MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

61. The sum of absolute maximum and minimum values of the function $f(x) = |x^2 - 5x + 6| - 3x + 2$ in the interval $[-1, 3]$ is equal to :

(1) 24 (2) 13

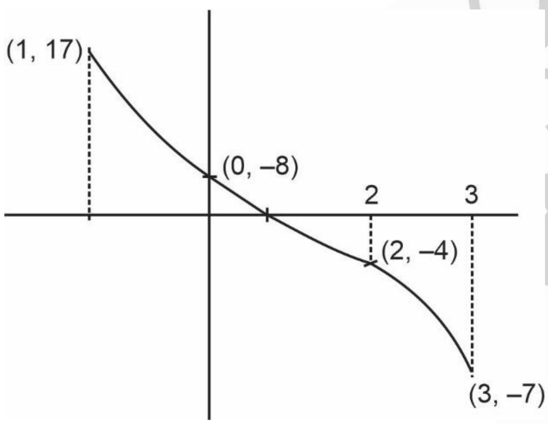
(3) 12 (4) 10

Answer (1)

Sol. $f(x) = |(x-2)(x-3)| - 3x + 2 \quad x \in [-1, 3]$

$\Rightarrow f(x) = x^2 - 8x + 8 \quad x \in [-1, 2]$

$-x^2 + 2x - 4 \quad x \in (2, 3)$



$\therefore$ Maximum value = 17

Minimum value of –7

$\therefore$ Sum = 24

62. The value of the integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x+\frac{\pi}{4}}{2-\cos 2x}dx$ is :

(1) $\frac{\pi^2}{6}$ (2) $\frac{\pi^2}{6\sqrt{3}}$

(3) $\frac{\pi^2}{12\sqrt{3}}$ (4) $\frac{\pi^2}{3\sqrt{3}}$

Answer (2)

Sol. $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x+\frac{\pi}{4}}{(2-\cos 2x)}dx$

Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{-x+\frac{\pi}{4}}{2-\cos 2x}\right)dx$$

$$\therefore \quad 2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\pi\, dx}{2(2-\cos 2x)}$$

$$\Rightarrow \quad I = \frac{2\pi}{4}\int_0^{\frac{\pi}{4}} \left(\frac{dx}{2 - \frac{1-\tan^2 x}{1+\tan^2 x}}\right)$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \left(\frac{1+\tan^2 x}{1+3\tan^2 x}\right)dx$$

Put $\tan x = t$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^1 \frac{dt}{1+3t^2} \qquad \Rightarrow \quad I = \frac{\pi}{6\sqrt{3}}$$

63. The sum $\sum_{n=1}^{\infty} \frac{2n^2+3n+4}{(2n)!}$ is equal to :

(1) $\frac{13e}{4} + \frac{5}{4e}$ (2) $\frac{11e}{2} + \frac{7}{2e} - 4$

(3) $\frac{13e}{4} + \frac{5}{4e} - 4$ (4) $\frac{11e}{2} + \frac{7}{2e}$

Answer (1)

Sol. $\sum_{n=1}^{\infty} \frac{2n^2+3n+4}{(2n)!}$

Put $2n = t \Rightarrow n = \frac{t}{2}$

$$\therefore \quad \sum_{t \to \text{even}} \frac{\frac{t^2}{2} + \frac{3t}{2} + 4}{t!}$$

$$\Rightarrow \sum_{t\to\text{even}} \frac{t^2+3t+8}{2t!}$$

$$\Rightarrow \frac{1}{2}\sum_{t\to\text{even}}\left(\frac{t-1}{(t-1)!}+\frac{1}{(t-1)!}+\frac{3}{(t-1)!}+\frac{8}{t!}\right)$$

$$\Rightarrow \frac{1}{2}\sum_{t\to\text{even}}\left(\frac{1}{(t-2)!}+\frac{4}{(t-1)!}+\frac{8}{t!}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{e+e^{-1}}{2}+\frac{4\left(e-e^{-1}\right)}{2}+\frac{8\left(e+e^{-1}\right)}{2}\right)$$

$$\Rightarrow \frac{1}{4}\left(13e+5e^{-1}\right)$$

64. Let a, b be two real numbers such that $ab < 0$. If the complex number $\frac{1+ai}{b+i}$ is of unit modulus and $a + ib$ lies on the circle $|z - 1| = |2z|$, then a possible value of $\frac{1+[a]}{4b}$, where $[t]$ is greatest integer function is :

(1) –1 (2) 1

(3) $-\frac{1}{2}$ (4) $\frac{1}{2}$

Answer (*)

Sol. $|1+ai| = |b+i|$

$\Rightarrow a^2+1=b^2+1 \Rightarrow a^2=b^2$

& $|a+ib-1| = |2a+2ib|$

$\Rightarrow a^2+1-2a+b^2 = 4a^2+4b^2$

$\Rightarrow 3a^2+3b^2+2a-1=0$

$\Rightarrow ba^2+2a-1=0$

$\therefore a = \frac{-2\pm\sqrt{4+24}}{2(6)}$

$= \frac{-1\pm\sqrt{7}}{6}$

$\therefore (a,\ b) \equiv \left(\frac{-1+\sqrt{7}}{6}, \frac{-1-\sqrt{7}}{6}\right)$ or

$\left(\frac{-1-\sqrt{7}}{6}, \frac{-1+\sqrt{7}}{6}\right)$

$\therefore \frac{1+[a]}{4b} = 0$ or $\frac{3}{2\left(-1-\sqrt{7}\right)}$

$\therefore$ No option matches

65. Let $f : \mathbb{R} - \{0, 1\} \to \mathbb{R}$ be a function such that $f(x)+f\left(\frac{1}{1-x}\right)=1+x$. then $f(2)$ is equal to

(1) $\frac{9}{2}$ (2) $\frac{9}{4}$

(3) $\frac{7}{3}$ (4) $\frac{7}{4}$

Answer (2)

Sol. $f(x)+f\left(\frac{1}{1-x}\right)=1+x$ …(i)

If $x \to \frac{1}{1-x}$

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{1}{1-\frac{1}{1-x}}\right)=1+\frac{1}{1-x}$$

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{1-x}{-x}\right)=\frac{2-x}{1-x} \quad \text{…(ii)}$$

If $x \to \frac{x-1}{x}$

$$f\left(\frac{x-1}{x}\right)+f(x)=\frac{2x-1}{x} \quad \text{…(iii)}$$

Putting $x = 2$

$f(2) + f(-1) = 3$

$f(-1)+f\left(\frac{1}{2}\right)=0$

$f\left(\frac{1}{2}\right)+f(2)=\frac{3}{2}$

Solving these $f(2)=\frac{9}{4}$

66. Let the plane P pass through the intersection of the planes $2x + 3y - z = 2$ and $x + 2y + 3z = 6$, and be perpendicular to the plane $2x + y - z + 1 = 0$. If d is the distance of P from the point $(-7, 1,1)$, then d^2 is equal to :

(1) $\frac{25}{83}$ (2) $\frac{15}{53}$

(3) $\frac{250}{82}$ (4) $\frac{250}{83}$

Answer (4)

Sol. Let the equation of plane P be

$$(2x+3y-z-2)+\lambda(x+2y+3z-6)=0$$

Now since P is $\perp^r$ to $2x+y-z+1=0$

$$\therefore\quad 2(2+\lambda)+1(3+2\lambda)-1(-1+3\lambda)=0$$

$$\boxed{\lambda=-8}$$

$$\therefore\quad P:\ 6x+13y+25z=46$$

Now distance from the point (–7, 1, 1)

$$d=\left|\frac{-42+13+25-46}{\sqrt{36+169+625}}\right|$$

$$\therefore\quad d^2=\frac{2500}{830}=\frac{250}{83}$$

67. Let $S=\left\{x\in\mathbb{R}:0<x<1 \text{ and } 2\tan^{-1}\left(\frac{1-x}{1+x}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$

If $n(S)$ denotes the number of elements in S then :

(1) $n(S)=0$

(2) $n(S)=2$ and only one element in S is less than $\frac{1}{2}$

(3) $n(S)=1$ and the element in S is less than $\frac{1}{2}$

(4) $n(S)=1$ and the elements is S is more than $\frac{1}{2}$

Answer (3)

Sol. $$2\tan^{-1}\left(\frac{1-x}{1+x}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put $\tan^{-1}x=\theta,\ \theta\in\left(0,\ \frac{\pi}{4}\right)$

$$2\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)=\cos^{-1}(\cos 2\theta)$$

$$2\left(\frac{\pi}{4}-\theta\right)=2\theta$$

$$\frac{\pi}{2}=4\theta \text{ or } \boxed{\theta=\frac{\pi}{8}}$$

$$x=\tan\left(\frac{\pi}{8}\right)=\sqrt{2}-1<\frac{1}{2}$$

68. Let $\vec{a}=2\hat{i}-7\hat{j}+5\hat{k}$, $\vec{b}=\hat{i}+\hat{k}$, $\vec{c}=\hat{i}+2\hat{j}-3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{a}=\vec{c}\times\vec{a}$ and $\vec{r}\cdot\vec{b}=0$, then $|\vec{r}|$ is equal to

(1) $\frac{11}{7}$ (2) $\frac{11}{5}\sqrt{2}$

(3) $\frac{\sqrt{914}}{7}$ (4) $\frac{11}{7}\sqrt{2}$

Answer (4)

Sol. $\vec{r}=\vec{c}+\lambda\vec{a}$

$$\vec{r}\cdot\vec{b}=\vec{c}\cdot\vec{b}+\lambda\vec{a}\cdot\vec{b}=0$$

$$-2+\lambda(7)=0\ \Rightarrow\ \lambda=\frac{2}{7}$$

$$\vec{r}=(\hat{i}+2\hat{j}-3\hat{k})+\frac{2}{7}(2\hat{i}-7\hat{j}+5\hat{k})$$

$$=\frac{11}{7}\hat{i}+0\hat{j}+\frac{11}{7}\hat{k}$$

$$|\vec{r}|=\frac{11}{7}\sqrt{2}$$

69. Let $P(x_0, y_0)$ be the point on the hyperbola $3x^2-4y^2=36$, which is nearest to the line $3x+2y=1$. Then $\sqrt{2}(y_0-x_0)$ is equal to

(1) 9 (2) –3

(3) –9 (4) 3

Answer (3)

Sol. If (x_0, y_0) is point on hyperbola then

tangent at (x_0, y_0) is parallel to $3x+2y=1$

Equation of tangent $\rightarrow \frac{xx_0}{12}-\frac{yy_0}{9}=2$

Slope of tangent $=\frac{-3}{2}$

Equation of tangent in slope form

$$y=\frac{-3}{2}x\pm\sqrt{12\cdot\frac{9}{4}-9}$$

$$y=\frac{-3}{2}x\pm 3\sqrt{2}$$

Or $3x+2y=6\sqrt{2}$

Comparing

$$\frac{\frac{x_0}{12}}{3}=\frac{\frac{-y_0}{9}}{2}=\frac{1}{6\sqrt{2}}$$

$$x_0=3\sqrt{2},\ y_0=\frac{-3}{\sqrt{2}}$$

$$\sqrt{2}\left(y_0-x_0\right)=-3-6=-9$$

70. The number of integral values of k, for which one root of the equation $2x^2 - 8x + k = 0$ lies in the interval (1, 2) and its other root lies in the interval (2, 3) is

(1) 1
(2) 2
(3) 3
(4) 0

Answer (1)

Sol. $f(1)>0 \Rightarrow k>6$

$f(2)<0 \Rightarrow k<8$

$f(3)>0 \Rightarrow k>6$

$k \in (6,\ 8)$

Only 1 integral value of k is 7

71. Which of the following statements is a tautology?

(1) $(p \wedge (p \to q)) \to \sim q$

(2) $p \vee (p \wedge q)$

(3) $(p \wedge q) \to (\sim(p) \to q)$

(4) $p \to (p \wedge (p \to q))$

Answer (3)

Sol. $\sim p \to q \equiv \sim(\sim p) \vee q \equiv p \vee q$

$p \wedge q \to (\sim p \to q)$

$\equiv p \wedge q \to (p \vee q)$

$\equiv \sim(p \wedge q) \vee (p \vee q)$

$\equiv (\sim p \vee \sim q) \vee (p \vee q)$

$\equiv (\sim p \vee (p \vee q)) \vee (\sim q \vee (p \vee q))$

$\equiv T \vee T$

$\equiv T$

72. Let $P(S)$ denote the power set of $S = \{1, 2, 3,\ldots,10\}$. Define the relations R_1 and R_2 on $P(S)$ as AR_1B if $\left(A \cap B^c\right) \cup \left(B \cap A^c\right) = \phi$ and AR_2B if $A \cup B^c = B \cup A^c$, $\forall A, B \in P(S)$. Then

(1) Only R_2 is an equivalence relation

(2) Both R_1 and R_2 are not equivalence relations

(3) Only R_1 is an equivalence relation

(4) Both R_1 and R_2 are equivalence relations

Answer (4)

Sol. $R_1 : (A \cap B^C) \cup (B \cap A^C) = \phi$

$\Rightarrow \boxed{A = B}$

$R_2 : (A \cup B^C) = (B \cup A^C)$

$\Rightarrow \boxed{A = B}$

Both R_1 and R_2 are equivalence.

73. Let $\alpha x = \exp\left(x^{\beta} y^{\gamma}\right)$ be the solution of differential equation $2x^2ydy - (1 - xy^2)dx = 0$, $x > 0$, $y(2) = \sqrt{\log_e 2}$. Then $\alpha + \beta - \gamma$ equals

(1) –1
(2) 1
(3) 3
(4) 0

Answer (2)

Sol. Given differential equation

$$2x^2ydy-(1-xy^2)dx=0,\ x>0$$

$$2xy\,dy+y^2dx=\frac{1}{x}dx$$

$$\int d(xy^2)=\int \frac{1}{x}dx$$

$$xy^2=\ln x+C \qquad \ldots(i)$$

$$y(2)=\sqrt{\log_e 2}$$

$$2\ln 2=\ln 2+C$$

$\therefore\ \boxed{C=\ln 2}$

$\therefore$ by (i)

$xy^2 = \ln 2x$

$2x = e^{xy^2}$

$\therefore\ \alpha = 2, \beta = 1, \gamma = 2$

$\therefore\ \alpha + \beta - \gamma = 1$

74. For the system of linear equations $ax+y+z=1$, $x+ay+z=1$, $x+y+az=\beta$, which one of the following is **NOT** correct?

(1) It has infinitely many solutions if α = 1 and β = 1

(2) $x+y+z=\frac{3}{4}$ if α = 2 and β = 1

(3) It has no solution if α = –2 and β = 1

(4) It has infinitely many solutions if α = 2 and β = –1

Answer (4)

Sol. For infinite solution $\Delta=\Delta_x=\Delta_y=\Delta_z=0$

$$\Delta=\begin{vmatrix}\alpha & 1 & 1\\ 1 & \alpha & 1\\ 1 & 1 & \alpha\end{vmatrix}=0 \Rightarrow (\alpha^3-3\alpha+2)=0 \Rightarrow \alpha=1,-2$$

If β = 1, then all planes are overlapping

$\therefore$ Option (1) is correct.

Option (2)

α = 2, β = 1

$2x + y + z = 1$

$x + 2y + z = 1$

$x + y + 2z = 1$

Adding all three equations

$$x+y+z=\frac{3}{4}$$

$\therefore$ option (2) is correct.

Option (3)

If α = –2 and β = 1, then Δ = 0, $\Delta_x \neq 0$

$\therefore$ No solution

$\therefore$ Option (3) is correct.

Option (4)

If $\alpha = 2 \Rightarrow \Delta \neq 0$

$\therefore$ Unique solution exist

$\therefore$ Option (4) is incorrect.

$\therefore$ Option (4) is answer.

75. Two dice are thrown independently. Let A be the event that the number appeared on the 1st die is less than the number appeared on the 2nd die, B be the event that the number appeared on the 1st die is even and that one the second die is odd, and C be the event that the number appeared on the 1st die is odd and that on the 2nd is even. Then

(1) A and B are mutually exclusive

(2) The number of favourable cases of the events A, B and C are 15, 6 and 6 respectively

(3) The number of favourable cases of the event $(A\cup B)\cap C$ is 6

(4) B and C are independent

Answer (3)

Sol. $$A=\begin{cases}(1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\\ (2, 3), (2, 4), (2, 5), (2, 6)\\ (3, 4), (3, 5), (3, 6)\\ (4, 5), (4, 6)\\ (5, 6)\end{cases}$$

$n(A) = 15$

$$B=\begin{cases}(2, 1), (2, 3), (2, 5)\\ (4, 1), (4, 3), (4, 5)\\ (6, 1), (6, 3), (6, 5)\end{cases}$$

$n(B) = 9$

Similarly, $n(C) = 9$

$(4, 5) \in A$ and $(4, 5) \in B$

$\therefore$ A and B are not exclusive events

$n((A\cup B)\cap C)=n(A\cap C)+n(B\cap C)-n(A\cap B\cap C)$

$= 3 + 3 – 0$

$= 6$

Option (3) is correct.

$$n(B)=\frac{9}{36}, n(C)=\frac{9}{36}, n(B\cap C)=0$$

$\Rightarrow$ $n(B)\cdot n(C)\neq n(B\cap C)$

$\therefore$ B and C are not independent

76. The area of the region given by $\{(x,y): xy\le 8, 1\le y\le x^2\}$ is :

(1) $8\log_e 2-\frac{13}{3}$

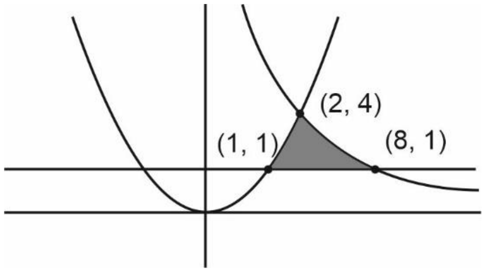
(2) $16\log_e 2+\frac{7}{3}$

(3) $16\log_e 2-\frac{14}{3}$

(4) $8\log_e 2+\frac{7}{6}$

Answer (3)

Sol.



Required area = $\int_1^2 (x^2 - 1)\,dx + \int_2^8 \left(\frac{8}{x} - 1\right) dx$

$= \left(\frac{x^3}{3} - x\right)\Bigg|_1^2 + (8\ln x - x)\Big|_2^8$

$= \left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right] + [8\ln 8 - 8 - (8\ln 2 - 2)]$

$= \frac{4}{3} + 8\ln 4 - 6$

$= 8\ln 4 - \frac{14}{3}$

$= 16\ln 2 - \frac{14}{3}$

77. If $y(x) = x^x$, $x > 0$, then $y''(2) - 2y'(2)$ is equal to:

(1) $4(\log_e 2)^2 - 2$

(2) $4(\log_e 2)^2 + 2$

(3) $4\log_e 2 + 2$

(4) $8\log_e 2 - 2$

Answer (1)

Sol. $y = x^x$

$y' = x^x(1 + \ln x)$

$y'' = x^x(1 + \ln x)^2 + \frac{x^x}{x}$

$f''(2) - 2f'(2) = \left(4(1 + \ln 2)^2 + 2\right) - (2)(4(1 + \ln 2))$

$= 4(1 + (\ln 2)^2) + 2 - 8$

$= 4(\ln 2)^2 - 2$

78. Let $\vec{a} = 5\hat{t} - \hat{j} - 3\hat{k}$ and $\vec{b} = i + 3\hat{j} + 5\hat{k}$ be two vectors. Then which one of the following statements is True?

(1) Projection of $\vec{a}$ on $\vec{b}$ is $\frac{-17}{\sqrt{35}}$ and the direction of the projection vector is same as of $\vec{b}$.

(2) Projection of $\vec{a}$ on $\vec{b}$ is $\frac{17}{\sqrt{35}}$ and the direction of the projection vector is opposite to the direction of $\vec{b}$.

(3) Projection of $\vec{a}$ on $\vec{b}$ is $\frac{17}{\sqrt{35}}$ and the direction of the projection vector is same as of $\vec{b}$.

(4) Projection of $\vec{a}$ on $\vec{b}$ is $\frac{-17}{\sqrt{35}}$ and the direction of the projection vector is opposite to the direction of $\vec{b}$

Answer (2)

Sol. $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}, \vec{b} = \hat{i} + 3\hat{j} + 5\hat{k}$

projection of $\vec{a} \cdot \vec{b} = \left|\frac{5 - 3 - 15}{\sqrt{35}}\right| = \frac{17}{\sqrt{35}}$

$\vec{a} \cdot \vec{b} < 0$

$\therefore$ Option (2) is correct.

79. Let $9 = x_1 < x_2 < \ldots < x_7$ be in an A.P. with common difference d. If the standard deviation of $x_1, x_2, \ldots, x_7$ is 4 and the mean is $\bar{x}$, then $\bar{x} + x_6$ is equal to:

(1) $18\left(1 + \frac{1}{\sqrt{3}}\right)$

(2) 34

(3) 25

(4) $2\left(9 + \frac{8}{\sqrt{7}}\right)$

Answer (2)

Sol. Let the series be $a - 3d$, $a - 2d$, $a - d$, a, $a + d$, $a + 2d$, $a + 3d$

Given $x_1 = 9 \Rightarrow a - 3d = 9$ ...(i)

Variance does not change of shifting origin

$\therefore$ Variance and mean of

$-3d, -2d, -d, 0, d, 2d, 3d$ is 16 and $\bar{x} - a$

$\Rightarrow 16 = \frac{2}{7}(9d^2 + 4d^2 + d^2) - (0)^2$

$\Rightarrow \quad 16 = \frac{2}{7} \times 14d^2$

$\Rightarrow \quad d = 2$ (A.P. is increasing)

Using (i)

$a = 15$

$x_6 = a + 2d$

$= 15 + 4 = 19$

$\bar{x} + x_6 = a + 19$

$= 15 + 19$

$= 34$

$\therefore$ option (2) is correct.

80. If $A = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$, then:

(1) $A^{30} - A^{25} = 2I$ (2) $A^{30} + A^{25} - A = I$

(3) $A^{30} + A^{25} + A = I$ (4) $A^{30} = A^{25}$

Answer (2)

Sol. $A = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$

Let $\theta = \frac{\pi}{3}$

$$A^2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$A^3 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$$

$$\therefore \quad A^{30} = \begin{bmatrix} \cos 30\theta & \sin 30\theta \\ -\sin 30\theta & \cos 30\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{25} = \begin{bmatrix} \cos 25\theta & \sin 25\theta \\ -\sin 25\theta & \cos 25\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} = A$$

$\therefore \quad A^{30} + A^{25} - A = I$

$\therefore$ option (2) is correct.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

81. If the term without x in the expansion of $\left(x^{\frac{2}{3}} + \frac{\alpha}{x^3}\right)^{22}$ is 7315, then $|\alpha|$ is equal to _______.

Answer (01)

Sol. Given expansion $\left(x^{\frac{2}{3}} + \frac{\alpha}{x^3}\right)^{22}$

$$T_{r+1} = {}^{22}C_r \left(x^{\frac{2}{3}}\right)^{22-r} \left(\frac{\alpha}{x^3}\right)^r$$

For constant term

$$\frac{44-2r}{3} - 3r = 0$$

$\boxed{r = 4}$

Now ${}^{22}C_4 \alpha^4 = 7315$

$$\frac{22 \times 21 \times 20 \times 19}{4 \times 3 \times 2 \times 1}\alpha^4 = 7315$$

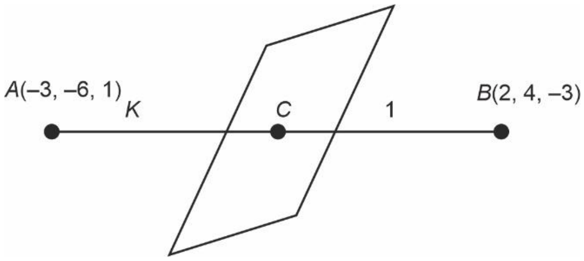
$\therefore \quad \alpha^4 = 1$

$\therefore \quad |\alpha| = 1$

82. The point of intersection C of the plane $8x + y + 2z = 0$ and the line joining the points $A(-3, -6, 1)$ and $B(2, 4, -3)$ divides the line segment AB internally in the ratio $k : 1$. If a, b, c ($|a|, |b|, |c|$ are coprime) are the direction ratios of the perpendicular from the point C on the line $\frac{1-x}{1} = \frac{y+4}{2} = \frac{z+2}{3}$, then $|a + b + c|$ is equal to _______.

Answer (10)

Sol.



Then $c \equiv \left(\frac{2K-3}{K+1}, \frac{4K-6}{K+1}, \frac{-3K+1}{K+1}\right)$

It lies on $8x + y + 2z = 0$

$\therefore$ $16K - 24 + 4K - 6 - 6K + 2 = 0$

$\therefore$ $\boxed{K = 2}$

$\therefore$ $C \equiv \left(\frac{1}{3}, \frac{2}{3}, \frac{-5}{3}\right)$

Given line : $\frac{x-1}{-1} = \frac{y+4}{2} = \frac{z+2}{3} = t$

$x = -t + 1,\ y = 2t - 4,\ z = 3t - 2$

for 1r

$-\left(1-t-\frac{1}{3}\right)+2\left(2t-4-\frac{2}{3}\right)+3\left(3t-2+\frac{5}{3}\right)=0$

$14t = 11 \Rightarrow t = \frac{11}{14}$

$\therefore$ $P.R' \left\langle \frac{-5}{3\times 14}, \frac{-130}{3\times 14}, \frac{85}{3\times 14} \right\rangle$

$\therefore$ $\langle a+b+c \rangle = 10$

83. If the x-intercept of a focal chord of the parabola $y^2 = 8x + 4y + 4$ is 3, then the length of the chord is equal to _______.

Answer (16)

Sol. $y^2 = 8x + 4y + 4$

$(y - 2)^2 = 8(x + 1)$

Focus $\equiv (1, 2)$

Equation of focal chord : $\frac{x}{3} + \frac{y}{b} = 1$ and $\frac{1}{3} + \frac{2}{b} = 1$

$\therefore$ $\boxed{b = 3}$

$\therefore$ $x + y = 3$

Intersection with parabola

$y^2 + 4 - 4y = 8(4 - y)$

$y^2 + 4y - 28 = 0$

$\therefore$ $(y_1 - y_2)^2 = 16 + 4 \times 28$

$(x_1 - x_2)^2 = 16 + 4 \times 28$

$\therefore$ length = $\sqrt{2 \times 16 \times 8} = 16$

84. Let $\alpha x + \beta y + \gamma z = 1$ be the equation of a plane passing through the point (3, –2, 5) and perpendicular to the line joining the points (1, 2, 3) and (–2, 3, 5). Then the value of $\alpha\beta\gamma$ is equal to _______.

Answer (06)

Sol. Plane :

$a(x-3) + b(y+2) + c(z-5) = 0$

Dr's of plane : $3\hat{i} - \hat{j} - 2\hat{k}$

$< 3, -1, -2 >$

$P : 3(x-3) - 1(y+2) - 2(z-5) = 0$

$3x - 9 - y - 2 - 2z + 10 = 0$

$3x - y - 2z = 1$

$\therefore$ $\alpha = 3, \beta = -1, \gamma = -2$

$\alpha\beta\gamma = 6$

85. The line $x = 8$ is the directrix of the ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the corresponding focus (2, 0). If the tangent to E at point P in the first quadrant passes through the point $\left(0, 4\sqrt{3}\right)$ and intersects the x-axis at Q, then $(3PQ)^2$ is equal to

Answer (39)

Sol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\left.\begin{array}{l} F(2,0) \equiv (ae, 0) \\ x = 8 \equiv x = \frac{a}{e} \end{array}\right\} \left.\begin{array}{l} a = 4 \\ e = \frac{1}{2} \end{array}\right\} \Rightarrow b^2 = 12$$

$E : \frac{x^2}{16} + \frac{y^2}{12} = 1$

$T : y = mx \pm \sqrt{16m^2 + 12}$

Passes through $\left(0, 4\sqrt{3}\right)$

$4\sqrt{3} = \pm\sqrt{16m^2 + 12}$

$\therefore \quad m = \pm\frac{3}{2}$

$T : y = \frac{-3}{2}x + \sqrt{48}$

$T : \frac{3}{2}x + y = \sqrt{48} \qquad \ldots(i)$

$T : \frac{xx_1}{16} + \frac{yy_1}{12} = 1 \qquad \ldots(ii)$

Comparing (i) and (ii)

$P\left(\frac{\sqrt{48}}{2}, \frac{\sqrt{48}}{4}\right)$

$Q\left(\frac{2\sqrt{48}}{3},\ 0\right)$

$3(PQ)^2 = 9(PQ)^2$

$= 9\left(\left(\frac{\sqrt{48}}{2} - \frac{2\sqrt{48}}{3}\right)^2 + \left(\frac{\sqrt{48}}{4}\right)^2\right)$

= 39

86. Number of integral solutions to the equation $x + y + z = 21$, where $x \geq 1, y \geq 3, z \geq 4$, is equal to ______.

Answer (105)

Sol. $x + y + z = 21$

$\because \quad x \geq 1, y \geq 3, y \geq 4$

$\therefore \quad x_1 + y_1 + z_1 = 13$

Number of solutions = $13 + 3 - {}^1C_{3-1}$

$= {}^{15}C_2 = \frac{151}{21,131} = 7 \times 15$

= 105

87. Let the sixth term in the binomial expansion of $\left(\sqrt{2^{\log_2(10-3^x)}} + \sqrt[5]{2^{(x-2)\log_2^3}}\right)^m$, in the increasing powers of $2^{(x-2)\log_2^3}$, be 21. If the binomial coefficients of the second, third and fourth terms in the expansion are respectively the first, third and fifth terms of an A.P., then the sum of the squares of all possible values of x is ________.

Answer (04)

Sol. ${}^mC_1, {}^mC_2, {}^mC_3$ are first, third and fifth term of AP

$\therefore \quad a = {}^mC_1$

$a + 2d = {}^mC_2$

$a + 4d = {}^mC_3$

$\therefore \quad 2\,{}^mC_2 - {}^mC_3 = m$

$\Rightarrow \quad m = 7$ or $m = 2$

$\because \quad m = 2$ is not possible

$\therefore \quad m = 7$

$\left(\sqrt{2^{\log_2(10-3^x)}} + \sqrt[5]{2^{(x-2)\log_2^3}}\right)^m$

$T_6 = 21$

${}^7C_5\left(\left(10 - 3^x\right)^{\frac{1}{2}}\right)^{7-5}(3)^{x-2} = 21$

$\frac{27}{9}3^x\left(10 - 3^x\right) = 27$

$3^x(10 - 3^x) = 9$

Let $3^x = t$

$t(10 - t) = 9$

$t^2 - 10t + 9 = 0$

$(t - 9)(t - 1) = 0$

$t = 9$ or $t = 1$

$3^x = 9$ or $3^x = 1$

$\therefore \quad x_1 = 2$ or $x_1 = 0$

$x_1^2 + x_2^2 = 4$

88. The sum of the common terms of the following three arithmetic progressions.

3, 7, 11, 15, …, 399,

2, 5, 8, 11, …, 359 and

2, 7, 12, 17, …, 197,

Is equal to ______.

Answer (321)

Sol. $S_1 \rightarrow 3, 7, 11, \ldots, 399$

$S_2 \rightarrow 2, 5, 8, \ldots, 359$

$S_3 \rightarrow 2, 7, 12, \ldots, 197$

Common terms of S_2 and S_3 are given by

$S_4 \to 2, 17, 32, \ldots, a_n$

$a_n \le 197$

$2+15(n-1)\le 197$

$n \le 14$

$S_4 \to 2, 17, 32, \ldots, 197$

Common terms of S_4 and S_1 are given by

47, 107, 167

Sum = 47 + 107 + 167 = 321

89. The total number of six digit numbers, formed using the digits 4, 5, 9 only and divisible by 6, is _______.

Answer (81)

Sol. Units, place must be occupied by 4 and hence, at least one 4 must be there.

Possible combination of 4, 5, 9 are as follows

4	5	9		No. of Number
1	1	4	$\to$	$\frac{5!}{4!}=5$
1	4	1	$\to$	$\frac{5!}{4!}=5$
2	2	2	$\to$	$\frac{5!}{2!2!}=30$
3	0	3		$\frac{5!}{2!3!}=10$
3	3	0		$\frac{5!}{2!3!}=10$
4	1	1		$\frac{5!}{3!}=20$
6	0	0		$\frac{5!}{5!}=1$
				$\text{Total}=81$

90. If $\int_0^{\pi}\frac{5^{\cos x}\left(1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x\right)dx}{1+5^{\cos x}}=\frac{k\pi}{16}$, then k is equal to ________.

Answer (13)

Sol. Let $g(x) = 1 + \cos x \cos 3x + \cos^2 x + \cos^3 x \cos 3x$

Clearly, $g(\pi + x) = g(x)$

$$I=\int_0^{\pi}\frac{5^{\cos x}(g(x))}{1+5^{\cos x}}dx \qquad \text{...(i)}$$

$$I=\int_0^{\pi}\frac{5^{\cos x}\times(g(x))}{1+5^{\cos x}}dx\left(\because \int_0^{\pi}f(x)dx=\int_0^{\pi}f(\pi-x)dx\right)$$

$$I=\int_0^{\pi}\frac{1}{1+5^{\cos x}}g(x) \qquad \text{...(ii)}$$

$$\text{(i)} + \text{(ii)} \Rightarrow 2I=\int_0^{\pi}g(x)dx$$

$$2I=\int_0^{\pi}1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x\,dx$$

$$=\int_0^{\pi}1+\frac{1}{2}(\cos 4x+\cos 2x)+\frac{1}{2}(1+\cos 2x)+\frac{1}{4}(\cos 3x+3\cos x)\cos 3x\,dx$$

$$=\pi+\frac{1}{2}(0+0)+\frac{\pi}{2}+\frac{1}{2}(0)+\frac{1}{4}\int\cos^2 3x+3\cos x\cos 3x\,dx$$

$$=\frac{3\pi}{2}+\frac{1}{4}\int\frac{1}{2}(1+\cos 6x)+\frac{3}{2}(\cos 4x+\cos 2x)dx$$

$$=\frac{3\pi}{2}+\frac{1}{4}\left(\frac{\pi}{2}+\frac{1}{2}\times 0+\frac{3}{2}(0+0)\right)=\frac{3\pi}{2}+\frac{\pi}{8}$$

$$=\frac{13\pi}{8}\Bigg|\Rightarrow I=\frac{13\pi}{16}\quad\Rightarrow k=13\Bigg|$$

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