Regd. Office: Aakash Tower, 8, Pusa Road, New Delhi-110005, Ph.011-47623456

ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
AMTI – NMTC - 2023 Jan. – JUNIOR – FINAL

Instructions:

1. Answer all questions. Each question carries 10 marks.
2. Elegant and innovative solutions will get extra marks.
3. Diagrams and justification should be given wherever necessary.
4. Before answering, fill in the FACE SLIP completely.
5. Your ‘rough work’ should be in the answer sheet itself.
6. The maximum time allowed is THREE hours.

1. $x,\ y,\ z$ are positive reals and $(x + y + z)^3 = 32\,xyz$. Find the numerical limits between which the expression $\frac{x^4+y^4+z^4}{(x+y+z)^4}$ lies?

Sol. $\frac{x^4+y^4+z^4}{(x+y+z)^4} = \frac{x^4}{(x+y+z)^4} + \frac{y^4}{(x+y+z)^4} + \frac{z^4}{(x+y+z)^4}$

$= \left(\frac{x}{x+y+z}\right)^4 + \left(\frac{y}{x+y+z}\right)^4 + \left(\frac{z}{x+y+z}\right)^4$

Now, apply $AM \geq GM$ for numbers $\left(\frac{x}{x+y+z}\right)^4, \left(\frac{y}{x+y+z}\right)^4, \left(\frac{z}{x+y+z}\right)^4$ we get

$$\frac{\left(\frac{x}{x+y+z}\right)^4+\left(\frac{y}{x+y+z}\right)^4+\left(\frac{z}{x+y+z}\right)^4}{3} \geq \sqrt[3]{\left(\frac{xyz}{(x+y+z)^3}\right)^4}$$

We know that $(x + y + z)^3 = 32\,xyz$

$$\left(\frac{x}{x+y+z}\right)^4 + \left(\frac{y}{x+y+z}\right)^4 + \left(\frac{z}{x+y+z}\right)^4 \geq 3\sqrt[3]{\left(\frac{1}{32}\right)^4}$$

$$\left(\frac{x}{x+y+z}\right)^4 + \left(\frac{y}{x+y+z}\right)^4 + \left(\frac{z}{x+y+z}\right)^4 \geq 3.\left(\frac{1}{32}\right)^{\frac{4}{3}}$$

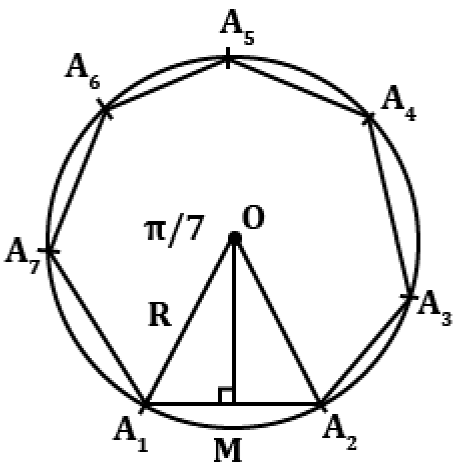
Or

$$\left(\frac{x}{x+y+z}\right)^4 + \left(\frac{y}{x+y+z}\right)^4 + \left(\frac{z}{x+y+z}\right)^4 \geq \frac{3}{(2)^{\frac{20}{3}}}$$

2. $A_1A_2A_3A_4A_5A_6A_7$ is a regular heptagon. Prove that

$$\frac{{A_1A_4}^3}{A_1A_{2^3}} - \frac{A_1A_7+2A_1A_6}{A_1A_5-A_1A_3} = 1$$

Sol. B



$A_1A_2 = A_1A_7$

$A_1A_3 = A_1A_6$

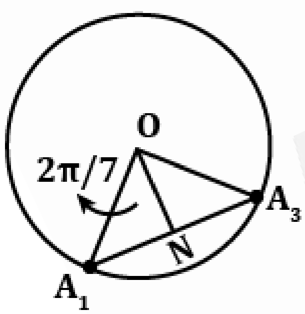
$A_1A_4 = A_1A_5$

Let radius of circumcircle $= R$

Centre $= O$

$\frac{A_1M}{R} = sin\,\frac{\pi}{7} \Rightarrow A_1M = R\,sin\frac{\pi}{7}$ $\angle A_iOA_2 = \frac{2\pi}{7}$

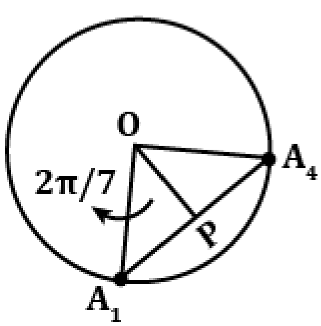
$\Rightarrow A_1A_2 = A_1A_7 = 2\,R\,sin/\pi\,\frac{\pi}{7}$ ………………(1)



$\angle A_1OA_3 = \frac{4\pi}{7}$

$A_1N = R\,sin\,\frac{2\pi}{7}$

$\Rightarrow A_1A_3 = 2d\,sin\,\frac{3\pi}{7} = A_1A_6$ ……………(2)



$\angle A_1OA_4 = \frac{4\pi}{7}$

$A_1P = R\,sin\,\frac{3\pi}{7}$

$\Rightarrow A_1A_4 = 2\,R\,sin\,\frac{3\pi}{7} = A_1A_5$ ……………(3)

Now To prove, $\frac{(A_1A_4)^3}{(A_1A_2)^3} - \frac{A_1A_1+2\,A_1A_6}{A_1A_5-A_1A_3} = 1$

To prove , $\frac{(A_1A_4)^3}{(A_1A_2)^3} = 1 + \frac{A_1A_7+2\,A_1A_6}{A_1A_5-A_1A_3}$

To prove $\left(\frac{A_1A_4}{A_1A_2}\right)^3 = 1 + \frac{A_1A_1+2\,A_1A_3}{A_1A_4-A_1A_3}$

To prove $\left(\frac{A_1A_4}{A_1A_2}\right)^3 = \frac{A_1A_2+A_1A_3+A_1A_4}{A_1A_4-A_1A_3}$

I.e., $\left(\frac{\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}\right)^3 = \frac{\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}}{\sin\frac{3\pi}{7}-\sin\frac{2\pi}{7}}$

R.H.S = $\frac{\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}}{\sin\frac{3\pi}{7}-\sin\frac{2\pi}{7}}$

$= \frac{\sin\frac{3\pi}{14}\sin\frac{2\pi}{7}}{2\sin\frac{\pi}{14}\sin\frac{2\pi}{14}\cos\frac{5\pi}{14}} = \frac{\sin\frac{3\pi}{14}\sin\frac{2\pi}{7}2\cos\frac{\pi}{14}\cos\frac{\pi}{14}}{2\sin\frac{\pi}{14}\sin\frac{\pi}{14}\sin\frac{\pi}{7}2\cos\frac{\pi}{14}\cos\frac{\pi}{14}}$

$= \frac{\sin\frac{3\pi}{14}\sin\frac{2\pi}{7}2\cos\frac{\pi}{14}\cos\frac{\pi}{14}}{\sin\frac{\pi}{7}\sin\frac{\pi}{7}\sin\frac{\pi}{7}} = \frac{2\sin\frac{3\pi}{14}\sin\frac{4\pi}{14}\cos\frac{\pi}{14}\cos\frac{\pi}{14}}{\sin\frac{\pi}{7}\sin\frac{\pi}{7}\sin\frac{\pi}{7}}$

$= \frac{\cos\frac{\pi}{14}\cos\frac{\pi}{14}\cos\frac{\pi}{14}}{\sin\frac{\pi}{7}\sin\frac{\pi}{7}\sin\frac{\pi}{7}} = \frac{\sin\frac{3\pi}{7}\sin\frac{3\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}\sin\frac{\pi}{7}\sin\frac{\pi}{7}}$ = L.H.S.

3. $ABCD$ is a square whose side is $1\,unit$. Let n be an arbitrary natural number. A figure is drawn inside the square consisting of only line segments, having a total length greater than $2n$. (This figure can have many pieces of single line segments intersecting or non-intersecting). Prove that for some straight-line L which is parallel to a side of the square must cross the figure at least $(n + 1)$ times.

Sol. Ambiguity in Question.

4. m is a natural number. If $(2m + 1)$ and $(3m + 1)$ are perfect squares, then prove that m is divisible by 40.

Sol. As 'm' is a natural number, to prove its divisibility by 40 , we prove that 'm' is divisible be 8 & 5.

Let $2m + 1 = k^2$……………..(i)

$3m + 1 = l^2$……………..(ii)

As $2m + 1$ is odd, $\therefore k^2$ is odd and thus k is odd.

So, let $k = 2n + 1 \Rightarrow k^2 = 4n^2 + 4n + 1$……………(iii)

$\Rightarrow 2m + 1 = 4n^2 + 4n + 1$

$\Rightarrow m = 2(n^2 + n)$

$\Rightarrow m$ = an even number

If 'm' is even, $3m + 1$ is odd.

$\Rightarrow l^2$ is odd & thus l is odd.

Let $l = 2p + 1 \Rightarrow l^2 = (2p + 1)^2$……..(iv)

Now, subtracting (ii) & (i),

$m = l^2 - k^2$

$= (2p + 1)^2 - (2n + 1)^2$………(v)

We know that, squares of two odd numbers are always divisible by 8

$\therefore m$ is divisible by 8 ……….(vi)
Also, from (i) & (ii)
$3k^2 - 2l^2 = 1$……….(vii)
As squares of odd numbers ends with $1, 5$ or 9
$\Rightarrow$ $3k^2$ ends with $3, 5$ or 7 & $2l^2$ ends with $2, 0, 8$
$\therefore k^2$ends with 1 & l^2 ends with 1
$\Rightarrow$ $m = l^2 - k^2$ (whose unit digit is zero)
$\therefore m$ is divisible by 5 …..(viii)
From (vii) & (viii), m is divisible by 40.

5. Given 69 distinct positive integers not exceeding 100, prove that one can choose four of them a, b, c, d such that $a < b < c$ and $a + b + c = d$. Is this statement true for 68?

Sol. $a + b + c = d \Rightarrow c + a = d - b$

Let $a_i + a_1 = f_i \ \forall \ i \in \{3, 4, ..., 69\}$

& $a_i + a_2 = g_i \ \forall \ i \in \{3, 4, ..., 69\}$

Now

$a_3 + a_1 < a_4 + a_1 < .... < a_{69} + a_1 \leq 132$

And

$1 \leq a_3 - a_2 < a_4 - a_2 < .... < a_{69} - a_2$

$\Rightarrow f_3 < f_4 < .... < f_{69} \leq 132$

& $1 \leq g_3 + g_4 < .... < g_{69}$

Now total counting of f_i & g_i are 134 which lies between 1 to 132

Because all f_i's are distinct and all g_i's are distinct, hence at least one f_i and g_i must be same

$\Rightarrow a_i + a_1 = a_j - a_2$

$\Rightarrow a_1 + a_2 + a_i = a_j$

$\Rightarrow$ First part is proved

This statement is not true for 68 e.g. let set of 68 numbers is $\{33, 34, ..., 100\}$

In this set sum of least 3 number is greater than 100 $\{because\ 33 + 34 + 35 = 102 > 100\}$

6. m, n are integers such that $n^2(m^2 + 1) + m^2(n^2 + 16) = 448$. Find all possible ordered pairs (m, n).

Sol. $m^2n^2 + n^2 + m^2n^2 + 16m^2 = 448$

$\Rightarrow 2m^2n^2 + n^2 + 16m^2 = 448$

$\Rightarrow m^2n^2 + \frac{n^2}{2} + 8m^2 = 224$

$\Rightarrow (m^2 + \frac{1}{2})(n^2 + 8) = 228$

$\Rightarrow (2m^2 + 1)(n^2 + 8) = 456$

$\Rightarrow (2m^2 + 1)(n^2 + 8) = 3 \times 152$ or 19×24

Odd even

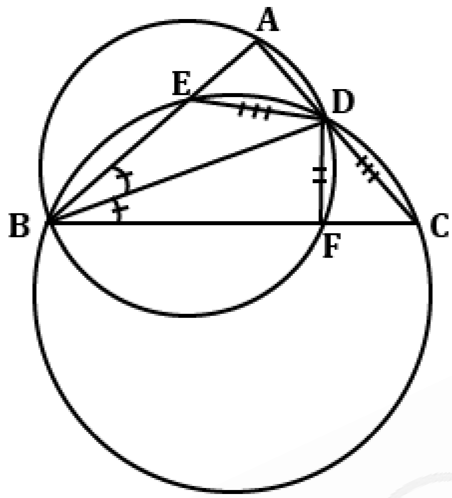
$\Rightarrow m^2 = 1, n^2 = 144$ or $m^2 = 9, n^2 = 16$

$\Rightarrow (m, n) = (1, 12), (1, -12), (-1, 12), (-1, -12)$

$(3, 4), (3, -4), (-3, 4), (-3, -4)$

7. BD is the bisector of $\angle ABC$ of triangle ABC. The circumcircles of triangle BCD and triangle ABD cut AB and BC at E and F respectively. Show that $AE = CF$.

Sol. Construction: Join DE and DF



Proof : $arc\ AD = arc\ DF$ for circumcircle of ΔABD

$\{\because \angle ABD = \angle DBF\}$

$\Rightarrow AD = DF$ ....(1)

Similarly $arc\ ED = arc\ DC$

For circumcircle of ΔBDC

$\Rightarrow ED = DC$ ....(2)

Now $\angle EDC = 180° - \angle ABC$

Similarly $\angle ADF = 180° - \angle ABC$

$\Rightarrow \angle EDC = \angle ADF$

$\Rightarrow \angle EDC - \angle EDF = \angle ADF - \angle EDF$

$\Rightarrow \angle FDC = \angle ADE$ ....(3)

Using (1),(2) & (3) $\Delta ADE \cong \Delta FDC$

$\Rightarrow AE = FC$ (using $CPCT$)

8. a is a two-digit number. b is a three-digit number. a increased by b percent is equal to b decreased by a percent. Find all possible ordered pairs $(a,\ b)$**.**

Sol. $a + \frac{ab}{100} = b - \frac{ab}{100}$

$\Rightarrow \frac{ab}{50} + a - b = 0$

$\Rightarrow ab + 50a - 50b = 0$

$\Rightarrow (a - 50)(b + 50) = -2500$

$= -4 \times 625$ or

$= -5 \times 500$ or

$= -10 \times 250$ or

$\Rightarrow (a, b) = (46, 575)$ or $(45, 450)$ or $(40, 200)$