Miscellaneous Exercise Page No: 29

1. Let f : R → R be defined as f(x) = 10x + 7. Find the function g : R → R such that g o f = f o g = I_R.

Solution:

Firstly, Find the inverse of f.
Let say, g is inverse of f and
y = f(x) = 10x + 7

y = 10x + 7

or x = (y-7)/10

or g(y) = (y-7)/10; where g : Y → N

Now, gof = g(f(x)) = g(10x + 7)

$= \frac{(10x+7)-7}{10}$

= x

= I_R

Again, fog = f(g(x)) = f((y-7)/10)

= 10((y-7)/10) + 7

= y – 7 + 7 = y

= I_R

Since g o f = f o g = I_R. f is invertible, and

Inverse of f is x = g(y) = (y-7)/10

2. Let f : W → W be defined as f(n) = n – 1, if n is odd and f(n) = n + 1, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

Solution:

f : W → W be defined as f(n) = n – 1, if n is odd and f(n) = n + 1, if n is even.

Function can be defined as:

$$f(n)=\begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

f is invertible, if f is one-one and onto.

For one-one:

There are 3 cases:
for any n and m two real numbers:

Case 1: n and m : both are odd

f(n) = n + 1
f(m) = m + 1
If f(n) = f(m)
=> n + 1 = m + 1
=> n = m

Case 2: n and m : both are even

f(n) = n - 1
f(m) = m - 1
If f(n) = f(m)
=> n - 1 = m - 1
=> n = m

Case 3: n is odd and m is even

f(n) = n + 1
f(m) = m - 1
If f(n) = f(m)
=> n + 1 = m - 1
=> m - n = 2 (not true, because Even – Odd ≠ Even)

Therefore, f is one-one

Check for onto:

$$f(n)=\begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

Say f(n) = y, and y ∈ W

Case 1: if n = odd

f(n) = n – 1

n = y + 1
Which show, if n is odd, y is even number.

Case 2: If n is even

f(n) = n + 1

y = n + 1

or n = y – 1
If n is even, then y is odd.

In any of the cases y and n are whole numbers.

This shows, f is onto.

Again, For inverse of f

f^{-1} : y = n – 1

or n = y + 1 and y = n + 1

⇨ n = y – 1

$$f^{-1}(n)=\begin{cases} n-1, & \text{if n is odd} \\ n+1, & \text{if n is even} \end{cases}$$

Therefore, f^{-1} (y) = y. This show inverse of f is f itself.

3. If f : R → R is defined by f(x) = x^2 – 3x + 2, find f (f(x)).

Solution:
Given: f(x) = $x^2 - 3x + 2$

f (f(x)) = f($x^2 - 3x + 2$)

= $(x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

= $x^4 - 6x^3 + 10x^2 - 3x$

4. Show that the function f : R → {x ∈ R : – 1 < x < 1} defined by f(x) = $\frac{x}{1+|x|}$,x ∈ R is one one and onto function.

Solution:
The function f : R → {x ∈ R : – 1 < x < 1} defined by f(x) = $\frac{x}{1+|x|}$,x ∈ R

For one-one:

Say x, y ∈ R
As per definition of |x|;

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

So f(x) = $\begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \geq 0 \end{cases}$

For x ≥ 0

f(x) = x/(1+x)

f(y) = y/(1+y)

If f(x) = f(y), then

x/(1+x) = y/(1+y)

x(1 + y) = y (1+x)
⇨ x= y

For x < 0

f(x) = x/(1-x)

f(y) = y/(1-y)

If f(x) = f(y), then

x/(1-x) = y/(1-y)

x(1 - y) = y (1- x)

⇨ x= y

In both the conditions, x = y.

Therefore, f is one-one.

Again for onto:

$$f(x) = \begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \geq 0 \end{cases}$$

For x < 0

y = f(x) = x/(1-x)

y(1-x) = x

or x(1+y) = y

or x = y/(1+y) …(1)

For x ≥ 0

y = f(x) = x /(1+x)

y(1+x) = x

or x = y/(1-y) …(2)

Now we have two different values of x from both the case.

Since $y \in \{x \in R : -1 < x < 1\}$
The value of y lies between -1 to 1.

If y = 1

x = y/(1-y) (not defined)

If y = -1

x = y/(1+y) (not defined)

So x is defined for all the values of y, and $x \in R$

This shows that, f is onto.

Answer: f is one-one and onto.

5. Show that the function f : R → R given by f(x) = x^3 is injective.

Solution:

The function f : R → R given by f(x) = x^3
Let x , y ∈ R such that f(x) = f(y)

This implies , $x^3 = y^3$

x = y
f is one-one. So f is injective.

6. Give examples of two functions f : N → Z and g : Z → Z such that g o f is injective but g is not injective.
(Hint : Consider f(x) = x and g (x) = | x |)

Solution:
Given: two functions are f : N → Z and g : Z → Z

Let us say, f(x) = x and g(x) = x

gof = (gof)(x) = f(f(x)) = g(x)

Here gof is injective but g is not.
Let us take a example to show that g is not injective: Since g(x) = |x|
g(-1) = |-1| = 1 and g(1) = |1| = 1
But -1 ≠ 1

7. Give examples of two functions f : N → Z and g : Z → Z such that g o f is injective but g is not injective.

(Hint : Consider f(x) = x +1 and g (x) = $\begin{cases} x-1 & if\ \ x>1 \\ 1 & if\ \ x=1 \end{cases}$)

Solution:

Given: Two functions f : N → Z and g : Z → Z

Say f(x) = x+ 1

And g (x) = $\begin{cases} x-1 & if\ \ x>1 \\ 1 & if\ \ x=1 \end{cases}$

Check if f is onto:

f : N →N be f(x) = x + 1

say y = x + 1

or x = y – 1

for y = 1, x = 0, does not belong to N

Therefore, f is not onto.

Find gof

For x = 1; gof = g(x + 1) = 1 (since g(x) = 1)
For x > 1 ; gof = g (x + 1) = (x + 1) – 1 = x (since g(x) = x – 1)

So we have two values for gof.

As gof is a natural number, as y = x. x is also a natural number. Hence gof is onto.

8. Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if A ⊂ B. Is R an equivalence relation on P(X)? Justify your answer.

Solution:

A ⊂ A ∴ R is reflexive.

A ⊂ B ≠ B ⊂ A ∴ R is not commutative.

If A ⊂ B, B ⊂ C, then A ⊂ C ∴ R is transitive

Therefore, R is not equivalent relation

9. Given a non-empty set X, consider the binary operation * : P(X) × P(X) → P(X) given by A * B = A ∩ B ∀ A, B in P(X), where P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation * .

Solution:

Let T be a non-empty set and P(T) be its power set. Let any two subsets A and B of T.

A ∪ B ⊂ T

So, A ∪ B ∈ P(T)

Therefore, ∪ is an binary operation on P(T).

Similarly, if A, B ∈ P(T) and A – B ∈ P(T), then the intersection of sets and difference of sets are also binary operation on P(T) and A ∩ T = A = T ∩ A for every subset A of sets

A ∩ T = A = T ∩ A for all A ∈ P(T)

T is the identity element for intersection on P(T).

10. Find the number of all onto functions from the set {1, 2, 3, ……….., n} to itself.

Solution:

Step 1: Compute the total number of one-one functions in the set {1, 2, 3}
As f is onto, every element of {1, 2, 3} will have a unique pre-image

Element	Number of possible pairings
1	3
2	2
3	1

Total number of one-one function
= 3 x 2 x 1
= 6

Step 2 - Compute the total number of onto functions in the given set
As f is onto, every element of {1, 2, 3, ... n} will have a unique pre-image

Element	Number of possible pairings
1	n
2	n - 1
3	n - 2
.	.
.	.
n - 1	2
n	1

Total number of one-one function
= n x (n - 1) x (n - 2) x .... x 2 x 1
= n!

Hence, the number of all onto functions from the set {1, 2, 3, ... n} to itself is n!.

11. Let S = {a, b, c} and T = {1, 2, 3}. Find F^{-1} of the following functions F from S to T, if it exists.

(i) F = {(a, 3), (b, 2), (c, 1)} **(ii) F = {(a, 2), (b, 1), (c, 1)}**

Solution:

(i) F = {(a, 3), (b, 2), (c, 1)}

F(a) = 3, F(b) = 2 and F(c) = 1

F^{-1} (3) = a, F^{-1} (2) = b and F^{-1} (1) = c

F^{-1} = {(3, a), (2, b), (1, c)}

(ii) F = {(a, 2), (b, 1), (c, 1)}

Since element b and c have the same image 1 i.e. (b, 1), (c, 1).

Therefore, F is not one-one function.

12. Consider the binary operations * : R × R → R and o : R × R → R defined as a * b = |a – b| and a o b = a, ∀ a, b ∈ R. Show that ∗ is commutative but not associative, o is associative but not commutative. Further, show that ∀ a, b, c ∈ R, a * (b o c) = (a * b) o (a * c). [If it is so, we say that the operation * distributes over the operation o]. Does o distribute over *? Justify your answer.

Solution:

Step 1: **Check for commutative and associative for operation** *.

a * b = |a – b| and b * a = |b – a| = (a, b)

Operation * is commutative.

a*(b*c) = a*|b-c| = |a-(b-c)| = |a-b+c| and

(a*b)*c = |a-b|*c = |a-b-c|

Therefore, a*(b*c) ≠ (a*b)*c

Operation * is associative.

Step 2: **Check for commutative and associative for operation** o.

aob = a ∀ a, b ∈ R and boa = b

This implies aob boa

Operation o is not commutative.

Again, a o (b o c) = a o b = a and (aob)oc = aoc = a
Here ao(boc) = (aob)oc

Operation o is associative.

Step 3: Check for the distributive properties

If * is distributive over o then, $a*(boc) = a*b = |a-b|$

RHS:

$(a*b)o(a*b) = (a-b)o(a-c) = |a-b|$

= LHS

And, $ao(b*c) = (aob)*(aob)$

LHS

$ao(b*c) = ao(|b-c|) = a$

$(aob)*(aob) = a*a = |a-a| = 0$

LHS ≠ RHS

Hence, operation o does not distribute over.

13. Given a non-empty set X, let * : P(X) × P(X) → P(X) be defined as A * B = (A – B) ∪ (B – A), ∀ A, B ∈ P(X). Show that the empty set ϕ is the identity for the operation * and all the elements A of P(X) are invertible with A^{-1} = A. (Hint : (A – ϕ) ∪ (ϕ – A) = A and (A – A) ∪ (A – A) = A * A = ϕ).

Solution: x ∈ P(x)

$\phi * A = (\phi - A) \cup (A - \phi) = \phi \cup A = A$

And

$A * \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$

ϕ is the identity element for the operation * on P(x).

Also A*A= $(A - A) \cup (A - A)$

$= \phi \cup \phi = \phi$

Every element A of P(X) is invertible with $A^{-1} = A$.

14. Define a binary operation * on the set {0, 1, 2, 3, 4, 5} as

a * b = $\begin{cases} a+b & if\ a+b<6 \\ a+b-6 & if\ a+b\ \geq 0 \end{cases}$

Show that zero is the identity for this operation and each element a ≠ 0 of the set is invertible with 6 – a being the inverse of a.

Solution:
Let x = {0, 1, 2, 3, 4, 5} and operation * is defined as

a * b = $\begin{cases} a+b & if\ a+b<6 \\ a+b-6 & if\ a+b\ \geq 0 \end{cases}$

Let us say, $e \in X$ is the identity for the operation *, if a*e = a = e*a $\forall a \in X$

$$\begin{cases} a+b=0=b+a, & \text{if } a+b<6 \\ a+b-6=0=b+a-6, & \text{if } a+b \geq 6 \end{cases}$$

That is a = -b or b = 6 – a, which shows a ≠ - b

Since x = {0, 1, 2, 3, 4, 5} and $a, b \in X$

Inverse of an element a $\in$ x, a ≠ 0, and a^{-1} = 6 – a.

15. Let A = {– 1, 0, 1, 2}, B = {– 4, – 2, 0, 2} and f, g : A → B be functions defined by f(x) = x^2 – x, x ∈ A and g(x) = 2|x – ½| - 1, x ∈ A. Are f and g equal?

Justify your answer. (Hint: One may note that two functions f : A → B and g : A → B such that f(a) = g (a) ∀ a ∈ A, are called equal functions).

Solution:
Given functions are: f(x) = x^2 – x and g(x) = 2|x – ½| - 1

At x = -1
f(-1) = 1^2 + 1 = 2 and g(-1) = 2|-1 – ½| - 1 = 2

At x = 0
F(0) = 0 and g(0) = 0
At x = 1
F(1) = 0 and g(1) = 0

At x = 2
F(2) = 2 and g(2) = 2

So we can see that, for each a ∈ A , f(a) = g(a)

This implies f and g are equal functions.

16. Let A = {1, 2, 3}. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

Solution:

Option (A) is correct.

As 1 is reflexive and symmetric but not transitive.

17. Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

Solution:
Option (B) is correct.

18. Let f : R → R be the Signum Function defined as

$$f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

and g : R → R be the Greatest Integer Function given by g (x) = [x], where [x] is greatest integer less than or equal to x. Then, does fog and gof coincide in (0, 1]?

Solution:
Given:
f : R → R be the Signum Function defined as

$$f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

and g : R → R be the Greatest Integer Function given by g (x) = [x], where [x] is greatest integer less than or equal to x.

Now, let say x ∈ (0, 1], then

[x] = 1 if x =1 and
[x] = 0 if 0< x < 1

Therefore:

$$fog(x) = f(g(x)) = f([x])$$

$$= \begin{cases} f(1), & \text{if } x = 1 \\ f(0), & \text{if } x \in (0,1) \end{cases}$$

$$= \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in (0,1) \end{cases}$$

Gof(x) = g(f(x)) = g(1) = [1] = 1
For x > 0

When x ∈ (0, 1), then fog = 0 and gof = 1
But fog (1) ≠ gof (1)

This shows that, fog and gof do not concide in 90, 1].

19. Number of binary operations on the set {a, b} are

(A) 10 (B) 16 (C) 20 (D) 8

Solution:
Option (B) is correct.

A = {a, b} and

A x A = {(a,a), (a,b),(b,b),(b,a)}

Number of elements = 4

So, number of subsets = 2^4 = 16.