JEE MAIN – 2021

26th Feb Shift – II

Section A

Multiple Choice Question:

1. Let L be a line obtained from the intersection of two planes $x + 2y + z = 6$ and $y + 2z = 4$. If point $P(\alpha, \beta, \gamma)$ is the foot of perpendicular from $(3, 2, 1)$ on L, then the value of $21(\alpha + \beta + \gamma)$ equals:

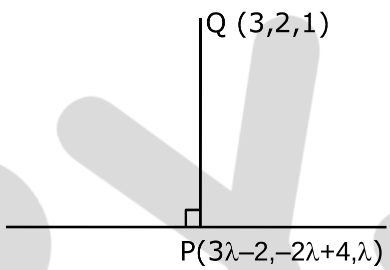
(1) 142 (2) 68

(3) 136 (4) 102

Ans. **(4)**

Sol. Equation of the line is $x + 2y + z - 6 = 0 = y + 2z = 4$

or $\frac{x+2}{3} = \frac{y-4}{-2} = \frac{z}{1} = \lambda$



Dr's of PQ : $3\lambda - 5, -2\lambda - 2, \lambda - 1$

Dr's of y lines are $(3, -2, 1)$

Since $PQ \perp$ line

$3(3\lambda - 5) - 2(-2\lambda + 2) + 1(\lambda - 1) = 0$

$\lambda = \frac{10}{7}$

$P\left(\frac{16}{7}, \frac{8}{7}, \frac{10}{7}\right)$

$21(\alpha + \beta + \gamma) = 21\left(\frac{34}{7}\right) = 102$

2. The sum of the series $\sum_{n=1}^{\infty} \frac{n^2+6n+10}{(2n+1)!}$ is equal to:

(1) $\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$ (2) $-\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$

(3) $\frac{41}{8}e - \frac{19}{8}e^{-1} - 10$ (4) $\frac{41}{8}e + \frac{19}{8}e^{-1} + 10$

Ans. **(3)**

Sol. $\sum_{n=1}^{\infty} \frac{n^2+6n+10}{(2n+1)!}$

Put $2n + 1 = r$, where $r = 3,5,7,\ldots$

$\Rightarrow n = \frac{r-1}{2}$

$\therefore \frac{n^2+6n+10}{(2n+1)!} = \frac{\left(\frac{r-1}{2}\right)^2+3r-3+10}{r!} = \frac{r^2+10r+29}{4\,(r!)}$

Now $\sum_{r=3,5,7\ldots} \frac{r(r-1)+11r+29}{4(r!)} = \frac{1}{4}\sum_{r=3,5,7\ldots}\left(\frac{1}{(r-2)!} + \frac{11}{(r-1)!} + \frac{29}{r!}\right)$

$= \frac{1}{4}\left\{\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots..\right) + 11\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots.\right) + 29\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\ldots..\right)\right\}$

$= \frac{1}{4}\left\{\frac{e-\frac{1}{e}}{2} + 11\left(\frac{e+\frac{1}{e}-2}{2}\right) + 29\left(\frac{e-\frac{1}{e}-2}{2}\right)\right\}$

$= \frac{1}{8}\left\{e - \frac{1}{e} + 11e + \frac{11}{e} - 22 + 29e - \frac{29}{e} - 58\right\}$

$= \frac{1}{8}\left\{41e - \frac{19}{e} - 80\right\}$

3. Let $f(x)$ be a differentiable function at $x = a$ with $f'(a) = 2$ and $f(a) = 4$.

Then $\lim_{x\to a} \frac{xf(a)-af(x)}{x-a}$ equals:

(1) $2a + 4$ (2) $2a - 4$

(3) $4 - 2a$ (4) $a + 4$

Ans. **(3)**

Sol. By L-H rule

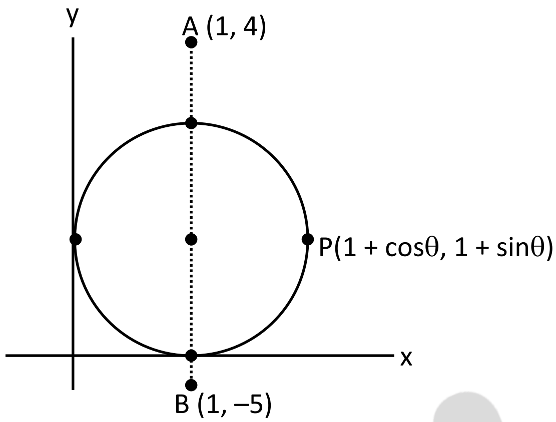
$L = \lim_{x\to a} \frac{f(a)-af'(x)}{1}$

$\therefore L = 4 - 2a$

4. Let $A\ (1, 4)$ and $B(1, -5)$ be two points. Let P be a point on the circle $(x - 1)^2 + (y - 1)^2 = 1$ such that $(PA)^2 + (PB)^2$ have maximum value, then the points P, A and B lie on:
(1) a parabola
(2) a straight line
(3) a hyperbola
(4) an ellipse

Ans. (2)

Sol.



$\therefore PA^2 = \cos^2\theta + (\sin\theta - 3)^2 = 10 - 6\sin\theta$

$PB^2 = \cos^2\theta + (\sin\theta + 6)^2 = 37 + 12\sin\theta$

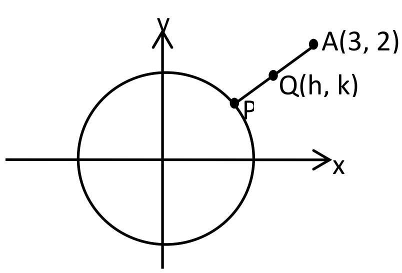
$PA^2 + PB^2|_{\max} = 47 + 6\sin\theta\,|_{\max} \Rightarrow \theta = \frac{\pi}{2}$

$\therefore P, A, B$ lie on a line $x = 1$

5. If the locus of the mid-point of the line segment from the point $(3, 2)$ to a point on the circle, $x^2 + y^2 = 1$ is a circle of the radius r, then r is equal to :
(1) $\frac{1}{4}$
(2) $\frac{1}{2}$
(3) 1
(4) $\frac{1}{3}$

Ans. (2)

Sol.



$\therefore P \equiv (2h - 3, 2k - 2) \rightarrow$on circle

$\therefore \left(h - \frac{3}{2}\right)^2 + (k - 1)^2 = \frac{1}{4}$

$\Rightarrow$ radius $= \frac{1}{2}$

6. Let slope of the tangent line to a curve at any point $P(x, y)$ be given by $\frac{xy^2+y}{x}$. If the curve intersects the line $x + 2y = 4$ at $x = -2$, then the value of y, for which the point $(3, y)$ lies on the curve, is:

(1) $-\frac{18}{11}$

(2) $-\frac{18}{19}$

(3) $-\frac{4}{3}$

(4) $\frac{18}{35}$

Ans. **(2)**

Sol. $\frac{dy}{dx} = \frac{xy^2+y}{x}$

$\Rightarrow \frac{xdy-ydx}{y^2} = xdx$

$\Rightarrow -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$

$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} + C$

Curve intersect the line $x + 2y = 4$ at $x = -2$

So, $-2 + 2y = 4 \Rightarrow y = 3$

So the curve passes through $(-2, 3)$

$\Rightarrow \frac{2}{3} = 2 + C$

$\Rightarrow C = \frac{-4}{3}$

$\therefore$ curve is $\frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$

It also passes through $(3, y)$

$\frac{-3}{y} = \frac{9}{2} - \frac{4}{3}$

$\Rightarrow \frac{-3}{y} = \frac{19}{6}$

$\Rightarrow y = -\frac{18}{19}$

7. Let A_1 be the area of the region bounded by the curves $y = \sin x$, $y = \cos x$ and y-axis in the first quadrant. Also, let A_2 be the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, x-axis and $x = \frac{\pi}{2}$ in the first quadrant. Then,

(1) $A_1 = A2$ and $A_1 + A_2 = \sqrt{2}$

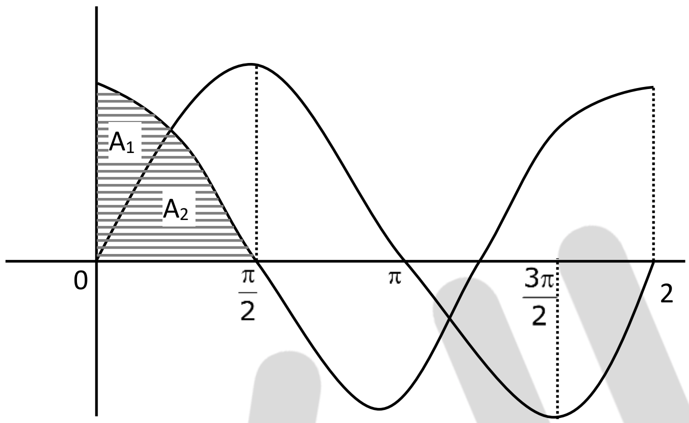
(2) $A_1 : A_2 = 1 : 2$ and $A_1 + A_2 = 1$

(3) $2A_1 = A_2$ and $A_1 + A_2 = 1 + \sqrt{2}$

(4) $A_1 : A_2 = 1 : \sqrt{2}$ and $A_1 + A_2 = 1$

Ans. **(4)**

Sol. $A_1 + A_2 = \int_0^{\pi/2} \cos x \, . dx = \sin x|_0^{\pi/2} = 1$



$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\, dx = (\sin x + \cos x)|_0^{\pi/4} = \sqrt{2} - 1$

$\therefore A_2 = 1 - \left(\sqrt{2} - 1\right) = 2 - \sqrt{2}$

$\therefore \frac{A_1}{A_2} = \frac{\sqrt{2}-1}{\sqrt{2}(\sqrt{2}-1)} = \frac{1}{\sqrt{2}}$

8. If $0 < a, b < 1$, and $\tan^{-1} a + \tan^{-1} b = \frac{\pi}{4}$, then the value of $(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + \cdots$ is :

(1) $\log_e 2$

(2) $\log_e\left(\frac{e}{2}\right)$

(3) e

(4) $e^2 - 1$

Ans. **(1)**

Sol. $\tan^{-1}\left(\frac{a+b}{1-ab}\right) = \frac{\pi}{4} \Rightarrow a + b = 1 - ab \Rightarrow (1 + a)(1 + b) = 2$

Now, $(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) + \ldots . \infty$

$= \left(a - \frac{a^2}{2} + \frac{a^3}{3} \ldots .\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} \ldots .\right)$

$= \log_e(1+a) + \log_e(1+b) = \log_e(1+a)(1+b) = \log_e 2$

9. Let $F_1(A, B, C) = (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$ and $F_2(A, B) = (A \vee B) \vee (B \rightarrow \sim A)$ be two logical expressions. Then:

(1) F_1 is not a tautology but F_2 is a tautology

(2) F_1 is a tautology but F_2 is not a tautology

(3) F_1 and F_2 both area tautologies

(4) Both F_1 and F_2 are not tautologies

Ans. **(1)**

Sol. $F_1(A, B, C) = (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$

Using the set theory

$(A \cap B') \cup \left(C' \cap (A \cup B)\right) \cup A' = (A - A \cap B) \cup (S - A) \cup [(S - C) \cap (A \cup B)]$

$= (S - A \cap B) \cup [A \cup B - C \cap (A \cup B)] = S - A \cap B \cap C$. Hence not a tautology.

Now $F_2(A, B) = (A \vee B) \vee (B \rightarrow \sim A) = (A \vee B) \vee (\sim B \vee A)$

Using set theory $(A \cup B) \cup (B' \cup A) = (A \cup B) \cup (S - A \cup B) = S$

Hence it is a tautology.

10. Consider the following system of equations:

$x + 2y - 3z = a$

$2x + 6y - 11z = b$

$x - 2y + 7z = c,$

Where a, b and c are real constants. Then the system of equations:

(1) has a unique solution when $5a = 2b + c$

(2) has infinite number of solutions when $5a = 2b + c$

(3) has no solution for all a, b and c

(4) has a unique solution for all a, b and c

Ans. **(2)**

Sol. $\Delta = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix}$

$= 20 - 2(25) - 3(-10)$

$= 20 - 50 + 30 = 0$

$\Delta_1 = \begin{vmatrix} a & 2 & -3 \\ b & 6 & -11 \\ c & -2 & 7 \end{vmatrix}$

$= 20a - 2(7b + 11c) - 3(-2b - 6c)$

$= 20a - 14b - 22c + 6b + 18c$

$= 20a - 8b - 4c$

$= 4(5a - 2b - c)$

$$\Delta_2 = \begin{vmatrix} 1 & a & -3 \\ 2 & b & -11 \\ 1 & c & 7 \end{vmatrix}$$

$= 7b + 11c - a(25) - 3(2c - b)$

$= 7b + 11c - 25a - 6c + 3b$

$= - 25a + 10b + 5c$

$= -5(5a - 2b - c)$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & a \\ 2 & 6 & b \\ 1 & -2 & c \end{vmatrix}$$

$= 6c + 2b - 2(2c - b) - 10a$

$= - 10a + 4b + 2c$

$= -2(5a - 2b - c)$

for infinite solution

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow 5a = 2b + c$

11. A seven-digit number is formed using digit $3, 3, 4, 4, 4, 5, 5$. The probability, that number so formed is divisible by 2, is:

(1) $\frac{6}{7}$
(2) $\frac{4}{7}$
(3) $\frac{3}{7}$
(4) $\frac{1}{7}$

Ans. **(3)**

Sol. $n(s) = \frac{7!}{2!3!2!}$

$n(E) = \frac{6!}{2!2!2!}$

$P(E) = \frac{n(E)}{n(S)} = \frac{6!}{7!} \times \frac{2!3!2!}{2!2!2!} = \frac{3}{7}$

12. If vectors $\vec{a_1} = x\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{a_2} = \hat{\imath} + y\hat{\jmath} + z\hat{k}$ are collinear, then a possible unit vector parallel to the vector $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is :

(1) $\frac{1}{\sqrt{2}}(-\hat{\jmath} + \hat{k})$
(2) $\frac{1}{\sqrt{2}}(\hat{\imath} - \hat{\jmath})$
(3) $\frac{1}{\sqrt{3}}(\hat{\imath} - \hat{\jmath} + \hat{k})$
(4) $\frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} - \hat{k})$

Ans. **(3)**

Sol. $\frac{x}{1} = -\frac{1}{y} = \frac{1}{z} = \lambda$ (let)

Unit vector parallel to $x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \pm \frac{\left(\lambda\hat{\imath} - \frac{1}{\lambda}\hat{\jmath} + \frac{1}{\lambda}\hat{k}\right)}{\sqrt{\lambda^2 + \frac{2}{\lambda^2}}}$

For $\lambda = 1$, it is $\pm \frac{(\hat{\imath} - \hat{\jmath} + \hat{k})}{\sqrt{3}}$

13. For $x > 0$, if $f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$, then $f(e) + f\left(\frac{1}{e}\right)$ is equal to:

(1) $\frac{1}{2}$ (2) -1

(3) 1 (4) 0

Ans. (1)

Sol. $f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ln t}{1+t} dt + \int_1^{1/e} \frac{\ln t}{1+t} dt = I_1 + I_2$

$I_2 = \int_1^{1/e} \frac{\ln t}{1+t} dt$ put $t = \frac{1}{z}, dt = -\frac{dz}{z^2}$

$= \int_1^e -\frac{\ln z}{1+\frac{1}{z}} \times \left(-\frac{dz}{z^2}\right) = \int_1^e \frac{\ln z}{z(z+1)} dz$

$f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ln t}{1+t} dt + \int_1^e \frac{\ln t}{t(t+1)} dt = \int_1^e \frac{\ln t}{1+t} + \frac{\ln t}{t(t+1)} dt$

$= \int_1^e \frac{\ln t}{t} dt$ $\left\{\ln t = u, \ \frac{1}{t} dt = du\right\}$

$= \int_0^1 u \, du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

14. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right), & if \ x < -1 \\ |ax^2 + x + b|, & if \ -1 \le x \le 1 \\ \sin(\pi x) & if \ x > 1 \end{cases}$. If $f(x)$ is continuous on $\mathbb{R}$,

then $a + b$ equals:

(1) 3 (2) -1

(3) -3 (4) 1

Ans. (2)

Sol. If f is continuous at $x = -1$, then

$f(-1^-) = f(-1)$

$\Rightarrow 2 = |a - 1 + b|$

$\Rightarrow |a + b - 1| = 2 \ldots\ldots (i)$

similarly

$f(1^-) = f(1)$

$\Rightarrow |a + b + 1| = 0$

$\Rightarrow a + b = -1$

15. Let $A = \{1,2,3 \cdots\cdots,10\}$ and $f: A \to A$ be defined as $f(k) = \begin{cases} k+1 & \text{if } k \text{ is odd} \\ k & \text{if } k \text{ is even} \end{cases}$. Then the number of possible functions $g : A \to A$ such that $gof = f$ is:

(1) 10^5

(2) ${}^{10}C_5$

(3) 5^5

(4) 5!

Ans. **(1)**

Sol. $g(f(x)) = f(x)$

$\Rightarrow g(x) = x$, when x is even.

5 elements in A can be mapped to any 10

So, $10^5 \times 1 = 10^5$

16. A natural number has prime factorization given by $n = 2^x 3^y 5^z$, where y and z are such that $y + z = 5$ and $y^{-1} + z^{-1} = \frac{5}{6}, y > z$. Then the number of odd divisors of n, including 1, is:

(1) 11

(2) $6x$

(3) 12

(4) 6

Ans. **(3)**

Sol. $y + z = 5 \ldots (1)$

$\frac{1}{y} + \frac{1}{z} = \frac{5}{6}$

$\Rightarrow yz = 6$

$\Rightarrow yz = 6$

Also $(y - z)^2 = (y + z)^2 - 4yz$

$\Rightarrow y - z = \pm 1 \quad \cdots (2)$

from (1) and (2), $y = 3$ or 2 and $z = 2$ or 3

for calculating odd divisor of $p = 2^x . 3^y . 5^z$

x must be zero

$P = 2^0 . 3^3 . 5^2$

$\therefore$ total odd divisors must be $(3 + 1)(2 + 1) = 12$

17. Let $f(x) = \sin^{-1} x$ and $g(x) = \frac{x^2-x-2}{2x^2-x-6}$. If $g(2) = \lim_{x\to 2} g(x)$, then the domain of the function fog is :

(1) $(-\infty,-2] \cup \left[-\frac{4}{3},\infty\right)$

(2) $(-\infty,-1] \cup [2,\infty)$

(3) $(-\infty,-2] \cup [-1,\infty)$

(4) $(-\infty,2] \cup \left[-\frac{3}{2},\infty\right)$

Ans. (1)

Sol. $g(2) = \lim_{x\to 2} \frac{(x-2)(x+1)}{(2x+3)(x-2)} = \frac{3}{7}$

For domain of $f\big(g(x)\big)$

$\left|\frac{x^2-x-2}{2x^2-x-6}\right| \le 1$ ($\because$ domain of $f(x)$ is $[-1,1]$)

$\Rightarrow (x+1)^2 \le (2x+3)^2 \Rightarrow 3x^2+10x+8 \ge 0$

$\Rightarrow (3x+4)(x+2) \ge 0$

$x \in (-\infty,-2] \cup \left[-\frac{4}{3},\infty\right]$

18. If the mirror image of the point $(1,3,5)$ with respect to the plane $4x-5y+2z = 8$ is (α,β,γ), then $5(\alpha+\beta+\gamma)$ equals:

(1) 47

(2) 39

(3) 43

(4) 41

Ans. (1)

Sol. Image of $(1,3,5)$ in the plane $4x-5y+2z=8$ is (α, β,γ)

$\Rightarrow \frac{\alpha-1}{4} = \frac{\beta-3}{-5} = \frac{\gamma-5}{2} = -2\left[\frac{4(1)-5(3)+2(5)-8}{4^2+5^2+2^2}\right] = \frac{2}{5}$

$\therefore \alpha = 1+4\left(\frac{2}{5}\right) = \frac{13}{5}$

$\beta = 3-5\left(\frac{2}{5}\right) = 1 = \frac{5}{5}$

$\gamma = 5+2\left(\frac{2}{5}\right) = \frac{29}{5}$

Thus, $5(\alpha+\beta+\gamma) = 5\left(\frac{13}{5}+\frac{5}{5}+\frac{29}{5}\right) = 47$

19. Let $f(x) = \int_0^x e^t f(t)dt + e^x$ be a differentiable function for all $x \in \mathbb{R}$. Then $f(x)$ equals.

(1) $2e^{(e^x-1)} - 1$ (2) $e^{(e^x-1)}$

(3) $2e^{e^x} - 1$ (4) $e^{e^x} - 1$

Ans. (1)

Sol. Given, $f(x) = \int_0^x e^t f(t)dt + e^x \cdots (1)$

Differentiating both sides w.r.t x

$f'(x) = e^x \cdot f(x) + e^x$ (Using Newton Leibnitz Theorem)

$\Rightarrow \frac{f'(x)}{f(x)+1} = e^x$

Integrating w.r.t x

$\int \frac{f'(x)}{f(x)+1} dx = \int e^x dx$

$\Rightarrow \ln(f(x)+1) = e^x + c$

Put $x = 0$

$\ln 2 = 1 + c$ (Q $f(0) = 1$, from equation (1))

$\therefore \ln(f(x)+1) = e^x + \ln 2 - 1$

$\Rightarrow f(x) + 1 = 2. e^{e^x - 1}$

$\Rightarrow f(x) = 2e^{e^x-1} - 1$

20. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is:

(1) A right-angle triangle having two of its sides of length $2r$ and r.

(2) An equilateral triangle of height $\frac{2r}{3}$.

(3) isosceles triangle with base equal to $2r$.

(4) An equilateral triangle having each of its side of length $\sqrt{3}\, r$.

Ans. (4)

Sol. Triangle of maximum area that can be inscribed in a circle is an equilateral triangle.

Let ΔABC be inscribed in circle,

Now, in $\Delta\, OBD$

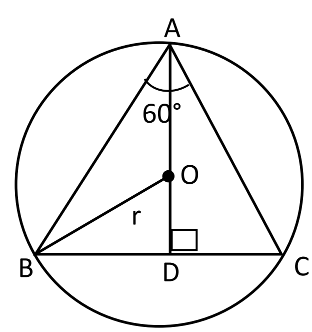
$OD = r \cos 60° = \frac{r}{2}$

Height $= AD = \frac{3r}{2}$

Again in ΔABD

Now $\sin 60° = \frac{3r/2}{AB}$

$\Rightarrow AB = \sqrt{3}r$



Section – B

Integer Type

1. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is _______

Ans. 1000

Sol. Since, required number has G.C.D with 18 as 3. It must be odd multiple of '3' but not a multiple of '9'.

(i) Now, 4-digit number which are odd multiple of '3' are,

$1005, 1011, 1017, \cdots, 9999 \rightarrow 1499$

(ii) 4–digit number which are odd multiple of 9 are,

$1017, 1035, \cdots, 9999 \rightarrow 499$

$\therefore$ Required numbers $= 1499 - 499 = 1000$

2. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $P_n = (\alpha)^n + (b)^n$, $P_{n-1} = 11$ and $P_{n+1} = 29$ for some integer n ³ 1. Then, the value of P_n^2 is____________.

Ans. 324

Sol. Given, $\alpha + \beta = 1, ab = -1$

$\therefore$ Quadratic equation with roots α, β is $x^2 - x - 1 = 0$

$\Rightarrow \alpha^2 = \alpha + 1$

Multiplying both sides by α^{n-1}

$\alpha^{n+1} = \alpha^n + \alpha^{n-1} \qquad \cdots (1)$

Similarly,

$\beta^{n+1} = b^n + b^{n-1} \qquad \cdots (2)$

Adding (1) & (2)

$\alpha^{n+1} + \beta^{n+1} = (\alpha^n + \beta^n) + (\alpha^{n-1} + \beta^{n-1})$

$\Rightarrow P_{n+1} = P_n + P_{n-1}$

$\Rightarrow 29 = P_n + 11$

$\Rightarrow P_n = 18$

$\therefore P_n^2 = 18^2 = 324$

3. Let $X_1, X_2, \cdots X_{18}$ be eighteen observation such that $\sum_{i=1}^{18}(X_i - \alpha) = 36$ and $\sum_{i=1}^{18}(X_i - \beta)^2 = 90$, where α and β are distinct real numbers. If the standard deviation of these observations is 1, then the value of $|\alpha - \beta|$ is____________.

Ans. **4**

Sol. Given, $\sum_{i=1}^{18}(X_i - \alpha) = 36$

$\Rightarrow \sum x_i - 18\alpha = 36$

$\Rightarrow \sum x_i = 18(\alpha + 2) \quad \cdots (1)$

Also, $\sum_{i=1}^{18}(X_i - \beta)^2 = 90$

$\Rightarrow \sum x_i^2 + 18\beta^2 - 2\beta \sum x_i = 90$

$\Rightarrow \sum x_i^2 + 18\beta^2 - 2\beta \times 18(\alpha + 2) = 90$ (using equation (1))

$\Rightarrow \sum x_i^2 = 90 - 18\beta^2 + 36\beta(\alpha + 2)$

Now

$\sigma^2 = 1 \Rightarrow \frac{1}{18}\sum x_i^2 - \left(\frac{\sum x_i}{18}\right)^2 = 1$ ($\because \sigma = 1$, given)

$\Rightarrow \frac{1}{18}(90 - 18\beta^2 + 36\alpha\beta + 72\beta) - \left(\frac{18(\alpha+2)}{18}\right)^2 = 1$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - (\alpha + 2)^2 = 1$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - \alpha^2 - 4 - 4\alpha = 1$

$\Rightarrow \alpha^2 - \beta^2 + 2\alpha\beta + 4\beta - 4\alpha = 0$

$\Rightarrow (\alpha - \beta)(\alpha - \beta + 4) = 0$

$\Rightarrow \alpha - \beta = -4$

Hence

$\therefore |\alpha - \beta| = 4 \ (\alpha \neq \beta)$

4. In $I_{m,n}=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for m, n $\geq$ 1 and $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx=\alpha I_{m,n}$, $\alpha \in \mathbb{R}$, then α equals____________

Ans. 1

Sol. $I_{m,n}=\int_0^1 x^{m-1}.(1-x)^{n-1}dx$

Put $x=\frac{1}{y+1} \Rightarrow dx=\frac{-1}{(y+1)^2}dy$

$1-x=\frac{y}{y+1}$

$\therefore I_{m,n}=\int_{\infty}^{0}\frac{y^{n-1}}{(y+1)^{m+n}}(-1)dy$

$=\int_0^{\infty}\frac{y^{n-1}}{(y+1)^{m+n}}dy \qquad \cdots(1)$

Similarly,

$I_{m,n}=\int_0^1 x^{n-1}.(1-x)^{m-1}dx$

$\Rightarrow I_{m,n}=\int_0^{\infty}\frac{y^{m-1}}{(y+1)^{m+n}}dy \qquad \cdots(2)$

From (1) & (2)

$2I_{m,n}=\int_0^{\infty}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}d\,y$

$\Rightarrow 2I_{m,n}=\int_0^{1}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy+\int_1^{\infty}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy$

I₁ I₂

Put $y=\frac{1}{z}$ in I_2

$dy=-\frac{1}{z^2}dz$

$\Rightarrow 2I_{m,n}=\int_0^{1}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}d\,y+\int_1^{0}\frac{z^{m-1}+z^{n-1}}{(z+1)^{m+n}}(-dz)$

$\Rightarrow I_{m,n}=\int_0^{1}\frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy \Rightarrow \alpha=1$

5. Let L be a common tangent line to the curves $4x^2 + 9y^2 = 36$ and $(2x)^2 + (2y)^2 = 31$. Then the square of the slope of the line L is ______________.

Ans. **3**

Sol. $E: \frac{x^2}{9} + \frac{y^2}{4} = 1 \qquad C: x^2 + y^2 = \frac{31}{4}$

equation of tangent to ellipse is

$y = mx \pm \sqrt{9m^2 + 4} \qquad \cdots (1)$

equation of tangent to circle is

$y = mx \pm \sqrt{\frac{31}{4}m^2 + \frac{31}{4}} \qquad \cdots (2)$

Comparing equation (1) & (2)

$9m^2 + 4 = \frac{31}{4}m^2 + \frac{31}{4}$

$\Rightarrow 36m^2 + 16 = 31m^2 + 31$

$\Rightarrow 5m^2 = 15 \Rightarrow m^2 = 3$

6. If the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ satisfies the equation $A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for some real numbers α and β, then $\beta - \alpha$ is equal to______________.

Ans. **4**

Sol. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\vdots \qquad \vdots \qquad \vdots$

$A^{19} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{19} & 0 \\ 3 & 0 & -1 \end{bmatrix}, A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

L. H. S$= A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 + \alpha + \beta & 0 & 0 \\ 0 & 2^{20} + \alpha 2^{19} + 2\beta & 0 \\ 3\alpha + 3\beta & 0 & 1 - \alpha - \beta \end{bmatrix}$

R.H.S $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \alpha + \beta = 0$ and $2^{20} + \alpha 2^{19} + 2\beta = 4$

$\Rightarrow 2^{20} + \alpha(2^{19} - 2) = 4$

$\Rightarrow \alpha = \frac{4 - 2^{20}}{2^{19} - 2} = -2 \quad \Rightarrow \beta = 2$

$\therefore \beta - \alpha = 4$

7. If the arithmetic mean and geometric mean of the p^{th} and q^{th} terms of the sequence $-16, 8, -4, 2, \cdots$ satisfy the equation $4x^2 - 9x + 5 = 0$, then $p + q$ is equal to ____________.

Ans. 10

Sol. Given, $4x^2 - 9x + 5 = 0$

$\Rightarrow (x - 1)(4x - 5) = 0$

$\Rightarrow$ A.M $= \frac{5}{4}$, G.M $= 1$ (A.M $\geq$ G.M)

Again, for the series

$-16, 8, -4, 2 \cdots$

p^{th} term $t_p = -16\left(\frac{-1}{2}\right)^{p-1}$

q^{th} term $t_p = -16\left(\frac{-1}{2}\right)^{q-1}$

Now, A.M $= \frac{t_p + t_q}{2} = \frac{5}{4}$

& G.M$= \sqrt{t_p t_q} = 1$

$\Rightarrow 16^2\left(-\frac{1}{2}\right)^{p+q-2} = 1$

$\Rightarrow (-2)^8 = (-2)^{p+q-2}$

$\Rightarrow p + q = 10$

8. Let the normals at all the points on a given curve pass through a fixed point (a, b). If the curve passes through $(3, -3)$ and $\left(4, -2\sqrt{2}\right)$, and given that $a - 2\sqrt{2}b = 3$, then $(a^2 + b^2 + ab)$ is equal to________________.

Ans. 9

Sol. Let the equation of normal is $Y - y = -\frac{1}{m}(X - x)$, where $m = \frac{dy}{dx}$

As it passes through (a, b)

$b - y = -\frac{1}{m}(a - x) = -\frac{dx}{dy}(\mathrm{a} - \mathrm{x})$

$\Rightarrow (b - y)dy = (x - a)dx$

$\Rightarrow by - \frac{y^2}{2} = \frac{x^2}{2} - ax + c \quad \cdots (i)$

It passes through $(3, -3)$ & $(4, -2\sqrt{2})$

$\therefore \quad -3b - \frac{9}{2} = \frac{9}{2} - 3a + c$

$\Rightarrow 3a - 3b - c = 9 \quad \ldots (ii)$

Also

$-2\sqrt{2}b-4=8-4a+c$

$\Rightarrow 4a\ -\ 2\sqrt{2}b-\ c\ =\ 12\quad \ldots(iii)$

Also $a-\ 2\sqrt{2}\,b\ =\ 3\ldots(iv)$(given)

From $(ii)-\ (iii)$

$\Rightarrow -a+\left(2\sqrt{2}-3\right)b=-3\quad \cdots(v)$

From $(iv\ )+(v)$

$\Rightarrow b=0,\quad a=3$

$\therefore a^2+b^2+ab=9$

9. Let z be those complex number which satisfy $|z+5|\le 4$ and $z(1+i)+\bar{z}(1-i)\ge -10$, $i=\sqrt{-1}$. If the maximum value of $|z+1|^2$ is $\alpha+\beta\sqrt{2}$, then the value of $(\alpha+\beta)$ is___________.

Ans. **48**

Sol. Given, $|z\ +\ 5|\le 4$

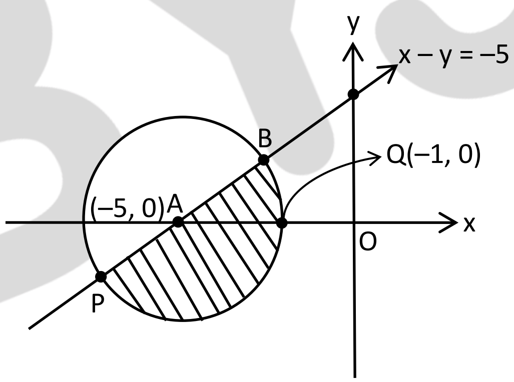
$\Rightarrow (x+5)^2+y^2\le 16$

Also, $z(1+i)+\ \bar{z}(1-i)\ge -10.$

$\Rightarrow x-\ y\ge -5\ldots(2)$

From (1) and (2)

Locus of z is the shaded region in the diagram.



$|z\ +\ 1|$ represents distance of 'z' from $Q(-1,0)$

Clearly 'p' is the required position of 'z' when $|z\ +\ 1|$ is maximum.

$\therefore P\equiv\left(-5-2\sqrt{2},-2\sqrt{2}\right)$

$\therefore PQ^2=32+16\sqrt{2}$

$\Rightarrow \alpha=\ 32$

$\Rightarrow \beta=\ 16$

Thus, $\alpha+\beta=\ 48$

10. Let a be an integer such that all the real roots of the polynomial $2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$ lie in the interval $(a, a+1)$. Then, $|a|$ is equal to______.

Ans. **2**

Sol. Let, $f(x) = 2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$

$\Rightarrow f'(x) = 10\,(x^4 + 2x^3 + 3x^2 + 2x + 1)$

$= 10x^2\left(x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) + 3\right)$

$= 10x^2\left(\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) + 1\right)$

$= 10x^2\left(\left(x + \frac{1}{x}\right) + 1\right)^2 > 0; \forall x \in \mathbb{R}$

$\therefore$ $f(x)$ is strictly increasing function. Since it is an odd degree polynomial it will have exactly one real root.

Now, by observation

$f(-1) = 3 > 0$

$f(-2) = -64 + 80 - 80 + 40 - 20 + 10$

$= -34 < 0$

$\Rightarrow f(x)$ has at least one root in $(-2, -1) \equiv (a, a+1)$

$\therefore$ $|a| = 2$