MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

61. Let N denote the number that turns up when a fair die is rolled. If the probability that the system of equations

$x + y + z = 1$

$2x + Ny + 2z = 2$

$3x + 3y + Nz = 3$

has unique solution is $\frac{k}{6}$, then the sum of value of k and all possible values of N is

(1) 18 (2) 20

(3) 21 (4) 19

Answer (2)

Sol. For unique solution $\Delta \neq 0$

i.e. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & N & 2 \\ 3 & 3 & N \end{vmatrix} \neq 0$

$\Rightarrow (N^2 - 6) - (2N - 6) + (6 - 3N) \neq 0$

$\Rightarrow N^2 - 5N + 6 \neq 0$

$\therefore$ $N \neq 2$ and $N \neq 3$

$\therefore$ Probability of not getting 2 or 3 in a throw of dice $= \frac{2}{3}$

As given $\frac{2}{3} = \frac{k}{6} \Rightarrow k = 4$

$\therefore$ Required value = 1 + 4 + 5 + 6 + 4 = 20

62. Let $\vec{u} = \hat{i} - \hat{j} - 2\hat{k}, \vec{v} = 2\hat{i} + \hat{j} - \hat{k}, \vec{v} \cdot \vec{w} = 2$ and $\vec{v} \times \vec{w} = \vec{u} + \lambda\vec{v}$. Then $\vec{u} \cdot \vec{w}$ is equal to

(1) 2 (2) 1

(3) $\frac{3}{2}$ (4) $-\frac{2}{3}$

Answer (2)

Sol. Given $\vec{v} \times \vec{w} = \lambda\vec{v} + \vec{u}$ ...(i)

Taking dot with $\vec{v}$ we get

$[\vec{v}\,\vec{v}\,\vec{w}] = \lambda|\vec{v}|^2 + \vec{u} \cdot \vec{v}$

Substituting values we have

$6\lambda + 3 = 0 \Rightarrow \lambda = -\frac{1}{2}$

$\therefore$ Equation (i) becomes

$\vec{v} \times \vec{w} = \vec{u} - \frac{\vec{v}}{2}$ ...(ii)

Taking dot with $\vec{w}$ of (ii) we get

$0 = \vec{u} \cdot \vec{w} - \frac{\vec{v} \cdot \vec{w}}{2}$

$\Rightarrow \vec{u} \cdot \vec{w} = \frac{2}{2} = 1$ (as $\vec{v} \cdot \vec{w} = 2$ given)

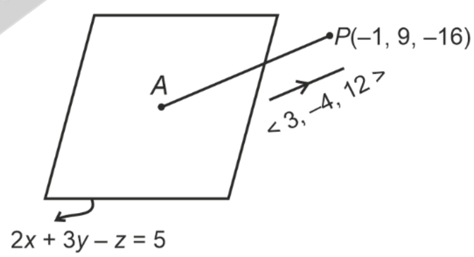
63. The distance of the point (–1, 9, –16) from the plane $2x + 3y - z = 5$ measured parallel to the line $\frac{x+4}{3} = \frac{2-y}{4} = \frac{z-3}{12}$ is

(1) $20\sqrt{2}$ (2) $13\sqrt{2}$

(3) 26 (4) 31

Answer (3)

Sol. $\frac{x+4}{3} = \frac{y-2}{-4} = \frac{z-3}{12}$



Equation of line AP

$\frac{x+1}{3} = \frac{y-9}{-4} = \frac{z+16}{12}$

Point $A(3\lambda - 1, -4\lambda + 9, 12\lambda - 16)$ lies on $2x + 3y - z = 5$, $\Rightarrow 6\lambda - 2 - 12\lambda + 27 - 12\lambda + 16 = 5 \Rightarrow \lambda = 2$

$\Rightarrow$ Point $A(5, 1, 8)$

$\Rightarrow AP^2 = 6^2 + 8^2 + 24^2 = 4(9 + 16 + 144) = 4 \times 169$

$AP = 26$

Option (3) is correct.

64. Let $y = y(x)$ be the solution of the differential equation $x^3\,dy + (xy - 1)dx = 0$, $x > 0$, $y\left(\frac{1}{2}\right) = 3 - e$. Then $y(1)$ is equal to

(1) e (2) $2 - e$

(3) 3 (4) 1

Answer (4)

Sol. $x^3dy + xy\,dx - dx = 0$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1 - xy}{x^3}$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{y}{x^2} = \dfrac{1}{x^3}$

I.F. $= e^{\int \frac{dx}{x^2}} = e^{-\frac{1}{x}}$

$\therefore \quad ye^{-\frac{1}{x}} = \int \dfrac{e^{-\frac{1}{x}}}{x^3}dx$

For RHS put $-\dfrac{1}{x} = t \Rightarrow \dfrac{dx}{x^2} = dt$

$\therefore \quad ye^{-\frac{1}{x}} = -\int te^t dt$

$\Rightarrow \quad ye^{-\frac{1}{x}} = -\left[te^t - e^t\right] + c$

$\Rightarrow \quad ye^{-\frac{1}{x}} = \dfrac{e^{-\frac{1}{x}}}{x} + e^{-\frac{1}{x}} + c$ ...(i)

$\downarrow y\left(\frac{1}{2}\right) = 3 - e$

$\Rightarrow (3 - e)e^{-2} = 2e^{-2} + e^{-2} + c$

$\Rightarrow \quad c = -\dfrac{1}{e}$ ...(ii)

For $y(1)$ put $x = 1$, $c = -e^{-1}$ in equation (i) we get

$ye^{-1} = e^{-1} + e^{-1} - e^{-1}$

$\Rightarrow \quad y = 1$

65. If A and B are two non-zero $n \times n$ matrics such that $A^2 + B = A^2B$, then

(1) $A^2B = BA^2$ (2) $A^2B = I$

(3) $A^2 = I$ or $B = I$ (4) $AB = I$

Answer (1)

Sol. Given : $A^2 + B = A^2B$ ...(i)

$\Rightarrow \quad A^2 + B - I = A^2B - I$

$\Rightarrow \quad A^2B - A^2 - B + I = I$

$\Rightarrow \quad A^2(B - I) - I(B - I) = I$

$\Rightarrow \quad (A^2 - I)(B - I) = I$

$\therefore \quad A^2 - I$ is the inverse matrix of $B - I$ and vice versa.

So, $(B - I)(A^2 - I) = I$

$\Rightarrow \quad BA^2 - B - A^2 + I = I$

$\therefore \quad A^2 + B = BA^2$ ...(ii)

So, by (i) and (ii)

$A^2B = BA^2$

$\therefore$ Option (1) is correct.

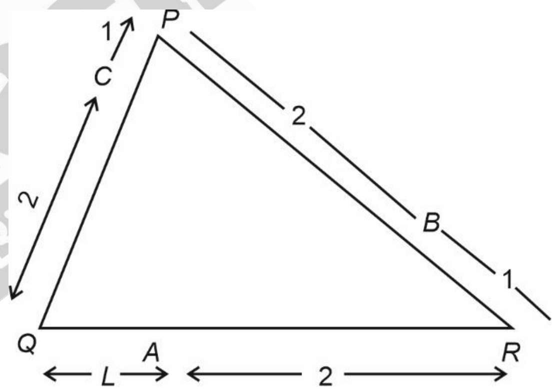
66. Let PQR be a triangle. The points A, B and C are on the sides QR, RP and PQ respectively such that $\dfrac{QA}{AR} = \dfrac{RB}{BP} = \dfrac{PC}{CQ} = \dfrac{1}{2}$. Then $\dfrac{Area(\Delta PQR)}{Area(\Delta ABC)}$ is equal to

(1) $\dfrac{5}{2}$ (2) 4

(3) 3 (4) 2

Answer (3)

Sol.



By PSY formula

$$\frac{\Delta ABC}{\Delta PQR} = \frac{(PC \times QA \times RB) + (CQ \times AR \times BP)}{PQ \times QR \times RP}$$

$$= \frac{8 + 1}{3 \times 3 \times 3} = \frac{1}{3}$$

67. For three positive integers p, q, r, $x^{pq^2} = y^{qr} = z^{p^2r}$ and $r = pq + 1$ such that 3, 3 $\log_y x$, 3$\log_z y$, 7 $\log_x z$ are in A.P with common difference $\dfrac{1}{2}$. Then $r - p - q$ is equal to

(1) 12 (2) 2

(3) –6 (4) 6

Answer (2)

Sol. $x^{pq^2} = y^{qr} = z^{p^2r}$

$$3\log_y x = \frac{7}{2}, 3\log_z y = 4, 7\log_x z = \frac{9}{2}$$

$$\Rightarrow \quad x = y^{\frac{7}{6}}, y = z^{\frac{4}{3}}, z = x^{\frac{9}{14}}$$

$$y^{\frac{7}{6}pq^2} = y^{qr} = y^{\frac{3}{4}p^2r}$$

$$\Rightarrow \quad \frac{7}{6}pq^2 = qr = \frac{3}{4}p^2r$$

$\therefore$ $7pq = 6r$, $4q = 3p^2$

$r = pq + 1$

$$r = \frac{6r}{7} + 1 \Rightarrow r = 7$$

$pq = 6$

$$p\left(\frac{3p^2}{4}\right) = 6$$

$p = 2$, $q = 3$

$r - p - q = 7 - 5 = 2$

68. The relation

$$R = \{(a,b) : gcd(a,b) = 1, 2a \neq b, a, b \in \mathbb{Z}\} \text{ is}$$

(1) Reflexive but not symmetric

(2) Neither symmetric nor transitive

(3) Symmetric but not transitive

(4) Transitive but not reflexive

Answer (3)

Sol. gcd $(a, a) = a$ so (a, a) so $\in R \Rightarrow$ not reflexive

If gcd $(a, b) = 1 \Rightarrow gcd$ $(b, a) = 1$

$\therefore$ $(b, a) \in R \Rightarrow$ Symmetric

If gcd $(a, b) = 1$ and gcd $(b, c) = 1$

$\Rightarrow$ gcd $(a, c) = 1$

$\therefore$ R is not transitive

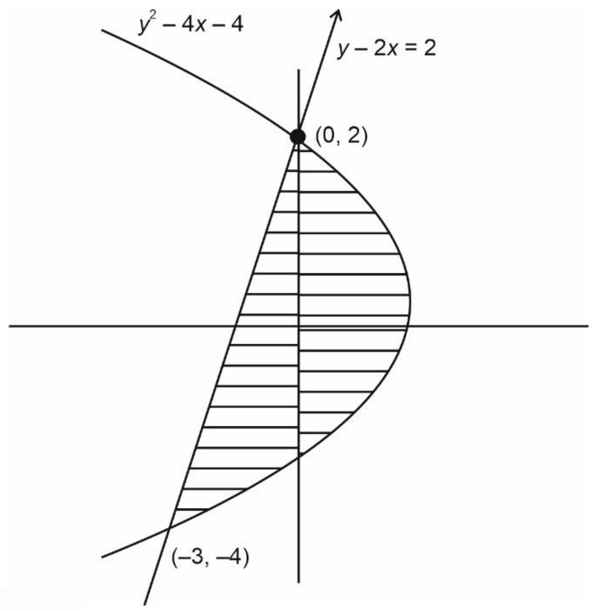
69. The area enclosed by the curves $y^2 + 4x = 4$ and $y - 2x = 2$ is

(1) $\frac{23}{3}$ (2) $\frac{22}{3}$

(3) $\frac{25}{3}$ (4) 9

Answer (4)

Sol.



$$\text{Required area} = \int_{-4}^{2}\left(\frac{4-y^2}{4} - \frac{y-2}{2}\right)dy$$

$$= \int_{-4}^{2} \frac{8-2y-y^2}{4} dy$$

$$= \frac{1}{4}\left\{8y - y^2 - \frac{y^3}{3}\right\}_{-4}^{2}$$

= 9 square units

70. $\lim\limits_{t \to 0}\left(1^{\frac{1}{\sin^2 t}} + 2^{\frac{1}{\sin^2 t}} + \ldots + n^{\frac{1}{\sin^2 t}}\right)^{\sin^2 t}$ is equal to

(1) n^2 (2) $n^2 + n$

(3) $\frac{n(n+1)}{2}$ (4) n

Answer (4)

Sol. $l = \lim\limits_{t \to 0} n\left(\left(\frac{1}{n}\right)^{\text{cosec}^2 t} + \left(\frac{2}{n}\right)^{\text{cosec}^2 t} + \ldots + 1\right)^{\sin^2 t}$

$l = n$

$$\because \quad \lim_{t \to 0}\left(\frac{r}{n}\right)^{\text{cosec}^2 t} = 0, \forall\ 1 \le r < n$$

71. The value of $\sum\limits_{r=0}^{22} {}^{22}C_r\ {}^{23}C_r$ is

(1) ${}^{44}C_{22}$ (2) ${}^{45}C_{24}$

(3) ${}^{44}C_{23}$ (4) ${}^{45}C_{23}$

Answer (4)

Sol. $(1+x)^{22} = {}^{22}C_0 + {}^{22}C_1x + {}^{22}C_2x^2 + ...$

$$+{}^{22}C_{21}x^{21} + {}^{22}C_{22}x^{22} \quad ...(i)$$

$$(x+1)^{23} = {}^{23}C_0x^{23} + {}^{23}C_1x^{22} + {}^{23}C_2x^{21} + ...$$

$$+{}^{23}C_{21}x^2 + {}^{23}C_{22}x + {}^{23}C_{23} \quad ...(ii)$$

Multiplying (i) & (ii) and comparing coefficients of x^{23} on both sides

$${}^{45}C_{23} = {}^{22}C_0 \cdot {}^{23}C_0 + {}^{22}C_1 \cdot {}^{23}C_1 + {}^{22}C_2 \cdot {}^{23}C_2 + ...$$

$$+{}^{22}C_{22} \cdot {}^{23}C_{22}$$

$$\sum_{r=0}^{22} {}^{22}C_r \cdot {}^{23}C_r = {}^{45}C_{23}$$

72. Let a tangent to the curve $y^2 = 24x$ meet the curve $xy = 2$ at the points A and B. Then the mid points of such line segments AB lie on a parabola with the

(1) Directrix $4x = 3$

(2) Length of latus rectum $\frac{3}{2}$

(3) Length of latus rectum 2

(4) Directrix $4x = -3$

Answer (1)

Sol. $y^2 = 24x$, $xy = 2$

Let the equation of tangent to $y^2 = 24x$ is $ty = x + 6t^2$

$ty = x + 6t^2$ meet the curve $xy = 2$ at points A and B.

Let mid-point of AB is $P(h, k)$.

$ty = \frac{2}{y} + 6t^2$	$t \cdot \frac{2}{x} = x + 6t^2$
$ty^2 - 6t^2y - 2 = 0$	$x^2 + 6t^2x - 2t = 0$
$y_1 + y_2 = 6t$	$x_1 + x_2 = -6t^2$

$\Rightarrow$ Mid-point P is $(-3t^2, 3t)$

$\Rightarrow$ $h = -3t^2$, $k = 3t$

$$\Rightarrow \left(\frac{h}{-3}\right) = \left(\frac{k}{3}\right)^2$$

$\Rightarrow$ $y^2 = -3x$

$\Rightarrow$ Length of L.R. = 3

Equation of directrix is $x = \frac{3}{4}$

73. The equation $x^2 - 4x + [x] + 3 = x[x]$, where $[x]$ denotes the greatest integer function, has

(1) Exactly two solutions in $(-\infty, \infty)$

(2) No solution

(3) A unique solution in $(-\infty, 1)$

(4) A unique solution in $(-\infty, \infty)$

Answer (4)

Sol. $x^2 - 4x + [x] + 3 = x[x]$

$(x - 1)(x - 3) = (x - 1)[x]$

$(x - 1)(x - 3 - [x]) = 0$

$x = 1$ or $x - 3 - [x] = 0$

$\{x\} = 3$

$x = \phi$

$\Rightarrow$ a unique solution in $(-\infty, \infty)$

74. The distance of a point $(7, -3, -4)$ from the plane passing through the points $(2, -3, 1)$, $(-1, 1, -2)$ and $(3, -4, 2)$ is

(1) $4\sqrt{2}$

(2) 4

(3) $5\sqrt{2}$

(4) 5

Answer (3)

Sol. $A(2, -3, 1)$, $B(-1, 1, -2)$, $C(3, -4, 2)$

$\overrightarrow{AB} = -3\hat{i} + 4\hat{j} - 3\hat{k} \qquad \overrightarrow{AC} = \hat{i} - \hat{j} + \hat{k}$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 4 & -3 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i} - \hat{k}$$

Let equation of plane is $x - z + \lambda = 0$ passes through point $A(2, -3, 1) \Rightarrow \lambda = -1$

Equation of plane is $x - z - 1 = 0$

Distance of point $(7, -3, -4)$ from the plane $x - z - 1 = 0$ is $5\sqrt{2}$

75. The compound statement $(\sim(P \wedge Q)) \vee ((\sim P) \wedge Q) \Rightarrow ((\sim P) \wedge (\sim Q))$ is equivalent to

(1) $((\sim P) \vee Q) \wedge (\sim Q)$

(2) $(\sim Q) \wedge P$

(3) $(\sim P) \vee Q$

(4) $((\sim P) \vee Q) \wedge ((Q) \vee P)$

Answer (4)

Sol. $(\sim(P \wedge Q)) \vee ((\sim P) \wedge Q) \Rightarrow ((\sim P) \wedge (\sim Q))$

$(\sim P \vee \sim Q) \vee (\sim P \wedge Q) \Rightarrow (\sim P \wedge \sim Q)$

$\Rightarrow (\sim P \vee \sim Q \vee \sim P) \wedge (\sim P \vee \sim Q \vee Q) \Rightarrow (\sim P \wedge \sim Q)$

$\Rightarrow (\sim P \vee \sim Q) \wedge (T) \Rightarrow (\sim P \wedge \sim Q)$

$\Rightarrow (\sim P \vee \sim Q) \Rightarrow (\sim P \wedge \sim Q)$

$\Rightarrow \sim(\sim P \vee \sim Q) \vee (\sim P \wedge \sim Q)$

$\Rightarrow (P \wedge Q) \vee (\sim P \wedge \sim Q) \Rightarrow (\sim P \vee Q) \wedge (\bullet Q \vee P)$

76. Let Ω be the sample space and $A \subseteq \Omega$ be an event.

Given below are two statements:

(S1) : If $P(A) = 0$, then $A = \varnothing$

(S2) : If $P(A) = 1$, then $A = \Omega$

Then

(1) Both (S1) and (S2) are false

(2) Only (S1) is true

(3) Only (S2) is true

(4) Both (S1) and (S2) are true

Answer (4)

Sol. Both statements are correct

77. Let α be a root of the equation $(a - c)x^2 + (b - a)x + (c - b) = 0$ where a, b, c are distinct real numbers such that the matrix $\begin{bmatrix} \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \\ a & b & c \end{bmatrix}$ is singular. Then, the value of

$$\frac{(a-c)^2}{(b-a)(c-b)} + \frac{(b-a)^2}{(a-c)(c-b)} + \frac{(c-b)^2}{(a-c)(b-a)} \text{ is}$$

(1) 12 (2) 6

(3) 9 (4) 3

Answer (4)

Sol. $\dfrac{(a-c)^3 + (b-a)^3 + (c-a)^3}{(a-c)(c-b)(b-a)}$

$= \dfrac{3(a-c)(b-a)(c-b)}{(a-c)(b-a)(c-b)}$

$= 3$

78. Let $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Then at $x = 0$

(1) f is continuous but not differentiable

(2) f and f' both are continuous

(3) f' is continuous but not differentiable

(4) f is continuous but f' is not continuous

Answer (4)

Sol. $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0) = 0$

$\therefore$ f is continuous at $x = 0$

Now, $R.H.D = \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h} = \dfrac{h^2 \sin\frac{1}{h}}{h} = 0$

at $x = 0$

and LHD

$= \lim\limits_{h \to 0} \dfrac{f(-h) - f(0)}{-h} = \dfrac{-h^2 \sin\left(\frac{1}{h}\right)}{-h} = 0$

$\therefore$ RHD = LHD $\therefore$ f is differentiable at $x = 0$

$\therefore \quad f'(x) = \begin{cases} 2x \sin\left(\dfrac{1}{x}\right) - \cos\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\lim\limits_{x \to 0^+} f'(x)$ is oscillatory

$\therefore$ f is continuous but f' is not at $x = 0$

79. $\tan^{-1}\left(\dfrac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\left(\sqrt{\dfrac{8+4\sqrt{3}}{6+3\sqrt{3}}}\right)$ is equal to:

(1) $\dfrac{\pi}{3}$ (2) $\dfrac{\pi}{6}$

(3) $\dfrac{\pi}{2}$ (4) $\dfrac{\pi}{4}$

Answer (1)

Sol. $\tan^{-1}\left(\dfrac{1+\sqrt{3}}{3+\sqrt{3}}\right) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

and $\sec^{-1}\left(\sqrt{\dfrac{4(2+\sqrt{3})}{3(2+\sqrt{3})}}\right) = \sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

$\therefore \quad \tan^{-1}\left(\dfrac{1+\sqrt{3}}{3+\sqrt{3}}\right) + \sec^{-1}\left(\sqrt{\dfrac{8+4\sqrt{3}}{6+3\sqrt{3}}}\right) = \dfrac{\pi}{6} + \dfrac{\pi}{6} = \dfrac{\pi}{3}$

80. Let $p, q \in \mathbb{R}$ and

$\left(1-\sqrt{3}i\right)^{200} = 2^{199}(p+iq), i=\sqrt{-1}$, then $p + q + q^2$ and $p - q + q^2$ are roots of the equation.

(1) $x^2 - 4x + 1 = 0$ (2) $x^2 - 4x - 1 = 0$

(3) $x^2 + 4x - 1 = 0$ (4) $x^2 + 4x + 1 = 0$

Answer (1)

Sol. Given $\left(1-\sqrt{3}i\right)^{200} = 2^{199}(p+iq) \ldots(1)$

$$\text{L.H.S} = 2^{200}\left[\cos\left(\frac{5\pi}{3}\right)+i\sin\frac{5\pi}{3}\right]^{200}$$

$$= 2^{200}\left[\cos\frac{1000\pi}{3}+i\sin\frac{1000\pi}{3}\right]$$

$$= 2^{200}\left[-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right]$$

So, by (1)

$P = -1, \quad q = -\sqrt{3}$

$\therefore \quad p+q+q^2 = -1-\sqrt{3}+3 = 2-\sqrt{3} = \alpha$

and $p-q+q^2 = -1+\sqrt{3}+3 = 2+\sqrt{3} = \beta$

$\therefore$ quadratic equation whose roots are α and β

$x^2 - 4x + 1 = 0$

Option (1) is correct.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

81. Suppose $\sum_{r=0}^{2023} r^2\ {}^{2023}C_r = 2023\times\alpha\times 2^{2022}$. Then the value of α is ____________

Answer (1012)

Sol. $\because \quad (1+x)^{2023} = \sum_{r=0}^{2023} {}^{2023}C_r x^r$

$$\Rightarrow (2023)(1+x)^{2022} = \sum_{r=0}^{2023} {}^{2023}C_r r x^{r-1}$$

$$\Rightarrow (2023)x(1+x)^{2022} = \sum_{r=0}^{2023} r\ {}^{2023}C_r x^r$$

$$\Rightarrow (2023)\left[x2022(1+x)^{2021}+(1+x)^{2022}\right] = \sum_{r=0}^{2023} r^2\,{}^{2023}C_r x^{r-1}$$

Put $x = 1$

$$\Rightarrow 2023\left[2022\cdot 2^{2021}+2^{2022}\right] = \sum_{r=0}^{2023} r^2\,{}^{2023}C_r$$

$$\therefore \sum_{r=0}^{2023} r^2\,{}^{2023}C_r = 2023\cdot 2^{2022}(1012)$$

$\therefore \quad \alpha = 1012$

82. Let C be the largest circle centred at (2, 0) and inscribed in the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$.

If $(1, \alpha)$ lies on C, then $10\alpha^2$ is equal to __________

Answer (118.00)

Sol. $\frac{x^2}{36}+\frac{y^2}{16}=1$

$r^2 = (x - 2)^2 + y^2$

Solving simultaneously

$-5x^2 + 36x + (9r^2 - 180) = 0$

$D = 0$

$r^2 = \frac{128}{10}$

Distance between $(1, \alpha)$ and (2, 0) should be r

$1+\alpha^2 = \frac{128}{10}$

$\alpha^2 = \frac{118}{10}$

$= 118.00$

83. The number of 9 digit numbers, that can be formed using all the digits of the number 123412341 so that the even digits occupy only even places, is ____________

Answer (60)

Sol. Given number 123412341

— — — — — — — — —
 ^ ^ ^ ^

Total number $= \frac{4!}{2!2!}\times\frac{5!}{3!2!} = 6\times 10 = 60$

84. Let a tangent to the curve $9x^2 + 16y^2 = 144$ intersect the coordinate axes at the points A and B. Then, the minimum length of the line segment AB is ____________

Answer (07)

Sol. Given curve : $9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$$

Let $P(4\cos\theta, 3\sin\theta)$ be any point on it.

Now tangent at P

$$\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1$$

$\therefore$ $A \equiv (4\sec\theta, 0)$ $B \equiv (0, 3\text{cosec}\theta)$

$$AB=\sqrt{16\sec^2\theta+9\text{cosec}^2\theta}$$

$$=\sqrt{16+9+16\tan^2\theta+9\cot^2\theta}$$

$$AB_{\min}=\sqrt{25+2\times12}$$

$= 7$

85. A boy needs to select five courses from 12 available courses, out of which 5 courses are language courses. If he can choose at most two language courses, then the number of ways he can choose five courses is ____________

Answer (546)

Sol. Case 1 If no language course is selected.

$$= {}^7C_5 \cdot$$

Case 2 If one language course is selected.

$${}^7C_4 \cdot {}^5C_1 \cdot$$

Case 3 If two language course is selected.

$${}^7C_3 \cdot {}^5C_2 \cdot$$

Total = ${}^7C_5 + {}^7C_4 \cdot {}^5C_1 + {}^7C_3 \cdot {}^5C_2$

$= 21 + 175 + 350$

$= 546$

86. The value of $12\int_0^3 \left|x^2-3x+2\right| dx$ is _______.

Answer (22)

Sol. $12\int_0^3 \left|x^2-3x+2\right| dx$

Let $I=\int_0^3 |(x-2)(x-1)| dx$

$$=\int_0^1 (x-1)(x-2)dx-\int_1^2 (x-1)(x-2)dx+\int_2^3 (x-1)(x-2)dx$$

$$=\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_0^1-\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_1^2+\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_2^3$$

$$=\left[\frac{1}{3}-\frac{3}{2}+2\right]-\left\{\left(\frac{8}{3}-6+4\right)-\left(\frac{1}{3}-\frac{3}{2}+2\right)\right\}+\left\{\left(9-\frac{27}{2}+6\right)-\left(\frac{8}{3}-6+4\right)\right\}$$

$$=\frac{11}{6}$$

$$12I=\frac{11}{6}\times12$$

$= 22$

87. The 4th term of GP is 500 and its common ratio is $\frac{1}{m}$, $m \in \mathbb{N}$. Let S_n denote the sum of the first n terms of this GP. If $S_6 > S_5 + 1$ and $S_7 < S_6+\frac{1}{2}$, then the number of possible values of m is

Answer (12)

Sol. $T_4 = 500$

$ar^3 = 500 \Rightarrow a=\frac{500}{r^3}$

Now,

$S_6 > S_5 + 1$

$$\frac{a(1-r^6)}{1-r}-\frac{a(1-r^5)}{1-r}>1$$

$ar^5 > 1$

Now, $r=\frac{1}{m}$ and $a=\frac{500}{r^3}$

$\Rightarrow$ $m^2 < 500$

$\because$ $m>0 \Rightarrow m \in \left(0, 10\sqrt{5}\right)$ ...(i)

$$S_7 < S_6+\frac{1}{2}$$

$$\frac{a(1-r^6)}{1-r}<\frac{a(1-r^6)}{1-r}+\frac{1}{2}$$

$$ar^6<\frac{1}{2}$$

$\because \quad r = \dfrac{1}{m}$ and $a = \dfrac{500}{r^5}$

$$\frac{1}{m^3} < \frac{1}{1000}$$

$\Rightarrow \quad m \in (10, \infty)$ ...(ii)

Possible values of m is {11, 12, ...22}

$\because \quad m \in N$

Total 12 values

88. The shortest distance between the lines $\dfrac{x-2}{3} = \dfrac{y+1}{2} = \dfrac{z-6}{2}$ and $\dfrac{x-6}{3} = \dfrac{1-y}{2} = \dfrac{z+8}{0}$ is equal to

Answer (14)

Sol. $\vec{a}_1 = 2\hat{i} - \hat{j} + 6\hat{k}$

$\vec{a}_2 = 6\hat{i} + \hat{j} - 8\hat{k}$

$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

$\vec{b} = 3\hat{i} - 2\hat{j} + 0\hat{k}$

$$S.D = \frac{\left|\left[\vec{a}_2 - \vec{a}_1 \quad \vec{a} \quad \vec{b}\right]\right|}{\left|\vec{a} \times \vec{b}\right|}$$

$$\left[\vec{a}_2 - \vec{a}_1\ \vec{a}\ \ \vec{b}\right] = \begin{vmatrix} 4 & 2 & -14 \\ 3 & 2 & 2 \\ 3 & -2 & 0 \end{vmatrix}$$

$= 4\times (4) - 2\ (-6) - 14\ (-12)$

$= 16 + 12 + 168 = 196$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 2 \\ 3 & -2 & 0 \end{vmatrix} = 4\hat{i} + 6\hat{j} - 12\hat{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{16 + 36 + 144} = \sqrt{196} = 14$$

$$S.D = \frac{196}{14} = 14$$

89. Let $\lambda \in \mathbb{R}$ and let the equation E be $|x|^2 - 2|x| + |\lambda - 3| = 0$. Then the largest element in the set $S = \{x + \lambda$: x is an integer solution of $E\}$ is ________.

Answer (05)

Sol. $D \ge 0 \Rightarrow 4 - 4|\lambda - 3| \ge 0$

$|\lambda - 3| \le 1$

$-1 \le \lambda - 3 \le 1$

$2 \le \lambda \le 4$

$$|x| = \frac{2 \pm \sqrt{4 - 4|\lambda - 3|}}{2}$$

$$= 1 \pm \sqrt{1 - |\lambda - 3|}$$

$x_{\text{largest}} = 1 + 1 = 2$, when $\lambda = 3$

Largest element of $S = 2 + 3 = 5$

90. The value of $\dfrac{8}{\pi}\displaystyle\int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}}\,dx$ is ________.

Answer (02)

Sol. $I = \dfrac{8}{\pi}\displaystyle\int_0^{\frac{\pi}{2}} \frac{(\cos x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}}\,dx$ ...(i)

$I = \dfrac{8}{\pi}\displaystyle\int_0^{\frac{\pi}{2}} \frac{(\sin x)^{2023}}{(\sin x)^{2023} + (\cos x)^{2023}}\,dx$ ...(ii)

$$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

(i) + (ii) $\Rightarrow 2I = \dfrac{8}{\pi}\displaystyle\int_0^{\frac{\pi}{2}} 1dx = \dfrac{8}{\pi} \times \dfrac{\pi}{2} = 4$

$I = 2$

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