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ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
AMTI – NMTC - 2023 Jan. – SUB-JUNIOR – FINAL

Instructions:

1. Answer all questions. Each question carries 10 marks.
2. Elegant and innovative solutions will get extra marks.
3. Diagrams and justification should be given wherever necessary.
4. Before answering, fill in the FACE SLIP completely.
5. Your 'rough work' should be in the answer sheet itself.
6. The maximum time allowed is THREE hours.

1. A tray contains of 40 toffees. Jaya and Uma take in turn some toffees from the tray. Each time they are allowed to take $1, 2$ or 3 toffees only. The person who gets the last toffee, wins. Jaya starts the game. Will she win? If so, how? If not, why?

Sol. Jaya and Uma both want to win.

So, Uma will try to pick the number of toffee in such a way, that Uma and Jaya get 4 toffee in every turn

$1 + 3$ or $2 + 2$ or $3 + 1$

So in 9 rounds

Uma and Jaya will pick $9 \times 4 = 36$ toffee

Now even Jaya pick 3 or 2 or 1, Uma will pick 1 or 2 or 3 toffee at last,

Hence, Uma will win.

2. Given $A = \left\{\dfrac{\frac{(a+b)^2+(a-b)^2}{b-a}-(a+b)}{\frac{1}{b-a}-\frac{1}{c-a}}\right\} \div \left\{\dfrac{(a+b)^3+(b-a)^3}{(a+b)^2+(a-b)^2}\right\}$

$B = \left\{\dfrac{c-b}{(a-b)(a-c)} - \dfrac{c-a}{(b-c)(b-a)} + \dfrac{b-a}{(c-a)(a-b)}\right\} \div \dfrac{2(a^2+b^2+c^2-ab-bc-ca)}{(a-b)(b-c)(c-a)}\}$

If $a = 2022, b = 2023$, find $(A + B)$.

Sol. $A = \left(\dfrac{\frac{2a^2+2b^2-(b^2-a^2)}{(b-a)}}{\frac{a+b-(b-a)}{(b-a)(b+a)}}\right) \div \{\dfrac{(a+b+b-a)[(a+b)^2+(b-a)^2+(a+b)(b-a)]}{2a^2+2b^2}]$

$A = \dfrac{3a^2+b^2}{(b-a)} \times \dfrac{(b-a)(b+a)}{2a} \times \dfrac{2(a^2+b^2)}{2b[3a^2+b^2]}$

$A = \dfrac{(a^2+b^2)(a+b)}{2ab}$

$B = \left\{\dfrac{b-c}{(a-b)(c-a)} + \dfrac{c-a}{(b-c)(a-b)} + \dfrac{(a-b)}{(c-a)(c-b)}\right\} \div \dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}$

$B = \dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)} \times \dfrac{(a-b)(b-c)(c-a)}{(a-b)^2+(b-c)^2+(c-a)^2}$

$B = 1$

$A = \dfrac{(a^2+b^2)(a+b)}{2ab}; B = 1$

So

$A + B = \dfrac{(2022^2+2023^2)(2022+2023)}{2\times2022\times2023} + 1$

$A + B = 4046.$

3. There are square papers of areas $1, 2, 3, . . .$ square millimeters. Asif started coloring them with red and green paint. He painted in red the papers whose areas can be written as the sum of two composite numbers and the rest in green. How many papers are painted green and what are their areas?

Sol. Areas of square papers are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10,. . . . .$

Opposite numbers are $4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, ...$

Papers with green paint have the areas that can't be the sum of the two composite numbers, so areas of green painted papers are-

$1, 2, 3, 4, 5, 6, 7, 8, 9, 11$ only rest all can be the sum of two composite number lines.

$12 = 8 + 4, 13 = 4 + 9, 14 = 8 + 6, 15 = 9 + 6. 16 = 10 + 6$ and 80 on

So required sum is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 11 = 56$

Number of green painted papers $= 10$.

4. Find natural numbers a, b, c such that their sum is 6, sum of their squares is 14 and the sum of the products of a and c, and b and c is equal to the square of one more than the product of a and b.

Sol. $a + b + c = 6$ ...(1)

$a^2 + b^2 + c^2 = 14$ ...(2)

$ac + bc = (ab + 1)^2$ ...(3)

$\Rightarrow ab + bc + ac = 11$

$\Rightarrow a^2b^2 + 1 + 2ab + ab = 11$

$\Rightarrow a^2b^2 + 3ab = 10$

$\Rightarrow a(ab^2 + 3b) = 10$

(1) $a = 5 \,\&\, ab^2 + 3b = 2$

$a, b \,\&\, c \in N$

$\Rightarrow ab^2 + 3b > 2$

$\therefore$ Rejected

OR

$a = 2 \,\&\, ab^2 + 3b = 5$

$\Rightarrow a = 2$

$(1) \Rightarrow b + c = 4, b^2 + c^2 = 10$

Sq.$b^2 + c^2 + 2bc = 16$

$\Rightarrow bc = 3 \Rightarrow b = 3 \,\&\, C = 1$

are $b = 1 \,\&\, C = 3$

(2) $a = 10 \,\&\, ab^2 + 3b = 1$

$a, b \,\&\, c \in N$

$\Rightarrow ab^2 + 3b > 1$

$\therefore$ Rejected

OR

$a = 1 \,\&\, ab^2 + 3b = 10$

$\Rightarrow a = 1$

$(1) \Rightarrow b + c = 5, b^2 + c^2 = 13$

Sq.$b^2 + c^2 + 2bc = 25$

$\Rightarrow 2bC = 12$

$\Rightarrow bc = 6$

$b = 1 \,\&\, C = 6$ or $b = 6 \,\&\, C = 1$

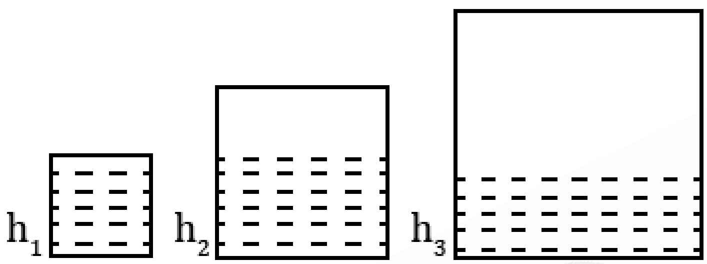
$\therefore$ Rejected as it does not satisfy equation 3

$b = 2 \,\&\, C = 3$ it satisfies or $b = 3 \,\&\, C = 2$ it does not satisfy

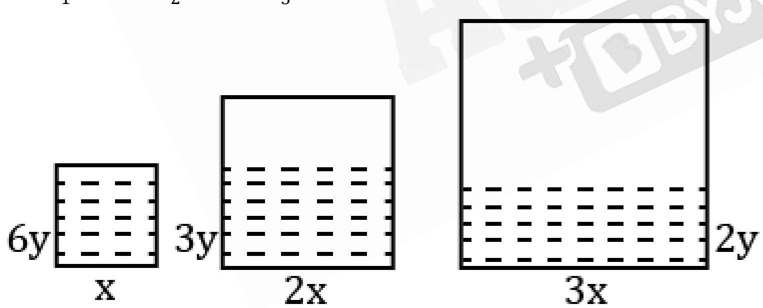
Two solutions: $a = 2, b = 1, C = 3$
$a = 1, b = 2, C = 3$

5. The volumes of three cubic containers C_1, C_2 and C_3 are in the ration $1:8:27$. The amount of water in them are in the ratio $1:2:3$. Water is poured from C_1 to C_2 and from C_2 to C_3 then the water level in all the containers is the same. Now $128\frac{4}{7}$ liters of water is poured out from C_3 to C_2 after which a certain amount is poured from C_2 to C_1 so that the depth of water in C_1 becomes twice that in C_2. This results in the amount of water in C_1 100 liters less than the original amount. How much water did each container contain originally?

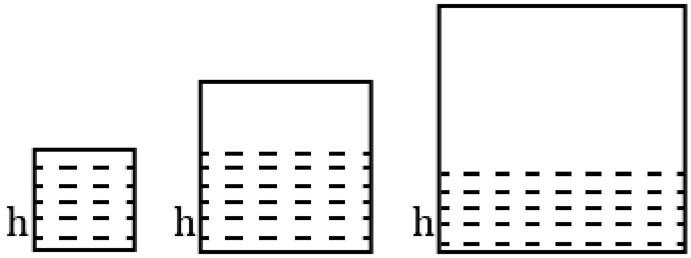
Sol. $l_1 : l_2 : l_3 = 1:2:3$



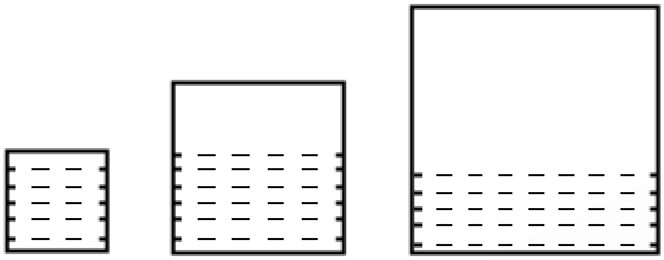
$x^2h_1 : (2x)^2h_2 : (3x)^2h_3 = 1:2:3$
$\Rightarrow h_1 : 4h_2 : 9h_3 = 1:2:3$
$\Rightarrow h_1 = 2h_2 : 3h_3$
$\Rightarrow h_1 : h_2 : h_3 = 6:3:2$
Let $h_1 = 6y,\ h_2 = 3y,\ h_3 = 2y$



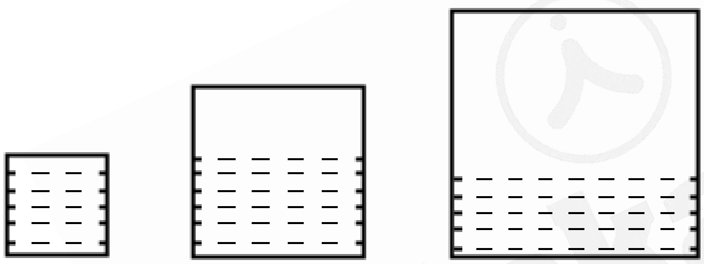
Total volume in container $= 6x^2y + 12x^2y + 18x^2y$
Now $hx^2 + h(4x^2) + h(9x^2) = 36x^2y$
$\Rightarrow 14x^2h = 36x^2y$
$\Rightarrow h = \frac{36}{14}y = \frac{18}{7}y$



$h = \frac{18}{7}y$

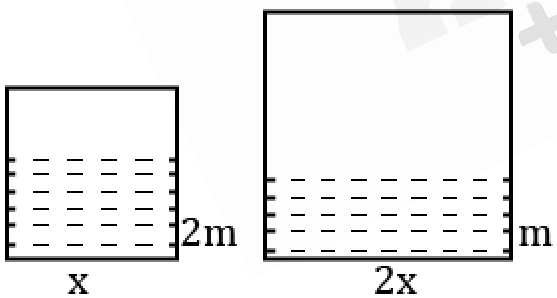


$\frac{18x^2y}{7}$ $\qquad$ $\frac{72x^2y}{7} + \frac{1000}{7}$ $\qquad$ $\frac{162x^2y}{7} - \frac{1000}{7}$



$\frac{18}{7}x^2 \times y + z$ $\qquad$ $\frac{72}{7}x^2y + \frac{1000}{7} - z$ $\qquad$ $\frac{162}{7}x^2y - \frac{1000}{7}$

$= V_1$ $\qquad$ $= V_2$



$V_1 = 2mx^2$ $\qquad$ $V_2 = 4x^2m$

$\Rightarrow V_2 = 2V_1$

$\Rightarrow \left(\frac{72}{7}x^2y + \frac{100}{7} - z\right) = 2\left(\frac{18}{7}x^2y + z\right)$

$\Rightarrow \frac{36}{7}x^2y + \frac{100}{7} = 3z$

$\Rightarrow z = \frac{12}{7}x^2y + \frac{100}{21}$

$\Rightarrow \frac{18}{7}x^2y + \frac{12}{7}x^2y + \frac{100}{21} = 6x^2y - 100$

$\Rightarrow \frac{100}{21} + 100 = \left(6 - \frac{30}{7}\right)x^2y$

$\Rightarrow \frac{2200}{21} = \frac{12}{7}x^2y$

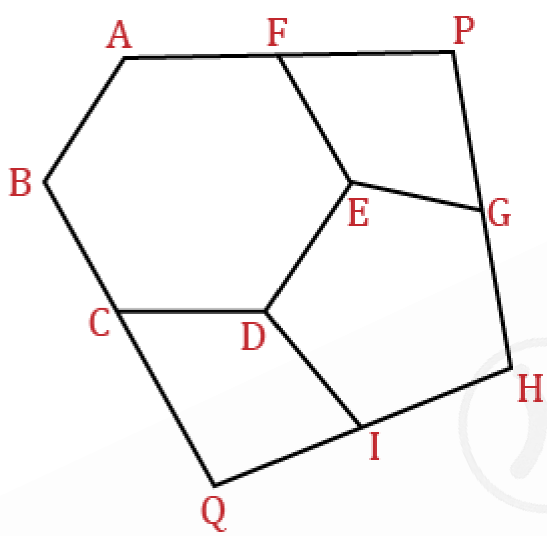
$\Rightarrow x^2y = \frac{2200\times7}{21\times12} = \frac{550}{9}$

Volume of water in C_1 is $6 \times \frac{550}{9} = \frac{1100}{3}$

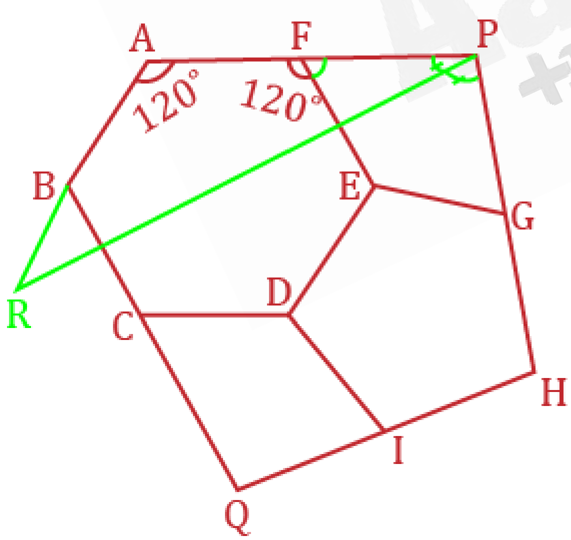
Volume of water in C_2 is $\frac{2200}{3}$

Volume of water in C_3 is $\frac{3300}{3}$.

6. In the adjoining figure, $ABCDEF$ and $DEGHI$ are respectively a regular hexagon and a regular pentagon. P is the intersection of AF and HG and similarly Q. The bisector of $\angle FPG$ cuts AB produced at R. Prove that $\angle Q = 8\angle R$.



Sol.



Each interior angle of a regular hexagon $= 120^0$

Each interior angle of a regular pentagon $= 108^0$

$\Rightarrow \angle PFE = 60^0$ & $\angle PGE = 72^0$ & $\angle FEG = 132^0$

$60^0 + \angle FPG + 72^0 + 132^0 = 360^0$

$\Rightarrow \angle FPG = 96^0$

$\Rightarrow \angle APR = 48^0$ (PR is angle bisector)

$\Rightarrow \angle R = 12^0$

By angle sum property in $APHQB$

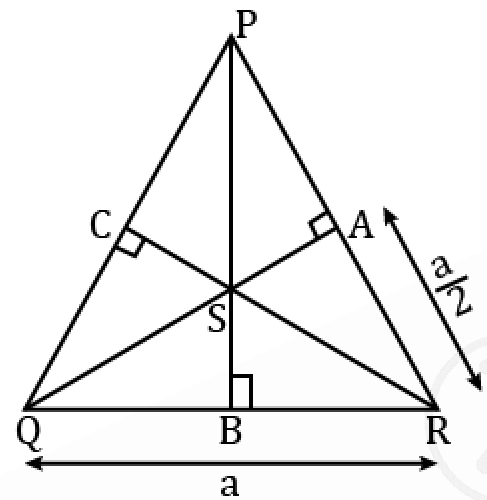
$\angle Q = 540^0 - (120^0 + 120^0 + 96 + 108^0)$

$= 96^0$

$\Rightarrow \angle Q = 8\angle R$

7. PQR is an equilateral triangle. S is any point inside the triangle. SA, SB, SC are respectively drawn perpendiculars to PR, RQ and PQ. Find the ratio of $\frac{SA+SB+SC}{QA+RB+PC}$.

Sol.



It is given that ΔPQR is an equilateral triangle. Let S be orthocentre of ΔPQR.

Let the side is a for equilateral triangle A, B, C are midpoint of sides PR, QR and PQ respectively.

$AR = \frac{a}{2} = BR = PC$

In ΔAQR, $AQ = \sqrt{a^2 - \left(\frac{a}{2}\right)^2}$

$= \sqrt{a^2 - \frac{a^2}{4}}$

$= \frac{\sqrt{3}}{2}a$

For equilateral triangle orthocentre and centroid are same point.

Therefore S divide QA in Radio $2:1$

Hence $AS = \frac{1}{3} \times \frac{\sqrt{3}}{2}a$

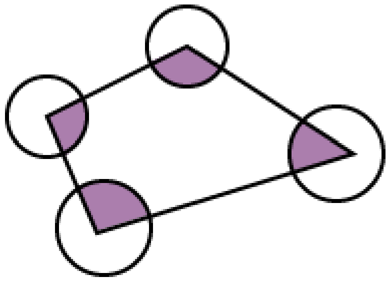
$= \frac{\sqrt{3}}{6}a$

Similarly $SB = SC = \frac{\sqrt{3}}{6}a$

Now $\frac{SA+SB+SC}{QA+RB+PC} = \frac{\frac{\sqrt{3}}{6}a+\frac{\sqrt{3}}{6}a+\frac{\sqrt{3}}{6}a}{\frac{\sqrt{3}}{2}a+\frac{a}{2}+\frac{a}{2}}$

$= \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}+1} = \frac{\sqrt{3}}{\sqrt{3}+2}$

8. In the adjoining figure, four equal circles of radius $7\,cm$ each are drawn with centers at the four vertices of a quadrilateral. Find the total area of the shaded sectors.



Sol. As the sum of all angles of a quadrilateral is 360^0,

$\therefore$ Sum of all the central angles is 360^0

Thus, total area of shaded sectors

$= \frac{360^0}{360^0} \times \pi r^2$

$= \frac{360^0}{360^0} \times \frac{22}{7} \times 7 \times 7\ cm^2$

$= 154\ cm^2$.