6(3x^2 + 4x + 5)^5 (6x + 4)$

OR (using direct method) without substitution.

$\frac{dy}{dx} = 6(3x^2 + 4x + 5) \cdot (6x + 4)$

2. e^{x^2}

Solution:

Let $y = e^{x^2}$

Put $u = x^2$ then $y = e^u$

$\frac{dy}{du} = e^u, \frac{du}{dx} = 2x$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot 2x = 2x \cdot e^{x^2}$

OR $\frac{dy}{dx} = e^{x^2} \cdot 2x$

3. log (x^2 – 2)

Solution:

Let $y = \log(x^2 - 2)$

put $u = x^2 - 2$ then $y = \log u$

$\therefore \quad \frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 2x$

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{u} \cdot 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2 - 2}$

OR $\frac{dy}{dx} = \frac{1}{x^2 - 2} \cdot (2x)$

Example 2 : Differentiate the following w.r.t. *x*

1. tan^4*x*

Solution: Let $y = \tan^4 x$

$\frac{dy}{dx} = 4\tan^3 x \frac{d}{dx}\tan x = 4\tan^3 x \cdot \sec^2 x$

2. sec (sec *x*)

Solution:

Let $y = \sec(\sec x)$

$$\frac{dy}{dx} = \sec(\sec x)\tan(\sec x)\frac{d}{dx}(\sec x)$$

$$= \sec(\sec x)\tan(\sec x).\sec x.\tan x$$

3. $\mathbf{\dfrac{1}{(2-\cos x)^3}}$

Solution:

$$\text{Let } y = \frac{1}{(2-\cos x)^3} = (2-\cos x)^{-3}$$

$$\frac{dy}{dx} = \frac{-3}{(2-\cos x)^4}\times(\sin x)$$

4. $\mathbf{\log(\sec x + \tan x)}$

Solution:

Let $y = \log(\sec x + \tan x)$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec x + \tan x}\ \frac{d}{dx}\left(\sec x + \tan x\right)$$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x}\quad(\sec x\tan x + \sec^2 x)$$

$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \sec x$$

Example 3 : Differentiate the following w.r.t. *x* :

1. $\mathbf{\sqrt{\tan\sqrt{x}}}$

Solution:

$$\text{Let } y = \sqrt{\tan\sqrt{x}}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan\sqrt{x}}}\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\frac{1}{2\sqrt{\tan\sqrt{x}}}\sec^2\sqrt{x}\,\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan\sqrt{x}}}\cdot\sec^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

2. tan [log (sin$x$)]

Solution:

Let $y=\tan\left[\log\left(\sin x\right)\right]$

$$\frac{dy}{dx}=\sec^2\left[\log\left(\sin x\right)\right]\cdot\frac{1}{\sin x}\cdot\cos x$$

$$=\cot x\cdot\sec^2\left[\log\left(\sin x\right)\right] \qquad \left[\because\frac{\cos x}{\sin x}=\cot x\right]$$

3. $\boldsymbol{\cot\left(x^2+\frac{1}{x^2}\right)}$

Solution:

Let $y=\cot\left(x^2+\frac{1}{x^2}\right)$

$$\frac{dy}{dx}=-\mathrm{cosec}^2\left(x^2+\frac{1}{x^2}\right)\left(2x-\frac{2}{x^3}\right)$$

4. $\boldsymbol{y=\log\left(\frac{1-x^2}{1+x^2}\right)}$

Solution:

$$y=\log\left(\frac{1-x^2}{1+x^2}\right)\Rightarrow y=\log\left(1-x^2\right)-\log\left(1+x^2\right) \qquad \left[\because\log\frac{A}{B}=\log A-\log B\right]$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}(-2x) - \frac{1}{1+x^2}(2x)$$

$$= -2x\left[\frac{1}{1-x^2} + \frac{1}{1+x^2}\right]$$

$$= -2x\left[\frac{(1+x^2)+(1-x^2)}{(1-x^2)+(1+x^2)}\right]$$

$$= \frac{-4x}{1-x^4}$$

Example 3:

1. **If $y = \log\left[x + \sqrt{1+x^2}\right]$. Prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$**

Solution:

$$y = \log\left[x + \sqrt{1+x^2}\right]$$

$$\frac{dy}{dx} = \frac{1}{\left(x+\sqrt{1+x^2}\right)}\left[\frac{1}{1} + \frac{1}{2\sqrt{1+x^2}} \cdot 2x\right] \text{ take L.C.M}$$

$$= \frac{1}{\left(x+\sqrt{1+x^2}\right)}\left[\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}}\right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

2. **If $y = \left(x + \sqrt{x^2+1}\right)^n$. Prove that $\left(x^2+1\right)\left(\frac{dy}{dx}\right)^2 = n^2y^2$**

Solution:

$$y = \left(x + \sqrt{x^2+1}\right)^n$$

$$\frac{dy}{dx} = n\left[x + \sqrt{x^2+1}\right]^{n-1} \left[1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x\right]$$

$$= \frac{n\left[x+\sqrt{x^2+1}\right]^n}{x+\sqrt{x^2+1}} \left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right].$$

$$= \frac{ny}{\sqrt{x^2+1}}$$

$$= \sqrt{x^2+1}\,\frac{dy}{dx} = ny \qquad \because y = \left(x+\sqrt{x^2+1}\right)^n$$

Squaring on both sides

$$\boxed{\left(x^2+1\right)\left(\frac{dy}{dx}\right)^2 = n^2y^2}$$

EXERCISE 18.2

One and Two marks questions:

Differentiate the following w.r.t. x

1. $\left(a^2-x^2\right)^{10}$
2. $\log\left[\log\left(\log x\right)\right]$
3. $\cos x^3$
4. $\sin^3\sqrt{x}$
5. $\left[\log\left(\cos x\right)\right]^2$
6. $\sec\left(x+\frac{1}{x}\right)$
7. $7^{\sin\sqrt{x}}$
8. $\sqrt{\cot\sqrt{x}}$
9. $\log\left(\sin\sqrt{x}\right)$
10. $\log\left[\log\left(\tan x\right)\right]$
11. $\cos 3x\cdot\sin 5x$
12. $\sin x\cdot\sin 2x$
13. $e^{\log e^{\left(x+\sqrt{x^2+a^2}\right)}}$
14. $e^{2x}\cdot\sin 3x$
15. $\cos^5 x\cdot\cos(x^5)$
16. $3^{x^2}\cdot\log x$
17. $\frac{x}{\sqrt{x^2-1}}$
18. $\frac{x}{\sqrt{2x-1}}$
19. $\frac{e^{\sin x}}{\sqrt{\log x}}$
20. $\log\left(\frac{1+\sin x}{1-\sin x}\right)$

Three marks questions:

1. If $y=\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)$, show that $\frac{dy}{dx}=\sec^2\left(x+\frac{\pi}{4}\right)$

2. If $y = \log\left[\frac{1-\cos x}{1+\cos x}\right]$, prove that $\frac{dy}{dx} = 2 \text{ cosec } x$

3. Differentiate e^{2x} w.r.t. x from first principles.

4. Differentiate $\sin 2x$ w.r.t. x from first principles.

5. Differentiate $\tan ax$ w.r.t. x from first principles.

ANSWERS 18.2

One and Two marks questions:

1. $-20x(a^2 - x^2)^9$

2. $\frac{1}{\log(\log x)} \cdot \frac{1}{\log x} \cdot \frac{1}{x}$

3. $-3x^2 \sin x^3$

4. $3\sin^2 \sqrt{x} \cdot \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$

5. $-2\tan x \cdot \log(\cos x)$

6. $\sec\left(x+\frac{1}{x}\right)\cdot\tan\left(x+\frac{1}{x}\right)\cdot\left(1-\frac{1}{x^2}\right)$

7. $\frac{(\log 7)\cos\sqrt{x}\cdot 7^{\sin\sqrt{x}}}{2\sqrt{x}}$

8. $\frac{-\text{cosec}^2\sqrt{x}}{4\sqrt{x}\sqrt{\cot\sqrt{x}}}$

9. $\frac{\cot\sqrt{x}}{2\sqrt{x}}$

10. $\frac{\sec^2 x}{\tan x \cdot \log(\tan x)}$

11. $5 \cos 3x \,.\, \cos 5x - 3 \sin 3x \,.\, \sin 5x$

12. $\frac{1}{2}(-\sin x + 3\sin 3x)$

13. $\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$

14. $e^{2x}(3\cos 3x + 2\sin 3x)$

15. $-5\cos^4 x\left[x^4\cos x\cdot\sin x^5 + \sin x\cos x^5\right]$

16. $\frac{3^{x^2}}{x}\left[1+2x^2\log 3\cdot\log x\right]$

17. $\frac{-1}{(x^2-1)^{3/2}}$

18. $\frac{x-1}{(2x-1)^{3/2}}$

19. $\frac{(2x\log x\cdot\cos x - 1)e^{\sin x}}{2x(\log x)^{3/2}}$

20. $2 \sec x$

Three marks questions:

3. $2e^{2x}$ 4. $2\sin x\cos x$ 5. $a\sec^2 ax$

18.10 Differentiation of implicit functions:

If x and y are connected by a relation of the type $y = 2x^4 - 3x^3 + 4x^2 + 7$ then we say that y is defined explicity in terms of x and the function $y = f(x)$ is said to be an explicit function of x.

However if x and y are connected by a relation of the type $x^4 - 2x^2y + y^3 = 2x^2 + 1$ then such an equation is called an **implicit form of a function.** Usually such functions are expressed as $f(x,y) = 0$. Here $f(x,y)$ denotes an expression involving the variables x and y.

WORKING RULE

Step 1 : Differentiate both sides of the equation $f(x,y) = 0$ w.r.t x.

Step 2 : Collect the terms involving $\frac{dy}{dx}$ on one side and the remaining on the other side.

we get $g(x,y)\frac{dy}{dx} = h(x,y)$

Step 3 : Solve for $\frac{dy}{dx}$ and write $\frac{dy}{dx} = \frac{h(x,y)}{g(x,y)}$

This process of finding $\frac{dy}{dx}$ by above method is called as implicit differentiation.

Example 1: Find $\frac{dy}{dx}$ if

a) $x^2 + y^2 = a^2$ **b)** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

c) $x^2 + y^2 - 3xy + 2x + 3y - 5 = 0$ **d)** $e^y = \sin(x+y)$

e) $\sin xy + \cos(x+y) = 4$

Solution:

a) $x^2 + y^2 = a^2$ Differentiating w.r.t. x both sides we get

$$2x + 2y\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow\ 2y\cdot\frac{dy}{dx} = -2x \qquad \Rightarrow\ \frac{dy}{dx} = -\frac{x}{y}$$

b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Differentiating w.r.t. x both sides we get

$$\frac{1}{a^2}\cdot 2x+\frac{1}{b^2}\cdot 2y\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{1}{b^2}\cdot 2y\cdot\frac{dy}{dx}=\frac{-2x}{a^2}$$

$$\frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

c) $x^2+y^2-3xy+2x+3y-5=0$

Differentiating w.r.t x both sides, we get

$$2x+2y\cdot\frac{dy}{dx}-3\left[x\cdot\frac{dy}{dx}+y\cdot 1\right]+2+3\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left[2y-3x+3\right]=-2x+3y-2$$

$$\therefore \quad \frac{dy}{dx}=\frac{3y-2x-2}{2y-3x+3}$$

d) $e^y=\sin(x+y)$

$$e^y\frac{dy}{dx}=\cos(x+y)\left[1+\frac{dy}{dx}\right]$$

$$e^y\frac{dy}{dx}-\cos(x+y)\frac{dy}{dx}=\cos(x+y)$$

$$\frac{dy}{dx}\left[e^y-\cos(x+y)\right]=\cos(x+y)$$

$$\frac{dy}{dx}=\frac{\cos(x+y)}{e^y-\cos(x+y)}$$

e) $\sin xy+\cos(x+y)=4$

$$\cos xy\left[\frac{d}{dx}(xy)\right]-\sin(x+y)\frac{d}{dx}(x+y)=0$$

apply product rule for xy

$$\cos xy\left[x\cdot\frac{dy}{dx}+y\right]-\sin(x+y)\left[1+\frac{dy}{dx}\right]=0$$

$$\frac{dy}{dx}[x\cos xy-\sin(x+y)]=\sin(x+y)-y\cos xy$$

$$\therefore\ \frac{dy}{dx}=\frac{\sin(x+y)-y\cos xy}{x\cos xy-\sin(x+y)}$$

Example 2:

If $x\sqrt{1+y}+y\sqrt{1+x}=0$ where $x\neq y$, show that $\frac{dy}{dx}=\frac{-1}{(1+x)^2}$.

Solution:

We have $x\sqrt{1+y}+y\sqrt{1+x}=0$, $x\sqrt{1+y}=-y\sqrt{1+x}$

squaring both sides we get $x^2(1+y)=y^2(1+x)$

$$\Rightarrow\ x^2+x^2y-y^2-y^2x=0$$

$$\Rightarrow\ (x^2-y^2)+xy(x-y)=0$$

$$\Rightarrow\ (x-y)\ [x+y+xy]=0$$

$$\therefore\ x+y+xy=0 \qquad [\because\ x\neq y]$$

$$\Rightarrow\ x+y(1+x)=0$$

$$y=\frac{-x}{1+x}\ \text{(By Quotient Rule)}$$

$$\frac{dy}{dx}=\frac{(1+x)(-1)-(-x)1}{(1+x)^2}$$

$$=\frac{-1-x+x}{(1+x)^2}$$

$$\frac{dy}{dx}=\frac{-1}{(1+x)^2}$$

Example 3 :

If $e^x + e^y = e^{x+y}$, show that $\dfrac{dy}{dx} = -e^{(y-x)}$

Solution:

Consider $e^x + e^y = e^{x+y}$

$$e^x + e^y \cdot \frac{dy}{dx} = e^{x+y}\left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow \quad e^x + e^y \cdot \frac{dy}{dx} = (e^x + e^y)\left[1 + \frac{dy}{dx}\right] \qquad \left(\because \ e^{x+y} = e^x + e^y\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left[e^y - e^x - e^y\right] = e^x + e^y - e^x$$

$$\Rightarrow \quad \frac{dy}{dx}\left(-e^x\right) = e^y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-e^y}{e^x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -e^{y-x}$$

Example 4 :

If $\sin y = x\sin(a+y)$. Prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$

Solution:

We have $\sin y = x\sin(a+y)$

$\therefore \quad x = \dfrac{\sin y}{\sin(a+y)}$ Differentiate w.r..t y

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y \cdot \cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin\left[(a+y) - y\right]}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx}=\frac{1}{dx/dy}$$

$$=\frac{\sin^2(a+y)}{\sin a}$$

Example 5:

If $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ then find $\frac{dy}{dx}$ at (a,b)

Solution:

We have $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$

Differentiating w.r.t. x

$$n\left(\frac{x}{a}\right)^{n-1}\cdot\frac{1}{a}+n\left(\frac{y}{b}\right)^{n-1}\frac{1}{b}\cdot\frac{dy}{dx}=0$$

At (a,b)

$$n\cdot(1)^{n-1}\frac{1}{a}+n\cdot(1)^{n-1}\cdot\frac{1}{b}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow\quad \frac{n}{a}+\frac{n}{b}\frac{dy}{dx}=0$$

$$\Rightarrow\quad \frac{n}{b}\frac{dy}{dx}=\frac{-n}{a}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{-b}{a}$$

EXERCISE 18.3

One and Two marks questions:

Differentiate the following w.r.t. x

1. $3x^2+4y^2=10$
2. $\sqrt{x}+\sqrt{y}=3$
3. $y^2=4ax$
4. $x^{2/3}+y^{2/3}=a^{2/3}$
5. $x^2=4ay$
6. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
7. $x^3+y^3=3axy$
8. $x-y=0$
9. $x^2-y^2=a^2$
10. $x+\sqrt{xy}=x^2$

Three marks questions:

Differentiate the following w.r.t. x

1. $\log(xy) = x^2 + y^2$
2. $2^x + 2^y = 2^{x+y}$
3. $x^y = y^x$
4. $\sin xy = \cos(x+y)$
5. $y = 4^{x+y}$

Five marks questions:

1. If $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = a$. Prove that $x \cdot \frac{dy}{dx} = y$
2. If $x^y = e^{y-x}$, show that $\frac{dy}{dx} = \frac{2 - \log x}{(1 - \log x)^2}$
3. If $\cos y = x\cos(a+y)$, show that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$
4. If $e^y = y^x$, show that $\frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}$
5. If $e^{x+y} = xy$, show that $\frac{dy}{dx} = \frac{y(1-x)}{x(y-1)}$
6. If $y^x = x^y$, show that $\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$

Answers

One mark questions:

1. $\frac{-3x}{4y}$
2. $-\sqrt{\frac{y}{x}}$
3. $\frac{2a}{y}$
4. $-\left(\frac{y}{x}\right)^{1/3}$
5. $\frac{x}{2a}$
6. $\frac{-b^2x}{a^2y}$
7. $\frac{ay - x^2}{y^2 - ax}$
8. 1
9. $\frac{x}{y}$
10. $-\left[\frac{y + 2\sqrt{xy}}{x + 2\sqrt{xy}}\right]$

Three marks questions:

1. $\dfrac{y(2x^2-1)}{x(1-2y^2)}$
2. $\dfrac{2^x(2^y-1)}{2^y(1-2^x)}$
3. $\dfrac{y[x\log y-y]}{x[y\log x-x]}$
4. $-\left[\dfrac{\sin(x+y)+y\cos xy}{x\cos xy+\sin(x+y)}\right]$
5. $\dfrac{y\log 4}{1-y\log 4}$

18.11 Differentiation of Infinite series

When we have to find $\dfrac{dy}{dx}$ in case y is given as infinite series then we use the fact "If a term is deleted from an infinite series, it remains unaltered".

Example 1:

If $y=x^{x^{x^{x^{\cdot^{\cdot^{\cdot\infty}}}}}}$. Prove that $\dfrac{dy}{dx}=\dfrac{y^2}{x(1-y\log x)}$

Solution:

We have $y=x^{x^{x^{x^{\cdot^{\cdot^{\cdot\infty}}}}}}$

Apply logarithm on both sides

$$\Rightarrow\quad y=x^y \qquad \Rightarrow\quad \log y=y\log x$$

Diffn. both sides w.r.t. x

$$\Rightarrow\quad \frac{1}{y}\cdot\frac{dy}{dx}=y\cdot\frac{1}{x}+\log x\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[\frac{1}{y}-\log x\right]=\frac{y}{x} \qquad \Rightarrow \frac{dy}{dx}\left[\frac{1-y\log x}{y}\right]=\frac{y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y^2}{x(1-y\log x)}$$

Example 2:

If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\text{..........}\infty}}}$ show that $(2y-1)\dfrac{dy}{dx}=\dfrac{1}{x}$

Solution:

We have $y=\sqrt{\log x+\sqrt{\log x+\text{............}\infty}}$

$$y=\sqrt{\log x+y}$$

Squaring on both sides

$$y^2 = \log x + y$$

Diffn. both sides w.r.t. x

$$2y \cdot \frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx} \quad \Rightarrow \quad (2y-1)\frac{dy}{dx} = \frac{1}{x}$$

Example 3:

If $y = \sqrt{\sin x + \sqrt{\sin x + ...\infty}}$. Prove that $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

Solution:

We have $y = \sqrt{\sin x + \sqrt{\sin x + ...\infty}}$

$$y = \sqrt{\sin x + y}$$

Squaring both sides

$$y^2 = \sin x + y$$

Diffn. both sides w.r.t. x

$$2y \cdot \frac{dy}{dx} = \cos x + \frac{dy}{dx} \qquad (2y-1)\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Example 4:

If $y = e^{x+e^{x+e^{x+.......to\ \infty}}}$ show that $\frac{dy}{dx} = \frac{y}{1-y}$

Solution:

We have $y = e^{x+e^{x+.........\infty}}$

$$y = e^{x+y}$$

Apply log on both sides

$$\log y = x + y \qquad \left[\because \log_e^e = 1\right]$$

Diffn. both sides w.r.t. x

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y} - 1\right) = 1$$

$$\frac{dy}{dx} = \frac{y}{y-1}$$

EXERCISE 18.4

1. If $y=\sqrt{x+\sqrt{x+\sqrt{x+.....\infty}}}$ then prove that $\dfrac{dy}{dx}=\dfrac{1}{2y-1}$

2. If $y=\sqrt{\tan x+\sqrt{\tan x+......\infty}}$ then prove that $\dfrac{dy}{dx}=\dfrac{\sec^2 x}{2y-1}$

3. If $y=(\sin x)^{(\sin x)^{\cdot^{\cdot^{\cdot^{\infty}}}}}$, show that $\dfrac{dy}{dx}=\dfrac{y^2\cot x}{1-y\log(\sin x)}$

4. If $y=(\tan x)^{(\tan x)^{\cdot^{\cdot^{\cdot^{\infty}}}}}$ show that $\dfrac{dy}{dx}=\dfrac{2y^2\ \operatorname{cosec}2x}{1-y\log\tan x}$

5. If $y=(e^x)^{(e^x)^{\cdot^{\cdot^{\cdot^{\infty}}}}}$, show that $\dfrac{dy}{dx}=\dfrac{y^2}{1-xy}$

18.12 Logarithmic Differentiation:

Many a times, the function whose derivative is required involves products, quotients and powers. In such a case differentiation can be carried out more conveniently if we take logarithms on the two sides and simplify before differentiation. This process is called logarithmic differentiation. This is especially useful when the variable occurs in the exponent.

Example 1:

Differentiate the following w.r.t. x.

1) x^x **2)** $(\sin x)^{\tan x}$ **3)** $(\log x)^{\cos x}$

Solution:

1) Let $y=x^x$

Taking log on both sides, we get $\log y=x\log x$

$$\frac{1}{y}\cdot\frac{dy}{dx}=x\cdot\frac{1}{x}+\log x\cdot 1$$

$$\frac{dy}{dx}=y[1+\log x]$$

$$\therefore\ \frac{dy}{dx}=x^x[1+\log x]$$

2) Let $y=(\sin x)^{\tan x}$

Taking log on both sides we get $\log y=\tan x\cdot\log(\sin x)$

Differentiating both sides w.r.t. x

$$\frac{1}{y}\frac{dy}{dx}=\tan x\cdot\frac{1}{\sin x}(\cos x)+\sec^2 x\log\sin x$$

$$\Rightarrow\quad\frac{dy}{dx}=y\left[\tan x\cdot\cot x+\sec^2 x\log\sin x\right]\qquad\left[\because\frac{\cos x}{\sin x}=\cot x\right]$$

$$\Rightarrow\quad\frac{dy}{dx}=(\sin x)^{\tan x}\left[1+\sec^2 x\cdot\log\sin x\right]$$

3) Let $y=(\log x)^{\cos x}$

Taking log on both sides, we get $\log y=\cos x\cdot\log(\log x)$

Differentiating w.r.t x we get

$$\frac{1}{y}\frac{dy}{dx}=\cos x\cdot\frac{1}{\log x}\cdot\frac{1}{x}+(-\sin x)\log\ (\log x)$$

$$\frac{dy}{dx}=y\left[\frac{\cos x}{x\log x}-\sin x\cdot\log(\log x)\right]$$

$$=\ (\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\cdot\log\ (\log x)\right]$$

Example 2:

Find $\frac{dy}{dx}$ if $y^x+x^y=a^b$

Solution:

Given that $y^x+x^y=a^b$

Putting $u=y^x,\ v=x^y$

we get $u+v=a^b$

$$\therefore\quad\frac{du}{dx}+\frac{dv}{dx}=0\qquad......(1)$$

Now $u=y^x$

Taking logarithm on both sides, we have $\log u = x \log y$

Differentiating both sides w.r.t. x we have

$$\frac{1}{u}\cdot\frac{du}{dx} = x\cdot\frac{d}{dx}(\log y) + \log y\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{y}\cdot\frac{dy}{dx} + \log y\cdot 1$$

$$\frac{du}{dx} = u\left[\frac{x}{y}\cdot\frac{dy}{dx} + \log y\right]$$

$$= y^x\left[\frac{x}{y}\cdot\frac{dy}{dx} + \log y\right] \quad .... (2)$$

Also $v = x^y$

Taking logarithm on both sides, we have $\log v = y \log x$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\cdot\frac{du}{dx} = y\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{dy}{dx}$$

$$= y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

$$\frac{dv}{dx} = v\left[\frac{y}{x} + \log x\cdot\frac{dy}{dx}\right]$$

$$= x^y\left[\frac{y}{x} + \log x\cdot\frac{dy}{dx}\right] \quad .... (3)$$

from (1), (2) and (3) we have

$$y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] + x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right]$$

$$(x\cdot y^{x-1} + x^y\cdot\log x)\frac{dy}{dx} = -y\cdot x^{y-1} - y^x\log y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-\left[y^x\log y + y\cdot x^{y-1}\right]}{x\cdot y^{x-1} + x^y\log x}$$

Example 3:

Differentiate $\sqrt{\dfrac{(x-3)(x^2+4)}{3x^2+4x+5}}$ **w.r.t.** x

Solution:

Let $y=\sqrt{\dfrac{(x-3)(x^2+4)}{3x^2+4x+5}}$

Taking logarithm on both sides

$$\log y=\frac{1}{2}\left[\log(x-3)+\log(x^2+4)-\log(3x^2+4x+5)\right]$$

Differentiating both sides w.r.t. x we get

$$\frac{1}{y}\,\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

$$\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

Example 4:

If $x^y=e^{x-y}$**, prove that** $\dfrac{dy}{dx}=\dfrac{\log x}{(1+\log x)^2}$

Solution:

Given : $x^y=e^{x-y}$

Taking logarithm on both sides

$$\Rightarrow\ y\log x=(x-y)\log e$$

$$\Rightarrow\ y\log x=x-y\quad\Rightarrow\ y(1+\log x)=x$$

i.e. $y=\dfrac{x}{1+\log x}$

Diffn. w.r.t. x by quotient rule

$$\frac{dy}{dx}=\frac{(1+\log x)\cdot 1-x\cdot\dfrac{1}{x}}{(1+\log x)^2}$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$$

Example 5:

If $x^m \cdot y^n = (x+y)^{m+n}$. Show that $\frac{dy}{dx} = \frac{y}{x}$

Solution:

We have $x^m \cdot y^n = (x+y)^{m+n}$

Taking log both sides we get $m\log x + n\log y = (m+n)\log(x+y)$

Differentiating both sides w.r.t. x, we get

$$m \cdot \frac{1}{x} + n \cdot \frac{1}{y}\,\frac{dy}{dx} = (m+n)\frac{1}{x+y}\left[1+\frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{m}{n} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y}\left[1+\frac{dy}{dx}\right]$$

$$\frac{dy}{dx}\left[\frac{n}{y} - \frac{m+n}{x+y}\right] = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{n(x+y)-y(m+n)}{y(x+y)}\right] = \frac{x(m+n)-m(x+y)}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx}(nx+ny-my-ny) = \frac{y}{x}(mx+nx-mx-my)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cdot y(nx-my)}{x(nx-my)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

Example 6:

If $(xe)^y = e^x$, show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution:

we have $(xe)^y = e^x$

Taking log on both sides $y\log\ (xe) = x\ \log\ e$ $\quad [\because \log xe = \log x + \log e = \log x + 1]$

$y(1+\log x) = x \cdot 1 \qquad [\because \log e = 1]$

$$\therefore \quad y = \frac{x}{1+\log x}$$

Diffn. w.r.t. x

$$\frac{dy}{dx}=\frac{(1+\log x)\cdot 1-x\cdot\frac{1}{x}}{(1+\log x)^2}$$

$$=\frac{1+\log x-1}{(1+\log x)^2}$$

$$\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$$

EXERCISE 18.5

Two and Three marks questions:

Differentiate the following w.r.t x:

1. $x^{\sqrt{x}}$
2. $x^{\sin x}$
3. $(\sin x)^x$
4. $x^{5+\log x}$
5. $x^{(\sin x\,-\,\cos x)}$

Five marks questions:

Differentiate the following w.r.t x:

1. $x^{\log x}+(\log x)^x$
2. $x^2\cdot e^{x^2}\cdot\log x$
3. $(x+1)^2(x+2)^3(x+3)^4$
4. $x^{2x}+x^{x^2}$
5. $x^{\left(1+\frac{1}{x}\right)}+\left(1+\frac{1}{x}\right)^x$
6. $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$
7. $x^3\cdot e^{2x}\cdot\sec^2 x$

ANSWERS 18.5

1. $\frac{x^{\sqrt{x}}}{2\sqrt{x}}(2+\log x)$
2. $x^{\sin x}\left[\frac{\sin x}{x}+\cos x\ \log x\right]$
3. $(\sin x)^x\left[x\cot x+\log\sin x\right]$
4. $x^{5+\log x}\left[\frac{1}{x}(5+2\log x)\right]$
5. $x^{\sin x-\cos x}\left[\frac{\sin x-\cos x}{x}+(\cos x+\sin x)\log x\right]$

Five mark questions:

1. $x^{\log x}\left(\frac{2\log x}{x}\right)+(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]$

2. $xe^{x^2}\left[2\log x+2x^2\log x+1\right]$

3. $(x+1)^2(x+2)^3(x+3)^4\left[\frac{2}{x+1}+\frac{3}{x+2}+\frac{4}{x+3}\right]$

4. $x^{2x}\left[2+\log x^2\right]+x\cdot x^{x^2}\left[1+\log x^2\right]$

5. $x^{\left(1+\frac{1}{x}\right)}\left[\frac{1}{x}\left(1+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]+\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right]$

6. $\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$

7. $\left(x^3.e^{2x}\cdot\sec^2 x\right)\left[\frac{3}{x}+2+2\tan x\right]$

18.13 Differentiation of parametric functions:

If the variable x and y of a function are expressed as the functions of another variable 't' or 'θ' then we say the function is defined in parametric form. The variable 't' or 'θ' is called a parameter.

If $x=f(t),\ \ y=g(t)$

In order to find the derivative of function is such form, we have

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\text{or}\quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}\qquad\left(\text{where }\frac{dx}{dt}\neq 0\right)$$

$$\text{Thus}\quad \frac{dy}{dx}=\frac{g'(t)}{f'(t)}\quad\left[\begin{array}{l} dy/dt=g'(t)\\ \textit{and}\ \ dx/dt=f'(t)\end{array}\right]$$

provided $f'(t)\neq 0$

Example 1: Find $\frac{dy}{dx}$ if

a) $x = a\ \cos\theta,\ y = a\ \sin\theta$ **b)** $x = at^2,\ y = 2at$ **c)** $x = \frac{1-t^2}{1+t^2},\ y = \frac{2t}{1+t^2}$

Solution:

a) We have $x = a\ \cos\theta,\ \ y = a\ \sin\theta$

$$\therefore\ \frac{dx}{d\theta} = -a\ \sin\theta,\ \frac{dy}{d\theta} = a\ \cos\theta$$

Hence $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$= \frac{a\ \cos\theta}{-a\ \sin\theta}$$

$$= -\cot\theta$$

b) We have $x = at^2,\ \ y = 2at$

$$\frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

$$\therefore\ \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2a}{2at}$$

$$= \frac{1}{t}$$

c) We have $x = \frac{1-t^2}{1+t^2},\ \ y = \frac{2t}{1+t^2}$

$$\frac{dx}{dt} = \frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}$$

$$= \frac{-4t}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{\left(1+t^2\right)(2)-2t\cdot 2t}{\left(1+t^2\right)^2}=\frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}$$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\dfrac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}}{\dfrac{-4t}{\left(1+t^2\right)^2}}=\frac{1-t^2}{-2t}$$

$$\frac{dy}{dx}=\frac{t^2-1}{2t}$$

Example 2:

If $x=a\left[\cos t+\log\tan\frac{t}{2}\right],\ y=a\ \sin\ t$. Show that $\frac{dy}{dx}=\tan t$

Solution:

We have $x=a\left[\cos t+\log\tan\frac{t}{2}\right]$

$$\frac{dx}{dt}=a\left[-\sin t+\frac{1}{\tan\frac{t}{2}}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}\right]$$

$$=a\left[-\sin t+\frac{\cos t/2}{2\sin t/2\cos^2 t/2}\right] \text{By Multiple angle} \begin{bmatrix}\sin 2t=2\sin t.\cos t\\ \sin t=2\sin t/2\cos t/2\\ \text{Half angle}\end{bmatrix}$$

$$=a\left[-\sin t+\frac{1}{2\sin t/2\cos t/2}\right]=a\left[-\sin t+\frac{1}{\sin t}\right]$$

$$=a\left[\frac{-\sin^2 t+1}{\sin t}\right]=\frac{a\cos^2 t}{\sin t}$$

$$\frac{dx}{dt}=a\cot t\cdot\cos t$$

Again $y = a\sin t \qquad \frac{dy}{dt} = a\cos t$

Now $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\cot t \cdot \cos t}$

$\therefore \frac{dy}{dx} = \tan t$

Example 3:

If $x = a\theta$, $y = \frac{a}{\theta}$, prove that $\frac{dy}{dx} + \frac{y}{x} = 0$

Solution:

Take, $xy = a\theta \cdot \frac{a}{\theta}$

$xy = a^2$

Using product rule

$\Rightarrow \; y \cdot 1 + x \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \; \frac{y}{x} + \frac{dy}{dx} = 0$

$\Rightarrow \; \frac{dy}{dx} + \frac{y}{x} = 0$

Example 4:

Differentiate $\sin^3 x$ w.r.t. $\cos^3 x$

Solution:

Let $u = \sin^3 x$ and $v = \cos^3 x$

$\frac{du}{dx} = 3\sin^2 x \; \cos x$

$\frac{dv}{dx} = -3\cos^2 x \; \sin x$

$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{3\sin^2 x\cos x}{-3\cos^2 x\sin x} = -\tan x$

EXERCISE 18.6

Two and Three marks questions:

1. Find $\frac{dy}{dx}$ if

a) $x=a\left(t-\frac{1}{t}\right),\ y=a\left(t+\frac{1}{t}\right)$

b) $x=e^{2t},\ y=\log(2t+1)$

c) $x=\log(1+t),\ y=\frac{1}{1+t}$

d) $x=\log t,\ y=\frac{1}{t}$

e) $x=4t,\ y=\frac{4}{t}$

f) $x=a\ \sec\theta,\ y=b\ \tan\theta$

g) $x=a(\theta-\sin\theta),\ y=a(1-\cos\theta)$

h) $x=a\ \cos(\log t),\ y=a\ \log(\cos t)$

2. Differentiate $\tan^2 x$ w.r.t. $\cos^2 x$

3. Differentiate $\sin^2 x$ w.r.t. x^2

4. Differentiate $\tan\sqrt{x}$ w.r.t. $\sqrt{x}$

5. Differentiate $\log x$ w.r.t. ${}^1\!/\!_x$

6. Differentiate $\log \sin x$ w.r.t. $\sqrt{\cos x}$

7. If $x=e^{\log\cos 4\theta}$, $y=e^{\log\sin 4\theta}$ show that $\frac{dy}{dx}=\frac{-x}{y}$

8. If $x=a\ \cos^4\theta,\ y=a\ \sin^4\theta$ show that $\frac{dy}{dx}=-\tan^2\theta$

9. If $x=e^t(\cos t+\sin t)$, $y=e^t(\cos t-\sin t)$. Show that $\frac{dy}{dx}=-\tan t$

10. If $x=a\log\sec\theta,\ \ y=a(\tan\theta-1)$ show that $\frac{dy}{dx}=2\ \text{cosec}2\theta$

ANSWERS 18.6

1) a) $\frac{t^2-1}{t^2+1}$ b) $\frac{1}{(2t+1)e^{2t}}$ c) $\frac{-1}{1+t}$

d) $\frac{-1}{t}$ e) $\frac{-1}{t^2}$ f) $\frac{b}{a}\ \text{cosec}\,\theta$

g) $\cot\left(\frac{\theta}{2}\right)$ h) $\frac{t\tan t}{\sin(\log t)}$

2) $-\sec^4 x$ 3) $\cos x^2$ 4) $\sec^2\sqrt{x}$ 5) $-x$

6) $\dfrac{-2(\cos x)^{3/2}}{\sin^2 x}$

18.14 Second order derivative

Let $y = f(x)$ then

$$\frac{dy}{dx} = f'(x) = y_1 \quad ......(1)$$

If $f'(x)$ is differentiable, we may differentiate (1) again w.r.t x. Then the left hand side becomes $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ which is called the second order derivative of y w.r.t x and is denoted by $\dfrac{d^2y}{dx^2}$. The second order derivative of $f(x)$ is denoted by $f''(x)$. It is also denoted by D^2y or y'' or y_2 if $y = f(x)$.

In general the process of differentiating the same function again and again is called successive differentiation, further $\dfrac{d^ny}{dx^n}, y_n$ or $f^{(n)}(x)$ denotes the n[th] order derivative (if it exists) of the function $y = f(x)$.

1. Find the second order derivative of the following functions w.r.t. x:

a) $x + x^{-1}$ **b)** $3\cos x + 4\sin x$

c) $x = at^2,\ y = 2at$ **d)** $x = ct,\ y = \dfrac{c}{t}$

Solution:

a) we have $y = x + x^{-1} \Rightarrow y = x + \dfrac{1}{x}$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} \qquad \frac{d^2y}{dx^2} = 0 + \frac{2}{x^3}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

b) we have $y = 3\cos x + 4\sin x$

$$\frac{dy}{dx} = -3\sin x + 4\cos x$$

$$\frac{d^2y}{dx^2} = -3\cos x - 4\sin x$$

c) we have $x = at^2, \; y = 2at$

$$\therefore \frac{dx}{dt} = 2at, \; \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at}$$

$$\frac{dy}{dx} = \frac{1}{t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-1}{t^2} \cdot \frac{dt}{dx} = \frac{-1}{t^2}\left(\frac{1}{2at}\right) = \frac{-1}{2at^3}$$

d) we have $x = ct, \; y = c/t$

$$\therefore \frac{dx}{dt} = c, \; \frac{dy}{dt} = \frac{-c}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-c/t^2}{c} = \frac{-1}{t^2}$$

$$\frac{d^2y}{dx^2} = -\left(\frac{-2}{t^3}\right) \cdot \frac{dt}{dx} = \frac{2}{t^3} \cdot \frac{1}{c}$$

$$= \frac{2}{ct^3}$$

Example 2 :

If $y^2 + 2y = x^2$. Show that $y_2 = \frac{1}{(1+y)^3}$

Solution:

We have $y^2 + 2y = x^2$

Diffn. w.r.t. x

$$2y \cdot y_1 + 2y_1 = 2x \;\Rightarrow\; y_1 = \frac{2x}{2y+2} \qquad \left(y_1 = \frac{dy}{dx}\right)$$

$$y_1 = \frac{x}{y+1}$$

Differentiate again w.r.t. x

$$y_2 = \frac{(y+1)\cdot 1 - x\cdot y_1}{(y+1)^2} = \frac{(y+1) - x\cdot\left(\frac{x}{y+1}\right)}{(y+1)^2}$$

$$\Rightarrow \frac{(y+1)^2 - x^2}{(y+1)^3} = \frac{y^2+1+2y-x^2}{(y+1)^3}$$

$$y_2 = \frac{1}{(y+1)^3} \qquad \left(\because\ y^2+2y=x^2\right)$$

Example 3 :

If $y = \mathrm{A}\sin x + \mathrm{B}\cos x$ then prove that $\frac{d^2y}{dx^2} + y = 0$

Solution:

We have $y = \mathrm{A}\sin x + \mathrm{B}\cos x$

$$\frac{dy}{dx} = \mathrm{A}\cos x - \mathrm{B}\sin x$$

$$\frac{d^2y}{dx^2} = -\mathrm{A}\sin x - \mathrm{B}\cos x \quad \Rightarrow \frac{d^2y}{dx^2} = -y$$

Hence $\frac{d^2y}{dx^2} + y = 0$

Example 4:

If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\cdot\frac{dy}{dx} + 6y = 0$

Solution:

we have $y = 3e^{2x} + 2e^{3x}$

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x} = 6\left(e^{2x} + e^{3x}\right)$$

Again diffn. w.r.t. x

$$\frac{d^2y}{dx^2} = 6\left(2e^{2x} + 3e^{3x}\right)$$

Hence $\frac{d^2y}{dx^2}-5\cdot\frac{dy}{dx}+6y$

$= 6\left(2e^{2x}+3e^{3x}\right)-30\left(e^{2x}+e^{3x}\right)+6\left(3e^{2x}+2e^{3x}\right)$

$=0$

Example 5:

If $y=\left[x+\sqrt{a^2+x^2}\right]^n$. Show that $\left(a^2+x^2\right)y_2+xy_1-n^2y=0$

Solution:

we have $y=\left[x+\sqrt{a^2+x^2}\right]^n$ ......(1)

$$y_1=n\left[x+\sqrt{a^2+x^2}\right]^{n-1}\left[1+\frac{1}{2\sqrt{a^2+x^2}}\cdot 2x\right]$$

$$= n\left[x+\sqrt{a^2+x^2}\right]^{n-1}\left[\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}}\right]=\frac{n\left(x+\sqrt{a^2+x^2}\right)^{n-1+1}}{\sqrt{a^2+x^2}}\text{(same base)}$$

$$\sqrt{a^2+x^2}\cdot y_1=n\left(x+\sqrt{a^2+x^2}\right)^n$$

$$\Rightarrow\sqrt{a^2+x^2}\;y_1=ny \qquad [\text{from}\,(1)]$$

Squaring both sides we get $\left(a^2+x^2\right)y_1^2=n^2y^2$

Differentiating both sides $\left(a^2+x^2\right)2y_1y_2+2x\cdot y_1^2=n^2\cdot 2y\cdot y_1$

Dividing throughout by $2y_1$, we get $\left(a^2+x^2\right)y_2+xy_1-n^2y=0$

Example 6:

If $y=e^x\log x$. Show that $xy_2-(2x-1)y_1+(x-1)y=0$

Solution:

We have $y=e^x\cdot\log x$

$$\frac{dy}{dx}=e^x\cdot\frac{1}{x}+e^x\cdot\log x$$

$$y_1=\frac{e^x}{x}+y \qquad \left[\because\ y=e^x\log x\right]$$

$xy_1=e^x+yx$ ....(1)

Differentiating both sides w.r.t x we get $xy_2 + y_1 \cdot 1 = e^x + \left(y_1 x + y\right)$

$$\Rightarrow \quad xy_2 + y_1 = xy_1 - yx + xy_1 + y \qquad \begin{pmatrix} \because \ e^x = xy_1 - yx \\ \text{from (1)} \end{pmatrix}$$

$$\Rightarrow \quad xy_2 + y_1 - 2xy_1 + xy - y = 0$$

$$\Rightarrow \quad xy_2 - (2x-1)y_1 + (x-1)y = 0$$

Example 7:

If $e^y(x+1)=1$. Show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

Solution:

we have $e^y(x+1)=1$ applying the product rule

$$e^y \cdot 1 + e^y \cdot \frac{dy}{dx}(x+1) = 0$$

$$e^y(x+1)\frac{dy}{dx} = -e^y$$

$$\frac{dy}{dx} = \frac{-e^y}{e^y(x+1)}$$

$$\frac{dy}{dx} = \frac{-1}{1+x} \qquad \text{........(1)}$$

Again diffn. w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{1}{(1+x)^2}$$

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \qquad \text{From (1)}$$

Example 8 :

If $y = \tan x + \sec x$. Show that $\dfrac{d^2y}{dx^2} = \dfrac{\cos x}{\left(1-\sin x\right)^2}$.

Solution:

we have $y = \tan x + \sec x$

$$\frac{dy}{dx} = \sec^2 x + \sec x \cdot \tan x$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1+\sin x}{\cos^2 x} = \frac{1+\sin x}{1-\sin^2 x} = \frac{1+\sin x}{(1+\sin x)(1-\sin x)}$$

$$= \frac{1}{1-\sin x} \quad \left[\because \ \cos^2 x = 1 - \sin^2 x\right]$$

Now $\dfrac{dy}{dx} = \dfrac{1}{1-\sin x}$

$$\therefore \quad \frac{d^2 y}{dx^2} = \frac{-1}{(1-\sin x)^2} \ (-\cos x)$$

$$\frac{d^2 y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$$

Example 9:

If $xy + 6y = 2x$. Show that $\dfrac{d^2 y}{dx^2} = \dfrac{-24}{(x+6)^3}$.

Solution:

Given: $xy + 6y = 2x$

$y(x + 6) = 2x$

$$\therefore \ y = \frac{2x}{x+6}$$

$$\frac{dy}{dx} = \frac{(x+6)(2) - 2x(1)}{(x+6)^2} \quad \text{(by quotient rule)}$$

$$= \frac{12}{(x+6)^2}$$

Again diffn. w.r.t. x

$$\frac{d^2 y}{dx^2} = \frac{(x+6)^2 (0) - 12.2(x+6)}{(x+6)^4} = \frac{-24(x+6)}{(x+6)^4}$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{-24}{(x+6)^3}$$

Example 10:

If $y = \log\left(x + \sqrt{x^2+1}\right)$. **Show that $(x^2 + 1)\, y_2 + xy_1 = 0$.**

Solution:

Given: $y = \log\left(x+\sqrt{x^2+1}\right)$

$$\frac{dy}{dx} = y_1 = \frac{1}{\left(x+\sqrt{x^2+1}\right)}\left[1+\frac{1}{2\sqrt{x^2+1}}.2x\right]$$

$$= \frac{1}{\cancel{\left(x+\sqrt{x^2+1}\right)}}\left[\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right]$$

$$y_1 = \frac{1}{\sqrt{x^2+1}}$$

$$\therefore \sqrt{x^2+1}.y_1 = 1$$

Again diffn. w.r.t. x

$$\sqrt{x^2+1}.y_2 + y_1\left(\frac{1}{2\sqrt{x^2+1}}.2x\right) = 0$$

$$\sqrt{x^2+1}.y_2 + \frac{xy_1}{\sqrt{x^2+1}} = 0$$

Multiply by $\sqrt{x^2+1}$

$$\therefore \left(x^2+1\right)y_2 + xy_1 = 0$$

Example 11:

If $x^2 + xy + y^2 = a^2$, show that $\dfrac{d^2y}{dx^2} = \dfrac{-6a^2}{(x+2y)^3}$

Solution:

Given : $x^2 + xy + y^2 = a^2$

Diffn. w.r.t. x on both sides we get

$$2x + \left(x\frac{dy}{dx} + y.1\right) + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x+2y)=-2x-y$$

$$\frac{dy}{dx}=\frac{-(2x+y)}{(x+2y)} \qquad \text{........(1)}$$

Again diffn. w.r.t. x

$$\frac{d^2y}{dx^2}=-\left[\frac{(x+2y)\left(2.1+\frac{dy}{dx}\right)-(2x+y)\left(1+2\frac{dy}{dx}\right)}{(x+2y)^2}\right]$$

$$=-\left[\frac{\cancel{2x}+x\frac{dy}{dx}+4y+2y\frac{dy}{dx}-\cancel{2x}-4x\frac{dy}{dx}-y-2y\frac{dy}{dx}}{(x+2y)^2}\right]$$

$$=-\left[\frac{\frac{dy}{dx}(x+2y-4x-2y)+(4y-y)}{(x+2y)^2}\right]$$

$$=-\left[\frac{\frac{dy}{dx}(-3x)+3y}{(x+2y)^2}\right]$$

$$=-\left[\frac{-\left(\frac{2x+y}{x+2y}\right)(-3x)+3y}{(x+2y)^2}\right] \text{ from (1)}$$

$$=\frac{\left[\frac{-6x^2-3xy-3y(x+2y)}{(x+2y)}\right]}{(x+2y^2)}$$

$$=\frac{-6x^2-3xy-3xy-6y^2}{(x+2y)^3}$$

$$=\frac{-6(x^2+xy+y^2)}{(x+2y)^3}$$

$$\frac{d^2y}{dx^2}=\frac{-6a^2}{(x+2y)^3} \qquad (\because x^2+xy+y^2=a^2)$$

Example 12:

If $y = (a^2 + x^2)^6$. Show that $(x^2 + a^2)y_2 - 10xy_1 - 12y = 0$

Solution:

$y = (a^2 + x^2)^6$

Diffn. w.r.t. x

$y_1 = 6(a^2 + x^2)^5.2x$

$\quad = 12x.(a^2 + x^2)^5$ ....(1)

Again diffn. w.r.t. x

$y_2 = 12x.5(a^2 + x^2)^4.2x + (a^2 + x^2)^5.(12)$

$\quad = 120x^2(a^2 + x^2)^4 + 12(a^2 + x^2)^5$ ....(2)

LHS $= (x^2 + a^2)y_2 - 10xy_1 - 12y_1$ From (1) and (2)

$= (x^2 + a^2)\ \{120x^2(a^2 + x^2)^4 + 12(a^2 + x^2)^5\} - 10x\ \{12x(a^2 + x^2)^5\} - 12(a^2 + x^2)^6$

$= \cancel{120x^2(a^2+x^2)^5} + \cancel{12(x^2+a^2)^6} - \cancel{120x^2(a^2+x^2)^5} - \cancel{12(a^2+x^2)^6}$

$= 0 =$ RHS.

EXERCISE 18.7

One mark questions:

1. Find $\dfrac{d^2y}{dx^2}$

1) $y = 3x^3 + 4x^2 + 7$	2) $y = \sqrt{2x+3}$	3) $y = e^{3x+2}$
4) $y = x^3 \cdot \log x$	5) $y = \log x + a^x$	6) $y = e^{-x}\sin 2x$
7) $y = \log(\log x)$	8) $y = \cos 4x \cos 2x$	9) $y = \sin 3x \sin 2x$
10) $y = \cos mx \sin nx$		

Two and Three marks questions:

1. Find $\dfrac{d^2y}{dx^2}$ if

   1) $x = a\ \cos\theta,\ y = a\ \sin\theta$ 2) $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$

   3) $x = a\ \cos^3 t,\ y = -a\ \sin^3 t$

2. If $y = \sin mx$, show that $\dfrac{d^2y}{dx^2} + m^2y = 0$.

3. If $y = 500e^{7x} + 600e^{-7x}$, show that $\dfrac{d^2y}{dx^2} = 49y$.

4. If $y = e^{ax} + e^{-ax}$, show that $y_2 - a^2 y = 0$.

5. If $y = 2 + \log x$, show that $xy_2 + y_1 = 0$

Five marks questions:

1. If $x^2 - xy + y^2 = a^2$, show that $\frac{d^2y}{dx^2} = \frac{6a^2}{(x-2y)^3}$

2. If $x^2 + 2xy + 3y^2 = 1$, show that $y_2 = \frac{-2}{(x+3y)^3}$

3. If $y = a\cos mx + b\sin mx$, show that $\frac{d^2y}{dx^2} + m^2 y = 0$

4. If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

5. If $y = \log\left(x - \sqrt{x^2+1}\right)$, show that $\left(x^2+1\right)y_2 + xy_1 = 0$

6. If $y = x + \sqrt{x^2-1}$, show that $\left(x^2-1\right)y_2 + xy_1 - y = 0$

7. If $y = \left(x + \sqrt{x^2+1}\right)^m$, show that $\left(x^2+1\right)y_2 + xy_1 - m^2 y = 0$

8. If $y = \sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

Answers

One mark questions:

1. $18x + 8$
2. $\frac{-1}{(2x+3)^{3/2}}$
3. $9y$
4. $x(5 + 6\log x)$
5. $-\frac{1}{x^2} + (\log a)^2 \cdot a^x$
6. $-e^x(3\sin 2x + 4\cos 2x)$
7. $\frac{-1}{x^2} \cdot \frac{1}{\log x}\left[\frac{1}{\log x} + 1\right]$
8. $-2(\cos 2x + 9\cos 6x)$
9. $\frac{1}{2}(-\cos x + 25\cos 5x)$
10. $\frac{1}{2}\left[(m+n)^2 \sin(m+n)x - (m-n)^2 \sin(m-n)x\right]$

Two marks questions:

1. $\frac{1}{a}\operatorname{cosec}^3\theta$
2. $\frac{1}{4a}\sec^4\left(\frac{\theta}{2}\right)$
3. $\frac{1}{3a}\sec^4 t \operatorname{cosec} t$

* * * * *

Chapter

19 APPLICATION OF DERIVATIVES

19.1 Introduction:

Derivative have wide range of application in science engineering, social science and also in the field of commerce. This application information about the behaviour function and hence it in an aid in sketching graph of functions. The derivatives used to define a scope of a tangent to the curve, to find the angle of intersection of two curves. The interpretation of derivate as a rate of change of one variable with respect to another leads to its importance in many field. In chemistry the rate of change of chemical reaction in physics velocity in rectilinear motion in biology the rate of growth of bacteria and in economics it physics concerned with marginal cost, marginal revenue and marginal profit etc.

Even though derivation find many application few application are discussed in this chapter.

19.2 Derivative as a Rate Measure:

If one variable depends on another variable, then the rate at which the first changes with respect to (w.r.t) the scond can be measured by means of derivative.

Note:

$y = f(x)$ be a function of x corresponds to an increment δx of x. Let δy be the increment in y.

$\therefore y + \delta y = f(x + \delta x)$

The average change of y w.r.t. x is given by

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

The $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$ exists and is called the instantaneous rate of change of 'y' w.r.t. 'x'.

hence $\boxed{\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx} = f'(x)}$ represent the rate of change of y w.r.t x

Note:

(Here rate measure w.r.t time (t) change mean = increase or decrease)

(i) If 's' is the displacement of a particle in time 't', v is the velocity of the particle then

v = the rate of change of displacement $\boxed{v = \frac{ds}{dt}}$

Intitial velocity mean t = 0, final velocity or particle at rest means v = 0.

(ii) In v is the velocity of a particle in time 't', 'a' is the acceleration of the particle then

a = the rate of chante of velocity $\therefore a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$

$\therefore \boxed{a = \frac{dv}{dt} \ or \ \frac{d^2s}{dt^2}}$

(iii) Similarly the rate of change of area 'A' of a circle w.r.t its radius 'r' is given by $\left(\frac{dA}{dr}\right)$

and w.r.t. time 't' = $\left(\frac{dA}{dt}\right)$

(iv) The rate of change of volume 'v' of a sphere w.r.t. its radius 'r' = $\left(\frac{dv}{dr}\right)$ and

w.r.t. time 't' $\left(\frac{dv}{dt}\right)$

(v) The rate of change of temperature w.r.t. 't' = $\frac{dT}{dt}$

(vi) The rate of change of price 'p' w.r.t. quantity $q = \frac{dp}{dq}$

Important formulae of plane / solid

(1) $\begin{cases} \text{Area of square} = A = (\text{side})^2 = x^2 \\ \text{Perimeter of square} = 4(\text{side}) = 4x \end{cases}$

(2) $\begin{cases} \text{Area of circle} = \pi r^2 \\ \text{Perimeter or circumference of circle} = 2\pi r \end{cases}$

(3) $\begin{cases} \text{Surface area of sphere} = 4\pi r^2 \\ \text{Volume of a sphere} = \frac{4}{3}\pi r^3 \end{cases}$

(4) Volume of a cylinder = $\pi r^2 h$

(5) Volume of a cone = $\frac{1}{3}\pi r^2 h$

(r = radius, h = height)

(6) Area of an equilateral triangle = $\frac{\sqrt{3}}{4}$ (side)^2

(7) $\begin{cases} \text{Volume of cube} = (\text{side})^3 = x^3 \\ \text{Surface area of cube} = 6(\text{side})^2 = 6x^2 \end{cases}$

WORKED EXAMPLES

Example 1 :

A particle moves a distance S = 6t² – t³ + 5 (S = mt, t = sec) then find

(i) the velocity and acceleration after 3 second

(ii) the time at which it is rest

Solution :

(i) Formula, $V = \frac{ds}{dt}$

$= 6(2t) - 3t^2 + 0$

$V = 12t - 3t^2$

at $t = 3$ sec, $V = 12(3) - 3\,(3)^2 = 9$ mt/sec

acceleration $= \frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$

$a = 12(1) - 3\,(2t)$

$a = 12 - 6t$

at t = 3sec acc $= 12 - 6(3) = -6$

$\therefore$ acc $= -6$ mt/sec^2 (retardation)

(ii) At rest $V = 0$

$\therefore \frac{ds}{dt} = 0$

$12t - 3t^2 = 0$

$3t\,(4 - t) = 0$

$3t = 0$ or $4 - t = 0$

$t = 0$ or $t = 4$ sec

$\therefore$ the particle became to rest at 0 or at 4 sec

Example 2 :

The distance ‘S’ feet travelled by a particle in time ‘t’ second is given by $s = t^3 - 6t^2 + 15t + 2$. Find when the aceleration is zero.

Solution :

Given acc = 0, t = ?

$$V = \frac{ds}{dt} = 3t^2 - 12t + 15$$

$$\therefore \quad acc = \frac{dv}{dt} = 6t - 12$$

$$0 = 6t - 12 \qquad \therefore 6t = 12$$

$$\boxed{t = 2\,\text{sec}}$$

Example 3 :

According to law of motion of a particle if $s = t^3 - 6t^2 + 9t + 8$, find its initial velocity?

Solution : $V = \frac{ds}{dt}$

$$= 3t^2 - 12t + 9$$

Initial velocity means at $t = 0$ sec

$\therefore$ Initial velocity $= 3(0) - 12(0) + 9 = 9$ unit

Example 4 :

If the displacement ‘s’ at any time ‘t’ is given by $s = \sqrt{1-t}$. Show that the velocity is inversely proportional to the displacement.

Solution : $V = \frac{ds}{dt}$

$$= \frac{1}{2\sqrt{1-t}} \frac{d}{dt}(1-t)$$

$$= \frac{1}{2\sqrt{1-t}}(-1) = \frac{1}{2\sqrt{1-t}} = \frac{1}{2} \times \frac{1}{\sqrt{1-t}} = -\frac{1}{2} \times \frac{1}{s}$$

$$\boxed{V \alpha \frac{1}{s}} \quad \left(\because -\frac{1}{2} = a \text{ constant} \right)$$

I. 3 marks question:

Example 5 :

The displacement 's' of a particle at time 't' is given by $S = 2t^3 - 5t^2 + 4t - 3$ find

(i) the time when the acceleration is $14ft\ sec^2$.

(ii) the velocity and displacement at that time.

Solution :

(i) $V = \frac{ds}{dt}$

$= 2(3t^2) - 5(2t) + 4(1)$

$= 6t^2 - 10t + 4$

$\therefore$ acc $= \frac{ds}{dt} = 6(2t) - 10$

acc $= 12t - 10$, given acc $= 14ft\ sec^2$

$14 = 12t - 10$

$24 = 12t \quad \therefore \boxed{t = 2sec}$

(ii) $\therefore$ at $t = 2$ sec, $V = 6(2)^2 - 10(2) + 4$

$= 24 - 20 - 4 = 8$ ft / sec

at $t = 2$ sec, $S = 2(2)^3 - 5(2)^2 + 4(2) - 3$

$= 16 - 20 + 8 - 3$

$= 24 - 23 = 1$ ft

Example 6 :

When brakes are applied to a moving car, the car travels a distance 's' feet in 't' sec given by $s = 20t - 40t^2$. When and where does the car stop?

Solution :

Given, car stop $v = 0$, $t = ?$ $s = ?$

(i) $v = \frac{ds}{dt}$

$0 = 20(1) - 40(2t)$

$0 = 20 - 80t$

$80t = 20 \qquad \therefore t = \frac{20}{80} = \frac{1}{4}$ sec or $\boxed{0.25sec}$

$\therefore$ when the brakes are applied the car take 0.25 sec to stop

(ii) $\therefore$ distance travelled during $t = 0.25$ sec

$$s = 20\,t - 40t^2$$

$$s = 20(0.25) - 40(0.25)^2$$

$$\boxed{s = 2.5 \text{ feet}}$$

Example 7 :

A particle moves according to the laws $s = t^3 + at^2 + bt$. Find 'a' and 'b' if the initial velocity is 5 unit and when $t = 1$ sec it is moving with a velocity which is 4 times its initial velocity.

Solution : $v = \dfrac{ds}{dt}$

$$= 3t^2 + a(2t) + b(1)$$

$$v = 3t^2 + 2at + b$$

Initial velocity at $t = 0$ sec is 5

$$\therefore\ 5 = 3(0) + 2a(0) + b$$

$$\boxed{b = 5}$$

And at $t = 1$ sec, velocity = 4 times initial velocity

$$3t^2 + 2at + b = 4 \times 5$$

$$3(1) + 2a(1) + b = 20$$

$$2a + b = 20 - 3$$

$$2a + b = 17 \quad \text{put } b = 5$$

$$2a + 5 = 17$$

$$2a = 17 - 5$$

$$2a = 12$$

$$\therefore\ \boxed{a = 6}$$

Example 8 :

A particle shot vertically upward rises 's' feet in 't' sec where $s = 40t - 16t^2$. Find the greatest height attained by the particle.

Solution :

The greatest height attained by the particle is $v = 0$

$$v = \frac{ds}{dt}$$

$$0 = 40(1) - 16(2t)$$

$$32t = 40 \quad \therefore\ t = \frac{40}{32} = \frac{5}{4} \text{ or } \boxed{1.25\,\text{sec}}$$

at t = 1.25sec, the greatest height reached by the particle

$$s = 40\,(1.25) - 16(1.25)^2$$

$$s = 50 - 16\left(\frac{25}{10}\right) = 50 - 25 = \boxed{25\text{ feet}}$$

Example 9 :

The radius of a circular plate increase at the rate $\frac{8}{3\pi}$ cm/sec. Find the

(i) rate of change in the area when the diameter in 12cm

(ii) and also find the rate of increase of the circumference of circle after 3 sec.

Solution :

Given $\frac{dr}{dt} = \frac{8}{3\pi}, \frac{dA}{dt} = ? \qquad \frac{dc}{dt} = ?$, r = 12cm

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi.2r\left(\frac{dr}{dt}\right)$$

$$= \pi.2r\left(\frac{8}{3\pi}\right) = 32cm/\sec^2$$

circumference of circle , C = 2πr $\qquad \therefore \frac{dc}{dt} = 2\pi(1)\frac{dr}{dt}$

$$= 2\times\frac{22}{7}\times\frac{16r}{3} = \frac{44\times16\times(6)^2}{21}$$

Rate of change of circumference $\frac{dc}{dt} = \frac{88\times16}{7} = \frac{1408}{7} = 201.1\ cm/\sec$

Example 10 :

A circular blot of ink increases in area in such a way that the radius at time 't' sec is given by $r = 2t^2 - \frac{t^3}{4}$. What rate the area of the blot increase when t = 4 sec.

Solution :

$$A = \pi r^2$$

$$\therefore \frac{dA}{dt} = \pi 2r\frac{dr}{dt} \qquad \text{Given, } r = 2t^2 - \frac{t^3}{4}$$

$$= 2\pi\ 16(4),$$

$$= 128\ \pi cm^2 / sec$$

$$[\ \therefore\ r = 2\ (4)^2 - \frac{4^3}{4} = 32 - 16 = 16]$$

$$\frac{dr}{dt} = 2(2t) - \frac{3t^2}{4}$$

$$= 4t - \frac{3t^2}{4}$$

at t = 4 sec

$$\frac{dr}{dt} = 4(4) - \frac{3(4)}{4} = 16 - 12$$

$$\therefore \frac{dr}{dt} = 4cm/\sec$$

Example 11 :

The volume of a sphere is increasing at the rate 4π c.c/sec. Find the rate at which the area of its surface increases when its radius is 10cm.

Solution : Given $\frac{dv}{dt} = 4\pi c.c./\sec$, r = 10cm, $\frac{dA}{dt} = ?$

Volume of a sphere = $V = \frac{4}{3}\pi r^3$

$$\frac{dv}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right)$$

$$4\pi = 4\pi(10)^2\frac{dr}{dt} \quad \Rightarrow \quad 4\pi = 400\pi\frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{4\pi}{400\pi} = \frac{1}{100}cm/\sec$$

A = surface are of the sphere = $4\pi r^2$

$$\frac{dA}{dt} = 4\pi\left(2r\frac{dr}{dt}\right) \Rightarrow 8\pi(10)\left(\frac{1}{100}\right) \therefore \boxed{\frac{dA}{dt} = \frac{4\pi}{5}cm^2/\sec}$$

Example 12 :

A spherical balloon is being inflated so that is volume is increasing at the rate of 30c.c/ min. How fast its surface area increasing when its volume is 36π c.c ?

Solution :

Given $\frac{dv}{dt} = 30c.c/\min$, $\frac{dA}{dt} = ?$, $v = 36\pi c.c$

Volume = $V = \frac{4}{3}\pi r^2 \Rightarrow 36\pi = 4\frac{\pi}{3}(r^3)$

$\frac{36\pi \times 3}{4\pi} = r^3 \quad \therefore r^3 = 27 \quad \therefore \boxed{r = 3cm}$

Again, $V = \frac{4}{3}\pi r^3$

$$\therefore \frac{dv}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right)$$

$$30 = 4\pi(3)^2\frac{dr}{dt} \Rightarrow 30 = 36\pi\frac{dv}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{30}{36\pi} = \frac{5}{6\pi}cm/\min$$

surface area $A = 4\pi r^2 \Rightarrow \frac{dA}{dt} = 4\pi\ 2r\frac{dr}{dt}$

$$= 8\pi(3)\left(\frac{5}{6\pi}\right) = \frac{40}{2}$$

$$\frac{dA}{dt} = 20\ cm^2/\min$$

Example 13 :

A squareplate in expanding uniformly, the side is increasing the rate of 5cm / sec. What is the rate at which the area and its perimeter is increasing when the side is 20cm long.

Solution : Let the side of the square be 'a' cm

Given, $\frac{da}{dt} = 5cm/\sec$, $\frac{dA}{dt} = ?$, $\frac{dP}{dt} = ?$, a = 20cm

Area of the square $A = a^2$

$$\therefore \frac{dA}{dt} = 2a\left(\frac{da}{dt}\right)$$

$$= 2(20)\,(5) = 200\text{cm}^2/\sec$$

Again, perimeter $= P = 4a$

$$\therefore \frac{dP}{dt} = 4(1)\frac{da}{dt} \quad = \quad 4(5) \qquad \therefore \boxed{20cm/\sec}$$

Example 14 :

The sides of an equilateral triangel are increasing at the rate 3 cm / sec. How fast is its area increasing when the side is 10 cm.

Solution:

Let us assume that the side equilateral triangle = 'a' cm

$$\frac{da}{dt} = 3cm/\sec, \quad a = 10\text{cm}, \quad \frac{dA}{dt} = ?$$

$$A = \frac{\sqrt{3}}{4}a^2 \qquad \therefore \frac{dA}{dt} = \frac{\sqrt{3}}{4}\left(2a\frac{da}{dt}\right)$$

$$= \frac{\sqrt{3}}{4}\times 2(10)(3) \quad = \quad \frac{60\sqrt{3}}{4} = 15\sqrt{3}$$

$$\therefore \boxed{\frac{dA}{dt} = 15\sqrt{3}\ cm^2/\sec}$$

Example 15 :

Water is flowing out of a vertical cylinderical tank at the rate of 15 cubicft / min. Find how the level water is decreasing if the radius of the tank is 3ft.

Solution :

$$\frac{dv}{dt} = 15\ C\ ft/\min, \qquad \frac{dh}{dt} = ?, \qquad r = 3\,ft$$

Volume of a cylindrical tank, $v = \pi r^2 h$

$$\therefore \frac{dv}{dt} = \pi r^2\left(\frac{dh}{dt}\right) \qquad (r = \text{constant})$$

$$15 = \pi(3)^2\frac{dh}{dt} \qquad \therefore \frac{dh}{dt} = \frac{15}{9\pi} = \frac{5}{3\pi}\text{ft}/\min$$

Example 16 :

The edge of a variable cube is increasing at the rate of 6cm / min. How fast is the volume and its surface increasing when the edge is 10cm long?

Solution :

Let the edge of the cube be 'x' cm.

Given, $\frac{dx}{dt} = 6cm/\text{min}$, $\frac{dv}{dt} = ?$, $\frac{dA}{dt} = ?$, $x = 10\text{cm}$

Volume of the cube = $\text{v} = x^3$

$$\therefore \frac{dv}{dt} = 3x^2 \frac{dx}{dt}$$

$$= 3(10)^2 (6) = 3 \times 100 \times 6$$

$$\boxed{\frac{dA}{dt} = 1800cm^3 / \text{min}}$$

surface area of the cube = A = $6x^2$

$$\therefore \frac{dA}{dt} = 6(2x)\frac{dx}{dt} = 12\,(10)\,(6)$$

$$\boxed{\frac{dA}{dt} = 720cm^2 / \text{min}}$$

Example 17 :

The height of a cone is 60cm and it is constant the radius of the base is increasing at the rate of 0.50 cm/sec. Find the rate of increase of volume of the cone when the radius is 10cm?

Solution :

Given, h = 60 cm (constant) , $\frac{dr}{dt} = 0.50\ cm/\text{sec}$ $\frac{dv}{dt} = ?$, r = 10 cm

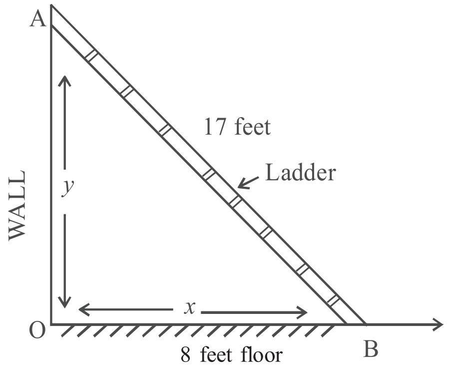
Volume of the cone = $\text{v} = \frac{1}{3}\pi r^2 h$

$$\therefore \frac{dv}{dt} = \frac{1}{3}\pi h\left(2r\frac{dr}{dt}\right) \quad (\therefore \text{h} = \text{constant})$$

$$= \frac{2}{3}\pi \times 60 \times 10 \times 0.50 \qquad \therefore \boxed{\frac{dv}{dt} = 200\pi\ \text{cm}^2 / \text{sec}}$$

Example 18 : A ladder 17 feet long leans against a smooth vertical wall. If the lower end is moving at the rate of 2 ft min. Find the rate at which the upper end is moving when the lower end is 8 ft from the wall.

Solution :



Given, $\frac{dy}{dt}$ = Rate at which the upper end moves = 2ft min.

$\frac{dx}{dt}$ = Rate at which the lower end moves = ?

ΔABC is a right angle triangle

$\therefore$ from pythogoras theorem

$$OB^2 + OA^2 = AB^2$$

$$x^2 + y^2 = 17^2 \quad ....(1)$$

$$8^2 + y^2 = 17^2 \Rightarrow 64 + y^2 = 289$$

$$\therefore y^2 = 289 - 64 = 225 \qquad \therefore y = 15 \text{ feet}$$

Again differentiate (1) w.r.t. t

$$\therefore\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\div 2, \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \qquad \therefore 8\frac{dx}{dt} + 15(2) = 0$$

$$8\frac{dx}{dt} = -30 \qquad \therefore\ \frac{dx}{dt} = \frac{-30}{8}$$

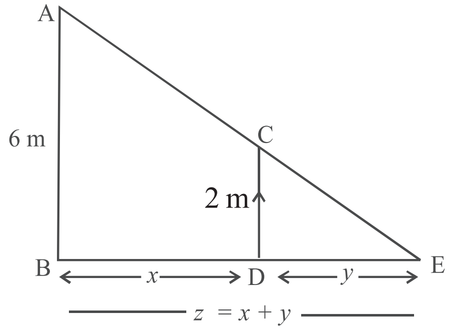
$$\therefore\ \boxed{\frac{dx}{dt} = -3.75 \text{ ft / min}}$$

(negative sign shows that the ladder in moving away from the wall)

Example 19:

A man 2mt height walks at a uniform speed of 6km / hour away from the lamp post 6m high. Find the rate at which the (i) length of his shadow increase (ii) the rate at which the tip of the shadow is moving.

Solution :



Given the rate of man walking $\frac{dx}{dt} = 6km/hr$

Rate of shadow length increase = $\frac{dy}{dt} = ?$

Rate tip of shadow nowing $\frac{dz}{dt} = ?$

ΔABE and ΔCDE are similar

$$\therefore \quad \frac{AB}{CD} = \frac{BE}{DE} \quad \Rightarrow \quad \frac{6}{2} = \frac{x+y}{y} \quad \Rightarrow \quad \frac{3}{1} = \frac{x+y}{y}$$

$$3y = x + y \qquad \therefore \boxed{2y = x} \text{ Diff. w.r.t. t}$$

$$\therefore \; 2\frac{dy}{dt} = \frac{dx}{dt} = 6$$

$\therefore$ Rate of increase of length of shadow = $\boxed{\frac{dy}{dt} = 3km/hr}$

Again, $z = x + y$ diff. w.r.t.

$$\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \quad \Rightarrow \quad 6 + 3 = 9 \text{ km/hr}$$

$\therefore$ Rate at which tip of shadow is moving = 9km / hr

Example 20 :

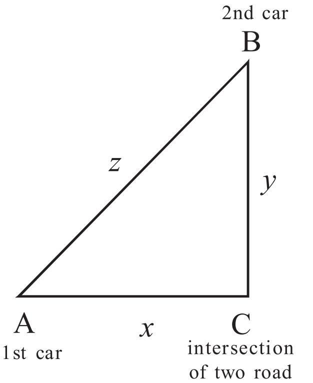
Two car, one going due east the rate 90 mt/min and the other going due south at the rate 60 mt/min all travelling towards an intersection of the two roads. Show that the two cars are approaching each other at the rate 108 mt/min. At the instant when the 1st car is 200 mt and the 2nd car is 150 mt from the intersection.

Solution :

Position of 1st car is at A,   2nd car is at B

Let x = distnace of 1st car from the point of intersection of two road (at C) = 200mt

y = 2nd car = 150 mt



$$\frac{dx}{dt} = -90 \text{ mt / min}, \quad \frac{dy}{dt} = -60 \text{ mt / min}$$

(negative sign show both are approaching towards the point of Intersection)

z = distance between two car

$\therefore$ ABC is a right angled triangle

$$z^2 = x^2 + y^2 ....(1)$$

$$= 200^2 + 150^2 = 40000 + 22500$$

$$z^2 = 62500 \qquad \therefore z = \sqrt{62500} \qquad \boxed{z = 250\text{mt}}$$

Diff (1) w.r.t 't'

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dz}{dt}$$

$$\div 2, \qquad z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dz}{dt}$$

$$250\left(\frac{dz}{dt}\right) = 200(-90) + 150(-60) = -18000 - 9000$$

$$250\left(\frac{dz}{dt}\right) = -27000$$

$$\therefore \frac{dz}{dt} = \frac{-27000}{250} = \boxed{-180 \text{ mt / min}}$$

$\therefore$ The two car approach at the rate 180 mt/min

EXERCISE 19.1

I. 2 marks questions:

1. The displacement 's' of a particle at time 't' is given by $S = 4t^3 - 6t^2 + t - 7$. Find the velocity and acceleration when t = 2 sec.
2. If $S = 5t^2 + 4t - 8$. Find the initial velocity and acceleration. (s = displacement, t = time)
3. A stone thrown vertically upward rises 's' ft. in 't' sec. where $s = 80t - 16t^2$. What its velocity after 2 sec? Find the acceleration?
4. A body is thrown vertically upwards its distance S feet is 't' sec is given by $S = 5 + 12t - t^2$. Find the greatest height reached by the body.
5. If $v = \sqrt{s^2 + 1}$ prove that acceleration is 'S'. (V = velocity, S = displacement)
6. If $S = at^3 + bt$. Find a and b given that when t = 3 velocity is '0' and the acceleration is 14 unit. (S = displacement, t = time).
7. When the brakes are applied to a moving car, the car travels a distance 's' ft in 't' see given by $s = 8t - 6t^2$ when does the car stop?

II. 3 marks questions:

1. The radius of a sphere is increasing at the rate of 0.5 mt/ sec. Find the rate of increase of its surface area and volume after 3 sec?
2. The surface area of a spherical bubble is increasing at the rate of a 0.8 cm^2 / sec find at what rate is its volume increasing when r = 2.5cm [r = radius of the sphere).
3. A spherical balloon is being inflated at the rate 35 cc / sec. Find the rate at which the surface area of the balloon increases when its diameter is 14 cm.
4. The radius of a circular plate is increasing at the rate of $\frac{2}{3\pi}$ cm/sec. Find the rate of change of its area when the radius is 6cm.
5. A circular patch of oil spreads on water the area growing at the rate of 16cm^2 / min. How fast are radius and the circumference increasing when the diameter is 12 cm.
6. A stone is dropped into a pond waved in the form of circles are generated and the radius of the outer most ripple increases at the rate 2 inches / sec. How fast is the area increasing when the a) radius is 5 inches b) after 5 sec ?
7. The side of an equilateral triangle is increasing at the rate $\sqrt{3}$ cm/sec. Find the rate at which its area is increasing when its side is 2 meters.

8. Water is being poured at the rate of 30 mt^3 /min into a cylindrical vessel whose base is a circle of radius 3 mt. Find the rate at which the level of water is rising ?
9. Sand is being dropped at the rate of 10 mt^3 /sec into a conical pile. If the heihgt of the pile twice the radius of the base, at what rate is the height to the pile is increasing when the sand in the pile is 8mt high.
10. A ladder of 15ft long leans against a smooth vertical wall. If the top slides downwards at the rate of 2 ft / sec. Find how fast the lower and is moving when the lower end is 12 ft from the wall.
11. An edge of a variable cube is increasing at the rate of 10cm/sec. How fast in the volume and also its surface area is increasing when the edge is 5cm long.
12. A man 6ft tall is moving directly away from a lamp post of height 10 ft above the ground. If he is moving at the rate 3 ft / sec. Find the rate at whcih the length of his shadow is increasing and also the tip of his shadow is moving?
13. The height of circular cone is 30 cm and it is constant. The radius of the base is increasing at the rate of 0.25 cm/ sec. Find the rate of increase of volume of the cone when the radius of base is 10cm.
14. The volume of a spherical ball in increasing at the rate 4πcc /sec. Find the rate of increase of the radius of the ball when the volume is 288 πcc.
15. A drop of ink spreads over a blotting paper so that the circumferenes of the blot is 4πcm and it changes 3cm/sec. Find the rate of increase of its radius and also find the rate of increase of its area?
16. A circular plate of metal is heated so that its radius increases at the rat of 0.1 mm / min. At what rate is the plate's area increasing when the radius is 25cm [1cm = 10mm]
17. The surface area of a spherical soap bubble increasing at the rate of 0.6cm^2 /sec. Find the rate at which its volume is increasing when its radius is 3cm.
18. A rod 13 feet long slides with it end A and B as two straight lines at right angles which meet at 'O' . If A is moved away from O with a uniform speed at 4ft / sec, find the speed of the end B move when A is 5 feet from O.
19. A street lamp is hung 12 feet above a straight horizontal floor on which a man of 5 feet is walking how fast his shadow lengthening when he is walking away from the lamp post at the rate of 175 ft/min.
20. Find a point on the parabola $y^2 = 4x$ at which the ordinate increases at twice the rate of the abscissa [ordinate = y, absissa = x].

Answers 19.1

I. 1) V = 25 unit, acc = 36 unit

2) I.V = 4 unit, acc = 10 unit

3) V = 10ft / sec, acc = –32ft/sec^2

4) S = 41 feet

6) $a = \frac{7}{9}$, $b = -21$

7) $t = \frac{+2}{3}$ sec

II. 1) 6πm^3/sec, 4.5πm^3/sec

2) 1.cc/sec

3) 10cm^2/sec

4) 8cm^2/sec

5) $\frac{4}{3\pi}$ cm / min, $\frac{8}{3}$ cm/min

6) 20π inch^2/sec, 40π inch^2/sec

7) 300 cm^2 /sec

8) $\frac{10}{3\pi}$ mt/min

9) $\frac{5}{8\pi}$ mt/sec

10) $\frac{3}{2}$ ft/sec

11) 750 cm^2 /sec, 600 cm^2 /sec

12) 4.5ft/ sec, 7.5 ft/sec

13) 50π cm^2 /sec

14) $\frac{1}{36}$ cm/sec

15) $\frac{3}{2\pi}$ cm/sec, 6cm^2 /sec

16) 50π mm^2 /min

17) 0.9 cc/sec

18) $\frac{5}{3}$ ft /sec

19) 125 ft/min

20) $(x, y) = \left(\frac{1}{4}, 1\right)$

19.3 Increasing and Decreasing function

Definition : A function $y = f(x)$ be a continuous function defined on an Interval I. Let x_1 and x_2 are any two points in the interval. Then $f(x)$ is said to

(i) Increasing on the interval if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

(ii) Strictly increasing on the interval if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

(iii) Decreasing on the interval if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

(iv) Strictly decreasing on the interval if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Graphical Representation:

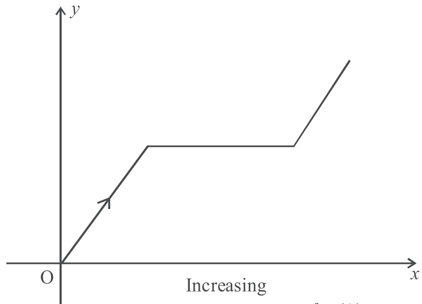


fig (1)

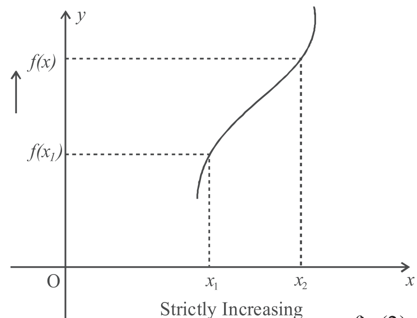


fig (2)

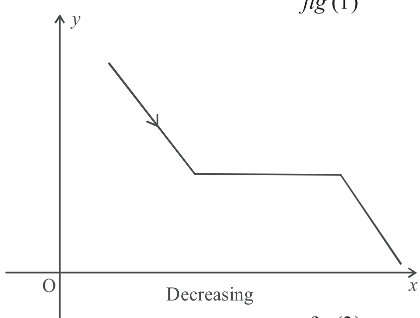


fig (3)

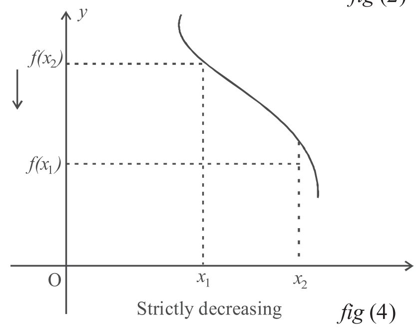


fig (4)

Note :(i) When the function $y = f(x)$ is increasing in an Interval then $\dfrac{dy}{dx} = f^1(x) > 0$ (positive)

(ii) When the function $y = f(x)$ is decreasing in an Interval then $\dfrac{dy}{dx} = f^1(x) < 0$ (negative)

(iii) If $y = f(x)$ is said to be **stationary at a point *f(x)* is neither increases nor decreases then** $\dfrac{dy}{dx} = 0$ i.e. the tangent is parallel x – axis ....fig (5)

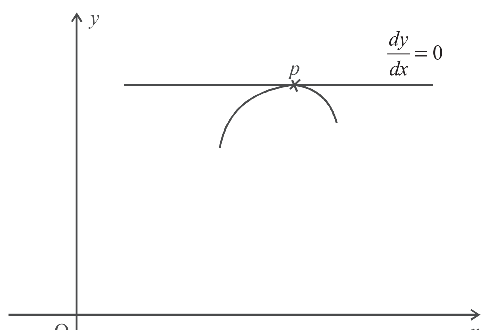


fig (5)

(iv) The value of x for which the function $y = f'(x)$ is stationary are more frequently called **Critial value** for the function.

Example 1 :

Find the interval in which $f(x) = 5 + 36x + 3x^2 - 2x^3$ is

(i) increasing (ii) decreasing

Solution :

Let $y = f(x) = 5 + 36x + 3x^2 - 2x^3$

$\therefore \frac{dy}{dx} = f^1(x) = 36(1) + 3(2x) - 2(3x^2)$

(i) for increasing $36 + 6x - 6x^2 > 0$

$\div$ 6, $6 + x - x^2 > 0$

$-(x^2 - x - 6) > 0$

multiplying by (–) sign $(x^2 - x - 6) < 0$

$(x-3)(x+2) < 0$

$\Rightarrow x - 3 < 0$ or $x + 2 > 0$ $\Rightarrow x < 3$ or $x > -2$

$\Rightarrow \boxed{-2 < x < 3}$

$\therefore f(x)$ is increasing in the interval (–2, 3)

(ii) for Decreasing $f^1(x) < 0$

$\therefore 36 + 6x - 6x^2 < 0$

$\div$ 6, $6 + x - x^2 < 0$

$-(x^2 - x - 6) < 0$

multiply by (–) sign $x^2 - x - 6 > 0$

$(x-3)(x+2) > 0 \Rightarrow x - 3 > 0$ or $x + 2 < 0$

$x > 3$ or $x < -2$ $\therefore$ *f(x)* is decreasing in the Interval $(-\infty, -2) \cup (3, \infty)$

Example 2 :

Find whether the function are increasing or decreasing or neither increasing nor decreasing.

Solution :

(i) $f(x) = 2x^2 + 3x - 7$ at $x = 2$

$$f^1(x) = 4x + 3$$

$$f^1(2) = 4(2) + 3 = 11 > 0(+)$$

$\therefore$ *f(x)* is increasing at $x = 2$

(ii) $f(x) = x^3 - 6x^2 + 12x - 1$

$\therefore f^1(x) = 3x^2 - 6(2x) + 12(1)$

$$= 3x^2 - 12x + 12 = 3(x^2 - 4x + 4)$$

$3(x-2)^2 > 0$ for all the value of 'x'

$\therefore$ *f(x)* is increasing at all points

(iii) $f(x) = x^2 - 6x + 5$

$f^1(x) = 2x - 6(1) = 2x - 6 = 2(x-3)$

$f^1(x) > 0 \Rightarrow x - 3 > 0 \Rightarrow x > 3$

$f^1(x) < 0 \Rightarrow x - 3 < 0 \Rightarrow x < 3$

$\therefore$ *f(x)* is increasing at all $x > 3$

decreasing for all $x < 3$

(iv) $f(x) = x^3 - 4x^2 + 5x + 1$ at $x = 1$

$f^1(x) = 3x^2 - 4(2x) + 5(1) = 3x^2 - 8x + 5$

$\therefore f^1(1) = 3(1)^2 - 8(1) + 5 = 3 - 8 + 5 = 0$

$\therefore f(x)$ is neither increasing or decreasing at $x = 1$

Example 3 :

Find the critical (stationary) points of the function $f(x) = 2x^3 - 9x^2 + 12x + 6$

Solution : $f(x) = 2(3x^2) - 9(2x) + 12(1)$

$$= 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$$

for critical point $f^1(x) = 0$

$\therefore 6(x^2 - 3x + 2) = 0 \Rightarrow (x-1)(x-2) = \frac{0}{6} = 0$

$\therefore x = 1$ and $x = 2$ $\quad\quad$ $\therefore$ the critical points are 1 and 2

Example 4 :

Find the stationary points and the corresponding value of the function $f(x) = x^3 - 3x^2 - 9x + 5$

Solution :

$f^1(x) = 3x^2 - 3(2x) - 9(1) = 3x^2 - 6x - 9$

At stationary point $f^1(x) = 0$

$\therefore 0 = 3(x^2 - 2x - 3) \qquad \therefore 0 = 3(x+1)(x-3)$

$\Rightarrow x + 1 = 0$ or $x - 3 = 0$

$x = -1$ or $x = 3$

the stationary point are –1 and 3.

the values are at $x = -1$

$f(x) = (-1) - 3(-1)^2 - 9(-1) + 5$

$= -1 - 3 + 9 + 5 = \boxed{10}$

at $x = 3$ $f(3) = (3)^3 - 3(3)^2 - 9(3) + 5$

$= 27 - 27 - 27 + 5 = \boxed{-22}$

$\therefore$ the stationary values are 10 and –22

Example 5 :

Find the domain of increasing and decreasing of $f(x) = x^2 - 4x + 3$

Solution :

$f(x) = x^2 - 4x + 3 \quad \Rightarrow \quad f^1(x) = 2x - 4(1)$

for increasing $\quad f^1(x) > 0 \quad \therefore 2x - 4 > 0$

$\therefore 2x > 4 \qquad \boxed{x > 2}$

for decreasing function $f^1(x) < 0 \quad \Rightarrow \quad 2x - 4 < 0$

$2x - 4 \qquad \therefore \quad \boxed{x < 2}$

$\therefore$ the given function increasing in $(2, \infty)$

given function decreasing in $(-\infty, 2)$

Example 6 :

Find whether the following are increasing or decreasing or neither. And also find the critical point if they exist.

(i) $f(x) = 2x^3 - 21x^2 + 36x - 20$ **at** $x = 0, -1$

Solution :

$$f^1(x) = 2(3x^2) - 21(2x) + 36(1) = 6x^2 - 42x + 36$$

at $x = 0$, $\quad f^1(0) = 6(0) - 42(0) + 36 = 36 > 0$

at $x = -1$, $\quad f^1(1) = 6(1)^2 - 42(1) + 36 = 6 + 42 + 36 = 84 > 0$

$\therefore$ *f(x)* is increasing at both $x = 0$ and $x = -1$

for critical point

$f^1(x) = 0$

$6x^2 - 42x + 36 = 0 \Rightarrow 6(x^2 - 7x + 6) = 0$

$x^2 - 7x + 6 = 0 \Rightarrow (x-1)(x-6) = 0$

$x - 1 = 0$ and $x - 6 = 0$

$\therefore \quad x = 1,$ 6 all critical point

(ii) $f(x) = x + \dfrac{1}{x}$ **at** $x = 2, 3$

Solution : $f^1(x) = (1) + (-1x^{-1-1}) = 1 - \dfrac{1}{x^2}$

at $x = 2$, $\quad f^1(2) = 1 - \dfrac{1}{(2)^2} = 1 - \dfrac{1}{4} = \dfrac{3}{4} > 0$

$x = 3$, $\quad f^1(3) = 1 - \dfrac{1}{3^2} = 1 - \dfrac{1}{9} = \dfrac{8}{9} > 0$

$\therefore$ *f(x)* is increasing at $x = 2$ and 3

for critical value, $f^1(x) = 0 \quad \therefore 1 - \dfrac{1}{x^2} = 0$

$\dfrac{1}{x^2} = 1 \qquad \therefore x^2 = 1$

$\boxed{x = \pm 1}$ are the critical points

(iii) $f(x) = \dfrac{x^2 - 7x + 6}{x - 10}$ **at** $x = 1, -1$

Solution : $f^1(x) = \dfrac{(x-10)(2x-7) - (x^2 - 7x + 6)(1)}{(x-10)^2}$

at $x = 1$ $f^1(1) = \dfrac{(1-10)(2-7) - (1-7+6)}{(1-10)^2}$

$$= \frac{(-9)\ (-5) - (0)}{(-9)^2} = \frac{45}{81} > 0$$

at $x = -1$ $f^1(-1) = \dfrac{(-1-10)(-2-7) - \left((-1)^2 - 7(-1) + 6\right)}{(-1-10)^2}$

$$= \frac{(-11)(-9) - (1+7+6)}{(-11)^2} = \frac{99-14}{121} = \frac{85}{121} > 0$$

$\therefore f(x)$ is increasing at $x = 1$ and -1

for critical value $f^1(x) = 0$

$$\frac{(x-10)(2x-7) - (x^2 - 7x + 6)}{(x-10)^2} = 0$$

$(x-10)(2x-7) - (x^2 - 7x + 6) = 0$

$2x^2 - \cancel{7x} - 20x + 70 - x^2 + \cancel{7x} - 6 = 0$

$x^2 - 20x + 64 = 0 \Rightarrow (x-16)(x-4) = 0$

$\therefore$ $\boxed{x = 16,\ 4}$ are critical value

(iv) $\boldsymbol{f(x) = e^x + 1}$ **at** $\boldsymbol{x = 0}$

$f^1(x) = e^x$

at $x = 0$ $f^1(0) = e^0 = 1 > 0$ $\therefore f(x)$ is increasing at $x = 0$

for critical point $f^1(x) = 0$ $\therefore e^x = 0$ but e^x can never be '0' for any value of 'x'

$\therefore f(x)$ has no critical point

EXERCISE 19.2

2 and 3 mark

I. Find whether the following functions are increasing or decreasing or neither.

(i) $f(x)=x^4-8x^3+22x^2-24x+5$ at $x=0,-2$

(ii) $f(x)=4x^3-15x^2+12x-2$ at $x=1,-1$

(iii) $f(x)=(x-1)(x-2)^2$ at $x=1,\ 3$

II. Find the value of x (Interval) for which the function is increasing or decreasing.

(i) $f(x)=2x^3-15x^2-84x+7$

(ii) $f(x)=x^4-2x^3+1$

(iii) $f(x)=x^3-3x^2+3x-100$

(iv) $f(x)=2x^2-96x+5$

(v) $f(x)=10-6x-2x^2$

(vi) $f(x)=2x^3+9x^2+12x+20$

ANSWER 19.2

I. (i) Decreasing at $x=0$, Increasing at $x=-2$

(ii) Decreasing at $x=1$, Increasing at $x=-1$

(iii) Increasing at $x=1, x=3$

II. (i) Increasing : $x<-2$ and $x>7$, Interval $(-\infty,-2)\cup(7,\infty)$

(ii) Increasing : $x>\frac{3}{2}$, Interval $\left(\frac{3}{2},\infty\right)$

(iii) Increasing for all x

(iv) Decreasing $-4<x<4$

(v) Decreasing $x>-\frac{3}{2}$, Increasing $x<-\frac{3}{2}$

(vi) Increasing $(-\infty,-2)\cup(-1,\infty)$,
Decreasing $(-2,-1)$

19.4 Maxima and Minima

Open Interval & Closed Interval : If $x\in R$ (R = set of finite real number) 'a' and 'b' the open interval with end point a and b is denoted by (a,b) is the set $\{x\in R, a<x<b\}$ for closed Interval $[a,b]$ is the set $\{x\in R, a\le x\le b\}$

Local Maxima (Relative Maxima)

In the fig(1) A is called a point of Local Maxima at $x = a$ because the value of the function at A is greater than at any point in the neighbourhood M and N ie., at $x = a - h$ and $x = a + h$

$\therefore$ *f(x)* is said to have a local maxima at $x = a$ there exist an interval $(a-h,\ b+h)$ neighbourhood of 'a' such that $f(a) > f(a+h)$ and $f(a) > f(a-h)$.

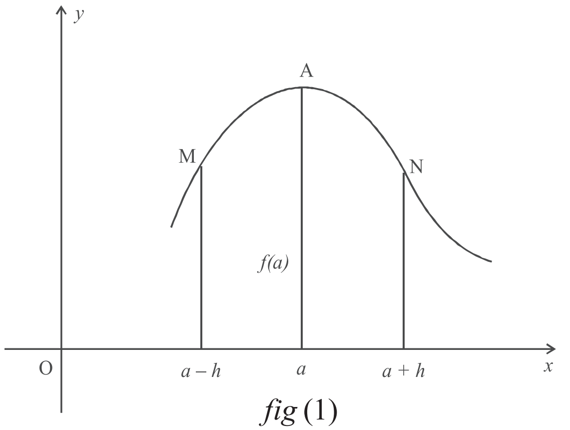


fig (1)

Local Minimum (Relative Minima)

In the *fig*(2) B is called a point of Local Minima at $x = b$ because the value of the function at B is small than at any point in the neighbourhood M and N ie., at $x = b - h$ and $x = b + h$

$\therefore$ *f(x)* is said to have a local minima at $x = b$ there exist an interval $(b - h,\ b + h)$ neighbourhood of b such that

$$f(b) < f(b-h)$$
$$f(b) < f(b+h)$$

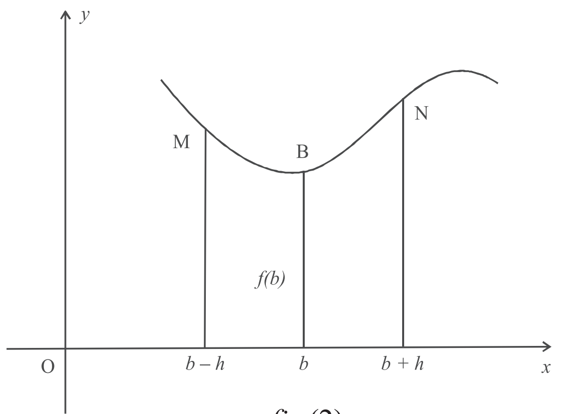


fig (2)

Note:

(i) A function may have many local mixima and many local minimal values. In *fig*(3) *f(x)* has attained maxima at A, C, E, G and minima at B, D, F, H.

(ii) Points of local maxima and local minima of a continuous function always occur alternately.

(iii) A maximal value in a neighbourhood can even be small than a minimal value in another neighbourhood ie., in the fig(3) D and G.

Maxima

Definition : A function $y = f(x)$ in said to have a maximum at $x = a$ is there is an open interval $(a-h,\ a+h)$ such that $f(x) \le f(a)$ for all $x \in (a-h, a+h)$ and *f(x)* is maximum value of the given function.

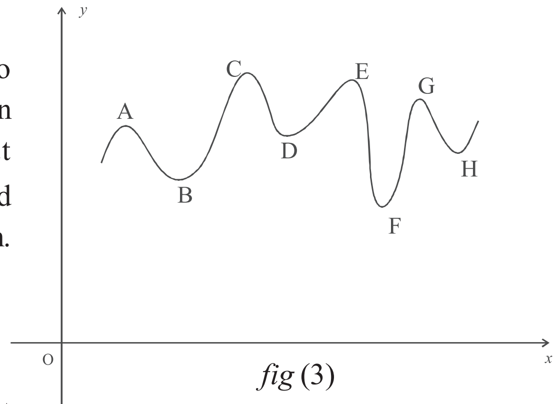


fig (3)

From the fig(4) it is clear at A. The tangent is parallel to x - axis

$\therefore$ slope = 0

$$\therefore \ \frac{dy}{dx} = f(x) = 0$$

at $x = a - h$, $\qquad f^1(x) > 0$

at $x = a + h$, $\qquad f^1(x) > 0$

Hence is *f(x)* is maximum at $x = a$ the $f^1(x)$ changes sign from +ve to –ve

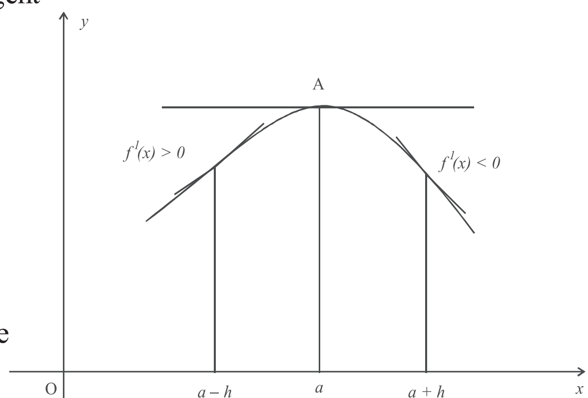


fig (4)

Minima :

A function $y = f(x)$ is said to have a minimum at $x = b$ is there exist an open intervel $(b-h,\ b+h)$ such that $f(x) \ge f(a)$ for all $x \in (b-h, b+h)$ and *f(b)* is the minimum value of the given function.

From fig(5) it is clear at B the tangent is parallel to x -axis

$\therefore$ slope = 0

$$\therefore \ \frac{dy}{dx} = f(x) = 0$$

at $x = b - h$, $\qquad f^1(x) > 0$

at $x = b + h$, $\qquad f^1(x) > 0$

Hence *f(x)* is minimum at $x = b$ then *f(x)* changes sign from –ve to +ve

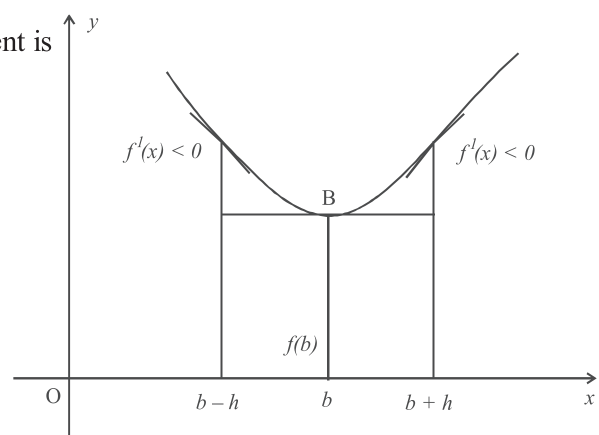


fig (5)

Working method for finding Maxima or Minima

Step (1) : find $\frac{dy}{dx} = f^1(x)$

Step (2) : put $\frac{dy}{dx} = 0$. Solve the equation. Let $x = a,\ x = b$ etc. be the value of x

Step (3) : Find $\frac{d^2y}{dx^2}$ (second derivativetent) = $f^1(x)$

for $x = a$, If $\frac{d^2y}{dx^2}$ = +ve then $f(x)$ is minimum

for $x = a$, If $\frac{d^2y}{dx^2}$ = –ve then $f(x)$ is miximum

for $x = a$, If $\frac{d^2y}{dx^2} = 0$ then $f(x)$ is neither maximum nor minimum then $x = a$ is said to be a **point of Inflexion**

Step (4) : To find the maximum or minimum value the corresponding value of $x = ab$ etc. is substitute in $y = f(x)$ to get y_{max} or y_{min}.

Example 1 :

Find the minimum value of $x^2 + \frac{250}{x}$

Solution :

Let $y = f(x) = x^2 + \frac{250}{x}$ ....(1)

$$\therefore\ \frac{dy}{dx} = 2x + 250\left(\frac{-1}{x^2}\right) \Rightarrow \frac{dy}{dx} = 2x - \frac{250}{x^2}$$

put $\frac{dy}{dx} = 0$, $\quad \therefore\ 0 = 2x - \frac{250}{x^2}$

$$\frac{250}{x^2} = 2x = 2x^3 = 250 \Rightarrow x^3 = 125 = (5)^3 \qquad \therefore\ \boxed{x = 5}$$

Again, $\frac{d^2y}{dx^2} = 2(1) - 250\left(\frac{-2}{x^3}\right) = 2 + \frac{500}{x^3}$

at $x = 5$, $\frac{d^2y}{dx^2} = 2 + \frac{500}{53} = \frac{2+500}{125} = 6(+)$ $\therefore f(x)$ is minimum

for minimum value sub $x = 5$ in (1)

$$\therefore y \text{ min} = (5)^2 + \frac{250}{2} = 25 + 50 = 75$$

Example 2 :

Find the maximum and minimum value of $x^3 - 9x^2 + 15x - 1$

Solution :

Let $x^3 - 9x^2 + 15x - 1$ .... (1)

$$\frac{dy}{dx} = 3x^2 - 9(2x) + 15(1) = 3x^2 - 18x + 15$$

put $\frac{dy}{dx} = 0$ $\therefore 0 = 3x^2 - 18x + 15$

$$0 = 3(x^2 - 6x + 5) \Rightarrow \frac{0}{3} = x^2 - 6x + 5$$

$$x^2 - 6x + 5 = 0$$

$$(x-5)(x-1) = 0 \Rightarrow x = 1, 5$$

Now, $\frac{d^2y}{dx^2} = 3(2x) - 18(1) = 6x - 18$

at $x = 1$, $\frac{d^2y}{dx^2} = 6(1) - 18 = -12$ (–ve) < 0. Hence the function attain maxima

at $x = 5$, $\frac{d^2y}{dx^2} = 6(5) - 18 = 30 - 18 = 12$ (+ve) > 0 $\therefore$ the function attain minima

for maximum value sub $x = 1$ in (1)

$$\therefore y \text{ max} = (1)^3 - 9(1)^2 + 15(1) - 1$$

$$= 1 - 9 + 15 - 1 = \boxed{6}$$

for minimum value substitute $x = 5$ in (1)

$$\therefore y \text{ min} = (5)^3 - 91(5)^2 + 15(5) - 1$$

$$= 125 - 225 + 75 - 1 = \boxed{-26}$$

Example 3 :

Find the maximum value of $\frac{\log x}{x}$ (x in real)

Solution :

Let $y = \frac{\log x}{x}$ ....(1) by quotient rule

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \log x(1)}{x^2} \Rightarrow 0 = \frac{1-\log x}{x^2}$$

$0 = 1 - \log x$

$\log_e x = 1 \qquad \therefore \boxed{x = e}$

Now, $$\frac{d^2y}{dx^2} = \frac{(x^2)\left(-\frac{1}{x}\right) - (1-\log x)(2x)}{(x^2)^2}$$

$$= \frac{-x - 2x + 2x \ \log x}{x^4} = \frac{-3x + 2x \ \log x}{x^4}$$

$$= \frac{x(-3 + 2\log x)}{x^4} \qquad \therefore \boxed{\frac{d^2y}{dx^2} = \frac{-3+2\log x}{x^3}}$$

at $x = e$, $\frac{d^2y}{dx^2} = \frac{-3 + 2\log_e^e}{e^3} \Rightarrow e = \frac{-3+2(1)}{e^3} \qquad (\because \log^e = 1)$

$$= -\frac{1}{e^3} < 0 \quad (\therefore \text{ e is +ve})$$

$\therefore$ the function is maximum at $x = e$

for maximum value put $x = e$ in (1)

$$\therefore y \max = \frac{\log_e^e}{e} = \boxed{\frac{1}{e}}$$

Example 4 : Show that $\frac{1}{x^x}$ is maximum at $x = e$

Solution : $y = \frac{1}{x^x}$ ....(1)

apply log on both side

$$\log y = \log \frac{1}{x^x} \quad \Rightarrow \quad \log y = \frac{1}{x} \log x$$

diff. w.r.t. x by produce rule

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + \log x\left(\frac{-1}{x^2}\right)$$

put $\frac{dy}{dx} = 0, \ \frac{1}{y}(0) = \frac{1}{x^2} - \frac{\log x}{x^2}$

$$0 = \frac{1-\log x}{x^2} \Rightarrow 0 = 1 - \log x$$

$$\therefore \ \log_e^x = 1 \qquad \therefore \ \boxed{x = e}$$

Again, $\frac{dy}{dx} = y\left[\frac{1-\log x}{x^2}\right]$ apply quotient rule

$$\frac{d^2y}{dx^2} = \frac{x^2\left(y\left(\frac{-1}{x}\right) + (1-\log x)\frac{dy}{dx}\right) - y(1-\log x)2x}{\left(x^2\right)^2}$$

at $x = e,$ $\frac{d^2y}{dx^2} = \frac{-e.e^{1/e}}{e^4} = \frac{-e^{1/e}}{e^3} < 0$

$\therefore$ the function has a maxima at $x = e$

Example 5 :

Show that x^x is minimum at $x = \frac{1}{e}$

Solution :

Let $y = x^x$ ....(1)

apply 'log' on both sides

$\therefore \log \ y = \log x^x \quad \Rightarrow \quad \log \ y = x \log x \qquad$ Diff. w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x(1)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$$

put $\frac{dy}{dx} = 0$ $\qquad \therefore \frac{1}{y}(0) = 1 + \log x \quad \Rightarrow \quad 0 = 1 + \log x$

$$\Rightarrow \quad \log_e^x = -1 \qquad \therefore x = e^{-1} \qquad \therefore \boxed{x = \frac{1}{e}}$$

Now $\boxed{\frac{dy}{dx} = y(1 + \log x)}$

Diffn. w.r.t. x

$$\frac{d^2y}{dx^2} = y\left(\frac{1}{x}\right) + (1 + \log x)\frac{dy}{dx}$$

$$= \frac{y}{x} + (1 + \log x)y(1 + \log x) \quad = y\left[\frac{1}{x} + (1 + \log x)^2\right]$$

$$= x^x\left(\frac{1}{x} + (1 + \log x)^2\right)$$

$$\left(\frac{d^2y}{dx^2}\right) \text{ at } x = \frac{1}{e} = \left(\frac{1}{e}\right)^{1/e}\left[e + \left(1 + \log_e^{1/e}\right)^2\right] > 0 (\because e > 0)$$

$\therefore$ the function has a minima at $x = \frac{1}{e}$

Example 6 :

Find the maxima and minima of the function $f(x) = 3x^3 - 9x^2 - 27x + 30$

Solution :

Given $\quad f(x) = y = 3x^3 - 9x^2 - 27x + 30$ ....(1)

$$\therefore \frac{dy}{dx} = 3(3x^2) - 9(2x) - 27(1)$$

$$\frac{dy}{dx} = 9x^2 - 18x - 27$$

$$0 = 9(x^2 - 2x - 3) \qquad x^2 - 2x - 3 = 0/9 = 0$$

$$(x - 3)(x + 1) = 0 \qquad \therefore x = 3, \ -1$$

Now, $\frac{d^2y}{dx^2} = 9(2x) - 18(1) = 18x - 18$

at $x = 3$, $\frac{d^2y}{dx^2} = 18(3) - 18 = 54 - 18 = 36 > 0$ $\therefore$ $f(x)$ is maximum at $x = -1$

at $x = -1$, $\frac{d^2y}{dx^2} = 18(-1) - 18 = 36 < 0$ $\therefore$ $f(x)$ is maximum at $x = -1$

$\therefore$ for minimum value put $x = 3$ in (1)

$$y_{min} = 3(3)^3 - 9(3)^2 - 27(3) + 30$$
$$= 81 - 81 - 81 + 30 = -51$$

for maximum value put $x = -1$ in (1)

$$y_{max} = 3(-1)^3 - 9(-1)^2 - 27(-1) + 30$$
$$= -3 - 9 + 27 + 30 = 45$$

Example 7 :

Find the maximum and minimum value of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution :

Let $y = x^5 - 5x^4 + 5x^3 - 1$ ....(1)

$$\frac{dy}{dx} = 5x^4 - 5(4x^3) + 5(3x^2) = 5x^4 - 20x^3 + 15x^2$$

put $\frac{dy}{dx} = 0$, $5x^4 - 20x^3 + 15x^2 = 0$

$$5x^2(x^2 - 4x + 3) = 0 \Rightarrow 5x^2 = 0,\ x^2 - 4x + 3 = 0$$

$$x^2 = \frac{0}{5},\ (x-3)(x-1) = 0$$

$$\boxed{x = 0,\ x = 3,\ x = 1}$$

Now $\frac{d^2y}{dx} = 5(4x^3) - 20(3x^2) + 15(2x) = 20x^3 - 60x^2 + 30x$

at $x = 0$, $\frac{d^2y}{dx^2} = 20(0)^3 - 60(0)^2 + 30(0) = 0$

$\therefore$ *f(x)* is neither maxima nor minima at $x = 0$

at $x = 1, \quad \frac{d^2y}{dx^2} = 20(1)^3 - 60(1)^2 + 30(1) = 20 - 60 + 30 = -10 < 0$

$\therefore$ *f(x)* is maximum at $x = 1$

at $x = 3, \quad \frac{d^2y}{dx^2} = 20(3)^3 - 60(3)^2 + 30(3) = 540 - 540 + 90 = 90 > 0$

$\therefore$ *f(x)* is minimum at $x = 3$

$\therefore$ for maximum value put $x = 1$ at (1)

$\therefore$ y max = $(1)^5 - 5(1)^4 + 5(1)^4 - 1 = 1 - 5 + 5 - 1 = 0$

for minimum value put $x = 3$ at (1)

$\therefore$ y min = $(3)^5 - 5(3)^4 + 5(3)^3 - 1 = 243 - 405 + 135 - 1 = 28$

Example 8 :

Show that $x^3 - 6x^2 + 12x - 3$ has neither a maximum nor a minimum at $x = 2$.

Solution : Let $y = f(x) = x^3 - 6x^2 + 12 - 3$

$$\frac{dy}{dx} = 3x^2 - 6(2x) + 12(1)$$

$$0 = 3x^2 - 12x + 12 \quad \Rightarrow \quad 0 = 3(x^2 - 4x + 4)$$

$$x^2 - 4x + 4 = \frac{0}{3} = 0 \quad \Rightarrow \quad (x-2)^2 = 0 \quad \therefore \boxed{x = 2}$$

$$\frac{d^2y}{dx^2} = 3(2x) - 12(1) = 6x - 12$$

at $\frac{d^2y}{dx^2} = 3(2x) - 12(1) = 6x - 12$

$\therefore$ *f(x)* has neither maxima nor minima at $x = 2$

Example 9 :

Divide the number 40 into two parts suh that their product is maximum.

Solution : Let the number be x and y

$\therefore x + y = 40$

$y = 40 - x$ ....(1)

product , $p = xy$

$p = x(40 - x)$

$p = 40x - x^2$

for the product to be maximum diffn.

$$\frac{dp}{dx} = 40(1) - 2x$$

$$0 = 40 - 2x \Rightarrow 2x = 40 \qquad \therefore \boxed{x = 20}$$

sub in (1) $\quad \therefore\ y = 40 - 20 \qquad \boxed{y = 20}$

$\therefore$ the numbers are 20 and 20

$$\frac{d^2p}{dx^2} = -2 < 0$$

$\therefore$ product is maximum at $x = 20,\ y = 20$

Example 10 :

The product of two natural number is 144, find the numbers if their sum is minimum.

Solution :

Let the numbers are x and y

Given $\quad x\,y = 144$

$$\therefore\ y = \frac{144}{x} \quad ....(1)$$

sum of numbers = $S = x + y$

$$S = x + \frac{144}{x} \quad \text{from (1)}$$

for the sum to be minimum differentiate w.r.t. x

$$\frac{ds}{dx} = 1 + 144\left(-\frac{1}{x^2}\right)$$

$$0 = 1 - \frac{144}{x^2}$$

$$\frac{144}{x^2} = 1 \quad \Rightarrow \quad x^2 = 144 \quad \therefore \boxed{x = 12}$$

Second derivative test for minimum

$$\frac{d^2y}{dx^2} = -144\left(-2/x^3\right)$$

at $x = 12,\ \dfrac{d^2s}{dx^2} = \dfrac{1}{6}$ (+ ve)

sub in (1) $\quad y = \dfrac{144}{2} \qquad \therefore y = 12$

$\therefore$ the two numbers are 12 and 12

Example 11 :

Show that $f(x)=4x^5-25x^4+40x^3-3$ has a point of Inflexion at $x=0$

Solution : Let $f(x)=4x^5-25x^4+40x^3-3$

$$\frac{dy}{dx}=4(5x^4)-25(4x^3)+40(3x^2) = 20x^4-100x^3+120x^2$$

$$\therefore \frac{dy}{dx}=20x^2(x^2-5x+6) \quad \Rightarrow \quad 0=20x^2(x^2-5x+6)$$

$$\therefore 20x^2=0,\ x^2-5x+6=0$$

$$x^2=\frac{0}{20},\ (x-2)(x-3)=0$$

$$\boxed{x=0,\ x=2,3}$$

Now, $\dfrac{d^2y}{dx^2}=20(4x^3)-100(3x^2)+120(2x) = 80x^3-300x^2+240x$

at $x=0,\ \dfrac{d^2y}{dx^2}=80(0)-300(0)+240(0)=0$

$\therefore$ *f(x)* in neither maxima nor minima

$\therefore$ *f(x)* has a point of inflexion at $x=0$

Example 12 :

Divide 64 into two pars such that the sum of the cubes of two parts is minimum

Solution :

Let the two parts are x and y

Given $x+y=64$

$\therefore y=64-x$ .....(1)

sum of their cubes, $s=x^3+y^3$

$s=x^3+(64-x)^3$ from (1)

for the sum to be minimum diffn. w.r.t. x

$$\frac{ds}{dx}=3x^2+3(64-x)^2(-1)$$

$$0=3x^2-3(64-x)^2$$

$$\therefore 3x^2=3(64-x)^2$$

$\Rightarrow x^2 = (64-x)^2 \ \Rightarrow \ x = 64 - x$

$\therefore \ 2x = 64, x = 32$

Second derivative test for minima

$$\frac{d^2s}{dx^2} = 3(2x) - 3.2(64-x)(-1) = 6x + 6\ (64 - x)$$

at $x = 32, \ \frac{d^2s}{dx^2} = 6(32) + 6\ (64-32) = 192 + 192 = 384$ (+ve)

$\therefore$ the sum is minimum

To find 2nd part sub x = 12 in (1)

$\therefore y = 64 - 32 \quad \therefore y = 32$

$\therefore$ the two parts are 32 and 32

Example 13 :

Find the maximum and minimum of the function $f(x) = x^3 - 12x^2 + 36x - 4$

Solution :

Let $y = f(x) = x^3 - 12x^2 + 36x - 4$ ....(1)

$$\frac{dy}{dx} = 3x^2 - 12(2x) + 36(1)$$

$$0 = 3x^2 - 24x + 36 \ \Rightarrow \ 0 = 3(x^2 - 8x + 12)$$

$\therefore \ x^2 - 8x + 12 = \frac{0}{3} = 0 \quad \Rightarrow (x - 6(x-2) = 0 \quad \therefore \ x = 2$ and 6

Now $\frac{d^2y}{dx^2} = 3(2x) - 24(1) = 6x - 24$

at $x = 2, \ \frac{d^2y}{dx^2} = 6(2) - 24 = 12 - 24 = -12(-ve)$

$\therefore$ *f(x)* is maximum at $x = 2$

at $x = 6, \ \frac{d^2y}{dx^2} = 6(6) - 24 = 36 - 24 = +12(+ve)$

$\therefore$ *f(x)* is manimum at $x = 6$

$\therefore$ for maximum value put $x = 2$ in (1)

$$y_{\max} = (2)^3 - 12(2)^2 + 36(2) - 4 = 8 - 48 + 72 - 4 = 80 - 52 = 28$$

for minimum value put $x = 6$ in (1)

$$y_{\min} = (6)^3 - 12(6)^2 + 36(6) - 4 = 216 - 432 + 214 - 4 = -6$$

EXERCISE 19.3

2 and 3 mark questions:

Find the maximum and minimum value of the following function.

1. (i) $f(x) = x^3 - 3x$ (ii) $f(x) = x^3 - 6x^2 + 9x + 15$, $(0 \le x \le 6)$

(iii) $f(x) = x^4 - 62x^2 + 120x + 9$ (iv) $f(x) = 2x^3 - 3x^2 - 12x + 12$

(v) $f(x) = 2x^3 - 3x^2 - 36x + 10$ (vi) $f(x) = 9x^2 + 12x + 2$

(vii) $f(x) = 2x^3 - 15x^2 + 36x + 10$ (viii) $f(x) = 2x^3 - 21x^2 + 36x - 20$

(ix) $f(x) = 12x^5 - 45x^4 + 40x^3 + 6$

2) The sum of two natural numbers is 48. find the numbers when their product is maximum.
3) Find two positive numbers whose sum is 14 and the sum of whose square is minimum.
4) Find two positive numbers whose sum is 30 and the sum of their cubes is minimum.
5) The product of two natural numbers is 64. Find the numbers is their sum is minimum.

ANSWERS 19.3

1. (i) Min at $x = 1$ is –2, Max at $x = 1$ is 2

(ii) Min at $x = 1$ is 19, Max at $x = 3$ is 15

(iii) Min at $x = -6$ is –1647, Max at $x = 1$ is 8 and Min at $x = 5$ is –316

(iv) Min at $x = 2$ is –8, Max at $x = -1$ is 19

(v) Min at $x = 3$ is –17, Max at $x = -2$ is 54

(vi) Min at $x = \frac{-2}{3}$ is –2

(vii) Min at $x = 3$ is 37, Max at $x = 2$ is 38

(viii) Min at $x = 6$ is –128, Max at $x = 1$ is –3

(ix) at $x = 0$, neither Max nor Min

Min at $x = 1$ is 13, Mx at $x = 2$ is –10

2. 24, 24 3. 7, 7 4. 15 and 15 5. 32 and 32

19.5 Total cost, Average cost and Marginal cost

Total cost (cost function) : The outflow usually raw material, rent, pay & salaries from total cost.

It is the sum total of all costs

Total cost = variable cost + fixed cost

$T\ C = ax + b$ [x = output / quantity produced]

Average cost : Average cost is the cost per unit of the output

$$A.C = \frac{\text{Total cost}}{\text{Quantity}} = \frac{T.C}{x}$$

Margtinal cost : Marginal cost is the additional cost incurred as a result of producing and selling one more unit of the product ie., Instantaneous rate of change of total cost with respect to output

$$\therefore M.C = \frac{d(T.C)}{dx} \quad [x = \text{output}]$$

19.6 Total Revenue, Average Revenue and Marginal Revenue

Total Revenue (Revenue function) : The money which flow into an organisation from either giving service or selling product is called as Revenue

$\therefore$ Total Revenue = price / unit × quantity sold

$T\ R = p \times x$ or $p \times q$

Average Revenue : Average Revenue refers to Revenue / unit Quantity sold

$$\therefore AR = \frac{T.C}{x}$$

Marginal Revenue : Change in total revenue bought about by infinitesimal change in quantity sold

$$\therefore M.R = \frac{d(TR)}{dx}$$

Profit function : It is the difference between total Revenue and Total cost

$\therefore$ profit = TR – TC

Profit Maximization : For many production situation the Marginal Revenue exceeds the marginal cost. As the level of output increases the amount by which M.R. exceeds M.C. become smaller.

$\therefore$ for profit maximization **Marginal Revenue = Marginal cost** at which the level of output can be indentified.

Total profit (TP) : Total profit is the difference between

Total Revenue and Total Cost

TP = TR – TC

REMEMBER

(1) Total Cost (TC) $= \begin{cases} \text{(i) variable cost + Fixed Cost} \\ \text{(ii) A.C} \times \text{Quantity} \end{cases}$

(2) Marginal Cost (MC) $= \dfrac{d}{dx}(\text{T.C})$

(3) Average Cost (AC) $= \dfrac{\text{T.C}}{\text{Quantity}}$

(4) Total Revenue (TR) $= \begin{cases} \text{(i) Price} \times \text{Quantity} \\ \text{(ii) A.R} \times \text{Quantity} \end{cases}$

(Revenue function)

(5) Average Revenue = AR $= \dfrac{\text{TR}}{\text{Quantity}}$

(Demand function)

(6) Marginal Revenue = MR $= \dfrac{d}{dx}(\text{TR})$

(7) Total Profit = TP = TR – TC

(8) Profit Maximization : MR = MC

(9) Fixed cost = F.C = T.C (at $x = 0$)

Example 1:

If the cost function of a firm is given by $c(x) = x^3 - 3x + 7$. Find Average cost and Marginal cost when the output $x = 6$ unit.

Solution : $\text{A.C} = \dfrac{\text{T.C}}{x}$

$$= \frac{x^3 - 3x + 7}{x} = x^2 - 3 + \frac{7}{x}$$

Marginal cost (MC) $= \dfrac{d}{dx}(T.C) \quad = \dfrac{d}{dx}(x^3 - 3x + 7)$

$= 3x^2 - 3$

at $x = 6$, MC $= 3\,(6)^2 - 3$

$= 108 - 3 = 105$

Example 2:

The total cost function is given by

$C = f(q) = q^3 - 3q^2 + 15q + 27$ (q = output). Find the average cost and the marginal cost. What is the fixed cost in the cost function $f(q)$

Solution :

$$\text{Average cost} = \text{A.C} = \frac{T.C}{q}$$

$$= \frac{q^3 - 3q^2 + 15q + 27}{q} = q^2 - 3q + 15 + \frac{27}{q}$$

$$\text{Marginal cost} = \text{M.C}\ \frac{d}{dq}(T.C)$$

$$= 3q^2 - 3(2q) + 15\ (1) = 3q^2 - 6q + 15$$

Fixed cost function is when $q = 0$ in $f(0)$

then $f(q) = 0 - 3\ (0) + 15\ (0) + 27.$

$$\therefore \boxed{f(0) = 27 = \text{fixed cost}}$$

Example 3:

If the total cost function $C = 9q - 3q^2 + \frac{q^3}{3}$, **find the level of output at which Average cost is minimized.**

Solution : $\text{A.C} = \dfrac{T.C}{q}$

$$= \frac{9q - 3q^2 + \frac{q^3}{3}}{q} \quad \Rightarrow \quad \text{AC} = 9 - 3q\frac{q^2}{3}$$

or A.C to be minimise differentiate w.r.t. q

$$\frac{d}{dq}(A.C) = -3(1) + \frac{2q}{3}$$

$$0 = -3 + \frac{2q}{3} = 3 = \frac{2q}{3}$$

$$\therefore \boxed{q = 9/2} \text{ unit.}$$

Example 4:

The Total Revenue function is given by $R = 400x - 2x^2$, and the total cost function is given by $C = 2x^2 + 40x + 4000$. Find

(i) the Marginal Revenue and Marginal cost functions

(ii) the Average Revenue and Average cost

(iii) the output at which Marginal Revenue = marginal cost.

Solution :

(i) $\text{MR} = \frac{d}{dx}(\text{TR})$

$= \frac{d}{dx}(400x - 2x^2)$

$= 400\ (1) - 2\ (2x) = 400 - 4x$

$\text{MC} = \frac{d}{dx}(\text{T.C})$

$= \frac{d}{dx}(2x^2 + 40x + 4000)$

$= 2(2x) + 40(1)$

$= 4x + 40$

(ii) $\text{AR} = \frac{\text{T.R}}{x} = \frac{400x - 2x^2}{x} = 400 - 2x$

$\text{AC} = \frac{\text{T.R}}{x} = \frac{2x^2 + 40x + 4000}{x}$

$= 2x + 40 + \frac{4000}{x}$

(iii) Given MR = MC

$400 - 4x = 4x + 40$

$400 - 40 = 4x + 4x$

$360 = 8x$

$x = \frac{360}{8}$

$\therefore \boxed{x = 45 \text{ unit}}$

Example 5:

If the demand function is given by $p = 50 - 2q$ (p = price, q = output). Find the value of 'p' and 'q' at which the revenue is maximum and the revenue corresponding to these values.

Solution :

Given $p = 50 - 2q$ .... (1)

Total Revenue = TR = price × quantity

$= p \times q$

$= (50 - 2q) \times q$

$= 50q - 2q^2$

$\therefore$ for Maximum Revenue diffn. TR w.r.t. q

$$\frac{dR}{dq} = 50(1) - 2(2q)$$

$$0 = 50 - 4q$$

$$4q = 50 \qquad \therefore q = \frac{50}{4} = \boxed{\frac{25}{2}} \text{ units}$$

$$\frac{d^2R}{dq^2} = -4 < 0 \qquad \therefore \text{ Revenue is Maximum}$$

$$\text{sub } q = \frac{25}{2} \, in(1) \qquad \therefore p = 50 - 2\left(\frac{25}{2}\right)$$

$$p = 50 - 25 \qquad \boxed{p = ₹25}$$

$$\therefore \quad TR = p \times q = \quad 25 \times {}^{25}\!/_{2} = ₹\frac{625}{2} = ₹312.0$$

Example 6:

The total cost of commodity is given by $C = x^2 - 7x + 2$ where 'x' is the number of units and the price / unit is ₹ 5.00. Find the profit function.

Solution :

$p(x)$ = profit function = $R(x) - C(x)$ ....(1)

T.R (x) = price × quantity

$= 5 \, . \, x$

T.C $(x) = x^2 - 7x + 2$ sub in (1)

$$p(x) = 5x - (x^2 - 7x + 2)$$
$$= 5x - x^2 + 7x - 2$$

$$\boxed{p(x) = -x^2 + 12x - 2}$$

Example 7:

The cost 'C' of manufacturing an article is given by $C = 5 + \frac{48}{q} + 3q^2$ **(q = number of article). find the number of article produced at the minimum cost and also find the minimum cost.**

Solution :

Given $C = 5 + \frac{48}{q} + 3q^2$ .... (1)

for minimum cost dittn. w.r.t. q

$$\therefore \frac{dc}{dq} = 0 + 48\left(\frac{-1}{q^2}\right) + 3(2q)$$

$$0 = \frac{-48}{q^2} + 6q \qquad 6q = \frac{48}{q^2} \qquad \therefore 6q^3 = 48$$

$$q^3 = 8 \qquad \therefore \boxed{q = 2}$$

$$\frac{d^2C}{dq^2} = -48\left(\frac{-2}{q^3}\right) + 6(1)$$

$$= \frac{96}{q^3} + 6$$

at $q = 2, \frac{d^2C}{dq^2} = \frac{96}{2^3} + 6 = \frac{96}{8} + 6 = 12 + 6 = 18(+)$

$\therefore$ C is minimum

for minimum cost sub $q = 2$ in (1)

$$C = 5 + \frac{48}{2} + 3(2)^2 = 5 + 24 + 12$$

$$\boxed{C = ₹41}$$

Example 8:

Let the demand function of an article be $p = 75 - 2x$ and the cost function be $C(x) = 350 + 12x + \frac{x^2}{4}$. Find the number of units and the price at which the total profit is maximum [p = price, x = output]

Solution :

Total profit = T.R – T.C or

= R (x) – C(x) .... (1)

R(x) = price × quantity

= $(75 - 2x)x$

= $75x - 2x^2$

$$C(x) = 350 + 12x + \frac{x^2}{4} \qquad \text{sub in (1)}$$

$$\therefore p(x) = \left[75x - 2x^2\right] - \left[350 + 12x + \frac{x^2}{4}\right] \qquad \text{........... (2)}$$

$$= 75x - 2x^2 - 350 - 12x - \frac{x^2}{4}$$

$$p(x) = -2x^2 - \frac{x^2}{4} + 63x - 350$$

for maximum profit dittn. w.r.t. x

$$\frac{d[p(x)]}{dx} = -2(2x) - \frac{2x}{4} + 63$$

$$0 = -4x - \frac{\cancel{2}x}{\cancel{4}^{2}} + 63 \quad \Rightarrow \quad 0 = -4x - \frac{x}{2} + 63$$

$$\frac{x}{2} = -4x + 63$$

$x = -8x + 126 \qquad \therefore 9x = +126$

$$x = \frac{+126}{9} = 14(+)$$

for total profit put $x = 14$ in p = $75 - 2x$

$\therefore$ $p = 75 - 2(14) = 75 - 28$

$\therefore$ $p =$ ₹ 47.

Example 9:

A manufacture gives the total cost $C = \frac{q^3}{3} - 7q^2 + 111q + 50$ and $q = 100p$. (q = output, p = price) find the level of output at which profit is maximised?

Solution :

$$\text{T.C} = \frac{q^3}{3} - 7q^2 + 111q + 50$$

$$\therefore \text{M.C} = \frac{dc}{dq} = \frac{\cancel{3}q^2}{\cancel{3}} - 7(2q) + 111(1)$$

$$= q^2 - 14q + 1111$$

T.R = price × quantity

$$= (100 - q)\,q \qquad \begin{pmatrix} q = 100 - p \\ \therefore p = 100 - q \end{pmatrix}$$

$$= 100\,q - q^2$$

$$\therefore \text{MR} = \frac{d}{dq}(\text{TR}) = 100(1) - 2q \quad = 100 - 2q$$

for profit maximese

MR = MC

$$100 - 2q = q^2 - 14q + 1111$$

$$q^2 - 14q + 2q + 111 - 100$$

$$q^2 - 12q + 11 = 0 \qquad \Rightarrow (q-1)(q-11) = 0$$

$$\therefore \boxed{q = 1 \text{ or } q = 11}$$

To find for which output the profit maximise

$$\therefore \text{ Take } \frac{d}{dq}(\text{MK}) < \frac{d}{dq}(\text{MC})$$

$$\frac{d}{dq}(100 - 2q) < \frac{d}{dq} \qquad (q^2 - 14q + 111)$$

$$-2 < 2q - 14 \qquad = \quad -2 + 14 < 2q$$

$$= \quad 12\,(2q)$$

at $q = 1$ $\quad 12 < 2\,(1)$

$12 < 2$ does not satisfied

$\therefore\ q = 11 \qquad 12 < 2\,(11)$

$\qquad\qquad\quad 12 < 22$ (satified)

$\therefore \boxed{q = 11}$

Example 10:

The demond function of a firm is $p = 500 - 0.2q$ and the total cost $C = 25q + 10000$ (p = price, q = output). Find the output at which the profit of the firm is maximised. What is the price charged.

Solution :

Given T.C = $25q = 10000$

T.R = price × quantity

$= (500 - 0.2q)q$

$= 500\,q - 0.2q^2$

for profit maximise : MR = MC .... (1)

$$\text{MR} = \frac{d}{dq}(\text{TR}) = \frac{d}{dx}\left(500q - 0.2q^2\right)$$

$$= 500\,(1) - 0.2(2q) = 500 - 0.4\text{q}$$

$$\text{MC} = \frac{d}{dx}(\text{TC})$$

$$= 25(1) = 25$$

sub is (1)

MR = MC

$500 - 0.4q = 25$

$0.4q = 500 - 25$

$0.4q = 475$

$$\therefore q = \frac{475}{0.4} = \frac{4750}{4} = 1187.5 \text{ unit}$$

sub is $p = 500 - 0.2\,(1187.5)$

$\boxed{p = ₹\ 262.50}$

Exercise 19.4

2 and 3 mark questions:

1) Find the Average cost and Marginal cost if the total cost function of an article is given by C = $5x^2 + 2x + 3$(x = quantity)

2) The total cost fo a commodity is given by $C = -x^2 + 5x + 7$ (x = number of unit) and the price per unit is ₹ 12. Find the profit function.

3) For the demond function $2x - 5y = 7$ (x = number of unit and y is the price / unit). Find the Total Revenue, Marginal Revenue and Average Revenue.

4) The total cost of C of output Q is given by $\mathrm{C} = 300\mathrm{Q} - 10\mathrm{Q}^2 + \frac{\mathrm{Q}^3}{3}$. find the output level at which the marginal cost and the Average cost attain their respective minimum.

5) The total cost of the production of a firm in given by the following function C = $0.7x + 18$ (x = output) Find
   (i) the Total cost for an output 10 unit
   (ii) the Average cost for an output 9 unit
   (iii) the Marginal cost for an output of 6 unit

6) If $R = 250x + 45x^2 - x^3$, (R = total Revenue, x = no. of unit) what will be the Marginal Revenue if x = 25 unit and the Average Revenue if x = 10 unit.

7) The total cost function of a manufacturer is C = $5x^2 + 500x + 50000$. Find the output (x) when AC = MC.

8) If $R = x\left(15 - \frac{x}{30}\right)$. What is the Marginal Revenue function and what will be the Marginal Revenue if 100 unit were produced?

9) The Total Revenue (R) and the total cost (C) function of a company are given by $\boldsymbol{R(q) = 300q - q^2}$ and $C(q) = 20 + 4q$ (q = output). Find the equilibrium output (Hint : equilbriums MR = MC)

Answers 19.4

1) $\mathrm{AC} = 5x + 2 + \frac{3}{x}$, $\mathrm{M.C.} = 10x + 2$

2) $p(x) = x^2 + 7x - 7$

3) Price = $y = \frac{2x-7}{5}$, TR $= \frac{2x^2-7x}{5}$, MR $= \frac{4x-7}{5}$, AR $= \frac{2x-7}{5}$

4) MC = ₹ 10, AC = ₹ 15

5) TC = ₹ 25, AC = 2.7, MC = 0.7

6) MR = ₹ 625, AR = ₹ 600

7) $x = 100$ units

8) MR(x) $= 15 - \frac{x}{15}$ and MR = ₹ 8.33

9) $q = 13$ unit.

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Chapter

20 INDEFINITE INTEGRALS

20.1 Introduction

Students are already familiar with differentiation and have to find out the differential coefficient of a given function. In this chapter we have to find out the function whenever its derivative is given. The required function is known as **anti derivative** (or primitive) of the given derivative. The anti derivatives are called the **Indefinite integral.**

Integration is the 'Inverse' or 'Reverse' or 'opposite' process of differentiation.

There are two forms of integrals (i) Indefinite and (ii) definite integral. Definite integral is used to find out the area bounded by the graph of a function under certain conditions. Definite integral has wide range of application in the field of engineering, science, economics, finance, commerce etc. Definite integral will be discussed later. In this chapter we shall study indefinite integral.

20.2 Primitive or Anti derivative

Definition : A function F(x) is called a primitive or an anti derivative or an integral of a function f(x) if $\frac{d}{dx}$ F(x) = f(x)

***Eg* :** (i) $\frac{x^2}{2}$ is primitive of x, because $\frac{d}{dx}\left(\frac{x^2}{2}\right) = x$

(ii) $\sin x$ is primitive of $\cos x$, because $\frac{d}{dx}(\sin x) = \cos x$

(iii) $\log x$ is primitive of $\left(\frac{1}{x}\right)$, because $\frac{d}{dx}(\log x) = \frac{1}{x}$

Note: that $\frac{x^2}{2} + c$ is also primitive of x, because $\frac{d}{dx}(\frac{x^2}{2} + c) = x$

($\sin x$ + c) is also primitive of $\cos x$, because $\frac{d}{dx}(\sin x + c) = \cos x$

($\log x$ + c) is also primitive of $\frac{1}{x}$, because $\frac{d}{dx}(\log x + c) = \frac{1}{x}$

From the above examples, we can observe that the primitives or anti derivatives or integrals of functions are not unique. There are infinitely many antiderivatives of each of these functions and can be find out by choosing C arbitrarly from the set of real numbers.

In general, let F*(x)* be primitive of *f(x)* and let C be arbitrary constant. Then

$$\frac{d}{dx}[\mathrm{F}(x)+c]=f(x)$$

$\therefore$ F*(x)* + *c,* where C$\in$ R denotes family of antiderivatives of *f.*

20.3 Indefinite Integral

Definition : Let *f(x)* be a function. Then the family of all its primitives (or antiderivatives) is called the indefinite integral of *f(x)* and is denoted by $\int f(x)dx$

$$\therefore \frac{d}{dx}[\mathrm{F}(x)+c]=f(x) \qquad \int f(x)dx=\mathrm{F}(x)+c$$

where F*(x)* is primitive of *f(x)* and c is an arbitrary constant called the constant of integration.

Symbols and their meanings

Symbols / terms	Meaning
$\int f(x)\ dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)\ dx$	Integrand
x in $\int f(x)\ dx$	variable of integration
Integrate	Find the integral
An integral of *f* or primitive of *f* or antiderivative of *f*	A function F such that $F^1(x) = f(x)$
Integration	The process of finding integral
Constant of integration	Any real number C, considered as constant function.

20.4 Standard integrals

Integrals (Anti derivatives)	Derivatives
(i) $\int x^n dx=\frac{x^{n+1}}{n+1}+C,\ n\neq -1$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$
<u>Note :</u> $\int dx=x+c$	$\frac{d}{dx}(x)=1$
(ii) $\int \frac{1}{x}dx=\log x+c$	$\frac{d}{dx}(\log x)=\frac{1}{x}$

(iii) $\int e^x dx = e^x + c$	$\frac{d}{dx}(e^x) = e^x$
(iv) $\int a^x dx = \frac{a^x}{\log a} + c$	$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$
(v) $\int \sin x \ dx = -\cos x + c$	$\frac{d}{dx}(-\cos x) = \sin x$
(vi) $\int \cos x \ dx = \sin x + c$	$\frac{d}{dx}(\sin x) = \cos x$
(vii) $\int \sec^2 x \ dx = \tan x + c$	$\frac{d}{dx}(\tan x) = \sec^2 x$
(viii) $\int \operatorname{cosec}^2 x \ dx = -\cot x + c$	$\frac{d}{dx}(-\cot x) = \operatorname{cosce}^2 x$
(ix) $\int \sec x \tan dx = \sec x + c$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
(x) $\int \operatorname{cosec} x \cot x \ dx = -\operatorname{cosec} x + c$	$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

20.5 Properties of the Idefinite integral

Property (i)

$$\frac{d}{dx}\int f(x)\ dx = f(x)$$

and $\int f^1(x)\ dx = f(x) + c$, where c is any arbitrary constant

Property (ii)

If $\frac{d}{dx}\int f(x)\ dx = \frac{d}{dx}\int g(x)dx$

Then $\int f(x)dx + c_1 = \int g(x)dx + c_2$

Property (iii)

If $\int [f(x) + g(x)]\ dx = \int f(x)dx + \int g(x)dx$

Property (iv)

$\int k\, f(x)dx = k\int f(x)dx,$ where k is any real number

Property (v)

$$\int\left[k_1 f_1(x)+k_2 f_2(x)+k_3 f_3(x)+---+k_n f_n(x)\right] dx =$$

$$k_1\int f_1(x)dx+k_2\int f_2(x)dx+k_3\int f_3(x)dx+--k_n\int f_n(x)dx$$

WORKED EXAMPLES

Example 1 :

Evaluate the following integrals:

(a) $\int x^5 dx$ (b) $\int 6\sqrt{x}\ dx$ (c) $\int \frac{1}{\sqrt[3]{x^5}}dx$ (d) $\int \frac{1}{5e^{-x}}dx$ (e) $\int 7.5^x dx$

(f) $\int \frac{3}{x}dx$ (g) $\int \frac{8}{\operatorname{cosec} x}dx$ (h) $\int \frac{9}{\sin^2 x}dx$ (i) $\int 4\sec^2 x\ dx$

Solution :

(a) $\int x^5 dx = \frac{x^{5+1}}{5+1}+c$

$= \frac{x^6}{6}+c$

(b) $\int 6\sqrt{x}\ dx = 6\int x^{\frac{1}{2}}dx$

$= 6\times\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c \quad = \quad 6\times\frac{2}{3}x^{\frac{3}{2}}+c$

$= 4x^{\frac{3}{2}}+c$

(c) $\int \frac{1}{\sqrt[3]{x^5}}dx = \int x^{-\frac{5}{3}}dx$

$= \frac{x^{\frac{-5}{3}+1}}{-\frac{5}{3}+1}+c \quad = \quad \frac{x^{-\frac{2}{3}}}{-\frac{2}{3}}+c$

$= \frac{-3}{2}x^{-\frac{2}{3}}+c$

(d) $\int \frac{1}{5e^{-x}} dx = \frac{1}{5}\int e^x dx \qquad = \frac{1}{5}e^x + c$

(e) $\int 7.5^x dx = 7\int 5^x dx \qquad = 7\frac{5^x}{\log 5} + c$

(f) $\int 3\frac{1}{x} dx = 3\int \frac{1}{x} dx \qquad = 3\log x + c$

(g) $\int \frac{8}{\text{cosec}\, x} dx = 8\int \sin x \, dx \qquad = -8 \cos x + c$

(h) $\int \frac{9}{\sin^2 x} dx = 9\int \text{cosec}^2 x dx \qquad = -9 \cot x + c$

(i) $\int 4\sec^2 x dx = 4\int \sec^2 x dx \qquad = 4 \tan x + c$

Example 2 :

Evaluate the following integrals:

(a) $\int \left(2x^2 - \frac{3}{x} + 2e^x\right) dx$

(b) $\int \left(\frac{x^4 + 3x^2 - 5x}{x^2}\right) dx$

(c) $\int \left(x + \frac{1}{x}\right)^3 dx$

(d) $\int \sqrt{x}\left(1 - \frac{1}{x}\right) dx$

(e) $\int \left(e^{5 \log x} + 5^{3 \log_5 x}\right) dx$

(f) $\int \left(\frac{7^x - 6.8^x}{5^x}\right) dx$

(g) $\int \frac{\left(3^x - 2^x\right)^2}{3^x 2^x} dx$

(h) $\int \left(3e^x + 5a^x - e^{\log a}\right) dx$

Solution:

(a) $\int \left(2x^2 - \frac{3}{x} + 2e^x\right) dx$

$$= 2\int x^2 dx - 3\int \frac{1}{x} dx + 2\int e^x dx$$

$$= 2\frac{x^3}{3} - 3 \log x + 2e^x + c$$

(b) $\int\left(\frac{x^4+3x^2-5x}{x^2}\right)dx$

$$= \int\frac{x^4}{x^2}dx+3\int\frac{x^2}{x^2}dx-5\int\frac{x}{x^2}dx$$

$$= \int x^2dx+3\int 1dx-5\int\frac{1}{x}dx$$

$$= \frac{x^3}{3}+3x-5\ \log x+c$$

(c) $\int\left(x+\frac{1}{x}\right)^3 dx \qquad \because (a+b)^3=a^3+b^3+3ab\ (a+b)$

$$= \int\left(x^3+\frac{1}{x^3}+3x\frac{1}{x}\left(x+\frac{1}{x}\right)\right)dx$$

$$= \int\left(x^3+\frac{1}{x^3}+3x+\frac{3}{x}\right)dx$$

$$= \int x^3dx+\int x^{-3}dx+3\int xdx+3\int\frac{1}{x}dx$$

$$= \frac{x^4}{4}-\frac{x^{-2}}{2}+\frac{3x^2}{2}+3\ \log x+c$$

(d) $\int\sqrt{x}\left(1-\frac{1}{x}\right)dx$

$$= \int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx \quad = \quad \int\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)dx$$

$$= \int x^{\frac{1}{2}}dx-\int x^{\frac{-1}{2}}dx \quad = \quad \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$= \frac{2}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}+c$$

(e) $\int\left(e^{5\ \log x}+5^{3\ \log_5 x}\right)dx$

$$= \int\left(e^{\log x^5}+5^{\log_5 x^3}\right)dx \qquad [\because m\ \log\ a=\log\ a^n]$$

$$= \int\left(x^5+x^3\right)dx \qquad \left[\because a^{\log_a^x}=x\right]$$

$$= \int x^5 dx+\int x^3 dx$$

$$= \frac{x^6}{6}+\frac{x^4}{4}+c$$

(f) $\int\left(\frac{7^x-6.8^x}{5^x}\right)dx$

$$= \int\left(\frac{7}{5}\right)^x dx-6\int\left(\frac{8}{5}\right)^x dx$$

$$= \frac{\left(\frac{7}{5}\right)^x}{\log\left(\frac{7}{5}\right)}-6\frac{\left(\frac{8}{5}\right)^x}{\log\left(\frac{8}{5}\right)}+c$$

(g) $\int\frac{(3^x-2^x)^2}{3^x 2^x}\ dx$

$$= \int\frac{(3^{2x}+2^{2x}-2.3^x.2^x)}{3^x 2^x}$$

$$= \int\left(\frac{3^x}{2^x}+\frac{2^x}{3^x}-2\right)dx$$

$$= \int\left(\frac{3}{2}\right)^x dx+\int\left(\frac{2}{3}\right)^x-\int 2dx$$

$$= \frac{\left(\frac{3}{2}\right)^x}{\log\frac{3}{2}}+\frac{\left(\frac{2}{3}\right)^x}{\log\frac{2}{3}}-2x+c$$

(h) $\int\left(3e^x + 5a^x - e^{\log a}\right)dx$ $\qquad [\because e^{\log_e a} = a]$

$= \quad 3\int e^x + 5\int a^x - \int a\ dx$

$= \quad 3e^x + 5\dfrac{a^x}{\log a} - ax + c$

Example 3 :

Integrate the following w.r.t.x.

(a) $\tan^2 x$ **(b)** $\dfrac{5}{3}\dfrac{\sin x}{\cos^2 x}$ **(c)** $\dfrac{1}{1+\cos x}$ **(d)** $\sqrt{1+\cos 2x}$

(e) $\sqrt{1-\sin 2x}$ **(f)** $\dfrac{\sin x}{1+\sin x}$ **(g)** $\dfrac{\cos^2 x}{1+\sin x}$ **(h)** $\sec x\sqrt{\sec^2 x - 1}$

(i) cosce x [cosce x + cot x] **(j)** $5x^3 - 3\operatorname{cosec}^2 x + 6^x$ **(k)** $\dfrac{5+7\cos x}{\sin^2 x}$

Solution:

(a) $\int \tan^2 x\ dx$

$= \int (\sec^2 x - 1)dx \qquad \because 1+\tan^2 x = \sec^2 x$

$= \tan x - x + c$

(b) $\int \dfrac{5\ \sin x}{3\ \cos^2 x}dx$

$= \dfrac{5}{3}\int \dfrac{\sin x}{\cos^2 x}dx \quad = \quad \dfrac{5}{3}\int \sec x \tan x\ dx$

$= \dfrac{5}{3}\sec x + c$

(c) $\int \dfrac{1}{1+\cos x}dx$

$= \int \dfrac{1}{1+\cos x} \times \dfrac{1-\cos x}{1-\cos x}dx$ (Rationalise the Denominator)

$= \int \dfrac{1-\cos x}{1-\cos^2 x}dx \quad = \quad \int \dfrac{1-\cos x}{\sin^2 x}dx$

$$= \int \frac{1}{\sin^2 x} dx - \int \frac{\cos x}{\sin^2 x} dx$$

$$= \int \operatorname{cosec}^2 x \, dx - \int \operatorname{cosec} x \cot x \, dx$$

$$= -\cot x + \operatorname{cosec} x + c$$

(d) $\int \sqrt{1+\cos 2x} \, dx$

$$= \int \sqrt{2\cos^2 x} \, dx \qquad \because \cos 2x = 2\cos^2 x - 1$$

Multiple angle

$$= \sqrt{2} \int \cos x \, dx$$

$$= \sqrt{2} \sin x + c$$

(e) $\int \sqrt{1-\sin 2x} \, dx$

$$= \int \sqrt{\left(\cos^2 x + \sin^2 x - 2\sin x \cos x\right)} \, dx \qquad [\because \cos^2 x + \sin^2 x = 1]$$

$$= \int \sqrt{(\cos x - \sin x)^2} \, dx$$

$$= \int (\cos x - \sin x) dx$$

$$= \int \cos x \, dx - \int \sin x \, dx$$

$$= \sin x + \cos x + c$$

(f) $\int \frac{\sin x}{1+\sin x} dx$

$$= \int \left(\frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} \right) dx$$

$$= \int \frac{\sin x - \sin^2 x}{1-\sin^2 x} dx$$

$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$= \int \frac{\sin x}{\cos^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$= \int \frac{1}{\cos x} \times \frac{\sin x}{\cos x} dx - \int \tan^2 x \, dx$$

$$= \int \sec x \tan x \, dx - \int (\sec^2 x - 1) dx$$

$$= \sec x - \int \sec^2 x dx + \int dx$$

$$= \sec x - \tan x + x + c$$

(g) $\int \frac{\cos^2 x}{1 + \sin x} dx$

$$= \int \frac{\cos^2 x}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x} dx$$

$$= \int \frac{\cos^2 x (1 - \sin x)}{1 - \sin^2 x} dx$$

$$= \int \frac{\cos^2 x (1 - \sin x)}{\cos^2 x} dx \quad = \quad \int (1 - \sin x) dx$$

$$= x + \cos x + c$$

(h) $\int \sec x \sqrt{\sec^2 x - 1} \, dx$

$$= \int \sec x \sqrt{\tan^2 x} \, dx$$

$$= \int \sec x \tan x \, dx \quad = \quad \sec x + c$$

(i) $\int \operatorname{cosec} x \, (\operatorname{cosec} x + \cot x) dx$

$$= \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x) dx$$

$$= \int \operatorname{cosec}^2 x \, dx + \int \operatorname{cosec} x \, \cot x \, dx$$

$$= -\cot x - \operatorname{cosec} x + c$$

(j) $\int (5x^3 - 3\operatorname{cosec}^2 x + 6^x) dx$

$$= 5\int x^3 dx - 3\int \operatorname{cosec}^2 x \, dx + \int 6^x dx$$

$$= 5\frac{x^4}{4} + 3 \cot x + \frac{6^x}{\log_e^6} + c$$

(k) $\int\left(\frac{5+7\cos x}{\sin^2 x}\right)dx$

$$= 5\int\frac{1}{\sin^2 x}dx+7\int\frac{\cos x}{\sin^2 x}dx$$

$$= 5\int \operatorname{cosec}^2 x\ dx+7\int \operatorname{cosec} x\cot x\ dx$$

$$= -5\ \cot x-7\operatorname{cosec} x+c$$

Exercise 20.1

I One Mark Questions:

(a) $x^2-\frac{6}{x}+5e^x$ (b) $7^x-10.9^x$ (c) $x^e+e^x-\log a$ (d) $7^{6\ \log_7 x}$ (e) $\cot^2 x$

(f) $\frac{9}{5}\frac{\cos x}{\sin^2 x}$ (g) $7x^2-4\sec^2 x$

II Two Marks Questions:

(a) $\frac{x^5+5x^2-7x}{\sqrt{x}}$ (b) $\left(x-\frac{1}{x}\right)^3$ (c) $\sqrt{x}\left(1-\frac{1}{\sqrt{x}}\right)$ (d) $x^{\frac{3}{2}}\left(x-\frac{1}{x^2}\right)$ (e) $\frac{6^x-3^x}{5^x}$

(f) $3^{3\log_3 x}-7^{7\log_7 x}$ (g) $\sqrt{1-\cos 2x}$ (h) $\frac{1+\sin x}{\cos^2 x}$ (i) $\sec x(\sec x-\tan x)$

III Three Marks Questions:

(a) $\frac{(a^x+b^x)^2}{a^x b^x}$ (b) $\operatorname{cosec} x\sqrt{\operatorname{cosec}^2 x-1}$ (c) $\sqrt{1+\sin 2x}$ (d) $\frac{1-\cos\ 2x}{1+\cos\ 2x}$

IV Five Marks Questions:

(a) $\frac{1}{1+\sin x}$ (b) $\frac{\sec x}{1+\sec x}$ (c) $\frac{\cos x}{1+\cos x}$ (d) $\frac{\sin^2 x}{1+\cos x}$

Answers 20.1

I (a) $\frac{x^3}{3}-6\log x+5e^x+c$ (b) $\frac{7^x}{\log 7}-10\frac{9^x}{\log 9}+c$ (c) $\frac{x^{e+1}}{e+1}+e^x-x\log\ a+c$

(d) $\frac{x^7}{7}+c$ (e) $-\cot x-x+c$ (f) $-\frac{9}{5}\operatorname{cosec}\ x+c$ (g) $\frac{7x^3}{3}-4\tan x+c$

II (a) $\frac{2}{11}x^{\frac{11}{2}}+2x^{\frac{5}{2}}-\frac{14}{3}x^{\frac{3}{2}}+c$ (b) $\frac{x^4}{4}+\frac{1}{2x^2}-\frac{3x^2}{2}+3\log x+c$

(c) $\frac{2}{3}x^{\frac{3}{2}}-x+c$ (d) $\frac{2}{7}x^{\frac{7}{2}}-2x^{\frac{1}{2}}+c$ (e) $\frac{\left(\frac{6}{5}\right)^x}{\log\left(\frac{6}{5}\right)}-\frac{\left(\frac{3}{5}\right)^x}{\log\left(\frac{3}{5}\right)}+c$

(f) $\frac{x^4}{4}-\frac{x^8}{8}+c$ (g) $-\sqrt{2}\cos x+c$ (h) $\tan x+\sec x+c$

(i) $\tan x-\sec x+c$

III (a) $\frac{\left(\frac{a}{b}\right)^x}{\log\left(\frac{a}{b}\right)}+\frac{\left(\frac{b}{a}\right)^x}{\log\left(\frac{b}{a}\right)}+2x+c$ (b) $-\operatorname{cosec} x+c$ (c) $\sin x-\cos x+c$

(d) $\tan x-x+c$

IV (a) $\tan x-\sec x+c$ (b) $-\cot x+\operatorname{cosec} x\cot x+c$

(c) $-\operatorname{cosec} x+\cot x+x+c$ (d) $x-\sin x+c$

20.6 Integration by Substitution:

In the previous topic we have evaluated the integrals by using the standard integrals directly. While the integrals of some functions cannot be obtained directly, but those functions can be reduced to standard integrals by using proper method of integration.

Important methods of integration are:

1. **Integration by substitution** 2. **Integration by partial fractions**
3. **Integration by parts**

In this topic, we will discuss the method of integration by substitution.

The method of evaluating an integral by reducing it to standard form by proper substitution is called Integration by substitution.

Some special forms of substitution

(i) If $\int f(x)dx=\phi(x)+c$, then $\int f(ax+b)dx=\frac{1}{a}\phi(ax+b)+c$

(ii) $\int \frac{f^1(x)}{f(x)}dx=\log[f(x)]+c \quad \left[\frac{d}{dx}f(x)=f'(x)\right]$

(iii) $\int [f(x)]^n f^1(x)dx=\frac{[f(x)]^{n+1}}{n+1}+c$

STANDARD THIFOREMS

(i) $\int f(x)\,dx = \phi(x) + c,\ \int f(ax+b)dx = \frac{1}{a}\phi(ax+b) + c,\ a \neq 0$

Proof: Put $ax + b = t$

$\Rightarrow adx = dt \quad \Rightarrow dx = \frac{1}{a}dt$

$$\therefore \int f(ax+b)dx = \int f(t)\frac{1}{a}dt$$

$$= \frac{1}{a}\int f(t)dt \quad = \frac{1}{a}\ \phi(t) + c$$

$$= \frac{1}{a}\phi(ax+b) + c,\ (a \neq 0)$$

(ii) $\int \frac{f^1(x)}{f(x)}\,dx = \log[f(x)] + c,\ f(x) \neq 0$

Proof: put $f(x) = t$

$\Rightarrow f^1(x)dx = dt \quad \therefore \int \frac{f^1(x)}{f(x)}\,dx = \int \frac{1}{t}dt$

$$= \log\ t + c$$

$$= \log\ [f(x)] + c$$

(iii) $\int [f(x)]^n f^1(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$

Proof: put $f(x) = t$

$\Rightarrow f^1(x)\ dx = dt$

$$\therefore \int [f(x)]^n f^1(x)\ dx = \int t^n dt = \frac{t^{n+1}}{n+1} + c = \frac{[f(x)]^{n+1}}{n+1} + c$$

Standard results:

(i) $\int (ax+b)^n\,dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c,\ n \neq -1 \qquad a \neq 0$

(ii) $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b| + c \qquad a \neq 0$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$ $\quad a \neq 0$

(iv) $\int a^{bx+c}dx = \frac{1}{b}\frac{a^{bx+c}}{\log a} + c, a > 0$ and $a \neq 1$ $\quad a \neq 0$

(v) $\int \sin(ax+b)dx = \frac{-1}{a}\cos(ax+b) + c$ $\quad a \neq 0$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$ $\quad a \neq 0$

(vii) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$ $\quad a \neq 0$

(viii) $\int \operatorname{cosec}^2(ax+b)dx = \frac{-1}{a}\cot(ax+b) + c$ $\quad a \neq 0$

(ix) $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + c$ $\quad a \neq 0$

(x) $\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = \frac{-1}{a}\operatorname{cosec}(ax+b) + c$ $\quad a \neq 0$

(xi) $\int \tan(ax+b)\ dx = \frac{1}{a}\log\ \sec\ |ax+b| + c$ $\quad a \neq 0$

(xii) $\int \cot(ax+b)dx = \frac{1}{a}\log|\sin(ax+b)| + c$ $\quad a \neq 0$

(xiii) $\int \sec(ax+b)dx = \frac{1}{a}\log|\sec(ax+b) + \tan(ax+b)| + c$ $\quad a \neq 0$

(xiv) $\int \operatorname{cosec}(ax+b)dx = \frac{1}{a}\log|\operatorname{cosec}(ax+b) - \cot(ax+b)| + c$ $\quad a \neq 0$

Some other results:

(i) $\int \tan x\ dx = \log|\sec x|$ (ii) $\int \cot x\ dx = \log|\sin x|$

(iii) $\int \sec x\ dx = \log|(\sec x + \tan x)| = \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|$

(iv) $\int \operatorname{cosec} x\ dx = \log|\operatorname{cosec} x - \cot(x)| = \log\left|\tan\frac{x}{2}\right|$

Type I : $\boxed{\int f(ax+b)dx = \frac{1}{a}\phi(ax+b) + c}$

Worked Examples

Example 4:

Integrate the following functions w.r.t 'x'

(a) $(3x+5)^2$ **(b)** $\sqrt{2x-3}$ **(c)** $\frac{1}{7x+8}$ **(d)** e^{5x+7} **(e)** 7^{3x+4}

(f) $\text{cosec}^2(5x-3)$ **(g)** $\sec(2x+3)\tan(2x+3)$

Solution :

(a) $\int (3x+5)^2\,dx$

$$= \frac{(3x+5)^{2+1}}{3(2+1)}+c \quad = \frac{(3x+5)^3}{9}+c$$

(b) $\int \sqrt{2x-3}\,dx$

$$= \int (2x-3)^{\frac{1}{2}}dx \quad = \frac{1}{2}\frac{(2x-3)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)}+c$$

$$= \frac{1}{2}\times\frac{2}{3}(2x-3)^{\frac{3}{2}}+c$$

$$= \frac{1}{3}(2x-3)^{\frac{3}{2}}+c$$

(c) $\int \frac{1}{7x+8}\,dx = \frac{\log\,(7x+8)}{7}+c$

(d) $\int e^{5x+7}dx = \frac{e^{5x+7}}{5}+c$

(e) $\int 7^{3x+4}dx = \frac{7^{3x+4}}{3\log 7}+c$

(f) $\int \text{cosec}^2(5x-3)dx = \frac{-\cot\,(5x-3)}{5}+c$

(g) $\int \sec(2x+3)\tan(2x+3)dx = \frac{\sec(2x+3)}{2}+c$

Example 5:

Integrate the following functions w.r.t. 'x'

(a) $\sin^2 x$ **(b)** $\tan^2(3-2x)$ **(c)** $\sin^3 x$

(d) $\sin 3x \cos 2x$ **(e)** $\cos 5x \cos 3x$ **(f)** $\sin 5x \sin 2x$

Solution :

(a) $\int \sin^2 x dx$

$$= \int \frac{1-\cos 2x}{2} dx \qquad \left[\begin{array}{l}\text{From multiple angle} \\ \cos 2x = 2\sin^2 x - 1 \\ \therefore \sin^2 x = \dfrac{1-\cos 2x}{2}\end{array}\right]$$

$$= \frac{1}{2}x - \frac{\sin 2x}{4} + c$$

(b) $\int \tan^2(3-2x)dx$

$$= \int \left[\sec^2(3-2x) - 1\right] dx$$

$$= \frac{\tan(3-2x)}{-2} - x + c$$

$$= \frac{-1}{2} \tan(3-2x) - x + c$$

(c) $\int \sin^3 x dx$

$$= \int \frac{(3\sin x - \sin 3x)}{4} dx \qquad [\because \sin 3x = 3\sin x - 4\sin^3 x]$$

$$= \frac{1}{4}\left[\int 3\sin x dx - \int \sin 3x \, dx\right]$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + c$$

$$= \frac{-3}{4}\cos x + \frac{1}{12}\cos 3x + c$$

(d) $\int \sin 3x \cos 2x \, dx$

$$= \frac{1}{2}\int [\sin(3x+2x)+\sin(3x-2x)]dx \quad \left[\begin{array}{l}\text{From Transformation formula} \\ \because \sin A \cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))\end{array}\right]$$

$$= \frac{1}{2}\int (\sin 5x + \sin x)dx$$

$$= \frac{1}{2}\left[\frac{-\cos 5x}{5} - \cos x\right] + c$$

$$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$$

(e) $\int \cos 5x \cos 3x \, dx$

$$= \frac{1}{2}\int \big[(\cos(5x+3x)+\cos(5x-3x)\big]dx \quad [\because \cos A \ \ \cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

$$= \frac{1}{2}\left[\int \cos 8x dx + \int \cos 2x dx\right]$$

$$= \frac{1}{2}\left[\frac{+\sin 8x}{8} + \frac{\sin 2x}{2}\right] + c$$

$$= \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + c$$

(f) $\int \sin 5x \sin 2x \, dx$

$$= \frac{1}{2}\int \big[\cos(5x-2x)-\cos(5x+2x)\big]dx \quad \left[\because \sin A \sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)\right]$$

$$= \frac{1}{2}\int (\cos 3x - \cos 7x)dx$$

$$= \frac{1}{2}\left[\frac{\sin 3x}{3} - \frac{\sin 7x}{7}\right] + c$$

$$= \frac{\sin 3x}{6} - \frac{\sin 7x}{14} + c$$

Example 6 :

Evaluate the following integrals:

(a) $\int \sqrt{1-\sin x}\, dx$ **(b)** $\int \frac{1+\cos x}{1-\cos x} dx$ **(c)** $\int \frac{\sin x}{1+\cos x} dx$ **(d)** $\int \frac{1}{1+\cos x} dx$

Solution :

(a) $\int \sqrt{1-\sin x}\, dx$

$$= \int \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2\sin \frac{x}{2} \cos \frac{x}{2}}\, dx \qquad \left[\because \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1\right]$$

$$= \int \sqrt{\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)^2} dx$$

$$= \int \left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) dx$$

$$= \frac{-\cos \frac{x}{2}}{\frac{1}{2}} - \frac{\sin \frac{x}{2}}{\frac{1}{2}} + c$$

$$= -2\cos \frac{x}{2} - 2\sin \frac{x}{2} + c$$

(b) $\int \frac{1+\cos x}{1-\cos x} dx$ $\qquad \left[\begin{array}{l} \cos 2x = 2\cos^2 x - 1 \\ \therefore \cos x = 2\cos^2 \frac{x}{2} - 1 \end{array}\right]$

$$= \int \frac{2\cos^2 \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx \quad = \quad \int \cot^2 \frac{x}{2} dx$$

$$= \int \left(\operatorname{cosec}^2 \frac{x}{2} - 1\right) dx \qquad \left(\therefore 1 + \cot^2 x = \operatorname{cosec}^2 x\right)$$

$$= \frac{-\cot \frac{x}{2}}{\frac{1}{2}} - x + c \quad = \quad -2\cot \frac{x}{2} - x + c$$

(c) $\int \frac{1}{1+\cos x} dx$

$= \int \frac{1}{2\cos^2 \frac{x}{2}} dx \quad = \frac{1}{2}\int \frac{1}{\cos^2 \frac{x}{2}} dx$

$= \frac{1}{2}\int \sec^2 \frac{x}{2} dx \quad = \frac{1}{2}\frac{\tan\frac{x}{2}}{\frac{1}{2}} dx$

$= \tan\frac{x}{2} + c$

Example 7:

Integrate the following functions w.r.t. x

(a) $\int \frac{1}{\sqrt{x}+\sqrt{1+x}} dx$ **(b)** $\frac{1}{\sqrt{3x+5}-\sqrt{3x+2}}$ **(c)** $\frac{3}{\sqrt{2x}+\sqrt{2x+3}}$

(d) $\frac{x}{\sqrt{x+9}}$ **(e)** $\frac{3x+2}{2x-5}$

Solution: (a) $\int \frac{1}{\sqrt{x}+\sqrt{1+x}} dx$

$= \int \frac{1}{\sqrt{x}+\sqrt{1+x}} \times \frac{\sqrt{x}-\sqrt{1+x}}{\sqrt{x}-\sqrt{1+x}} dx = \int \frac{\sqrt{x}-\sqrt{1+x}}{\left(\sqrt{x}\right)^2-\left(\sqrt{1+x}\right)^2} dx$

$= \int \frac{\sqrt{x}-\sqrt{1+x}}{x-1-x} dx$

$= \int \frac{\sqrt{x}-\sqrt{1+x}}{-1} dx$

$= \int \left(\sqrt{1+x}-\sqrt{x}\right) dx$

$= \frac{2}{3}\left[(1+x)^{\frac{3}{2}} - x^{\frac{3}{2}}\right] + c$

(b) $\int \frac{1}{\sqrt{3x+5}-\sqrt{3x+2}}dx$

$$= \int \frac{1}{\sqrt{3x+5}-\sqrt{3x+2}} \times \frac{\sqrt{3x+5}+\sqrt{3x+2}}{\sqrt{3x+5}+\sqrt{3x+2}}dx$$

$$= \int \frac{\sqrt{3x+5}+\sqrt{3x+2}}{3x+5-3x-2}dx$$

$$= \frac{1}{3}\int\left(\sqrt{3x+5}+\sqrt{3x+2}\right)dx$$

$$= \frac{1}{3}\left[\frac{2}{3}\frac{(3x+5)^{\frac{3}{2}}}{3}+\frac{2}{3}\frac{(3x+2)^{\frac{3}{2}}}{3}\right]+c$$

$$= \frac{2}{27}(3x+5)^{\frac{3}{2}}+\frac{2}{27}(3x+2)^{\frac{3}{2}}+c$$

(c) $\int \frac{3}{\sqrt{2x}+\sqrt{2x+3}}dx = \int \frac{3}{\sqrt{2x}+\sqrt{2x+3}} \times \frac{\sqrt{2x}-\sqrt{2x+3}}{\sqrt{2x}-\sqrt{2x+3}}\ dx$

$$= \int \frac{3\left(\sqrt{2x}-\sqrt{2x+3}\right)}{2x-2x-3}dx$$

$$= \int \frac{3\left(\sqrt{2x}-\sqrt{2x+3}\right)}{-3}dx$$

$$= \int \left(\sqrt{2x+3}-\sqrt{2x}\right)dx$$

$$= \frac{2}{3}\frac{(2x+3)^{\frac{3}{2}}}{2}-\frac{2}{3}\frac{(2x)^{\frac{3}{2}}}{2}+c$$

$$= \frac{1}{3}(2x+3)^{\frac{3}{2}}-\frac{1}{3}(2x)^{\frac{3}{2}}+c$$

(d) $\int \frac{x}{\sqrt{x+9}} dx$

$$= \int \frac{(x+9-9)}{\sqrt{x+9}} dx = \int \frac{(x+9)}{\sqrt{x+9}} dx - \int \frac{9}{\sqrt{x+9}} dx$$

$$= \int \sqrt{x+9}\, dx - \int \frac{9}{\sqrt{x+9}} dx$$

$$= \frac{2}{3}(x+9)^{\frac{3}{2}} - \frac{9(x+9)^{\frac{-1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{2}{3}(x+9)^{\frac{3}{2}} - 18(x+9)^{\frac{1}{2}} + c$$

(e) $\int \frac{3x+2}{2x-5} dx$

Multiply and divide by 3

$$= 3\int \frac{x+\frac{2}{3}}{2x-5} dx$$

Multiply and divide by 2

$$= \frac{3}{2}\int \frac{2x+\frac{4}{3}}{2x-5} dx \quad = \quad \frac{3}{2}\int \frac{2x-5+5+\frac{4}{3}}{2x-5} dx$$

$$= \frac{3}{2}\int \frac{(2x-5)+\frac{19}{3}}{2x-5} dx$$

$$= \frac{3}{2}\left[\int \frac{2x-5}{2x-5} dx + \frac{19}{3}\int \frac{1}{2x-5} dx\right]$$

$$= \frac{3}{2}\left[\int 1dx + \frac{19}{3}\frac{\log(2x-5)}{2}\right] + c$$

$$= \frac{3}{2}\left[x + \frac{19}{6}\log(2x-5)\right] + c$$

EXERCISE 20.2

Integrate the following functions w.r.t. x

I Three and Five mark questions:

(a) $(7x-3)^4$ (b) $(2x+5)^{\frac{3}{2}}$ (c) $\frac{1}{10x+3}$ (d) e^{3-4x} (e) 3^{5x-3}

(f) $\sec^2(x-5)$ (g) $\operatorname{cosec}(3-5x)\ \cot(3-5x)$

II Two marks questions:

(a) $\cos^2 x$ (b) $\cot^2(5x+3)$ (c) $\cos^3 x$ (d) $(3x+4)^3+5^{7-3x}$

III Three marks questions:

(a) $\sin 2x \cos 3x$ (b) $\cos 9x \sin 4x$ (c) $\cos 7x \cos 6x$ (d) $\sin 11x \sin 7x$

(e) $\int \frac{x}{\sqrt{x-5}}$ (f) $\frac{2x}{2x+3}$ (g) $\frac{3x}{5x-1}$ (h) $\frac{2x+5}{3x+4}$

IV Four marks questions:

(a) $\sqrt{1+\sin x}$ (b) $\int \frac{1}{1-\cos x}dx$ (c) $\frac{1}{\sqrt{x}-\sqrt{2+x}}$

(d) $\frac{4}{\sqrt{x+1}+\sqrt{x+2}}$ (e) $\frac{5}{\sqrt{3x+1}-\sqrt{3x+4}}$

ANSWERS 20.2

I (a) $\frac{(7x-3)^5}{35}+c$ (b) $\frac{1}{5}(2x+5)^{\frac{5}{2}}+c$ (c) $\frac{\log\ (10x+3)}{10}+c$

(d) $\frac{-e^{3-4x}}{4}+c$ (e) $\frac{3^{5x-3}}{5\ \log\ 3}+c$ (f) $\tan(x-5)+c$

(g) $\frac{\operatorname{cosec}\ (3-5x)}{5}+c$

II (a) $\frac{1}{2}x+\frac{1}{4}\sin 2x+c$ (b) $\frac{-\cot(5x+3)}{5}-x+c$ (c) $\frac{\sin 3x}{12}+\frac{3\sin x}{4}+c$

(d) $\frac{(3x+4)^4}{12}-\frac{5^{7-3x}}{3\log 5}+c$

III (a) $\frac{-1}{10}\cos 5x+\frac{1}{2}\cos x+c$ (b) $\frac{-1}{26}\cos 13x+\frac{1}{10}\cos 5x+c$

(c) $\frac{-1}{26}\sin 3x-\frac{1}{2}\sin x+c$ (d) $\frac{-\sin 4x}{8}+\frac{\sin 18x}{36}+c$

(e) $\frac{2}{3}(x-5)^{\frac{3}{2}}+10(x-5)^{\frac{1}{2}}+c$ (f) $x-\frac{3}{2}\log(2x+3)+c$

(g) $\frac{3}{5}x+\frac{3}{25}\log(5x-1)+c$ (h) $\frac{2}{3}x+\frac{14}{18}\log(3x+4)+c$

IV (a) $2\sin\frac{x}{2}-2\cos\frac{x}{2}+c$ (b) $-\cot\frac{x}{2}+c$

(c) $\frac{-1}{3}x^{\frac{3}{2}}-\frac{1}{3}(2+x)^{\frac{3}{2}}+c$ (d) $\frac{-8}{3}(x+1)^{\frac{3}{2}}+\frac{8}{3}(x+2)^{\frac{3}{2}}+c$

(e) $\frac{-10}{27}(3x+1)^{\frac{3}{2}}-\frac{10}{27}(3x+4)^{\frac{3}{2}}+c$

TYPE II $\boxed{\int \frac{f^1(x)}{f(x)}dx=\log[f(x)]+c}$

WORKED EXAMPLES

Example 8:

Evaluate the following:

(a) $\int \frac{2x}{1+x^2}\,dx$ (b) $\int \frac{e^x}{e^x+1}\,dx$ (c) $\int \frac{1}{x(3+\log x)}\,dx$

(d) $\int \frac{2x+5}{x^2+5x+3}\,dx$ (e) $\int \frac{7x^6+7^x\log 7}{x^7+7^x}\,dx$

Solution: (a) Let $I=\int \frac{2x}{1+x^2}dx$

put $1+x^2=t$

$= 2x\,dx = dt \quad \therefore\ I=\int \frac{dt}{t}$

$= \log t + c \quad = \quad \log(1+x^2)+c$

(b) Let $I = \int \frac{e^x}{e^x + 1} dx$

put $e^x + 1 = t \quad \Rightarrow e^x dx = dt$

$\therefore \; I = \int \frac{dt}{t}$

$= \log t + c = \log(e^x + 1) + c$

(c) Let $I = \int \frac{1}{x(3 + \log x)} dx$

put $3 + \log x = t \quad \Rightarrow \frac{1}{x} dx = dt$

$\therefore \; I = \int \frac{1}{t} dt$

$= \log t + c = \log(3 + \log x) + c$

(d) Let $I = \int \frac{2x+5}{x^2 + 5x + 3} dx$

put $x^2 + 5x + 3 = t$

$\Rightarrow (2x+5)\, dx = dt$

$\therefore \; I = \int \frac{dt}{t}$

$= \log t + c = \log(x^2 + 5x + 3) + c$

(e) Let $I = \int \frac{7x^6 + 7^x \log 7}{x^7 + 7^x} dx$

put $x^7 + 7^x = t$

$\Rightarrow 7x^6 + 7^x \log 7 = \frac{dt}{dx}$

$\therefore \; I = \int \frac{dt}{t}$

$= \log t + c$

$= \log(x^7 + 7^x) + c$

Example 9:

Prove that:

(a) $\int \tan x \, dx = \log(\sec x) + c$

(b) $\int \cot x \, dx = \log(\sin x) + c$

(c) $\int \sec x \, dx = \log(\sec x + \tan x) + c$

(d) $\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) + c$

Solution :

(a) Let $\mathrm{I} = \int \tan x \, dx \Rightarrow \mathrm{I} = \int \frac{\sin x}{\cos x} dx$

put $\cos x = \mathrm{t}$

$\Rightarrow -\sin x \, dx = dt$

$\sin x \, dx = -dt$

$\therefore \mathrm{I} = -\int \frac{dt}{t}$

$= -\log \mathrm{t} + \mathrm{c} = \log \frac{1}{\mathrm{t}} + \mathrm{c}$

$= \log \frac{1}{\cos x} + c = \log (\sec x) + c$

(b) Let $I = \int \cot x \, dx$

$= \int \frac{\cos x}{\sin x} dx$

put $\sin x = \mathrm{t}$ $\quad \therefore \cos x \, dx = dt$

$\therefore \mathrm{I} = \int \frac{dt}{t}$

$= \log \mathrm{t} + \mathrm{c} = \log (\sin x) + \mathrm{c}$

(c) Let $I = \int \sec x \, dx$

multiplying and dividing by $\sec x + \tan x$, we get

$$\mathrm{I} = \int \frac{\sec x}{(\sec x + \tan x)} (\sec x + \tan x) \, dx$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} dx$$

put sec x + tan x = t

$\Rightarrow (\sec x \tan x + \sec^2 x)dx = dt \qquad \therefore I = \int \frac{dt}{t}$

$= \log t + c$

$= \log(\sec x + \tan x) + c$

(d) Let I = $\int \operatorname{cosec} x \, dx$

Multiplying and dividing by (cosec x - cot x), we get

$$I = \int \frac{\operatorname{cosec} x}{(\operatorname{cosec} x - \cot x)}(\operatorname{cosec} x - \cot x)dx$$

$$= \int \frac{(\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x)}{\operatorname{cosec} x - \cot x} dx$$

put cosec x - cot $x = t$

$(-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x)dx = dt$

$$\therefore I = \int \frac{dt}{t}$$

$= \log t + c \quad = \log(\operatorname{cosec} x - \cot x) + c$

Example 10 :

Evaluate the following:

(a) $\int \frac{(3x+2)}{3x^2+4x-5}dx$ **(b)** $\int \frac{e^{2x}-1}{e^{2x}+1}dx$ **(c)** $\int \frac{1}{x-\sqrt{x}}dx$ **(d)** $\int \frac{3^x \log 3}{3^x+1}dx$

Solution :

(a) Let I = $\int \frac{(3x+2)}{3x^2+4x-5}dx$

put $3x^2 + 4x - 5 = t$

$(6x + 4)\, dx = dt$

$(3x + 2)\, dx = \frac{dt}{2} \qquad \therefore I = \frac{1}{2}\int \frac{dt}{t}$

$= \frac{1}{2} \log t + c \quad = \frac{1}{2}\log(3x^2 + 4x - 5) + c$

(b) Let $I = \int \frac{e^{2x}-1}{e^{2x}+1}\,dx$

$$= \int \frac{e^{x}\left(e^{x}-\frac{1}{e^{x}}\right)}{e^{x}\left(e^{x}+\frac{1}{e^{x}}\right)}dx \quad = \int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx$$

put $e^{x}+e^{-x}=t \quad \Rightarrow \quad (e^{x}-e^{-x})dx = dt$

$\therefore I = \int \frac{dt}{t}$

$= \log t + c \quad = \log((e^{x}+e^{-x})+c$

(c) Let $I = \int \frac{1}{x-\sqrt{x}}dx$

$$= \int \frac{1}{\sqrt{x}(\sqrt{x}-1)}dx$$

put $\sqrt{x}-1=t \quad \Rightarrow \quad \frac{1}{2\sqrt{x}}dx = dt$

$\frac{1}{\sqrt{x}}dx = 2dt$

$I = 2\int \frac{dt}{t}$

$= 2\,\text{log}t + \text{c} \quad = \quad 2\log\left(\sqrt{x}-1\right) + \text{c}$

(d) Let $I = \int \frac{3^{x}\log 3}{3^{x}+1}dx$

Put $3^x + 1 = t$

$(3^x \log 3)\,dx = dt$

$\therefore \quad I = \int \frac{dt}{t}$

$= \log t + c$

$= \log(3^x + 1) + c$

Example 11 :

Evaluate the following:

(a) $\int \frac{\sin x}{1+\cos x}dx$ (b) $\int \frac{\sin 2x}{1+\sin^2 x}dx$

(c) $\int \frac{\tan x}{2+\log(\sec x)}dx$ (d) $\int \frac{1+\tan x}{1-\tan x}dx$

(e) $\frac{\sec^2 x\tan x}{3+\sec^2 x}dx$ (f) $\int \frac{\sec^2 x+\text{cosec}^2 x}{\tan x-\cot x}dx$

Solution :

(a) Let I $= \int \frac{\sin x}{1+\cos x}dx$

put $1 + \cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow \sin x\, dx = -dt$

$\therefore$ I$= -\int \frac{dt}{t}$

$= -\log t + c = -\log(1+\cos x) + c$

(b) Let I $= \int \frac{\sin 2x}{1+\sin^2 x}dx$

$= \int \frac{2\sin x\cos x}{1+\sin^2 x}dx$

Put $1 + \sin^2 x = t \Rightarrow 2\sin x\cos x\, dx = dt$

I $= \int \frac{dt}{t}$

$= \log t + c = \log(1+\sin^2 x) + c$

(c) Let I $= \int \frac{\tan x}{2+\log(\sec x)}dx$

Put $2 + \log(\sec x) = t \Rightarrow \frac{1}{\sec x}\sec x\tan x\, dx = dt$

$\tan x\, dx = dt$

$\therefore$ I $= \int \frac{dt}{t}$

$= \log t + c = \log[2 + \log(\sec x)] + c$

(d) Let $I = \int \frac{1+\tan x}{1-\tan x} dx$

$$= \int \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}} dx \quad = \quad \int \frac{\cos x + \sin x}{\cos x - \sin x} dx$$

Put $\cos x - \sin x = t$

$(-\sin x - \cos x)\, dx = dt$

$(\sin x + \cos x)\, dx = -dt$

$$= -\int \frac{dt}{t} \quad = \quad -\log t + c$$

$= -\log(\cos x - \sin x) + c$

(e) Let $I = \int \frac{\sec^2 x \tan x}{3+\sec^2 x} dx$

Put $3 + \sec^2 x = t$

$\Rightarrow 2\sec x \sec x \tan x\, dx = dt$

$$\sec^2 x \tan x\, dx = \frac{dt}{2}$$

$$I = \frac{1}{2}\int \frac{dt}{t} \quad = \quad \frac{1}{2}\log t + c$$

$$= \frac{1}{2}\log(3 + \sec^2 x) + c$$

(f) Let $I = \int \frac{\sec^2 x + \cos ce^2 x}{\tan x - \cot x} dx$

Put $\tan x - \cot x = t$

$\Rightarrow \sec^2 x + \operatorname{cosec}^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{t}$$

$= \log t + c = \log(\tan x - \cos x) + c$

EXERCISE 20.3

Evaluate the following:

I. Two marks questions:

(a) $\int \frac{3x^2}{1+x^3} dx$ (b) $\int \frac{4x+3}{2x^2+3x+5} dx$ (c) $\int \frac{e^x-1}{e^x-x} dx$

(d) $\int \frac{9x^8+9^x \log 9}{x^9+9^x} dx$ (e) $\int \frac{\cos x}{2+\sin x} dx$ (f) $\int \frac{1}{x(2\log x+5)} dx$

(g) $\int \frac{3\sin x}{3+4\cos x} dx$

II. Five marks questions:

(a) $\int \frac{1}{\sqrt{x}+x} dx$ (b) $\int \frac{\sin 2x}{1+\cos^2 x} dx$ (c) $\int \frac{e^{2x}+1}{e^{2x}-1} dx$

(d) $\int \frac{1+\cot x}{1-\cot x} dx$ (e) $\int \frac{\cot x}{3+\log(\sin x)} dx$ (f) $\int \frac{\operatorname{cosec}^2 x \cot x}{4+5\operatorname{cosec}^2 x} dx$

ANSWERS 20.3

I. (a) $\log(1+x^3)+c$ (b) $\log(2x^2+3x+5)+c$

(c) $\log(e^x-x)+c$ (d) $\log(x^9+9^x+c$

(e) $\log(2+\sin x)+c$ (f) $\frac{1}{2}\log(2\log x+5)+c$

(g) $\frac{-3}{4}\log(3+4\cos x)+c$

II. (a) $2\log\left(1+\sqrt{x}\right)+c$ (b) $-\log(1+\cos^2 x)+c$

(c) $\log(e^x-e^{-x})+c$ (d) $\log(\sin x - \cos x)+c$

(e) $\log(3+\log\sin x)+c$ (f) $\frac{-1}{10}\log(4+5\operatorname{cosec}^2 x)+c$

TYPE III : $\boxed{\int [f(x)]^n f^1(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}+c}$

WORKED EXAMPLES

Example 12 :

Evaluate the following:

(a) $\int (2x+3)\left(x^2+3x+5\right)^{\frac{3}{2}} dx$ (b) $\int \frac{1}{x}(\log x)^3\, dx$

(c) $\int \frac{\left(\sqrt{x}+1\right)^3}{\sqrt{x}} dx$ (d) $\int e^x\left(e^x+2\right)^{\frac{5}{2}} dx$

(e) $\int \frac{5}{x(3+2\log x)^5} dx$ (f) $\int x\sqrt{3x^2+5}\; dx$

Solution :

(a) Let I $= \int (2x+3)(x^2+3x+5)^{\frac{3}{2}} dx$

put $x^2+3x+5=t \Rightarrow (2x+3)dx = dt$

$$\mathrm{I} = \int t^{\frac{3}{2}}\, dt = \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1}+c$$

$$= \frac{2}{5}\, t^{\frac{5}{2}} + c$$

$$= \frac{2}{5}\,(x^2+3x+5)^{\frac{5}{2}} + c$$

(b) Let I $= \int \frac{1}{x}(\log x)^3\, dx$

put $\log x = t$

$$\Rightarrow \frac{1}{x}dx = dt$$

$$\therefore\ I = \int t^3 dt$$

$$= \frac{t^4}{4}+c = \frac{(\log x)^4}{4}+c$$

(c) Let $I=\int \frac{(\sqrt{x}+1)^3}{\sqrt{x}}\,dx$

put $\sqrt{x}+1=t \quad \Rightarrow \quad \frac{1}{2\sqrt{x}}dx=dt \quad \Rightarrow \quad \frac{1}{\sqrt{x}}dx=2dt$

$\therefore \quad I=2\int t^3 dt$

$=\frac{2t^4}{4}+c \quad = \frac{(\sqrt{x}+1)^4}{2}+c$

(d) Let $I=\int e^x(e^x+2)^{\frac{5}{2}}dx$

put $e^x+2=t \ \Rightarrow \ e^x dx=dt$

$\therefore \ I=\int t^{\frac{5}{2}}dt$

$= \frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1}+c \quad = \frac{2}{7}t^{\frac{7}{2}}+c=\frac{2}{7}(e^x+2)^{\frac{7}{2}}+c$

(e) Let $I=\int \frac{5}{x(3+2\log x)^5}\,dx$

put $3+2\log x=t$

$\frac{2}{x}dx=dt \quad \Rightarrow \quad \frac{1}{x}dx=\frac{dt}{2}$

$\therefore \ I=\frac{5}{2}\int \frac{dt}{t^5}$

$=\frac{5}{2}\int t^{-5}dt \quad = \quad \frac{5}{2}\times\frac{t^{-5+1}}{-5+1}+c$

$=\frac{5}{2}\times\frac{t^{-4}}{-4}+c \quad = \quad \frac{-5}{8}(3+2\log x)^{-4}+c$

$=\frac{-5}{8\,(3+2\log x)^4}+c$

(f) Let $I = \int x\sqrt{3x^2+5}\ dx$

put $3x^2+5=t \Rightarrow 6x\ dx = dt$

$x\ dx = \frac{dt}{6}$

$\therefore\ I = \frac{1}{6}\int \sqrt{t}\ dt$

$= \frac{1}{6}\times\frac{2}{3}t^{\frac{3}{2}}+c = \frac{1}{9}t^{\frac{3}{2}}+c = \frac{1}{9}(3x^2+5)^{\frac{3}{2}}+c$

Example 13 :

Integrate the following w.r.t. 'x'

(a) $\sin^3 x\cos x$ (b) $\sec^3 x\tan x$ (c) $\frac{\cos x}{(1+\sin x)^3}$

(d) $\sec^2 x\sqrt{1+\tan x}$ (e) $\sin 2x\sqrt{1+\sin^2 x}$

(f) $\frac{\sec^2 x+\sec x\cdot\tan x}{(\sec x+\tan x)^{\frac{3}{2}}}$ (g) $\frac{1-\sin 2x}{(x+\cos^2 x)^2}$

Solution :

(a) Let $I = \int \sin^3 x\ \cos x\ dx$

put $\sin x = t \Rightarrow \cos x\ dx = dt$

$\therefore\ I = \int t^3 dt$

$= \frac{t^4}{4}+c = \frac{\sin^4 x}{4}+c$

(b) Let $I = \int \sec^3 x\tan x\ dx$

put $\sec x = t$

$\sec x\tan x\ dx = dt$

$\therefore\ I = \int t^2 dt$

$= \frac{t^3}{3}+c = \frac{\sec^3 x}{3}+c$

(c) Let $I = \int \frac{\cos x}{(1+\sin x)^3}\, dx$

put $1 + \sin x = t$

$\cos x\, dx = dt$

$\therefore\; I = \int \frac{dt}{t^3}$

$= \frac{t^{-3+1}}{-3+1} + c \quad = \frac{-1}{2t^2} + c$

$= \frac{-1}{2(1+\sin x)^2} + c$

(d) Let $I = \int \sec^2 x \sqrt{1+\tan x}\, dx$

put $1 + \tan x = t$

$\sec^2 x\, dx = dt$

$\therefore\; I = \int \sqrt{t}\, dt$

$= \frac{2}{3} t^{\frac{3}{2}} + c \quad = \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + c$

(e) Let $I = \int \sin 2x \sqrt{1+\sin^2 x}\, dx$

put $1 + \sin^2 x = t$

$2 \sin x \cos x\, dx = dt$

$\sin 2x\, dx = dt$

$\therefore\; I = \int \sqrt{t}\, dt$

$= \frac{2}{3} t^{\frac{3}{2}} + c \quad = \frac{2}{3}(1+\sin^2 x)^{\frac{3}{2}} + c$

(f) Let $I = \int \frac{\sec^2 x + \sec x + \tan x}{(\sec x + \tan x)^{\frac{3}{2}}}\, dx$

put $\sec x + \tan x = t \quad \Rightarrow (\sec x \tan x + \sec^2 x)\, dx = dt$

$\therefore \quad I = \int \frac{dt}{t^{\frac{3}{2}}}$

$$= \frac{t^{\frac{-3}{2}+1}}{\frac{-3}{2}+1} + c \quad = \quad -2t^{\frac{-1}{2}} + c \quad = \quad \frac{-2}{\sqrt{t}} + c$$

$$= \frac{-2}{\sqrt{\sec x + \tan x}} + c$$

(g) Let $I = \int \frac{1 - \sin 2x}{(x + \cos^2 x)^2} dx$

put $\quad x + \cos^2 x = t$

$1 - 2 \cos x \sin x \, dx = dt$

$(1 - \sin 2x) \, dx = dt$

$\therefore \quad I = \int \frac{dt}{t^2}$

$$= \frac{t^{-2+1}}{-2+1} + c \quad = \quad -\frac{1}{t} + c$$

$$= \frac{-1}{x + \cos^2 x} + c$$

Exercise 20.4

Integrate the following *w.r.t. x:*

I. Two marks questions:

(a) $\int 2x(x^2 + 2)^{\frac{2}{3}} dx$

(b) $\int \frac{2}{x} (\log x)^2 dx$

(c) $\int e^x \sqrt{e^x + 1} \, dx$

(d) $\int \cos^2 x \sin x \, dx$

(e) $\int \operatorname{cosec}^4 x \cot x \, dx$

II. Three marks questions:

(a) $\int 6x + 5(3x^2 + 5x - 4)^{\frac{5}{3}} dx$

(b) $\int \frac{3}{x(2 + 3\log x)^{\frac{2}{3}}} dx$

(c) $\int \frac{1+e^x}{(x+e^x)^5}dx$

(d) $\int \frac{1}{2\sqrt{x}\left(\sqrt{x}-1\right)^{\frac{2}{5}}}dx$

(e) $\int \frac{5^x \log 5}{(5^x+3)^7}dx$

(f) $\int \operatorname{cosec}^2 x\sqrt{1+\cot x}\ dx$

(g) $\int \frac{\sin 2x}{(1-\cos^2 x)^3}\ dx$

(h) $\int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{(\operatorname{cosec} x - \cot x)^{\frac{1}{4}}}\ dx$

ANSWERS 20.4

I. (a) $\frac{3}{5}(x^2+2)^{\frac{5}{3}}+c$ (b) $\frac{2}{3}(\log x)^3+c$ (c) $\frac{2}{3}(e^x+1)^{\frac{3}{2}}+c$

(d) $\frac{-\cos^3 x}{3}+c$ (e) $\frac{-\operatorname{cosec}^4 x}{4}+c$

II. (a) $\frac{3}{8}(3x^2+5x-4)^{\frac{8}{3}}+c$ (b) $3(2+3\log x)^{\frac{1}{3}}+c$

(c) $\frac{-1}{4(x+e^x)^4}+c$ (d) $\frac{5}{3}\left(\sqrt{x}-1\right)^{\frac{3}{5}}+c$

(e) $\frac{-1}{6(5^x+3)^6}+c$ (f) $\frac{-2}{3}(1+\cot x)^{\frac{3}{2}}+c$

(g) $\frac{-1}{2(1-\cos^2 x)^2}+\text{c}$ (h) $\frac{4}{3}(\operatorname{cosec} x-\cot x)^{\frac{3}{4}}+c$

20.7 Integration by Resolving into Partial Fractions

In this topic students will learn to evaluate the integrals of proper rational functions and improper rational functions by resolving into partial fractions.

WORKED EXAMPLES

Example 14 : Evaluate

(a) $\int \frac{3}{(x+1)(x+2)}dx$ **(b)** $\int \frac{x+2}{(2x-1)(x-3)}dx$

(c) $\int \frac{2x+3}{(x+1)^2(x-3)}dx$ **(d)** $\int \frac{2x}{(x^2+4x+4)(x-1)}dx$

(e) $\int \frac{x-1}{(x^2-9)x}dx$

Solution: (a) Let $I = \int \frac{3}{(x+1)(x+2)}dx$

Consider , $\frac{3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ .... (1)

$3 = A(x+2) + B(x+1)$ .... (2)

put $x = -2$ in (2), we get

$3 = A(-2+2) + B(-2+1)$

$3 = -B$

$\therefore B = -3$

put $x = -1$ (2) we get

$3 = A(-1+2) + B(-1+1)$

$3 = A$

$\therefore A = 3$

Now, (1) $\Rightarrow \frac{3}{(x+1)(x+2)} = \frac{3}{x+1} - \frac{3}{x+2}$

$\therefore I = \int \left(\frac{3}{x+1} - \frac{3}{x+2} \right) dx$ $\left(\because \int \frac{1}{ax+b}dx = \frac{\log(ax+b)}{a} + c \right)$

$= 3\int \frac{1}{x+1}dx \quad -3\int \frac{1}{x+1}dx$

$= 3\log(x+1) - 3\log(x+2) + c$

(b) Let $I = \int \frac{x+2}{(2x-1)(x-3)}dx$

Consider , $\frac{x+2}{(2x-1)(x-3)} = \frac{A}{2x-1} + \frac{B}{x-3}$ ....(1)

$x+2 = A(x-3) + B(2x-1)$ ....(2)

Substituting $x = 3$ in (2), we get

$$3 + 2 = A(3 - 3) + B(6-1)$$

$$5 = 5B$$

$$B = 1$$

Substituting $x = \frac{1}{2}$ in (2), we get

$$\frac{1}{2} + 2 = A\left(\frac{1}{2} - 3\right) + B\left(2 \times \frac{1}{2} - 1\right)$$

$$\frac{5}{2} = A\left(\frac{-5}{2}\right) \qquad \therefore\ A = -1$$

Now, (1) $\Rightarrow \dfrac{x+2}{(2x-1)(x-3)} = \dfrac{-1}{2x-1} + \dfrac{1}{x-3}$

$$\therefore\ I = \int\left(\frac{-1}{2x-1} + \frac{1}{x-3}\right)dx$$

$$= -\int \frac{1}{2x-1}\,dx + \int \frac{1}{x-3}\,dx$$

$$= -\frac{\log(2x-1)}{2} + \log(x-3) + c$$

(c) Let $I = \int \dfrac{2x+3}{(x+1)^2(x-3)}\,dx$

Consider

$$\frac{2x+3}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3} \qquad \text{....... (1)}$$

$$2x + 3 = A(x+1)(x-3) + B(x-3) + (x+1)^2 \qquad \text{....... (2)}$$

put $x = -1$ in (2), we get

$$2(-1) + 3 = A(-1 + 1)(-1 - 3) + B(-1 - 3) + C(-1 + 1)^2$$

$$1 = -4B$$

$$B = \frac{-1}{4}$$

put $x = 3$ in (2), we get

$2(3) + 3 = A(3 + 1)(3 - 3) + B(3 + 3) + C(3 + 1)^2$

$6 + 3 = C(4)^2$

$9 = 16C$

$C = \frac{9}{16}$

put $x = 0$ in (2), we get

$2(0) + 3 = A(0 + 1)(0 - 3) + B(0 - 3) + C(0 + 1)^2$

$3 = -3A - 3B + C$

$\Rightarrow \quad 3 = -3A - 3\times\frac{-1}{4}+\frac{9}{16}$

$3A = -3+\frac{3}{4}+\frac{9}{16}$

$A = -1 + \frac{1}{4}+\frac{3}{16} \quad = \frac{-16+4+3}{16}$

$\therefore \quad A = \frac{-9}{16}$

Now, (1) $\Rightarrow$ $\underline{\frac{2x+3}{(x+1)^2(x-3)} = \frac{-9}{16(x+1)} - \frac{1}{4(x+1)^2} + \frac{9}{16(x-3)}}$

$$= \frac{-9}{16}\int\frac{1}{x+1}dx - \frac{1}{4}\int\frac{1}{(x+1)^2}dx + \frac{9}{16}\int\frac{1}{x-3}dx$$

$$= \frac{-9}{16}\log(x+1) - \frac{1}{4(x+1)} + \frac{9}{16}\log(x-3) + c$$

(d) Let $I = \int\frac{2x}{(x^2+4x+4)(x-1)}dx$

Consider

$$\frac{2x}{(x^2+4x+4)(x-1)} = \frac{2x}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1} \quad ......(1)$$

$2x = A(x+2)(x-1) + B(x-1) + C(x+2)^2$ ......(2)

put $x = 1$ in (2), we get

$2(1) = A(1+2)(1-1) + B(1-1) + C(1+2)^2$

$2 = C(3)^2$

$\therefore \quad C = \frac{2}{9}$

put $x = -2$ in (2), we get

$2(-2) = A(-2+2)(-2-1) + B(-2-1) + C(-2+2)^2$

$-4 = -3B$

$\therefore \; B = \frac{4}{3}$

put $x = 0$ in (2), we get

$2(0) = A(0+2)(0-1) + B(0-1) + C(0+2)^2$

$0 = -2A - B + 4C$

$0 = -2A - \frac{4}{3} + 4\left(\frac{2}{9}\right)$

$2A = \frac{-4}{3} + \frac{8}{9}$

$2A = \frac{-12+8}{9}$

$2A = \frac{-4}{9}$

$A = \frac{-2}{9}$

(1) $\Rightarrow$

$$\frac{2x}{(x+2)^2(x-1)} = \frac{-2}{9(x+2)} + \frac{4}{3(x+2)^2} + \frac{2}{9(x-1)}$$

$$\therefore \; I = \int \left[\frac{-2}{9(x+2)} + \frac{4}{3(x+2)^2} + \frac{2}{9(x-1)}\right] dx$$

$$= \frac{-2}{9}\int\frac{1}{x+2}dx+\frac{4}{3}\int\frac{1}{(x+2)^2}+\frac{2}{9}\int\frac{1}{x-1}dx$$

$$= \frac{-2}{9}\log(x+2)-\frac{4}{3(x+2)}+\frac{2}{9}\log(x-1)+c$$

(e) Let $I=\int\frac{x-1}{(x^2-9)x}dx$

Consider,

$$\frac{x-1}{(x^2-9)(x)}=\frac{x-1}{(x+3)(x-3)(x)}$$

Now, consider

$$\frac{x-1}{(x+3)(x-3)x}=\frac{A}{x+3}+\frac{B}{x-3}+\frac{C}{x} \qquad \text{........(1)}$$

$x-1 = \text{A}(x-3)\,x + \text{B}\,(x+3)\,x + \text{C}\,(x+3)\,(x-3)$ .... (2)

put $x = 0$ in (2), we get

$-1 = \text{C}\,(3)\,(-3)$

$-1 = -9\text{C}$

$\therefore\ \text{C} = \frac{1}{9}$

put $x = 3$ in (2), we get

$3-1 = \text{A}\,(3-3)\,3 + \text{B}\,(3+3)\,3 + \text{C}\,(3+3)\,(3-3)$

$2 = \text{B}(6)\,3$

$2 = 18\text{B}$

$\therefore\ \text{B} = \frac{2}{18} = \frac{1}{9}$

put $x = -3$ in (2), we get

$-3-1 = \text{A}\,(-3-3)\,(-3) + \text{B}\,(-3+3)\,(-3) + \text{C}(-3+3)\,(-3-3)$

$-4 = \text{A}(-6)\,(-3)$

$-4 = +18\text{A}$

$\therefore\ \text{A} = \frac{-4}{18} = \frac{-2}{9}$

Now,

(1) $\Rightarrow$

$$\frac{x-1}{(x+3)(x-3)x}=\frac{-2}{9(x+3)}+\frac{1}{9(x-3)}+\frac{1}{9x}$$

$$\therefore\ I=\int\left(\frac{-2}{9(x+3)}+\frac{1}{9(x-3)}+\frac{1}{9x}\right)dx$$

$$=\ \frac{-2}{9}\int\frac{1}{x+3}dx+\frac{1}{9}\int\frac{1}{x-3}dx+\frac{1}{9}\int\frac{1}{x}dx$$

$$=\ \frac{-2}{9}\log(x+3)+\frac{1}{9}\log(x-3)+\frac{1}{9}\log x+c$$

Example 15 :

(a) $\displaystyle\int\frac{1}{x\left[(\log x)^2-3\log x+2\right]}dx$ **(b)** $\displaystyle\int\frac{e^x}{4e^{2x}-4e^x-3}dx$

(c) $\displaystyle\int\frac{2x^2+5}{2x^2+5x+3}dx$ **(d)** $\displaystyle\int\frac{6x^2+2x+3}{(x+1)(3x+2)}dx$

Solution:

(a) Let $\mathrm{I}=\displaystyle\int\frac{1}{x\left[(\log x)^2-3\log x+1\right]}dx$

put $\log x=t \quad \Rightarrow \frac{1}{x}dx=dt$

$$\therefore \mathrm{I}=\int\frac{dt}{t^2-3t+2}$$

Now consider, $\dfrac{1}{t^2-3t+2}=\dfrac{1}{(t-1)(t-2)}=\dfrac{\mathrm{A}}{t-1}+\dfrac{\mathrm{B}}{t-2}$

$$1=\mathrm{A}(t-2)+\mathrm{B}(t-1)$$

put $t=1$, (2) we get, $1=\mathrm{A}(1-2)$

$\mathrm{A}=-1$

put $t=2$ in (2), we get, $1=\mathrm{B}\ (2-1)$

$\mathrm{B}=1$

$$\therefore (1) \Rightarrow \frac{1}{(t-1)(t-2)} = \frac{-1}{t-1} + \frac{1}{t-2}$$

Now, $\mathrm{I} = \int \left(-\frac{1}{t-1} + \frac{1}{t-2} \right) dt$

$$= -\log(t-1) + \log(t-2) + c$$

$$= -\log(\log x - 1) + \log(\log x - 2) + c$$

$$= \log\left[\frac{\log x - 2}{\log x - 1}\right] + c$$

(b) Let $\mathrm{I} = \int \frac{e^x}{4e^{2x} - 4e^x - 3}\, dx$

Put $e^x = t \quad \Rightarrow e^x dx = dt$

$$\therefore \mathrm{I} = \int \frac{dt}{4t^2 - 4t - 3}$$

Consider

$$\frac{1}{4t^2 - 4t - 3} = \frac{1}{(2t+1)(2t+3)} = \frac{\mathrm{A}}{2t+1} + \frac{\mathrm{B}}{2t-3} \quad .....(1)$$

1 = A(2t – 3) + B(2t +1)

Put $t = \frac{3}{2}$, we get $1 = \mathrm{A}\left(2\cdot\frac{3}{2} - 3\right) + \mathrm{B}\left(2\cdot\frac{3}{2} + 1\right)$

$$1 = 4\mathrm{B} \qquad \therefore \mathrm{B} = \frac{1}{4}$$

put $t = -\frac{1}{2}$, we get

$$1 = A\left[2\cdot\left(\frac{-1}{2}\right) - 3\right] + B\left[2.\left(\frac{-1}{2}\right) + 1\right]$$

$$1 = -4\mathrm{A} \qquad \therefore \mathrm{A} = \frac{-1}{4}$$

$\therefore (1) \Rightarrow \dfrac{1}{(2t+1)(2t-3)} = \dfrac{-1}{4(2t+1)} + \dfrac{1}{4(2t-3)}$

Now, $\mathrm{I} = \int \left[\dfrac{-1}{4(2t+1)} + \dfrac{1}{4(2t-3)} \right] dt = -\dfrac{1}{4}\int \dfrac{dt}{2t+1} + \dfrac{1}{4}\int \dfrac{1}{2t-3}dt$

$$= -\frac{1}{4}\frac{\log(2t+1)}{2} + \frac{1}{4}\cdot\frac{1}{2}\log(2t-3) + c$$

$$= -\frac{1}{8}\log\left[2e^x+1\right] + \frac{1}{8}\log\left(2e^x-3\right) + c$$

(c) Let $\mathrm{I} = \int \dfrac{2x^2+5}{2x^2+5x+3}dx$

Consider $\dfrac{2x^2+5}{2x^2+5x+3}$ This is an improper rational function.

$\therefore$ convert improper into proper by division

$$\begin{array}{r|l} & 1 = \text{quotient} \\ \hline \therefore 2x^2+5x+3 & 2x^2+5 \\ & 2x^2+5x+3 \\ & (-) \quad (-) \quad (-) \\ \hline & -5x+2 = \text{remainder} \end{array}$$

$\therefore \quad \dfrac{x^2+5}{2x^2+5x+3} = 1 + \dfrac{2-5x}{2x^2+5x+3}$ ......(1)

Now consider

$\dfrac{2-5x}{2x^2+5x+3} = \dfrac{2-5x}{(x+1)(2x+3)} = \dfrac{\mathrm{A}}{x+1} + \dfrac{\mathrm{B}}{2x+3}$ ..... (2)

$2 - 5x = \mathrm{A}(2x+3) + \mathrm{B}(x+1)$ ...... (3)

put $x = 1$ in (3), we get

$2 + 5 = \mathrm{A}(-2+3) + \mathrm{B}(-1+1)$

$7 = \mathrm{A}$

$\therefore \mathrm{A} = 7$

put $x = \frac{-3}{2}$ in (3), we get

$$2-5\left(\frac{-3}{2}\right) = A\left(2\times\frac{-3}{2}+3\right)+B\left(\frac{-3}{2}+1\right)$$

$$2+\frac{15}{2} = B\left(-\frac{3}{2}+1\right)$$

$$\frac{19}{\cancel{2}} = B\left(\frac{-1}{\cancel{2}}\right)$$

$\therefore$ B = –19

$$\therefore \text{ (1) } \Rightarrow \frac{x^2+5}{2x^2+5x+3} = 1+\frac{7}{x+1}-\frac{19}{2x+3}$$

Now,

$$\text{I} = \int\left(1+\frac{7}{x+1}-\frac{19}{2x+3}\right)dx$$

$$= \int dx + 7\int\frac{1}{x+1}dx - 19\int\frac{1}{2x+3}+c$$

$$= x + 7\log(x+1)\ \frac{-19}{2}\ \log(2x+3)+c$$

(d) Let $\text{I} = \int \frac{6x^2+2x+3}{(x+1)\ (3x+2)}dx$

Consider $\frac{x^2+2x+3}{(x+1)(3x+2)} = \frac{x^2+2x+3}{3x^2+5x+2}$

This is an improper rational function

$$\begin{array}{r|l} & \quad 2 \\ \hline 3x^2+5x+2 & 6x^2+2x+3 \\ & 6x^2+10x+4 \\ & (-)\quad(-)\quad(-) \\ \hline & \quad -8x-1 \end{array}$$

$$\therefore \quad \frac{6x^2+2x+3}{3x^2+5x+2} = 2 - \frac{8x+1}{3x^2+5x+2}$$

$$\frac{6x^2+2x+3}{3x^2+5x+2} = 2 - \frac{(8x+1)}{(x+1)(3x+2)} \quad \text{........(1)}$$

Now consider,

$$\frac{8x+1}{(x+1)(3x+2)} = \frac{A}{x+1} + \frac{B}{3x+2} \quad \text{.........(2)}$$

$$8x + 1 = A(3x+2) + B(x+1) \quad \text{.........(3)}$$

Substituting $x = -1$ in (3), we get

$$8(-1) + 1 = A[3(-1) + 2] + B(-1 + 1)$$

$$-7 = -A \quad \therefore \; A = 7$$

Substituting $x = \frac{-2}{3}$ in (3), we get

$$8\left(\frac{-2}{3}\right) + 1 = A\left[3\left(\frac{-2}{3}\right) + 2\right] + B\left[\frac{-2}{3} + 1\right]$$

$$\frac{-16+3}{3} = B\left(\frac{-2+3}{3}\right)$$

$$-13 = B$$

$$B = -13$$

$$\therefore \; (I) \Rightarrow \frac{6x^2+2x+3}{3x^2+5x+2} = 2 - \frac{7}{x+1} + \frac{13}{3x+2}$$

$$I = \int\left(2 - \frac{7}{x+1} + \frac{13}{3x+2}\right)dx$$

$$= 2\int dx - 7\int \frac{1}{x+1}dx + 13\int \frac{1}{3x+2}dx$$

$$= 2x - 7\log(x+1) + \frac{13}{3}\log(3x+2) + c$$

EXERCISE 20.5

Evaluate the following:

I. Five marks questions:

(a) $\int \frac{4x+5}{(x-1)(x+2)}dx$ (b) $\int \frac{3x+2}{(2x+3)(3x-1)}dx$

(c) $\int \frac{1}{x(x+1)(x+2)}dx$ (d) $\int \frac{5x+7}{(x-2)^2(x+3)}dx$

(e) $\int \frac{5}{(x^2+6x+9)(x-3)}dx$ (f) $\int \frac{2x-1}{(x^2-4)(x+1)}dx$

(g) $\int \frac{3e^x}{e^{2x}+5e^x+6}dx$ (h) $\int \frac{6}{x(2(\log x)^2+7\log x+5)}dx$

(i) $\int \frac{3x^2+2x+3}{(x+1)(3x+2)}dx$ (j) $\int \frac{x^2+3x-2}{x^2-4x-12}dx$

ANSWERS 20.5

I. (a) $3\log(x-1)+\log(x+2)+c$

(b) $\frac{5}{22}\log(2x+3)+\frac{9}{33}\log(3x-1)+c$

(c) $\frac{1}{2}\log x-\log(x+1)+\frac{1}{2}\log(x+2)+c$

(d) $\frac{8}{25}\log(x-2)-\frac{17}{5}(x-2)-\frac{8}{25}\log(x+3)+c$

(e) $\frac{-5}{36}\log(x+3)+\frac{5}{6(x+3)}+\frac{5}{36}\log(x-3)+c$

(f) $\frac{-5}{4}\log(x+2)+\frac{1}{4}\log(x-2)+\log(x+1)+c$

(g) $-6\log(e^x+2)+9\log(e^x+3)+c$

(h) $2\log(\log x+1)-2\log(2\log x+5)+c$

(i) $x - 4\log(x+1) + 3\log(3x+2) + c$

(j) $x + \frac{1}{2}\log(x+2) + \frac{13}{2}\log(x-6) + c$

20.8 Integration by parts

Theorem: If u and v are two functions of x, then

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{dx}{dx}\int v\,dx\right)dx$$

Proof : If u and w are functions of x, then

$$\frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx} \quad \text{(product rule in diffn.)}$$

Integrating both sides w.r.t. 'x', we get

$$\text{uw} = \int\left(u\frac{dw}{dx}\right)dx + \int\left(w\frac{du}{dx}\right)dx$$

$$\therefore \int u\frac{dw}{dx}dx = uw - \int\left(\frac{du}{dx}w\right)dx \quad ....(1)$$

Let $\frac{dw}{dx} = v$, then $w = \int v\,dx$

$\therefore$ From (1)

$$\boxed{\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx}$$

Note : Choose the first and second function in the order 'LATE' [i.e., Logarthmic, Algebraic, Trigonometric and Exponential]

WORKED EXAMPLES

Example 16 : Evaluate

(a) $\int xe^x\,dx$ **(b)** $\int \log x\,dx$ **(c)** $\int x\log x\,dx$

Solution: We know that

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \quad ....(1)$$

Now consider, $\int xe^x dx$

Here, $u = x$ $\qquad \dfrac{dv}{dx} = e^x$

$\dfrac{du}{dx} = 1 \qquad v = \int e^x dx = e^x$

$\therefore \quad \int xe^x dx = xe^x - \int e^x (1) dx \qquad$ [From (1)]

$= xe^x - e^x + c$

(b) $\int \log x (1) dx$

$= \quad \log x\,(x) - \int x.\dfrac{1}{x} dx = x \log x - \int 1\, dx$

$= \quad x \log x - x + c$

(c) $\int \log x.\, x\, dx$

$= \quad \log x \times \dfrac{x^2}{2} - \int \dfrac{x^2}{2} . \dfrac{1}{x} dx$

$= \quad \dfrac{x^2}{2} \log x - \dfrac{1}{2} \int x dx$

$= \quad \dfrac{x^2}{2} \log x - \dfrac{x^2}{4} + c$

Example 17 : Evaluate

(a) $\int x^2 \log x dx$ $\qquad$ **(b)** $\int x^2 e^x dx$ $\qquad$ **(c)** $\int (1+x) \log x dx$

Solution :

(a) $\int x^2 \log x dx = \log x.\dfrac{x^3}{3} - \int \dfrac{x^3}{3}.\dfrac{1}{x} dx$

$= \dfrac{x^3}{3} \log x - \dfrac{1}{3} \int x^2 dx$

$= \dfrac{x^3}{3} \log x - \dfrac{x^3}{9} + c$

(b) $\int x^2 e^x dx = x^2 e^x - \int e^x.(2x)dx$

$$= x^2 e^x - 2\int xe^x dx$$

$$= x^2 e^x - 2\left[xe^x - \int e^x (1) dx\right]$$

$$= x^2 e^x - 2xe^x + 2e^x + c$$

(c) $\int (1+x)\log x dx = \int \log x(1+x)dx$

$$= \log x\left(x+\frac{x^2}{2}\right) - \int \left(x+\frac{x^2}{2}\right)\frac{1}{x}dx$$

$$= \log x\left(x+\frac{x^2}{2}\right) - \int \left(1+\frac{x}{2}\right)dx$$

$$= \left(x+\frac{x^2}{2}\right)\log x - x - \frac{x^2}{4} + c$$

Example 18 : Evaluate

(a) $\int x\cos x\ dx$ **(b)** $\int x\sec^2 x\ dx$ **(c)** $\int x^2 \sin x\ dx$

Solution:

(a) $\int x\cos x\ dx$

$$= x(\sin x) - \int (\sin x)(1)dx$$

$$= x\sin x - \cos x + c$$

(b) $\int x\sec^2 x\ dx$

$$= x\tan x - \int \tan x(1)dx$$

$$= x\tan x - \log(\sec x) + c$$

(c) $\int x^2 \sin x\ dx$

$$= x^2(-\cos x) - \int -\cos x(2x)dx$$

$$= -x^2\cos x + 2\int x\cos x\, dx$$

$$= -x^2\cos x + 2\int x(+\sin x - \int(+\sin x)(1)dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

Example 19: Evaluate

(a) $\int xe^{2x}dx$ **(b)** $\int x\log 3x dx$ **(c)** $\int x\cos(2x+3)dx$

Solution : (a) $\int xe^{2x}dx$

$$= x\frac{e^{2x}}{2} - \int\frac{e^{2x}}{2}(1)dx \quad = x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + c$$

(b) $\int x\log 3x\, dx$

$$= \int \log 3x.x\, dx$$

$$= \log 3x.\frac{x^2}{2} - \int\frac{x^2}{2}.\frac{1}{3x}.3dx$$

$$= \frac{x^2}{2}\log 3x - \frac{1}{2}\int x\, dx$$

$$= \frac{x^2}{2}\log 3x - \frac{x^2}{4} + c$$

(c) $\int x\cos(2x+3)dx$

$$= x\frac{\sin(2x+3)}{2} - \int\frac{\sin(2x+3)}{2} \quad .....(1)\, dx$$

$$= \frac{x\sin(2x+3)}{2} + \frac{\cos(2x+3)}{4} + c$$

Exercise 20.6

I. Three marks questions:

Evaluate:

(a) $\int 2\log x\, dx$ (b) $\int x^3\log x$ (c) $\int\frac{1}{x^2}\log x\, dx$

(d) $\int xe^{3x+5}dx$ (e) $\int x\sin x\, dx$ (f) $\int x\operatorname{cosec}^2 x\, dx$

(g) $\int x^2 \cos x\, dx$ (h) $\int x\sin(5x+7)dx$

ANSWERS 20.6

(a) $2x\log x - 2x + c$

(b) $\frac{x^4}{4}\log x - \frac{x^4}{16} + c$

(c) $\frac{-1}{x}\log x - \frac{1}{x} + c$

(d) $\frac{xe^{3x+5}}{3} - \frac{e^{3x+5}}{9} + c$

(e) $-x\cos x + \sin x + c$

(f) $-x\cot x + \log(\sin x) + c$

(g) $x^2 \sin x + 2x\cos x - 2\sin x + c$

(h) $\frac{-x\cos(5x+7)}{5} + \frac{\sin(5x+7)}{25} + c$

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Chapter 21

DEFINITE INTEGRAL AND ITS APPLICATIONS TO AREAS

21.1 Fundamental theorem of integral calculus.

Let *f(x)* be a continuous function on the closed interval $[a, b]$ and $\mathrm{F}(x)$ be an anti-derivative of *f(x)* than

$$\int_a^b f(x)dx = \mathrm{F}(b - \mathrm{F}(a)$$

$\mathrm{F}(b) - \mathrm{F}(a)$ is also denoted by $[\mathrm{F}(x)]_a^b$

$$\therefore \int_a^b f(x)dx = [\mathrm{F}(x)]_a^b = \mathrm{F(b)} - \mathrm{F(a)}$$

Here **'a'** is called as lower limit or inferior limit and **'b'** is called as **upper limit** or superior limit.

Note: The value of a definite integral of a function is unique and it does not depend on different forms of indefinite integral.

If $\int f(x)\,d\,x = \mathrm{F}(x) + c$ then,

$$\int_a^b f(x)dx = [\mathrm{F}(x)+c]_a^b = [\mathrm{F}(b)+c] - [\mathrm{F}(a)+c]$$

$$= \mathrm{F}(b) - \mathrm{F}(a)$$

Thus the value of $\int_a^b f(x)dx$ is same even if we take $\int f(x)dx = \mathrm{F}(x) + \mathrm{C}$

WORKED EXAMPLES

Example 1 :

Evaluate

(a) $\int_1^2 x\,dx$ **(b)** $\int_0^1 e^{2x}$ **(c)** $\int_0^{\frac{\pi}{2}} Sin x\,dx$ **(d)** $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} Sin^2 x\,dx$

Solution: (a) $\int_1^2 x\,dx = \left.\frac{x^2}{2}\right]_1^2$

$$= \frac{2^2}{2} - \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

(b) $\int_0^1 e^{2x}\,dx = \left.\frac{e^{2x}}{2}\right]_0^1$

$= \frac{e^2}{2} - \frac{e^0}{2} = \frac{e^2}{2} - \frac{1}{2}$

$= \frac{e^2 - 1}{2}$

(c) $\int_0^{\frac{\pi}{2}} Sinx\,dx = -\cos x\Big]_0^{\frac{\pi}{2}}$

$= -\cos\frac{\pi}{2} - (-\cos 0)$

$= -0 + 1 = 1$

(d) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} Sec^2 x\,dx = \tan x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right) = \tan\frac{\pi}{4} + \tan\frac{\pi}{4}$

$= 1 + 1 = 2$

Example 2 : Evaluate

(a) $\int_0^1 \left(2x^2 + \frac{1}{x}\right) dx$ **(b)** $\int_0^3 \frac{x+3}{x+2}\,dx$

(c) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\,dx$ **(d)** $\int_0^1 \frac{e^x + 1}{e^x}\,dx$

Solution :

(a) $\int_0^1 \left(2x^2 + \frac{1}{x}\right) dx$

$= \left.\frac{2x^3}{3} + \log x\right]_0^1 = \left(\frac{2}{3} + \log 1\right) - \left(\frac{-2}{3} + \log 0\right)$

$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

(b) $\int_0^3 \frac{x+3}{x+2}\ dx$

$= \int_0^3 \frac{x+2+1}{x+2}\ dx \qquad = \int_0^3 \frac{x+2}{x+2}\ dx + \int_0^3 \frac{1}{x+2}\ dx$

$= x]_0^3 + \log(x+2)]_0^3 = \ 3 - 0 + \log 5 - \log 2$

$= 3 + \log\left(\frac{5}{2}\right)$

(c) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)}\ dx$

from half angle

$= \int_0^{\frac{\pi}{2}} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\ dx \ = \ \int_0^{\frac{\pi}{2}} \tan\frac{x}{2}\ dx \qquad \begin{bmatrix} \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\ \cos x = 2\cos^2\frac{x}{2} - 1 \end{bmatrix}$

$= 2\log\left(\sec\frac{x}{2}\right)\Big]_0^{\frac{\pi}{2}} \ = \ 2\log\left(\sec\frac{\pi}{4}\right) - 2\log(\sec 0)$

$= 2\log\sqrt{2} - 2\log(1)$

$= 2\left(\log\sqrt{2}\right) \qquad [\therefore\ \log 1 = 0]$

(d) $\int_0^1 \frac{e^x+1}{e^x}\ dx$

$= \int_0^1 1 + \frac{1}{e^x}\ dx \ = \ x - e^{-x}\Big]_0^1 \ = \ 1 - e^{-1} - (0 - e^{-0})$

$= 1 - \frac{1}{e} + 1 \ = 2 - \frac{1}{e}$

$= \frac{2e-1}{e}$

Example 3 : Evaluate

(a) $\int_0^3 \frac{2x}{(x^2-4)^2}\ dx \qquad$ **(b)** $\int_0^1 (6x+1)\sqrt{3x^2+x+5}\ dx$

(c) $\int_{1}^{2} xe^{x}\, dx$ **(d)** $\int_{0}^{\frac{\pi}{2}} x \cos x \, dx$

Solution :

(a) Let $I = \int_{0}^{3} \frac{2x}{\left(x^2-4\right)^2}\, dx$

Put $x^2 - 4 = t \Rightarrow 2x\, dx = dt$

when $x = 0$, $t = -4$

$x = 3$, $t = 5$

$$I = \int_{-4}^{5} \frac{dt}{t^2} = \left.\frac{-1}{t}\right]_{-4}^{5}$$

$$= \frac{-1}{5} - \frac{1}{4} = \frac{-4-5}{20}$$

$$= \frac{-9}{20}$$

(b) $\int_{0}^{1} (6x+1)\sqrt{3x^2+x+5}\, dx$

Put $3x^2 + x + 5 = t \Rightarrow (6x+1)\, dx = dt$

when $x = 0$ $t = 5$

when $x = 1$ $t = 9$

$$I = \int_{5}^{9} \sqrt{t}\, dt = \left.\frac{2}{3} t^{\frac{3}{2}}\right]_{5}^{9}$$

$$= \frac{2}{3}\left[9^{\frac{3}{2}} - 5^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[27 - 5\sqrt{5}\right]$$

(c) $\int_{1}^{2} xe^{x}\, dx$

$$= \int_{1}^{2} xe^{x}\, dx$$

$$= \left. xe^{x} - e^{x}\right]_{1}^{2} = (2e^2 - e^2) - (e - e)$$

$$= e^2$$

(d) $\int_0^{\frac{\pi}{2}} x \cos x \; dx$ (by parts)

$$= x\sin x - \int \sin x\, dx \;= x \sin x + \cos x$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x \cos\, dx = x\sin x + \cos x \Big]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - \left(\frac{-\pi}{2}\sin\left(\frac{-\pi}{2}\right) + \cos\frac{-\pi}{2}\right)$$

$$= \left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{2} + 0\right) = 0$$

Exercise 21.1

Evaluate:

I. One mark questions:

(a) $\int_0^1 x^2\, dx$ (b) $\int_1^2 \left(x + e^x\right) dx$ (c) $\int_0^{\frac{\pi}{2}} \sin 2x \; dx$

(d) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos ec^2 x \; dx$ (e) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x) dx$

II. 3 and 5 mark questions:

(a) $\int_1^2 \frac{2x+5}{\left(x^2+5x+3\right)^2} dx$ (b) $\int_{-1}^{1} (10x+3)\sqrt{5x^2+3x+7}\,dx$ (c) $\int_0^1 \frac{2x+3}{3x+5}\, dx$

(d) $\int_0^{\frac{\pi}{2}} x\sin x\; dx$ (e) $\int_2^3 \frac{1}{(x+1)(x+2)} dx$

Answers 21.1

I. (a) $\frac{1}{3}$ (b) $\frac{3}{2} + e^2 - e$ (c) 1 (d) –2 (e) 2

II. (a) $\frac{8}{153}$ (b) $\frac{2}{3}\left[15\sqrt{15} - 27\right]$ (c) $\frac{2}{3}\left[1 - \frac{1}{2}\log\left(\frac{8}{5}\right)\right]$ (d) 1 (e) $\log\frac{16}{15}$

21.2 Application of definite integrals to area bonded by curves and lines with the axes.

Introduction:

In this topic, we shall use definite integrals to find the areas enclosed by curves. Here we shall study areas enclosed by parabolas, straight lines, etc.,

Area enclosed by simple curves:

Theorem: Let $f(x)$ be a continuous non-negative function in interval $[a, b]$. Then the area of the region bounded by the curve $y = f(x)$, the x – axis and the ordinates $x = a$ and $x = b$, is given by $\int_a^b f(x)dx$ or $\int_a^b y\, dx$

Proof :

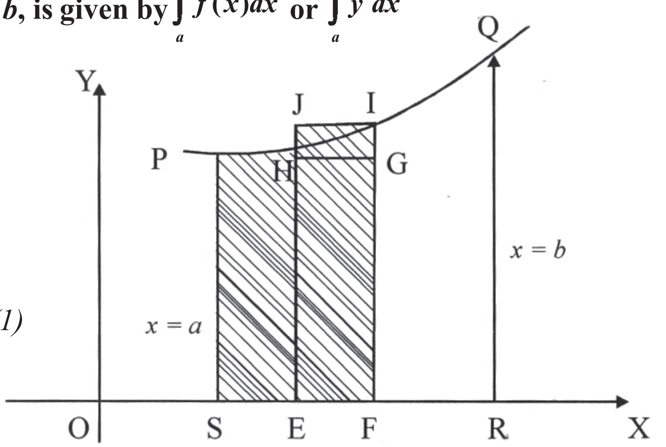

fig (1)

Let H(x, y) be any point on the curve and S ($x + \delta x$, $y + \delta y$) be a neighboring point on the curve HE and GF be perpendicular to x-axis. Let A be the area of SEHPS and A+ δA be the area of SFIPS. Thus the area EFIHE = (A + δA) – A = SA

From fig 1, we have

Area EFGHE < Area EFIHE < Area EFITE

$y.\delta x < \delta A < (y + \delta y)\, \delta x$

Dividing throughout by δx

$$y < \frac{\delta A}{\delta x} < y + \delta y$$

Applying the limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} y < \lim_{\delta x \to 0} \frac{\delta A}{\delta x} < \lim_{\delta x \to 0} (y + \delta y)$$

$$\Rightarrow\ y \le \frac{dA}{dx} \le y \quad (\therefore\ \delta y \to 0 \text{ as } \delta x \to 0)$$

$$\Rightarrow \frac{dA}{dx} = y \quad \Rightarrow dA = y\,dx \quad \Rightarrow A = \int y\,dx$$

$$A = F(x) + C$$

When $x = a$ then HE coincides with SP and the area A becomes zero

$$0 = F(a) + C \text{ ——— } (1)$$

Again, when $x = b$, than HE coincides with QR, then the area becomes, the required area A thus, we have

$$A = F(b) + C \text{ ——— } (2)$$

From (1) and (2) we have

$$F(b) - F(a) = A$$

$$F(b) - F(a) = \int_a^b f(x)dx$$

Thus the area a bounded by the curve, y = *f(x), x* –axis and the ordinates $x = a$ and $x = b$ is given by

$$\boxed{A = \int_a^b y dx = \int_a^b f(x) dx}$$

Note: Note: If the function $f(x)$ is continuous non- positive in the closed interval $[a, b]$ then the curve $y = f(x)$ lies below the x-axis and the definite integral $\int_a^b f(x)dx$ is negative since the area of a region is always non- negative the area of the region bounded by the curve $y = f(x)$ the x-axis the ordinates $x = a, x = b$ is given by $\left|\int_a^b f(x)\,dx\right|$ or $\left|\int_a^b y\,dx\right|$

Similarly,

The area of the region bounded by the curve $x = g(y)$, the y - axis and the abscissae $y = c, y = d$ is given by

$\left|\int_c^d g(y)\,dy\right|$ or $\left|\int_c^d x\,dy\right|$

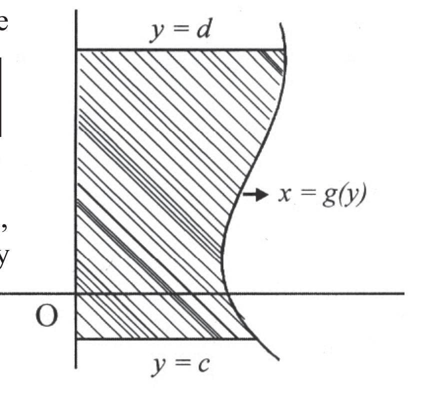

fig (2)

21.3 Areas between curves

If $f(x)$, $g(x)$ are both continuous in $[a,b]$ and $0 \leq g(x) \leq f(x)$ for all $x \in [a,b]$, then the area of the region between the graphs of $y = f(x)$, $y = g(x)$ and the ordinates $x = a, x = b$ is given by $\int_a^b [f(x) - g(x)]\, dx$

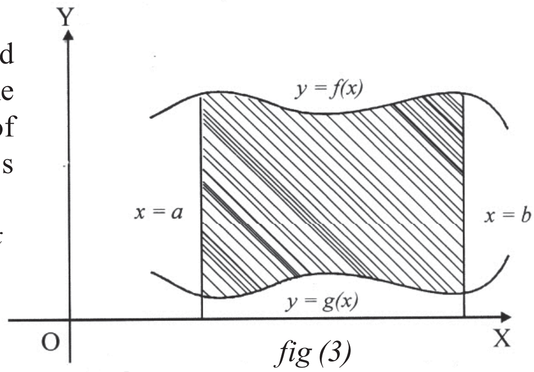

fig (3)

Similarly the area of the region between the graphs of $x = f(y)$, $x = g(y)$ and the abscissae $y = c, y = d$ is given $\int_c^d [f(y) - g(y)]\, dy$

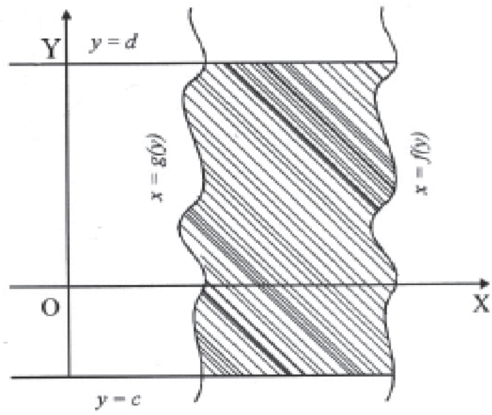

fig (4)

WORKED EXAMPLES

Example 1 :

Find the area bounded by the curve $y = x^2$, x - axis and the lines $x = 1$ and $x = 3$.

Solution:

$$\text{Required area } [A] = \int_1^3 y\, dy$$

$$= \int_1^3 x^2\, dy = \left.\frac{x^3}{3}\right]_1^3$$

$$= \frac{27}{3} - \frac{1}{3}$$

$$= \frac{26}{3} \text{ sq. units}$$

Example 2 :

Find the area bounded by the curve $x = 2y^2$, y - axis and the abscissae $y = 2$ and $y = 4$.

Solution :

$$\text{Required area } = \int_2^4 x\, dy$$

$$= \int_2^4 2y^2\, dy = \left.\frac{2y^3}{3}\right]_2^4$$

$$= \frac{2(64)}{3} - \frac{2(8)}{3}$$

$$= \frac{128}{3} - \frac{16}{3} = \frac{112}{3} \text{ sq. units}$$

Example 3 :

Find the area bounded by the parabola $y^2 = 16x$ and its latus rectum

Solution :

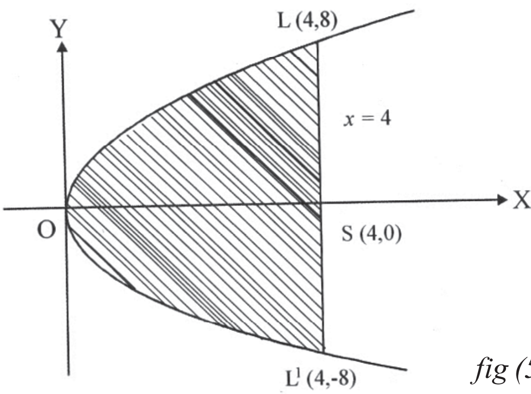

fig (5)

Since the parabola is symmetrical about x - axis

Required area = 2 area OSLO

$$= 2\int_0^4 y\, dx = 2\int_0^4 4\sqrt{x}\, dx = 8\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$= \frac{16}{3}\left[4^{\frac{3}{2}} - 0\right] = \frac{16}{3}\left(2^2\right)^{\frac{3}{2}}$$

$$= \frac{16}{3} \times 8$$

$$= \frac{128}{3} \text{ sq. units}$$

Example 4 :

Find the area bounded by the parabola $y^2 = 4x$ and the line $x - y = 0$

Solution :

The given curve is $y^2 = 4x$ ——— (1)

and the line is $x - y = 0$ ——— (2)

Solving (1) and (2) we get the points of intersection

Put $y = x$ is (1)

$x^2 = 4x$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, \ x = 4$

$\therefore$ The required area is

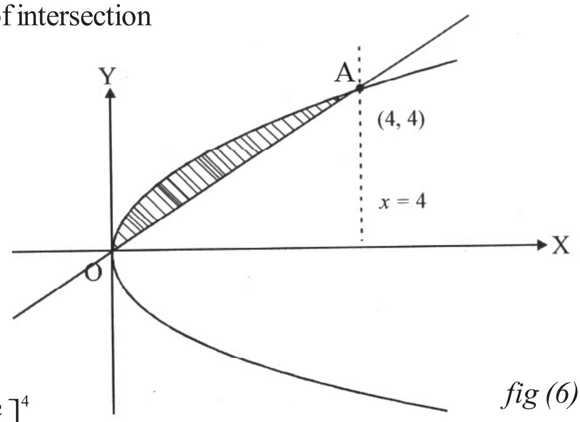

fig (6)

$$A = \int_0^4 \left[f(x) - g(x)\right] dx$$

$$= \int_0^4 \left[2\sqrt{x} - x\right] dx = \left. 2.\frac{2}{3}.x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^4$$

$$= \frac{4}{3}.4^{\frac{3}{2}} - \frac{16}{2} = \frac{4}{3}\left(2^2\right)^{\frac{3}{2}} - \frac{16}{2}$$

$$= \frac{32}{3} - \frac{16}{2} = \frac{32}{3} - 8 = \frac{32-24}{3}$$

$$= \frac{8}{3} \text{ sq. units.}$$

Example 5 :

Find the area of the region between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$

Solution :

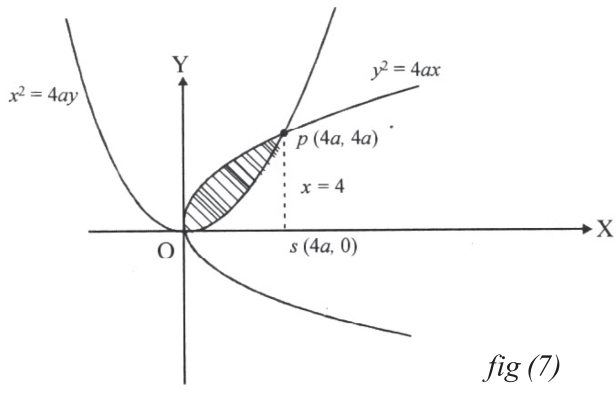

fig (7)

Given curve are $y^2 = 4ax$ ——— (1)

$x^2 = 4ay$ ——— (2)

To find the points of intersection, solve (1) and (2)

Put $y = \dfrac{x^2}{4a}$ in (1)

$\Rightarrow \left(\dfrac{x^2}{4a}\right)^2 = 4ax$

$x^4 = 64a^3 x \Rightarrow x^4 - 64a^3 x = 0$

$x(x^3 - 64a^3) = 0$

$x = 0, \; x = 4a$

$\therefore$ Required area is

$$A = \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= 2\sqrt{a}\,\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{12a}\Bigg]_0^{4a} \quad = \frac{4}{3}\sqrt{a}.(4a)^{\frac{3}{2}} - \frac{(4a)^3}{12a}$$

$$= \frac{4}{3}\sqrt{a}.\,8.\,a^{\frac{3}{2}} - \frac{64a^2}{12} \quad = \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3} \text{ sq. units}$$

Example 6 :

Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$

Solution :

The given parabola is $y^2 = 4x$

$\Rightarrow x = \dfrac{y^2}{4}$ ——— (1)

and the given line is

$y = 2x - 4$

$\Rightarrow x = \dfrac{y+4}{2}$ ——— (2)

solving (1) and (2) we get

$\dfrac{y^2}{4} = \dfrac{y+4}{2}$

$\Rightarrow y^2 = 2y + 8 \Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow (y+2)(y-4) = 0 \Rightarrow y = -2, 4$

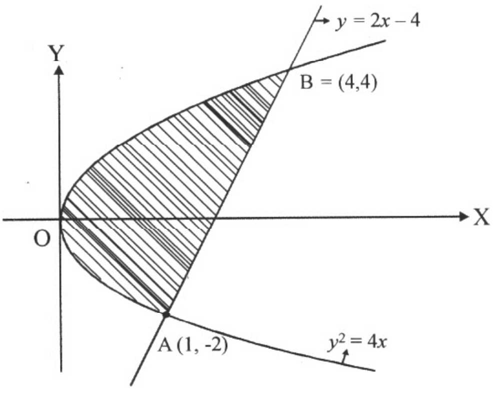

fig (8)

$\therefore$ The points of intersection are $A(1, -2)$ and $B(4, 4)$

$\therefore$ The required area

$$A = \int_{-2}^{4}\left(\frac{y+4}{2} - \frac{y^2}{4}\right)dy$$

$$= \frac{1}{2}\frac{y^2}{2} + 2y - \frac{1}{4}\cdot\frac{y^3}{3}\Bigg]_{-2}^{4}$$

$$= \left(\frac{1}{2}\cdot\frac{4^2}{2} + 2(4) - \frac{1}{4}\cdot\frac{(4)^3}{3}\right) - \left(\frac{1}{2}\cdot\frac{(-2)^2}{2} + 2(-2) - \frac{1}{4}\cdot\frac{(-2)^3}{3}\right)$$

$$= \left(4 + 8 - \frac{16}{3}\right) - \left(1 - 4 + \frac{2}{3}\right) = 15 - \frac{16}{3} - \frac{2}{3}$$

$$= \frac{27}{3} = 9 \text{ sq. units}$$

Example 7 :

Find the area of the region included between the curve $4y = 3x^2$ and the line $3x - 2y + 12 = 0$

Solution :

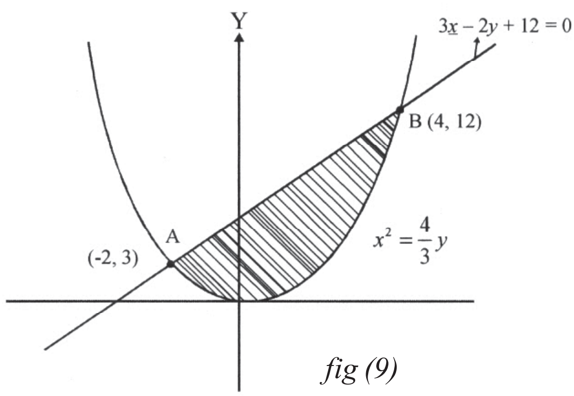

fig (9)

The given curve is $4y = 3x^2$

i.e $y = \frac{3}{4}x^2$ ———— (1)

The given line is

$3x - 2y + 12 = 0$

$\Rightarrow y = \frac{3x+12}{2}$ ———— (2)

solving (1) and (2) we get

$$\frac{3x+12}{2} = \frac{3}{4}x^2$$

$\Rightarrow 3x^2 - 6x - 24 = 0 \Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow (x + 2)(x - 4) = 0 \Rightarrow x = -2, x = 4$

$\therefore$ The points of intersection are $A(-2, 3)$ and $B(4, 12)$

$\therefore$ Required area

$$A = \int_{-2}^{4}\left(\frac{3x+12}{2} - \frac{3x^2}{4}\right)dx$$

$$= \left(\frac{3}{2}\cdot\frac{x^2}{2} + 6(x) - \frac{3}{4}\cdot\frac{x^3}{3}\right)_{-2}^{4}$$

$$= \left.\frac{3x^2}{4} + 6x - \frac{x^3}{4}\right]_{-2}^{4}$$

$$= \left(\frac{3}{2}\cdot\frac{16}{4} + 6(4) - \frac{64}{4}\right) - \left(3.\frac{4}{4} + 6(-2) + \frac{8}{4}\right)$$

$= [12 + 24 - 16] - [3 - 12 + 2]$

$= 20 - (-7) = 20 + 7$

$= 27$ sq. units

EXERCISE 21.2

I. Two marks questions:

1. Find the area bounded by the curve $y = x^2$ x - axis and the ordinates $x = 0$, $x = 1$.
2. Find the area bounded by the curve $y^2 = 8x$, x - axis and the lines $x = 0$, $x = 2$.
3. Find the area bounded by the curve $x^2 = 8y$, y - axis and abscissas $y = 3$, $y = 6$.
4. Find the area bounded by the curve $3x^2 = 4y$, y - axis and the lines $y = 1$, $y = 2$.

II. Five marks questions:

1. Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.
2. Find the area bounded by the parabola $y^2 = 4x$ and $x^2 = 4y$.
3. Find the area bounded by the curve $y^2 = 5x$ and the line $y = x$.
4. Find the area enclosed between the parabola $x^2 = 4y$ and the line $x = 4y - 2$.
5. Find the area enclosed between the parabola $y^2 = x$ and the line $x + y = 2$.
6. Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$.

ANSWERS 21.2

I. (1) $\frac{1}{3}$ sq. units (2) $\frac{16}{3}$ sq. units

(3) $\sqrt{3}\left(16 - 4\sqrt{2}\right)$ sq. units (4) $\frac{4\sqrt{3}}{9}\left(2\sqrt{2} - 1\right)$ sq. units

II. (1) $\frac{8a^2}{3}$ sq. units (2) $\frac{16}{3}$ sq. units (3) $\frac{25}{6}$ sq. units

(4) $\frac{9}{8}$ sq. units (5) $\frac{3}{2}$ sq. units (6) $\frac{8a^2}{3m^3}$ sq. units

21.4 Application definite integrals to cost and revenue function.

Students have already learnt to calculate marginal cost and marginal revenue. In this topic we will learn to calculate total cost and marginal revenue

$\boxed{MC = \frac{dc}{dQ}}$ where 'C' is the total cost, Q is the quantity produced.

$\boxed{\text{Total cost (C)} = \int_a^d MC\, dx}$ [MC = marginal cost]

Where 'a' & 'b' are the lower and upper limits respectively of the total cost function with respect to marginal cost.

In case of an indefinite integral is used in finding the total cost function, then the constant 'c' is considered as fixed cost.

$\boxed{MC = \frac{dR}{dQ}}$ where R is the total revenue

$\boxed{\text{Total revenue (R)} = \int_a^b MR\, dx}$

Where 'a' & 'b' are the lower and upper limits respectively of the total revenue with respect to marginal revenue. In case of indefinite integrals, the constant will always be taken as zero.

APPLICATION TO COMMERCE

Example 1 :

Compute the total cost for the marginal cost function $f^1(x) = 6x^2 - 6x + 12$ as summing that the fixed cost is ₹500.

Solution :

$f^1(x) = 6x^2 - 6x + 12 = \text{M.C}$

Fixed cost = ₹500

$$\text{Total cost } = \int f^1(x)\,.\,dx$$

$$= \int 6x^2 - 6x + 12.\,dx$$

$$= \frac{6x^3}{3} - \frac{6x^2}{2} + 12x + C$$

$\therefore$ T.C $= 2x^3 - 3x^2 + 12x + 500$

Example 2 :

The marginal cost function is given by $f^1(x) = 15x^2 + 2x + 1$ where x is the level of output. Find the total cost, Average cost, total variable cost, given that the fixed cost ₹100.

Sol. Given

$f^1(x) = 15x^2 + 2x + 1 = \text{M.C}$

Total cost $= \int f^1(x)\,.\,dx \;=\int 15x^2 + 2x + 1.\,dx$

$$\text{T. C} \;= \frac{15x^3}{3} + \frac{2x^2}{2} + x + \text{C}$$

C = ₹ 100

T. C $= 5x^3 + x^2 + x + 100$

Total variable cost $= 5x^3 + x^2 + x$.

Average cost $= \dfrac{\text{Total cost}}{x} \;= 5x^2 + x + 1 + \dfrac{100}{x}$

Average variable cost $= 5x^2 + x + 1$.

Example 3 :

The marginal cost function of a first is $3x^2 - 20x + 100$ where 'x' is the level of output. Final the total cost function if the total fixed cost is ₹ 500 what is the average cost.

Solution :

Total cost $= \int 3x^2 - 20x + 100\,dx$

$$= \frac{3x^3}{3} - \frac{20x^2}{2} + 100x + \text{C}$$

T. C $= x^3 - 10x^2 + 100x + 500$

A.C $= x^2 - 10x + 100 + \dfrac{500}{x}$

Example 4 :

If the marginal revenue $= 76 - x^2$ find the maximum total revenue. Also find the total and the average revenues. Write the demand function.

Solution :

$f^1(x) = \text{M.R} = 76 - x^2$

Total Revenue $\int 76 - x^2.\,dx$

$$76x - \frac{x^3}{3} + \text{C}$$

when Output = 0 Total revenue = 0 $\therefore$ C = 0

Total revenue R = $76x - \dfrac{x^3}{3}$

Average revenue = $\dfrac{\text{Total Revenue}}{x}$

$$= \frac{76x - \frac{x^3}{3}}{x} = 76 - \frac{x^2}{3}$$

Average revenue = price per unit which is nothing but the demand function

$\therefore$ the demand function = $76 - \dfrac{x^2}{3}$

Example 5 : The marginal cost = 8 + 0.08*x* and the marginal revenue = 16. Find the total revenue, total cost and total profit. Assume that the fixed cost is nil .

Solution :

Under conditions of perfect competition, MR is constant when MR = MC; profit is maximized

$16 = 8 + 0.08x$

$8 = 0.08x$

$$x = \frac{8}{0.08}$$

$$x = \frac{800}{8} = 100 \text{ units}$$

Total revenue $= \int \text{M.R } dx = \int 16\,dx$

$= 16x$

Total cost $= \int \text{M.C. } dx = \int (8 + 0.08x)dx$

$= 8x + 0.08\dfrac{x^2}{2} + C$

It is given fixed cost 0 $\therefore$ C = 0

T.C = $8x + 0.04x^2$

Total profit $= 16x - (8x + 0.04x^2)$

$= 8x - 0.04x^2$

Total profit $= 16x - (8x + 0.04x^2)$

Putting $x = 100$ in the total profit function, we have

Total proof $= -0.04\,(100)^2 + 8(100)$

$= -0.04\,(10000) + 800$

$= 400 + 800 =$ ₹400

Example 6 :

The marginal cost curve for a super Bazaar is 1.16 – 04x. Find the total cost for C(0) = 14.4.

Solution:

C(0) = 14.4 refers to fixed cost or means the total cost. When output is zero. In the short rum analysis of cost, the only cost that anise even when there is no production is

$$\text{Total cost} = \int (1.16 - 0.04x)\,dx$$

$$= 1.16x - \frac{0.04x^2}{2} + C$$

$$\text{T.C.} = 1.16x - 0.02x^2 + 14.4$$

Example 7 :

A manufacturer produces x units of a product per week is marginal cost function $f^1(x) = 6x^2 + 2x$. Find the total cost when fixed cost = ₹50.

Solution:

$$\text{Marginal cost} = f^1(x) = 6x^2 + 2x$$

$$\text{Total cost} = \int f^1(x)\,dx$$

$$= \int 6x^2 + 2x.dx$$

$$= \frac{6x^3}{3} + \frac{2x^2}{2} + C$$

$$\text{T. C} = 2x^3 + x^2 + C$$

$\because$ C = ₹50

$$\text{T. C} = 2x^3 + x^2 + 50$$

Example 8 :

The marginal cost function is $f^1(x) = 6x^2 + 2x + 1$ where x is the level of output. Find the total cost, average cost, total variable cost, average variable cost. Given that fixed cost = ₹70.

Solution:

$$\text{Marginal cost} = f^1(x) = 6x^2 + 2x + 1$$

$$\text{Total cost} = \int f^1(x)\,dx$$

$$= \int 6x^2 + 2x + 1.\,dx = \frac{6x^3}{3} + \frac{2x^2}{2} + x + C$$

T. C = $2x^3 + x^2 + x + 70$ (As C = ₹70)

Total variable cost $= 2x^3 + x^2 + x$

Average cost $= 2x^3 + x + 1 + \dfrac{70}{x}$

Average Variable cost $= 2x^3 + x + 1$

Example 9 :

Find the marginal revenue when the quantity sold is 5 units for the demand function $P = 300 - Q^2$. If fixed cost is 50 and variable cost is 3 at what quantity will profit be maximum

Solution :

M.R = price × quantiy

$= (300 - Q^2) \times Q = 300Q - Q^3$

at $Q = 5$

M.R $= 300\,(5) - 125$

$= 1500 - 125 = ₹1375$

T.C = V.C + F.C

$= 3Q + 50$

$$\text{M.C} = \frac{d(\text{T.C})}{dQ} = 3$$

Profit is maximum when M.R = M.C

$300Q - Q^3 = 3$

Example 10 :

Find the marginal cost is $f^1(x) = 1 - 2x + 12x^3$ where x is the output. Find the total cost, average cost, total variable cost and average variable cost given that fixed cost is ₹ 50.

Solution :

Given $f^1(x) = 1 - 2x + 12x^3$, Fixed cost = ₹ 50

$$\text{Total cost} = \int f^1(x)\,dx$$

$$= \int \left(1 - 2x + 12x^3\right) dx$$

$$= x - x^2 + 3x^4 + C$$

$$= x - x^2 + 3x^4 + 50 \text{ (Fixed cost = 50)}$$

$$\text{Average cost} = \frac{\text{Total cost}}{x}$$

$$= \frac{x - x^2 + 3x^4 + 50}{x} = 1 - x + 3x^4 + \frac{50}{x}$$

Total variable cost = $x - x^2 + 3x^4$

Average variable cost = $1 - x + 3x^3$

Example 11 :

If the marginal revenue is $f^1(x) = 30 - \frac{x}{30}$. Find the total revenue and average revenue obtained from on output of 60 units.

Solution : Total revenue = $\int f^1(x)dx = \int\left(30 - \frac{x}{30}\right)dx$

$$= \int 30x - \frac{x^2}{60}$$

Total revenue for an output of 60 units

$$= \left. 30x - \frac{x^2}{60} \right]_0^{60}$$

$$= 30\,(60) - \frac{(60)^2}{60} = 1800 - 60$$

Total revenue = ₹ 1740

Average revenue = $\frac{\text{Total revenue}}{x}$

$$= \frac{30x - \frac{x^2}{60}}{x}$$

$$= 30 - \frac{x}{60}$$

Average revenue for an output of 60 units $= 30 - \frac{60}{60} =$ ₹ 29

Example 12 :

Find the total variable cost of producing 10 units given that the marginal cost function is $f^1(x) = 275 - x - 0.3x^2$, where x is the number of units produced.

Solution :

Total cost $= \int f^1(x)dx$

$$= \int\left(275 - x - 0.3x^2\right)dx$$

Total cost $= 275x\frac{-x^2}{2} - 0.1x^3 + C$

Total variable cost $= 275x - \frac{x^2}{2} - 0.1x^3$

Total variable cost for producing 10 units

$$= 275(10) - \frac{(10)^2}{2} - (0.1)(10)^3$$

$$= 2750 - 50 - 100 = 2600$$

Total variable cost for producing 10 units = ₹ 2,600.

Example 13 :

If the marginal cost of a product is $2x^2 - 4x$, where x is output, find the total cost of producing 16 units.

Solution :

Given M.C $= 2x^2 - 4x$

$$\text{T.C} = \int \text{M.C}\, dx = \int_0^{16} \left(2x^2 - 4x\right) dx$$

$$= \frac{2x^3}{3} - \frac{4x^2}{2}$$

$$= \frac{2(16)^3}{3} - \frac{4(16)^2}{2}$$

Total cost $= 2730.66 - 512$

$= ₹\ 2218.66$

Example 14 :

Find the total revenue in rupees by raising the output from 20 units to 40 units when the marginal revenue function is $\frac{5q^2}{12} + 16q - 100$ where q is the output.

Solution :

Total revenue $= \text{TR} = \int \text{MR}\, dq$

$$= \int_{20}^{40} \left(\frac{5q^2}{12} + 16q - 100\right) dq$$

$$= \left[\frac{5q^3}{36} + 8q^2 - 100q\right]_{20}^{40}$$

$$= \left[\frac{5(40)^3}{36} + 8(40)^2 - 100(40)\right] - \left[\frac{5(20)^3}{36} + 8(20)^2 - 100(20)\right]$$

$= [8888.89 + 12800 - 4000] - [1111.11 + 3200 - 2000]$

$=$ ₹ 15, 377. 78

Example 15 :

If the marginal revenue = 10x – x^2, find the maximum revenue. Write the demand function.

Solution :

Given MR = $10x - x^2$

Total revenue (R) $= \int \text{MR}\, dx$

$$= \int (10x - x^2)dx \quad = 10x^2 - \frac{x^3}{3} + \text{C}$$

when output = nil, total revenue = nil

$\therefore$ C = 0 Hence R = $10x^2 - \dfrac{x^3}{3}$

Demand function (P) = Average revenue $= \dfrac{\text{Total revenue}}{x}$

$$\text{P} = \frac{10x^2}{x} - \frac{x^3}{3}$$

$$\text{P} = 10x - \frac{x^2}{3}$$

EXERCISE 21.3

I. Two marks questions:

1. If the marginal cost of a firm is $f^1(x) = 10 + 6x - 6x^2$ where x is the output find the total cost amusing that that the fixed cost is ₹125.
2. If the marginal revenue is given by $f^1(x) = \dfrac{30 - x^2}{30}$. Find the revenue obtained from an output of 50 units.
3. Find the total revenue in rupees by raising the output from 10 units to 20 units when the marginal revenue function is $2q^2 - q$ where q is the output

II. Three and Five marks questions:

1. If the marginal cost function is $3x^2 - x + 5$ where x is the output, then find the total cost, average cost, total variable cost and average variable cost given that the fixed cost is ₹25.

2. If the marginal revenue is $f^1(x) = 20 - \frac{x}{20}$. Find the total revenue and average revenue obtained from an output of 30 units.

ANSWERS 21.3

I. (1) $10x + 3x^2 - 2x^3 + 125$ (2) 111.1 (3) 4516.6

II. (1) Total cost $= x^3 - \frac{x^2}{2} + 5x + 25$, Average cost $= x^2 - \frac{x}{2} + 5 + \frac{25}{x}$

Total variable cost $= x^3 - \frac{x^2}{2} + 5x$, Average variable cost $= x^2 - \frac{x}{2} + 5$

(2) Total revenue = ₹1110

Average revenue = ₹185

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II P.U.C BASIC MATHEMATICS (CODE - 75)
Blue print of the Question paper (New syllabus 2014-15 onwards)
MODEL - 1

Unit / Chapter	Title of the Chapter	No. of Teaching hrs	Part A	Part B	Part C	Part D	Part E		Total Marks
			1 Mark	2 Mark	3 Mark	5 Mark	4 Mark	6 Mark	
UNIT-I	**ALGEBRA (42 hrs)**								
1	Matrices & Determinants	13	1	1	2	-	-	1	15
2	Permutation & Combination	08	1	1	1	-	-	-	06
3	Probability	05	-	1	1	-	-	-	05
4	Binomial theorem	06	-	-	-	1	1	-	09
5	Partial fraction	04	-	-	-	1	-	-	05
6	Mathematical logic	06	1	1	-	1	-	-	08
UNIT-II	**COMMERCIAL ARITHMETIC (34 hrs)**								
7	Ratios & proportions	10	1	1	1	1	-	-	11
8	Bill Discounting	06	-	1	1	-	-	-	05
9	Stocks & shares	04	1	-	1	-	-	-	04
10	Learning curve	04	-	-	-	1	-	-	05
11	Linear programming problem	06	-	-	-	1	-	-	05
12	Sales tax & value added tax	04	-	-	1	-	-	-	03
UNIT-III	**TRIGONOMETRY (12 hrs)**								
13	Heights & Distances	04	-	-	-	-	1	-	04
14	Compound angle, multiple angle, sub multiple angle & transformation formule	08	1	2	-	1	-	-	10
UNIT-IV	**ANALYTICAL GEOMETRY (10 hrs)**								
15	Circles	06	1	-	-	1	-	-	06
16	Parabola	04	-	1	1	-	-	-	05
UNIT-V	**CALCULUS (42 hrs)**								
17	Limit & Continuity of a function	08	1	1	-	-	-	1	09
18	Differential Calculus	10	1	1	1	1	-	-	11
19	Application of Derivative	08	-	1	2	-	-	-	08
20	Indefinite Integrals	08	1	1	2	-	-	-	09
21	Definite Integrals & its Application to areas	08	-	1	-	1	-	-	07
	Total No. of Teaching hrs / Marks	140 hrs	10	14	14	10	02	02	150

II P.U.C BASIC MATHEMATICS (CODE - 75)
Blue print of the Question paper (New syllabus 2014-15 onwards)
MODEL - 2

Unit / Chapter	Title of the Chapter	No. of Teaching hrs	Part A	Part B	Part C	Part D	Part E		Total Marks
			1 Mark	2 Mark	3 Mark	5 Mark	4 Mark	6 Mark	
UNIT-I	**ALGEBRA (42 hrs)**								
1	Matrices & Determinants	13	1	1	2	1	-	-	14
2	Permutation & Combination	08	1	1	1	-	-	-	06
3	Probability	05	-	1	1	-	-	-	05
4	Binomial theorem	06	-	-	-	1	1	-	09
5	Partial fraction	04	-	-	-	1	-	-	05
6	Mathematical logic	06	1	1	-	1	-	-	08
UNIT-II	**COMMERCIAL ARITHMETIC (34 hrs)**								
7	Ratios & proportions	10	1	1	1	1	-	-	11
8	Bill Discounting	06	-	1	1	-	-	-	05
9	Stocks & shares	04	-	-	1	-	-	-	03
10	Learning curve	04	-1	-	-	1	-	-	06
11	Linear programming problem	06	-	-	-	1	-	-	05
12	Sales tax & value added tax	04	-	-	1	-	-	-	03
UNIT-III	**TRIGONOMETRY (12 hrs)**								
13	Heights & Distances	04	-	-	-	-	1	-	04
14	Compound angle, multiple angle, sub multiple angle & transformation formule	08	1	2	-	1	-	-	10
UNIT-IV	**ANALYTICAL GEOMETRY (10 hrs)**								
15	Circles	06	1	-	-	-	-	1	07
16	Parabola	04	-	1	-1-	-	-	-	05
UNIT-V	**CALCULUS (42 hrs)**								
17	Limit & Continuity of a function	08	1	1	-	-	-	1	09
18	Differential Calculus	10	1	1	1	1	-	-	11
19	Application of Derivative	08	-	1	2	-	-	-	08
20	Indefinite Integrals	08	1	1	2	-	-	-	09
21	Definite Integrals & its Application to areas	08	-	1	-	1	-	-	07
	Total No. of Teaching hrs / Marks	140 hrs	10	14	14	10	02	02	150

II PUC BASIC MATHEMATICS
WEIGHTAGE GIVEN TO THE CURRICULUM

1. Knowledge	:	35%	=	52.5Mark
2. Understanding	:	30%	=	45 Mark
3. Applications	:	25%	=	37.5 Mark
4. Skill	:	10%	=	15 Mark
		Total		**150 Marks**

UNIT WISE WEIGHTAGE

Unit		
I Algebra	=	48 Marks
II Commercial arithmetic	=	33 Marks
III Trigonometry	=	14 Marks
IV Analytical geometry	=	11 Marks
V Calculus	=	44 Marks
Total		**150 Marks**

INSTRUCTION TO QUESTION PAPER SETTERS

(New syllabus 2014-15 onwards)

Note: In the chapter (4) Binomial Theorem the proof of the Binomial Theorem for Positive Integral power is Excluded

PART - E

6 MARKS QUESTIONS MUST BE SELECTED FROM THE FOLLOWING TOPICS ONLY:

1. **Application of matrix:** (3 × 3) order about statement problem, formation of linear equation and solve them by matrix method
2. **Circle:** Problems on concylic (circle passes through 4 points)
3. **Limits:** standard theorem

   (i) prove that $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$ for all n ∈ Q (Q = set of rational n is + ve, – ve and fraction)

   (ii) Prove that $\lim_{x \to 0}\left[\frac{\sin x}{x}\right] = 1,$ (x = angle mesuered in radian) and deduce that

   $\lim_{x \to 0}\left[\frac{\tan x}{x}\right] = 1,$

4 MARKS QUESTION MUST BE SELECTED FROM THE FOLLOWING TOPIC ONLY.

1. **Binomial theorem**: Application problems like Evaluate : $(0.\ 98)^5$, $(1.01)^5$, $(102)^4$, $(0.97)^4$... ect upto 4 decimal.
2. **Heights and distances** : Application problems
3. **Linear programming problem (L.P.P)** : Statement problems on L.P.P for mation-tion of linear equations.
4. **Cost and revenue function** : Problems on total cost, total revenue, Marginal cost, Marginal revenue, Profit Maximization. ..... etc.

MODEL QUESTION PAPER - 1
II PUC : BASIC MATHEMATICS [CODE : 75]
(NEW SYLLABUS 2014-15)

Time: 3.15 hours **Max. Marks: 100**

Instructions:

(i) *The question paper has 5 parts A, B, C, D, and E. Answer all the parts.*

(ii) *Part - A carries 10 marks, part - B carries 20 marks, part - C carries, part - D carries 30 marks and part - E carries 10 marks.*

(iii) *Write the question number properly as indicated in the question paper.*

PART – A

I. Answer ALL the questions: **(1 × 10 = 10)**

1. If $A = \begin{bmatrix} 5 & -6 & 3 \end{bmatrix}, B = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ find AB
2. If ${}^{n}P_{4} = 360$ find n.
3. Write the truth value of the proposition "If 2 is not a prime numbers then $\sqrt{2}$ is an irrational number".
4. Find the mean proportion of 9 and 16
5. Define the 'Index of learning'.
6. If Sin $A = \frac{3}{5}$, find Sin 2A
7. Find the radius of the circle $x^2 + y^2 - 4x - 5 = 0$.
8. Evaluate : $\underset{x \to 3}{\text{Lt}} \left[\frac{x^3 - 27}{x - 3} \right]$
9. If $y = 2\sqrt{x} - \cos 2x + 2$, find $\frac{dy}{dx}$.
10. Evaluate : $\int \left(\frac{1}{x} - \sin x + 3 \right) dx$.

PART – B

II. Answer any 10 questions: **2 × 10 = 20**

11. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Show that $A^2 - 4A + 3I = 0$ (I = Identify matrix of 2^{nd} order)
12. There are 15 points in a plane of which 5 are collinear. Find the number of straight lines can be formed.
13. A boy drawn at random 3 balls from a bag containing 9 red and 5 while balls. What is the probability of getting (i) all red balls (ii) 2 red and 1 white.
14. Write the converse and contrapositive of the compound proposition "If prashanth got first class in mathematics then prof. John will gift him a watch".
15. A certain number is subtracted from each of the two term of the ratio 21 : 35 to give a new ratio 3 : 10. Find the number which is subtracted?
16. Find the Banker's discount on ₹ 1000 due 6 months hence at 10% per annum.
17. Derive the value for cos 105°.
18. Transform 2sin 40° × cos 20° into sum.
19. Find the focus and the equation of Directrix of the parabola $x^2 - 16y = 0$.
20. Evaluate : $\lim_{x \to 0}\left(\frac{\sin 3x . \tan 4x}{x^2}\right)$.
21. If $y = x^x$, find $\frac{dy}{dx}$.
22. If $S = 4t^3 - 6t^2 + t - 7$ (S = distance in mt, t = time sec)
    Find the velocity and acceleration at t = 2 sec.
23. Evaluate : $\int \frac{x}{x^2 + 4} dx$.
24. Evaluate : $\int_0^{\pi/4} \sec^2 3x \, dx$.

PART – C

III. Answer any 10 questions: **(3 × 10 = 30)**

25. If $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ verify A. (adj A) = (adj A). A = |A|.I and (I=Identity Matrix of 2^{nd} order).
26. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$.

27. Find the number of permutation of the letters of the word 'COMMISSION'. If the word

    (i) Start with M and end with M

    (ii) 2S's are together

    (iii) 2O's are not together

28. In a class of 80 student 40 are taking Mathematics 25 are taking Statistics. If each student has taken at least one of these subject. Find the probability that the student is taking

    (i) both Mathematics and Statistics

    (ii) only Mathematics

    (iii) Only Statistics

29. A sum of ₹. 2415 has to be divided among three person A, B and C in such proportion that A's share to B's is 4 : 5 and B's share to C's share is 9 : 16. How much does each get?

30. A bill of ₹ 5000 was drawn an 10/4/2013 at 3 months and was discounted on 1/5/ 2013 at 12% p.a. For what sum was the bill Discounted and also find the Banker's gain.

31. Find the Interest earned on ₹ 4897.50 cash invested in 15% stock at 81.50, given that brokerage is 0.125%.

32. The price of a washing machine, inclusive of sales tax is ₹ 13,530. If the sales tax is 10%, find its basic price?

33. Find the equation of the parabola given that the ends of the latus rectum are L (3, 6) is Focus is (0,–3) and Directrix is y = 3

34. If $x = a\cos^4 t$, $y = b\sin^4 t$. find $\frac{dy}{dx}$ at $t = \pi/4$.

35. The height of a cone is 30 cm and it is constant the radius of the base is increasing at the rate 0.25cm / sec. Find the rate of increase of volume of the cone when the radius is 10 cm?

36. Find the maximum and minimum value of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$.

37. Evaluate: $\int \frac{5x}{(x-3)(x+4)}\,dx$.

38. Evaluate: $\int x^2 .\cos 3x\ dx$.

PART – D

IV. Answer any 6 questions: **(6 × 5 = 30)**

39. Find the term Independent of 'x' in the expansion of $\left(\sqrt{x}+\frac{1}{3x^2}\right)^{10}$.

40. Resolve $\frac{2x+1}{(x-1)(x+2)(3x-1)}$ into partial fractions.

41. Verify: $\sim(p \to q) \vee [(\sim p \wedge q) \leftrightarrow \sim q]$ is a Tautology or contradiction or neither.

42. A jar contains two liquids X and Y in the ratio 7 : 5. When 6 litres of the mixture is drawn and the jar in filled with the same quantity of Y, the ratio of X and Y becomes 7 : 9. Find the quantity X in the jar initially.

43. Samsung company which manufacture LCD TV. The 1st lot of 10 unit was completed in 1400 labour hrs. Find each subsequent lot, the commutative production was doubled. And it has observed that 90% learing effect applies to all labour related cost. The anticipated production is 320 unit of LCD TV find total labour cost required to manufacture 320 unit and also find the total labour cost at ₹ 20/per hrs.

44. Using the graphical method, solve the following. LPP:-

    Objective function : minimize $z = 1.5x + 2.5y$

    Subject to constraints : $x + 3y \geq 3$

    $x + y \geq 2$

    and $x \geq 0, y \geq 0.$

45. Show that $\frac{\sin^3\theta + \sin 3\theta}{\sin\theta} + \frac{\cos^3\theta - \cos 3\theta}{\cos\theta} = 3.$

46. A circle has its centre on x-axis and passess through (5, 1) and (3, 4). Find its equation.

47. If $y = (x^2 + a^2)^6$, show that $(x^2 + a^2)\frac{d^2y}{dx^2} - 10x\frac{dy}{dx} - 12y = 0.$

48. Find the area enclosed between the curve $y = 11x - 24 - x^2$ and the line $y = x$.

PART – E

V. Answer any 1 questions: **(1 × 10 = 10)**

49.(a) Transport corporation operate bus service between two villages. Data regarding the passenger traffic during the 1st three days of the week is given below along with the total revenue. (6)

Day	No. Passenger of travelled			Total Revenue (₹)
	Children	Senior citizen	Adult	
1	10	10	20	90
2	30	20	10	100
3	10	20	30	140

Find the bus fare charged per children, senior citizen and per adult. By using matrix method.

(b) The angle of elevations of the top of an unfinished tower at a point distance 120 mt from its base in 45°. How much higher must the tower be raised so that the angle of elevation at the same point many be 60°? (4)

50.(a) Prove that $\underset{x \to a}{\text{Lt}} \left(\dfrac{x^n - a^n}{x - a} \right) = n.a^{n-1}$,

for all 'n' belongs to rational (n is + ve, – ve or a fration). (6)

(b) Find the total Revenue obtained by raising the output from 10to 20 units. Where the marginal revenue function in given by $\text{MR} = 3\left(\dfrac{x^2}{20}\right) - 10x + 100$ (x = output). (4)

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MODEL QUESTION PAPER - 2

II PUC : BASIC MATHEMATICS [CODE : 75]

(NEW SYLLABUS 2014-15)

Time: 3.15 hours **Max. Marks: 100**

Instructions:

(i) The question paper has 5 parts A, B, C, D, and E. Answer all the parts.

(ii) Part - A carries 10 marks, part - B carries 20 marks, part - C carries, part - D carries 30 marks and part - E carries 10 marks.

(iii) Write the question number properly as indicated in the question paper.

PART – A

I. Answer all the questions: **10 × 1 = 10**

1. Evaluate : $\begin{vmatrix} 4000 & 4001 \\ 4002 & 4003 \end{vmatrix}$.
2. Find n if ${}^{n}C_{8} = {}^{n}C_{9}$
3. Write symbolically "If gold is an elemont then water is a com pound.
4. Find the mean proportion of 36 and 4.
5. Write the formula for yield Percentage.
6. Write the value for sin 75°.
7. Find the centre of the of the circle $x^2 + y^2 = 25$
8. Evaluate : $\lim_{x \to 2} \left[\dfrac{x^2 - 2^2}{x - 2}\right]$.
9. Find $\dfrac{dy}{dx}$ if $y = \sqrt[3]{x}$.
10. Evaluate : $\int e^{3x}\, dx$.

PART B

II. Answer any Ten question. **(2 × 10 = 20)**

11. If $\begin{bmatrix} x+y & 3 \\ y & x-y \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$ find x and y.

12. Find the number of words formed by the letters of the word "DELHI" How many of them. start with D and end with I
13. Find the probability of getting a black Jack from a pack of 52 cards.?
14. If the truth value of p is true, q is false. Find the truth value of $\sim(p \to \sim q) \vee \sim p$.
15. If a : b = 3 : 5, b : c = 15 : 23 find a : c.
16. Find the legally due date for a bill date 22/04/2014 due 6 months hence.
17. Tanya bought a coat for ₹ 220 inclusive of sales tax 10%. How much was the sales tax?
18. If $\text{Tan A} = \frac{1}{2}$ & $\text{Tan B} = \frac{1}{3}$ prove that $A + B = \frac{\pi}{4}$.
19. If $y^2 = 16x$ find the (i) Focus (ii) Latus Rectum of the parabola.
20. Evaluate : $\lim\limits_{\theta \to 0} \frac{\tan 5\theta . \sin^2 \theta}{\theta^3}$.
21. If $S = \sqrt{t-1}$ find velocity (S = distance, t = time)
22. If Total cost $C(x) = x^2 + 2x + 1$ find (i) marginal cost & (ii) Average cost.
23. Evaluate: $\int_1^2 \frac{1}{x}\, dx$.
24. Find the area enclosed by the curve $y = x^2 + 2x$ between the ordinates $x = 0$ and $x = 2$.

PART C

III. Answer any 10 questions: **$3 \times 10 = 30$**

25. Show that, if in a determinant scalar multiple of the elements of any row or column is added to any other row or column. The value of the determinant remains unchanged.
26. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
27. Find the no. of parallelogram that can be formed from a set of 6 parallel lines intersecting another on the set of 11 parallel line.
28. A bag contains 7 white, 3 red and 4 black balls, one ball is picked up at random. What is the probability that (i) none is black (ii) ball is red (iii) ball is white.
29. Two quantities are in the ratio 3 : 4 If 10 is subtracted from each of them, the remainder are in the ratio 1 : 3. find the quantities.
30. Calculate the banker's discount on face value ₹ 1000 of the period is 73 days at 5% p.a banker's commissions.

31. What is the market value of 9.5 stock when an investment of ₹12,400 produce income ₹1472.5.

32. Ananya went to the grocery shop to purchase biscuits for ₹ 40, Rice for ₹ 50, and wheat for ₹ 50 sales tax on each item is 10%. How much should she pay to the shop keepers?

33. If $y = -4$ is the equation of the drectrix, axis $x = 3$ and the length of lotus rectum in 8 find the equation of the parabola.

34. Differentiate 'e^x' w.r.t. x by Ist priciples.

35. Divide 20 into two parts so that the product is maximum.

36. The side of a square is increasing at the rate of 10 cm/sec. If the side 20 cm. Find the rate of increase of its area.

37. Evaluate : $\int \frac{1}{1+\cos x} dx.$

38. Evaluate : $\int e^{\sin x}.\cos x \; dx.$

PART D

IV. Answer any SIX questions **(5 ×6 = 30)**

39. Evaluate : $\left(1+\sqrt{5}\right)^4 - \left(1-\sqrt{5}\right)^4.$

40. Resolve in to partial fractions $\frac{2x+1}{(x-1)(x-2)(x-3)}.$

41. Define Tautology and verify for tautology $\sim(p \wedge q) \rightarrow (\sim p \vee \sim q).$

42. ₹. 5625 is Divided among A, B and C so that A receives one half as much as B and C together receive and B receives one fourth of what A and C together receive. Find the share of A, B and C.

43. XYZ company supplies water tankers to the government the 1^{st} water tanker takes 20,000 labour hours. The government auditors suggest that there should be 90% learning effect rate the management expects an order of 8 water tankers in the next year what will be total labour hrs and the labour cost the company will incur at the rate of ₹. 20 per hour.

44. Solve the following L.P.P by graphically

Maximise : $z = 400x + 150y$

Subject to the constraints, $3x + y \leq 600$

$x \leq 150$

$y \leq 400$

$x \geq 0, y \geq 0$

45. Prove that $\sin 3A = 3\sin A - 4\sin^3 A$

46. Find the equation of the circle passing through the points (0, 0) (1, 1) and has its centre on x-axis.

47. If $e^y = \sin(x + y)$ prove that $\frac{dy}{dx} = \frac{\cos(x+y)}{e^y - \cos(x+y)}$.

48. Find the area between the curves $x^2 = y$ and $y^2 = x$.

PART E

V. Answer any one question **(10 × 1 = 10)**

49. (a) Evaluate : $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$ for all rationals. (n is + ve, – ve & fraction)

(6)

(b) Expand $(0.99)^5$ using Binomial theorem upto 4 decimal. (4)

50.(a)A sales person has the following record of sales for the month of Jan, Feb and March 2014 for 3 products A, B and C. He has paid a commission at fixed rate per unit but at varying rates for products A, B and C.

Months	Sales (units)			Commission (₹)
	A	B	C	
January	9	12	2	800
February	15	5	4	900
March	6	10	3	950

Find the rate of commission payable on A, B & C per unit sold by using matrix method. (6)

(b) A person standing on the bank of a river observes, that the angle subtended by a tree on the opposite bank is 60°. When he returns 40 meters from the bank he finds the angle to be 30°. Find the height of the tree and the breadth of the river.

(4).

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MODEL QUESTION PAPER - 3
II PUC : BASIC MATHEMATICS [CODE : 75]
(NEW SYLLABUS 2014-15)

Time: 3.15 hours **Max. Marks: 100**

Instructions:

(i) *The question paper has 5 parts A, B, C, D, and E. Answer all the parts.*

(ii) *Part - A carries 10 marks, part - B carries 20 marks, part - C carries, part - D carries 30 marks and part - E carries 10 marks.*

(iii) *Write the question number properly as indicated in the question paper.*

PART – A

I. Answer all the questions: **10 × 1 = 10**

1. If $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, find AB.
2. If $5P_r = 60$, find the value of r.
3. Negate : "4 is an even integer of 7 is a prime number".
4. Find the third proportional to 6 and 24.
5. Define yield.
6. If $\sin A = \frac{1}{2}$, find sin 2A.
7. If the length of latus rectum of the parabola $x^2 = 4ky$ is 8, find k.
8. Evaluate : $\underset{x \to -1}{\text{Lt}} \left[\frac{x^3 + 1}{2x^2 + 5x + 3} \right]$.
9. Find $\frac{dy}{dx}$ if $x^2 - y^2 = a^2$.
10. Evaluate : $\int \frac{1}{7x + 8} dx$.

PART B

II. Answer any Ten question. **(10 × 2 = 20)**

11. If $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, show that $A^2 - 4A + 3I = 0$. (I = Identify matrix of 2nd order)
12. In how many ways can 6 gentlemen and 4 ladies be seated round a table so that no 2 ladies are together?

13. If $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}, P(A\cup B)=\frac{7}{12}$, find $P\left(B/A\right)$.

14. Write the converse and contrapositive of "If two straight lines are parallel then they do not intersect".

15. 2 numbers are in the ratio 3 : 5. If 5 is added to each term, then their ratio becomes 2:3. Find the numbers.

16. A banker pays ₹ 2380 on a bill of ₹ 2500, 73 days before the legal due date. Find the rate of discount charged by the banker.

17. Sanju purchases a bicycle costing ₹ 12,000. If the rate of sales tax is 9%, then calculate the total amount paid by him.

18. Prove that $\frac{\cos 2A - \cos 12A}{\sin 12A - \sin 2A} = \tan 7A$.

19. Find the equation of a circle whose end points of it's diameter are (3, 4) and (1, –2).

20. If $y=\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$, prove that $\frac{dy}{dx}=\sec^2 x$.

21. Evaluate : $\lim\limits_{\theta\to 0}\frac{\sin 3\theta}{\tan 2\theta}$.

22. The demand function of a firm is $2x - 5y = 7$(x is the output, y is price / unit) Find the marginal revenue.

23. Find : $\int \frac{3x^2}{1+x^3}\,dx$.

24. Evaluate $\int_1^2 x\,e^x\,dx$.

PART - C

III. Answer any TEN questions : (10 × 3 = 30)

25. Solve using cramer's rule : $3x + 4y = 7$ and $7x = y + 6$.

26. Using properties of determinants, prove : $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$.

27. A team of 8 players has to be selected from 14 players. In how many ways the selections can be made if
(i) 2 particular players are always included
(ii) 2 particular players are always excluded.

28. A box contains 4 defective and 6 non-defective bulbs. Find the probability that at least 3 bulbs are defective when 4 bulbs are selected at random.

29. 5 carpenters can earn ₹ 540 in 6 days, working 9 hours a day. How much will 8 carpenters earn in 12 days working 6 hours a day?

30. A bill for ₹ 2920 was drawn on september 11 for 3 months after date and was discounted at 16% p.a for ₹ 2875.20. On what date was the bill discounted?

31. What is the market value of 6% stock if it earns an interest of 4.5% after deducting the income tax of 4%.

32. The owner of a departmental store purchased an article of ₹ 1500 at 4% VAT and sells it at ₹ 1700 to the customer at 4% VAT. How much amount did the shopkeeper deposit to the Government as VAT?

33. Find the equation of a parabola whose focus is (–5, 0) and directrix $x = 5$.

34. If $x = a \sec\theta$, $y = b \tan\theta$, find $\frac{dy}{dx}$ at $\theta = \pi/4$.

35. The surface area of a spherical bubble is increasing at the rate of 0.8 cm^2/sec. Find the rate at which its volume is increasing when its radius is 2.5 cms.

36. Find the maximum and minimum values of the function $f(x) = 9x^2 + 12x + 2$.

37. Evaluate : $\int \frac{4x+3}{(x-1)(x+2)} dx$.

38. Evaluate : $\int \frac{1}{\sqrt{x}+x} dx$.

PART - D

IV. Answer any SIX questions: (6 × 5 = 30)

39. If $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, verify : A adj A = adj A. A = |A| . I (where I is the identify matrix of 3rd order).

40. Find the middle terms in the expansion of $\left(\sqrt{x} - \frac{4}{x^2}\right)^{11}$.

41. Resolve into partial fractions : $\frac{1+2x}{(x+2)^2(x-1)}$.

42. Examine whether $p \leftrightarrow q$ and $[(p \to q) \wedge (q \to p)]$ are logically equivalent.

43. Two taps fill a cistern separately in 20 mins and 40 mins respectively. Another pipe can drain off 30 litres per minute from the cistern. If all 3 pipes are opened together, the cistern fill in 72 mins what is the capacity of the cistern?

44. A company requires 1000 hours to produce the first 30 engines. If the learning effect is 90%, then find the total labour cost to produce a total of 120 engines, at the rate of ₹ 20 per hour.

45. Solve the following (LPP) graphically:

Maximize : $z = 60x + 15y$

Subject to Constraits : $x + y \leq 50$

$3x + y \leq 90$

$x, y \geq 0.$

46. Prove : sin 20° × sin 40° × sin 60° × sin 80° $= \frac{3}{16}$.

47. If $y = x + \sqrt{x^2 - 1}$, prove that : $(x^2 - 1)y_2 + xy_1 - y = 0$.

48. Find the area enclosed between the parabola $y^2 = x$ and the line $x + y = 2$.

PART - E

V. Answer any ONE question : (1 × 10 = 10)

49. (a) Show that the points (2, – 4), (0, 0), (3, –1) and (3, –3) are conyclic. (6)

(b) From the top of a house 32 m high, the angle of elevation of the top of a tower is 45° and the angle of depression of the foot of the tower is 30°. Find the height of the tower.

(4)

50. (a) Prove: $\underset{\theta \to 0}{\text{Lt,}} \frac{\sin\theta}{\theta} = 1,$ θ is in radians and hence show that $\underset{\theta \to 0}{\text{Lt}} \left[\frac{\tan\theta}{\theta}\right] = 1,$.

(6)

(b) Find the value of $(1.05)^5$ upto 4 places of decimals, using Binomial theorem.

(4)

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MODEL QUESTION PAPER - 4
II PUC : BASIC MATHEMATICS [CODE : 75]
(NEW SYLLABUS 2014-15)

Time: 3.15 hours **Max. Marks: 100**

Instructions:

(i) *The question paper has 5 parts A, B, C, D, and E. Answer all the parts.*

(ii) *Part - A carries 10 marks, part - B carries 20 marks, part - C carries, part - D carries 30 marks and part - E carries 10 marks.*

(iii) *Write the question number properly as indicated in the question paper.*

PART – A

I. Answer all the questions: **10 × 1 = 10**

1. Without expanding evaluate : $\begin{vmatrix} 500 & 503 \\ 506 & 509 \end{vmatrix}$.

2. Find r if ${}^{15}C_{r+3} = {}^{15}C_{2r-3}$.

3. Negate : If Ramya study hard then she will get the rank.

4. If a : b = 2 : 5 and b : c = 3 : 5 find a : c.

5. what rate of interest is realised by investing in 14.5% stock at 81?

6. find the value of sin 70° cos 20° + cos 70° sin 20°.

7. Find the equation of the point circle with centre at (4, – 3).

8. Evaluate : $\underset{x \to -4}{\text{Lt}} \dfrac{x^4 - 256}{x+4}$.

9. Find $\dfrac{dy}{dx}$ if $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

10. Evaluate : $\int \left(\dfrac{1+x^2}{x} \right) dx$.

PART - B

II. Answer any TEN questions: (2 × 10 = 20)

11. 11. Prove that if any two rows (or columns) of a determinant are interchanged then the value of the determinant changes only in sign.

12. If a convex polygon has 170 diagonals, find the number of sides of the polygon.

13. In a single throw of two dice, what is the probability of obtaining a total of 9?

14. If p, q, r are three propositions with truth values T, T, F respectively find the truth value of $p \to (\sim q \wedge r)$.

15. Two numbers are in the ratio 3 : 5. If 9 is subtracted from each the new numbers are in the ratio 12 : 23. Find the smaller number.

16. The Banker's gain on a certain Bill due six months hence is ₹ 10 the rate of interest being 10% p.a. Find the face value of the Bill.

17. If $\sin\theta = \frac{3}{5}$ and θ is acute, find the value of $\sin 3\theta$.

18. Show that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.

19. Find the equation of the parabola whose focus is (0, –6) and directrix $y = 6$.

20. Find the value of k if $f(x) = \begin{cases} \frac{e^{5x} - 1}{2x}, & \text{for } x \neq 0 \\ \frac{k}{2}, & \text{for } x = 0 \end{cases}$ is continuous at $x = 0$.

21. If $x = a\sec\theta$, $y = b\tan\theta$ find $\frac{dy}{dx}$.

22. Find a point on the parabola $y^2 = 18x$ at which ordinate increases at twice the rate of the abscissa.

23. Find two positive numbers whose sum is 14 and the sum of the squares of the numbers is minimum.

24. Integrate w.r.t x $\int (4x^2 - 2x + 7)^{3/2}(4x - 1)\,dx$.

PART - C

III. Answer any TEN questions **(3 × 10 = 30)**

25. If $A=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}$ show that $A^2 - 7A - 2I = 0$. (I = Idtenity matrix of 2nd order)

26. Find the adjoint of the matrix $A=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

27. In how many ways 3 boys and 5 girls can be arranged in a row so that no two boys are together?

28. If A and B are events with $P(A)=\frac{5}{8}, P(B)=\frac{3}{8}$ and $P(A\cup B)=\frac{3}{4}$ find (i) P(B/A) (ii) P(A/B).

29. The driver of a car is travelling at the speed of 36kmph and spot a bus at 80mt ahead of him, after 1hr the bus 120mt behind the car. What is the speed of the bus.

30. A banker pays ₹. 2380 on a bill of ₹. 2500, 73 days before the legally due date. Find the rate of discount charged by the banker.

31. Find the interest earned on ₹ 4897.50 cash invested in 15% stock at 81.5 the brokerage given is 0.125.

32. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots\infty}}}$ find $\frac{dy}{dx}$.

33. Find the co-ordinates of the vertex, focus and equation of directrix of the parabola $5x^2 + 24y = 0$.

34. Raju goes to purchase a motor cycle which is priced ₹ 35,640 including 10% on sales tax how ever the actual rate of sales tax at the time of purchase is 7%. Find the extra profit made by the shop keeper if he still charges the original listed price.

35. A circular plate of metal is heated so that its radius increases at the rate of 0.1mm per minute. At what rate is the plate's area increasing when the radius is 25cm.

36. The cost function $C=500x-20x^2+\frac{x^3}{3}$ where x stands for output. Calculate the output when the marginal cost is equal to average cost.

37. Evaluate : $\int\frac{1}{e^x+e^{-x}}dx$.

38. Evaluate : $\int_0^{\pi/2}\sin 5x.\cos 3x\, dx$.

PART - D

IV. Answer any SIX questions: **(5 × 6 = 30)**

39. Find the coefficient of x^{-11} in the expansion of $\left(\sqrt{x}-\frac{2}{x}\right)^{17}$?

40. Resolve $\frac{2x^2+16x+29}{(x+3)^2(x+4)}$ into partial fractions.

41. Prove that, $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$.

42. If 15 men working 12 hours per day perform job in 16 days how long will it take for 21 men working 10 hours daily to do the same job.

43. The time required to produce the first unit of a product is 1000 hours. If the manufacturer experience 80% learning effect. Calculate the cumulative average time per unit and the total time taken to produce altogether 8 units. Also find the labour charges for the production of 8 units at the rate of ₹ 10 per hour.

44. Solve L.P.P graphically, maximize $z = 4x + 3y$ subject to the constraints $x+2y \le 5$, $x+y \le 3$, $3x+y \le 7$, $x, y \ge 0$.

45. The angle of elevation of an object from a point 100m above a lake is 30° and angle of depression of its image in the lake is 45°. Find the height of the object above the lake.

46. Find the equation of the circle passing through the points (1, 1), (2, –1) and (3, 2).

47. If $y=\left(x+\sqrt{1+x^2}\right)^m$, prove that $\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0$.

48. Find the area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$.

PART - E

V. Answer any ONE question: **(1 × 10 = 10)**

49. (a) Salesman Venki has the following record of sales during 3 months of July, August and September for three products. A, B, C which have different rates of commission.

Month	Sales in Units			Total
	A	B	C	Commission (₹)
July	100	100	100	700
August	200	300	200	1700
September	400	900	100	3700

Using matrix method, find out the rates of commission on items A, B and C received by Venki. (6)

(b) Find the value of $(0.98)^3$ using binomial theorem upto 5 places of decimals. (4)

50. (a) Prove that $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ and hence deduce that $\lim\limits_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$. (θ in radian) (6)

(b) A company produces two types of leather belts A and B. A is of superior quality and B is of inferior quality. The respective profits are ₹ 10 and ₹ 5 per belt. The supply of raw material is sufficient for making 850 belts per day. For belt A, a special type of buckle is required and 500 are available per day. There are 700 buckles available for belt B per day. Belt A needs twice as much time as that required for belt B and the company can produce 500 belts if all of them were of the type A. Formulate a L.P.P model for the above problem. (4)

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KARNATAKA STATE WOMEN COMMISSION

Karnataka State Woman Commission is the Legislative Commission established on 06-08-1996 and is the important component of Karnataka State Women and Child Welfare Department. The different facilities provided to exploited women and the essential information to public about women are as follows:

FACILITIES:

1. Women are facilitated to get solution and help regarding the family problems such as ‘dowry’, ‘divorce’, ‘property disputes’, sexual harassment.
2. Provide counseling at ‘Parivarik Lok Adalath’ to solve those family litigations which are not solved in Family Court for a long time.
3. Women are provided help/suggestion to file a case in family court.
4. A relief fund of a minimum Rs. 20,000 to maximum Rs.2, 00,000 is provided to women who have attacked been by a person throwing acid on them causing grievous wounds.

INFORMATION TO PUBLIC ABOUT WOMEN:

1. Giving and taking ‘Dowry’ is an offence.
2. Minimum age for the girl’s marriage is 18.
3. Better to keep the records of photos (with negatives) which are taken at the time of marriage, gifts, an amount should a woman get by right, etc. which would help at the time of any problems in future and they will serve the purpose.
4. When a woman is raped, she should be given first-aid by the government doctor and her clothes should be kept carefully for scrutiny later.
5. Do not forget to take medical attestation from government doctor when woman is exploited physically.
6. When a woman dies unnaturally, post mortem must be done by a government doctor.
7. Whenever, death, accident or dispute take place in a family, do not sign on any paper given by the relatives of the women as the woman was not sound mentally. It may lead to exploitation or cheating.
8. Since 1989, women have the right to get equal share in the parental property.
9. Immovable assets must be registered in both the names of husband and wife.
10. According to the judgment given by the Supreme Court, there should be a committee to listen to complaints in order to prevent sexual harassment on women at each working place.
11. At each taluk centre, there is a C.D.P office to provide suggestion/help/counseling for women subjected to injustice.

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