MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

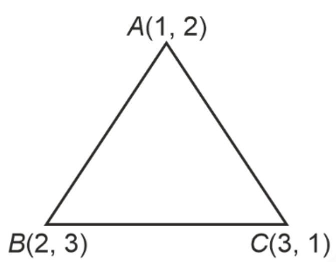
61. If the orthocentre of the triangle, whose vertices are (1, 2), (2, 3) and (3, 1) is (α, β), then the quadratic equation whose roots are $\alpha + 4\beta$ and $4\alpha + \beta$, is

(1) $x^2 - 20x + 99 = 0$ (2) $x^2 - 19x + 90 = 0$

(3) $x^2 - 22x + 120 = 0$ (4) $x^2 - 18x + 99 = 0$

Answer (1)

Sol.



Altitude of BC is $y - 2 = \frac{1}{2}(x - 1) \Rightarrow x - 2y + 3 = 0$

Altitude of AB is $y - 1 = (-1)(x - 3) \Rightarrow x + y = 4$

$\therefore$ Orthocentre $\left(\frac{5}{3}, \frac{7}{3}\right)$

$\therefore$ $\alpha + 4\beta = 11$ and $4\alpha + \beta = 9$

Equation is $x^2 - 20x + 99 = 0$

62. The mean and variance of 5 observations are 5 and 8 respectively. If 3 observations are 1, 3, 5, then the sum of cubes of the remaining two observations is

(1) 1456

(2) 1216

(3) 1792

(4) 1072

Answer (4)

Sol. Let observations 1, 3, 5, a, b

$\Rightarrow \frac{9 + a + b}{5} = 5$ & $\frac{a^2 + b^2 + 35}{5} - 25 = 8$

$\Rightarrow$ $a + b = 16$ & $a^2 + b^2 = 130$

$\therefore$ a & b are 7 & 9

$\therefore$ $a^3 + b^3 = 7^3 + 9^3 = 1072$

63. If the centre and radius of the circle $\left|\frac{z - 2}{z - 3}\right| = 2$ are respectively (α, β) and γ, then $3(\alpha + \beta + \gamma)$ is equal to

(1) 10 (2) 12

(3) 11 (4) 9

Answer (2)

Sol. $(x - 2)^2 + y^2 = 4(x - 3)^2 + 4y^2$

$\Rightarrow$ $3x^2 + 3y^2 - 20x + 32 = 0$

$\therefore$ $C \equiv \left(\frac{10}{3}, 0\right)$ & $r = \sqrt{\left(\frac{10}{3}\right)^2 - \frac{32}{3}} = \frac{2}{3}$

$\therefore$ $3(\alpha + \beta + \gamma) = 3\left(\frac{12}{3}\right) = 12$

64. If $y = y(x)$ is the solution curve of the differential equation $\frac{dy}{dx} + y \tan x = x \sec x$, $0 \le x \le \frac{\pi}{3}$, $y(0) = 1$, then $y\left(\frac{\pi}{6}\right)$ is equal to

(1) $\frac{\pi}{12} - \frac{\sqrt{3}}{2}\log_e\left(\frac{2\sqrt{3}}{e}\right)$

(2) $\frac{\pi}{12} + \frac{\sqrt{3}}{2}\log_e\left(\frac{2}{e\sqrt{3}}\right)$

(3) $\frac{\pi}{12} + \frac{\sqrt{3}}{2}\log_e\left(\frac{2\sqrt{3}}{e}\right)$

(4) $\frac{\pi}{12} - \frac{\sqrt{3}}{2}\log_e\left(\frac{2}{e\sqrt{3}}\right)$

Answer (4)

Sol. $\frac{dy}{dx} + y\tan x = x\sec x$

$\therefore$ $\text{I.F} = e^{\int \tan x dx} = \sec x$

$\Rightarrow$ $y\sec x = \int x\sec^2 x\, dx$

$\Rightarrow$ $y\sec x = x\tan x - \ln|\sec x| + c\cos x$

$\downarrow y(0) = 1$

$\Rightarrow$ $1 = e$

$\therefore$ $y = x\sin x - \cos x \ln|\sec x| + \cos x$

$\therefore$ $y\left(\frac{\pi}{6}\right) = \frac{\pi}{12} - \frac{\sqrt{3}}{2}\ln\left(\frac{2}{\sqrt{3}e}\right)$

65. The sum to 10 terms of the series $\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+....$ is

(1) $\frac{58}{111}$ (2) $\frac{59}{111}$

(3) $\frac{55}{111}$ (4) $\frac{56}{111}$

Answer (3)

Sol. $S=\sum_{r=1}^{10}\frac{r}{1+r^2+r^4}=\frac{1}{2}\sum\left(\frac{1}{r^2-r+1}-\frac{1}{r^2+r+1}\right)$

$T_1=\frac{1}{2}\left(\frac{1}{1^2-1+1}-\frac{1}{1^2+1+1}\right)$

$T_2=\frac{1}{2}\left(\frac{1}{2^2-2+1}-\frac{1}{2^2+2+1}\right)$

$T_3=\frac{1}{2}\left(\frac{1}{3^2-3+1}-\frac{1}{3^2+3+1}\right)$

$\vdots$

$T_{10}=\frac{1}{2}\left(\frac{1}{10^2-10+1}-\frac{1}{10^2+10+1}\right)$

$S=\frac{1}{2}\left(1-\frac{1}{111}\right)=\frac{55}{111}$

66. The combined equation of the two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ can be written as $(ax + by + c)(a'x + b'y + c') = 0$

The equation of the angle bisectors of the lines represented by the equation $2x^2 + xy - 3y^2 = 0$ is

(1) $3x^2 + 5xy + 2y^2 = 0$ (2) $x^2 - y^2 + 10xy = 0$

(3) $3x^2 + xy + 2y^2 = 0$ (4) $x^2 - y^2 - 10xy = 0$

Answer (4)

Sol. $\frac{x^2-y^2}{2-(-3)}=\frac{xy}{\frac{1}{2}}$

OR $x^2 - y^2 = 10xy$

67. Let S be the set of all solutions of the equation $\cos^{-1}(2x)-2\cos^{-1}\left(\sqrt{1-x^2}\right)=\pi,\ x\in\left[-\frac{1}{2},\frac{1}{2}\right]$.

Then $\sum_{x\in S}2\sin^{-1}\left(x^2-1\right)$ is equal to

(1) $\frac{-2\pi}{3}$ (2) 0

(3) $\pi-\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (4) $\pi-2\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

Answer (*)

Sol. $\cos^{-1}(2x)-2\cos^{-1}\left(\sqrt{1-x^2}\right)=\pi$

This is possible only when

$\cos^{-1}(2x)=\pi$ ...(i)

And $2\cos^{-1}\sqrt{1-x^2}=0$ ...(ii)

From (i)

$x=-\frac{1}{2}$

Which does not satisfy (ii)

So no such x exist

68. The value of

$\frac{1}{1!50!}+\frac{1}{3!48!}+\frac{1}{5!46!}+....+\frac{1}{49!2!}+\frac{1}{5!1!}$ is :

(1) $\frac{2^{50}}{51!}$ (2) $\frac{2^{51}}{51!}$

(3) $\frac{2^{50}}{50!}$ (4) $\frac{2^{51}}{50!}$

Answer (1)

Sol. $\frac{1}{(51)!}\left({}^{51}C_1+{}^{51}C_3+...+{}^{51}C_{51}\right)$

$=\frac{2^{50}}{(51)!}$

69. Let S denote the set of all real values of λ such that the system of equations

$\lambda x + y + z = 1$

$x + \lambda y + z = 1$

$x + y + \lambda z = 1$

is inconsistent, then $\sum_{\lambda\in S}\left(|\lambda|^2+|\lambda|\right)$ is equal to

(1) 4 (2) 2

(3) 6 (4) 12

Answer (3)

Sol. $\begin{vmatrix}\lambda & 1 & 1\\ 1 & \lambda & 1\\ 1 & 1 & \lambda\end{vmatrix}=0$

$\lambda(\lambda^2 - 1) - 1(\lambda - 1) + 1(1 - \lambda) = 0$

$\lambda^3-\lambda-\lambda+1+1-\lambda=0$

$\lambda^3-3\lambda+2=0$

$(\lambda-1)\left(\lambda^2+\lambda-2\right)=0$

$\lambda = 1, -2$

For $\lambda = 1 \Rightarrow \infty$ solution

$\lambda = -2 \Rightarrow$ no solution

$\sum_{\lambda \in S} |\lambda|^2 + |\lambda| = 6$

70. For a triangle ABC, the value of $\cos 2A + \cos 2B + \cos 2C$ is least. If its inradius is 3 and incentre is M, then which of the following is NOT correct?

(1) $\overrightarrow{MA} \cdot \overrightarrow{MB} = -18$

(2) perimeter of ΔABC is $18\sqrt{3}$

(3) area of ΔABC is $\frac{27\sqrt{3}}{2}$

(4) $\sin 2A + \sin 2B + \sin 2C = \sin A + \sin B + \sin C$

Answer (3)

Sol. We know that

$\cos 2A + \cos 2B + \cos 2C \geq \frac{-3}{2}$ where equality holds for equilateral triangle

$$r = \frac{\Delta}{s} = \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3}{2}a} = \frac{a}{2\sqrt{3}}$$

$$a = 2\sqrt{3}r = 6\sqrt{3}$$

$$\text{Area} = \frac{\sqrt{3}}{4}a^2 = 27\sqrt{3}$$

71. Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$, $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$. If α and β respectively are the maximum and the minimum values of f, then

(1) $\beta^2 + 2\sqrt{\alpha} = \frac{19}{4}$

(2) $\alpha^2 + \beta^2 = \frac{9}{2}$

(3) $\alpha^2 - \beta^2 = 4\sqrt{3}$

(4) $\beta^2 - 2\sqrt{\alpha} = \frac{19}{4}$

Answer (4)

Sol. $C_1 \to = C_1 + C_2 + C_3$

$$(2+\sin 2x)\begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 1 & 1+\cos^2 x & \sin 2x \\ 1 & \cos^2 x & 1+\sin 2x \end{vmatrix}$$

$R_2 \to R_2 \to R_1$; $R_3 \to R_3 \to R_1$

$$(2+\sin 2x)\begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$f(x) = 2 + \sin 2x;\ x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$$

$$f(x)_{max} = 2 + 1 = 3 \text{ for } x = \frac{\pi}{4}$$

$$f(x)_{min} = 2 + \frac{\sqrt{3}}{2} \text{ for } x = \frac{\pi}{6}, \frac{\pi}{3}$$

$$\beta^2 - 2\sqrt{\alpha} = 4 + \frac{3}{4} + 2\sqrt{3} - 2\sqrt{3}$$

$$= \frac{19}{4}$$

72. The area enclosed by the closed curve C given by the differential equation $\frac{dy}{dx} + \frac{x+a}{y-2} = 0, y(1) = 0$ is 4π.

Let P and Q be the points of intersection of the curve C and the y-axis. If normals at P and Q on the curve C intersect x-axis at points R and S respectively, then the length of the line segment RS is

(1) 2

(2) $\frac{2\sqrt{3}}{3}$

(3) $2\sqrt{3}$

(4) $\frac{4\sqrt{3}}{3}$

Answer (4)

Sol. $\frac{dy}{dx} + \frac{x+a}{y-2} = 0$

$(y - 2)dy + (x + a)dx = 0$

Integrating

$$\frac{y^2}{2} - 2y + \frac{x^2}{2} + ax = C$$

Or $x^2 + 2ax + y^2 - 4y = C$

At $x = 1$, $y = 0$

$1 + 2a = C$

Equation of circle

$x^2 + 2ax + y^2 - 4y = 1 + 2a$

$x^2 + y^2 + 2ax - 4y - (1 + 2a) = 0$

$r = \sqrt{a^2 + 4 + 1 + 2a} = 2$

$a^2 + 2a + 5 = 4 \Rightarrow \boxed{a = -1}$

Curve is $x^2 + y^2 - 2x - 4y + 1 = 0$

Intersection with y-axis

$P = (0, 2 + \sqrt{3}) \quad Q \equiv (0, 2 - \sqrt{3})$

For normal at P & Q

$R = \left(1 + \frac{2}{\sqrt{3}}, 0\right), S = \left(1 - \frac{2}{\sqrt{3}}, 0\right)$

$RS = \frac{4\sqrt{3}}{3}$

73. Let $f(x) = 2x + \tan^{-1} x$ and $g(x) = \log_e\left(\sqrt{1 + x^2} + x\right)$, $x \in [0, 3]$. Then

(1) min $f'(x) = 1 +$ max $g'(x)$

(2) there exist $0 < x_1 < x_2 < 3$ such that $f(x) < g(x)$, $\forall x \in (x_1, x_2)$

(3) there exists $\hat{x} \in [0, 3]$ such that $f'(\hat{x}) < g'(\hat{x})$

(4) max $f(x) >$ max $g(x)$

Answer (4)

Sol. $f'(x) = 2 + \frac{1}{1 + x^2}, g'(x) = \frac{1}{\sqrt{x^2 + 1}}$

$f''(x) = -\frac{2x}{(1 + x^2)^2} < 0$

$g''(x) = -\frac{1}{2}(x^2 + 1)^{-3/2} \cdot 2x < 0$

$f'(x)|_{\min} = f'(3) = 2 + \frac{1}{10} = \frac{21}{10}$

$g'(x)|_{\max} = g'(0) = 1$

$f'(x)|_{\max} = f(3) = 2 + \tan^{-1} 3$

$g(x)|_{\max} = g(3) = \ln(3 + \sqrt{10}) < \ln < 7 < 2$

74. In a binomial distribution $B(n, p)$, the sum and the product of the mean and the variance are 5 and 6 respectively, then $6(n + p - q)$ is equal to

(1) 52 (2) 50

(3) 53 (4) 51

Answer (1)

Sol. $np + npq = 5$

$np(1 + q) = 5 \quad \ldots(i)$

$np(npq) = 6 \quad \ldots(ii)$

$\Rightarrow np = 3, npq = 2$

$\Rightarrow q = \frac{2}{3}, p = \frac{1}{3}, n = 9$

$6(n + p - q) = 6\left(9 + \frac{1}{3} - \frac{2}{3}\right) = 6\left(9 - \frac{1}{3}\right)$

$= 52$

75. The shortest distance between the lines $\frac{x-5}{1} = \frac{y-2}{2} = \frac{z-4}{-3}$ and $\frac{x+3}{1} = \frac{y+5}{4} = \frac{z-1}{-5}$ is

(1) $5\sqrt{3}$ (2) $6\sqrt{3}$

(3) $4\sqrt{3}$ (4) $7\sqrt{3}$

Answer (2)

Sol. $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 1 & 4 & -5 \end{vmatrix} = \hat{i}(2) - \hat{j}(-2) + \hat{k}(2)$

$\therefore \vec{b_1} \times \vec{b_2} = \hat{i} + \hat{j} + \hat{k}$

$\vec{a_1} - \vec{a_2} = 8\hat{i} + 7\hat{j} + 3\hat{k}$

$d = \left|\frac{(\vec{a_1} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}\right| = \left|\frac{8 + 7 + 3}{\sqrt{3}}\right| = \frac{18}{\sqrt{3}} = 6\sqrt{3}$

76. $\lim_{n \to \infty}\left[\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \ldots + \frac{1}{2n}\right]$ is equal to

(1) 0 (2) $\log_e\left(\frac{3}{2}\right)$

(3) $\log_e 2$ (4) $\log_e\left(\frac{2}{3}\right)$

Answer (3)

Sol. $\lim_{n \to \infty}\left(\frac{1}{n+1} + \frac{1}{n+2} \ldots \frac{1}{n+n}\right)$

$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n}\left(\frac{1}{1 + \left(\frac{r}{n}\right)}\right)$

$= \int_0^1 \frac{dx}{1+x} = \log(1+x)_0^1 = \log 2$

77. Let R be a relation on $\mathbb{R}$, given by

$R = \{(a, b) : 3a - 3b + \sqrt{7}$ is an irrational number$\}$.

Then R is

(1) reflexive but neither symmetric nor transitive

(2) an equivalence relation

(3) reflexive and symmetric but not transitive

(4) reflexive and transitive but not symmetric

Answer (1)

Sol. For reflexive:

$3a - 3a + \sqrt{7}$ is an irrational number $\forall a \in R$ R is reflexive

For symmetric

Let $3a - 3b + \sqrt{7}$ is an irrational number

$\Rightarrow$ $3b - 3a + \sqrt{7}$ is an irrational number

For e.g., Let $3a - 3b = \sqrt{7}$

$\sqrt{7} + \sqrt{7}$ is irrational but $-\sqrt{7} + \sqrt{7}$ is not.

$\therefore$ R is not symmetric

For transitive:

Let $3a - 3b + \sqrt{7}$ is irrational and $3b - 3c + \sqrt{7}$ is irrational

$\Rightarrow$ $3a - 3c + \sqrt{7}$ is irrational

For e.g., take $a = 0$, $b = -\sqrt{7}, c = \dfrac{\sqrt{7}}{3}$

R is not transitive

78. The negation of the expression $q \vee ((\sim q) \wedge p)$ is equivalent to

(1) $p \wedge (\sim q)$

(2) $(\sim p) \vee (\sim q)$

(3) $(\sim p) \vee q$

(4) $(\sim p) \wedge (\sim q)$

Answer (4)

Sol. $q \vee (\sim q \wedge p)$

$\Rightarrow$ $(q \vee \sim q) \wedge (q \vee p)$

$\Rightarrow$ $T \wedge (q \vee p)$

$\Rightarrow$ $q \vee p$

Now,

$\sim (q \vee p)$

$= \sim q \wedge \sim p$

79. Let $S = \left\{ x : x \in \mathbb{R} \text{ and } (\sqrt{3} + \sqrt{2})^{x^2-4} + (\sqrt{3} - \sqrt{2})^{x^2-4} = 10 \right\}$.

Then $n(S)$ is equal to

(1) 2

(2) 4

(3) 0

(4) 6

Answer (4)

Sol. Let $\left(\sqrt{3} + \sqrt{2}\right)^{x^2-4} = t$

$t + \dfrac{1}{t} = 10$

$\Rightarrow$ $t^2 - 10t + 1 = 0$

$\Rightarrow$ $t = \dfrac{10 \pm \sqrt{100 - 4}}{2} = 5 \pm 2\sqrt{6}$

Case-I

$t = 5 + 2\sqrt{6}$

$\Rightarrow$ $\left(\sqrt{3} + \sqrt{2}\right)^{x^2-4} = \left(\sqrt{3} + \sqrt{2}\right)^2$

$\Rightarrow$ $x^2 - 4 = 2 \Rightarrow x^2 = 6 \Rightarrow x = \pm\sqrt{6}$

Case-II

$t = 5 - 2\sqrt{6}$

$\left(\sqrt{3} + \sqrt{2}\right)^{x^2-4} = \left(\sqrt{3} - \sqrt{2}\right)^2$

$\Rightarrow$ $\left(\left(\sqrt{3} - \sqrt{2}\right)^{-1}\right)^{x^2-4} = \left(\sqrt{3} - \sqrt{2}\right)^2$

$\Rightarrow$ $4 - x^2 = 2$

$\Rightarrow$ $x^2 = 2$

$\Rightarrow$ $x = \pm\sqrt{2}$

80. Let the image of the point $P(2, -1, 3)$ in the plane $x + 2y - z = 0$ be Q. Then the distance of the plane $3x + 2y + z + 29 = 0$ from the point Q is

(1) $2\sqrt{14}$

(2) $\dfrac{22\sqrt{2}}{7}$

(3) $\dfrac{24\sqrt{2}}{7}$

(4) $3\sqrt{14}$

Answer (4)

Sol. $P(2, -1, 3)$ Plane: $x + 2y - z = 0$

Let $Q(\alpha, \beta\ \gamma)$

Then,

$$\frac{\alpha-2}{1}=\frac{\beta+1}{2}=\frac{\gamma-3}{-1}=\frac{-2(-3)}{6}$$

$\therefore$ $\alpha = 3, \beta = 1, \gamma = 2$

Now distance of Q from the plane $3x + 2y + z + 29 = 0$

$$\left(d=\frac{9+2+2+29}{\sqrt{14}}=\frac{42}{\sqrt{14}}=3\sqrt{14}\right)$$

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

81. Let $a_1 = 8, a_2, a_3, \ldots, a_n$ be an A.P. If the sum of its first four terms is 50 and the sum of its last four terms is 170, then the product of its middle two terms is ________.

Answer (754)

Sol. Given, $a_1 = 8, a_2, a_3 \ldots a_n$ are in A.P.

Now $2(16 + 3d) = 50$

$3d = 9 \Rightarrow \boxed{d=3}$

Now $2(2a_n - 9) = 170$

$a_n = 47$

$8 + (n - 1)\,3 = 47$

$\boxed{n=14}$

Product of middle two terms = $a_7 \times a_8$

$= (8 + 18)(8 + 21)$

$= 26 \times 29$

$= 754$

82. If $\int_0^1 \left(x^{21}+x^{14}+x^7\right)\left(2x^{14}+3x^7+6\right)^{\frac{1}{7}} dx=\frac{1}{l}(11)^{\frac{m}{n}}$ where $l, m, n \in \mathbb{N}$, m and n are coprime then $l + m + n$ is equal to ________.

Answer (63)

Sol. $I=\int_0^1\left(x^{21}+x^{14}+x^7\right)\left(2x^{14}+3x^7+6\right)^{1/7}dx$

$I=\int_0^1\left(x^{20}+x^{13}+x^6\right)\left(2x^{21}+3x^{14}+6x^7\right)^{1/7}dx$

Let $2x^{21}+3x^{14}+6x^7=t$

$\Rightarrow\ 42\left(x^{20}+x^{13}+x^6\right)dx=dt$

$I=\frac{1}{42}\int_0^{11}t^{1/7}dt=\frac{1}{42}\frac{7}{8}\left[t^{8/7}\right]_0^{11}$

$=\frac{1}{48}11^{817}$

$\therefore$ $l = 48, m = 8, n = 7$

$\therefore$ $l + m + n = 63$

83. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that $f'(x)+f(x)=\int_0^2 f(t)dt$. If $f(0) = e^{-2}$, then $2f(0) - f(2)$ is equal to ________

Answer (01)

Sol. $f'(x) + f(x) = k$

$\Rightarrow\ e^x f(x)=ke^x+c$

$f(x)=k+ce^{-x}$

$k=\int_0^2\left(k+ce^{-t}\right)dt$

$k=2k+c\cdot\left.\frac{e^{-t}}{-1}\right|_0^2$

$k=2k+c\left(\frac{e^{-2}}{-1}+1\right)$

$\Rightarrow -k=c\left(1-\frac{1}{e^2}\right)$

$f(x)=ce^{-x}-c\left(1-\frac{1}{e^2}\right)$

$f(0)=c-c+\frac{c}{e^2}=\frac{1}{e^2}\ \Rightarrow c=1$

$f(2)=e^{-2}-r\left(1-e^{-2}\right)$

$=2e^{-2}-1$

$2f(0)-f(2)=1$

84. If $f(x) = x^2 + g'(1)x + g''(2)$ and $g(x) = f(1)\,x^2 + xf'(x) + f''(x)$, then the value of $f(4) - g(4)$ is equal to ______.

Answer (14)

Sol. Let $g'(1) = a$ and $g''(2) = b$

$\Rightarrow f(x) = x^2 + ax + b$

Now, $f(1) = 1 + a + b$; $f'(x) = 2x + a$; $f''(x) = 2$

$g(x) = (1 + a + b)x^2 + x(2x + a) + 2$

$\Rightarrow g(x) = (a + b + 3)x^2 + ax + 2$

$\Rightarrow g'(x) = 2x(a + b + 3) + a \Rightarrow g'(1) = 2(a + b + 3) + a = a$

$\Rightarrow a + b + 3 = 0$ …(i)

$g''(x) = 2(a + b + 3) = b$

$\Rightarrow 2a + b + 6 = 0$ …(ii)

Solving (i) and (ii), we get

$a = -3$ and $b = 0$

$f(x) = x^2 - 3x$ and $g(x) = -3x + 2$

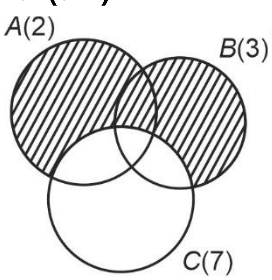
$f(4) = 4$ and $g(4) = -12 + 2 = -10$

$\Rightarrow f(4) - g(4) = 16 - 2 = 14$

85. The number of 3-digit numbers, that are divisible by either 2 or 3 but not divisible by 7, is ______.

Answer (514)

Sol.



A = Numbers divisible by 2

B = Numbers divisible by 3

C = Numbers divisible by 7

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= n(2) + n(3) - n(6)$

$n(A) = n(2) = 100, 102, ..., 998, = 450$

$n(B) = n(3) = 102, 105, ..., 999 = 30$

$n(A \cap B) = n(6) = 102, 108, ..., 996 = 150$

$n(2 \text{ or } 3) = 450 + 300 - 150 = 600$

Now,

$n(A \cap C) = n(14) = 112, 126, ..., 994 = 64$

$n(A \cap B \cap C) = n(42) = 126, 168, ..., 966 = 21$

$n(B \cap C) = n(21) = 105, 126, ..., 987, = 43$

$n(2 \text{ or } 3 \text{ not by } 7) = 600 - [64 + 43 - 21]$

$= 514$

86. The remainder, when $19^{200} + 23^{200}$ is divided by 49, is ______

Answer (29)

Sol. $19^{200} + 23^{200}$

$(21 - 2)^{200} + (21 + 2)^{200} = 49\lambda + 2^{201}$

$2^{201} = 8^{67} = (7 + 1)^{67} = 49\lambda + 7 \times 67 + 1$

$= 49\lambda + 470$

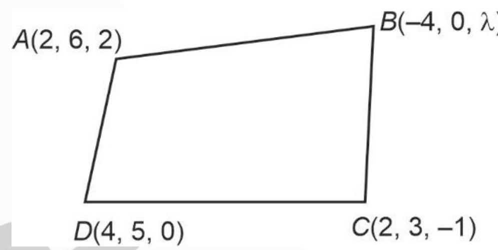
$= 49(\lambda + 9) + 29$

Remainder = 29

87. $A(2, 6, 2)$, $B(-4, 0, \lambda)$, $C(2, 3, -1)$ and $D(4, 5, 0)$, $|\lambda| \le 5$ are the vertices of a quadrilateral $ABCD$. If its area is 18 square units, then $5 - 6\lambda$ is equal to ______.

Answer (11)

Sol.



$\vec{d}_1 = 3\hat{j} + 3\hat{k}$

$\vec{d}_2 = 8\hat{i} + 5\hat{j} - \lambda\hat{k}$

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 3 \\ 8 & 5 & -\lambda \end{vmatrix}$$

$= (-3\lambda - 15)\hat{i} + 24\hat{j} - 24\hat{k}$

$\frac{1}{2}|\vec{d}_1 \times \vec{d}_2| = 18$

$\sqrt{(3\lambda + 15)^2 + 24^2 + 24^2} = 36$

$(3\lambda + 15)^2 = 1296 - 1152$

$3\lambda + 15 = \pm 12$

$3\lambda = -3$ | $3\lambda + 15 = -12$

$\lambda = -1$ | $\lambda = -\frac{27}{3}$

$\lambda = -9$

$\because \lambda \in [-5, 5]$

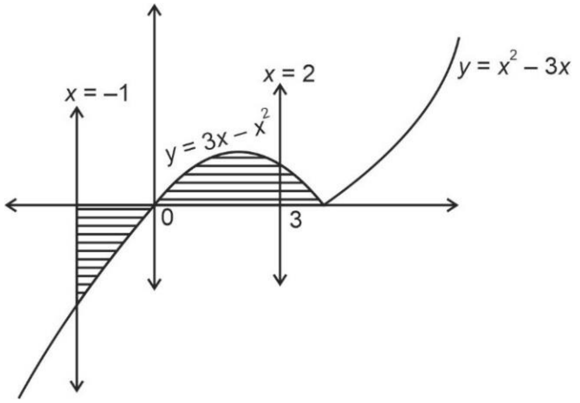
$\therefore \lambda = -1$

$5 - 6(-1) = 11$

88. Let A be the area bounded by the curve $y = x|x - 3|$, the x-axis and the ordinates $x = -1$ and $x = 2$. Then $12A$ is equal to ______.

Answer (62)

Sol.



$$\text{Area} = \int_{-1}^{2} \left|3x - x^2\right|$$

$$A = \int_{-1}^{0} x^2 - 3x\, dx + \int_{0}^{2} 3x - x^2 dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_{-1}^{0} + \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_{0}^{2}$$

$$= 0 - \left(\frac{-1}{3} - \frac{3}{2}\right) + \left(6 - \frac{8}{3}\right) - 0$$

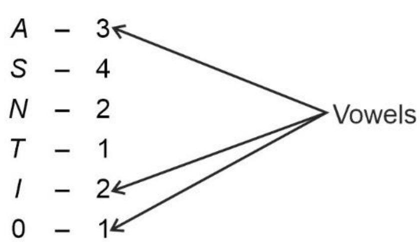
$$= \frac{31}{6}$$

$\therefore \quad 12A = 62$

89. The number of words, with or without meaning, that can be formed using all the letters of word ASSASSINATION so that vowels occur together, is ______.

Answer (50400)

Sol.



Number of arrangements $= \frac{8!}{4!2!} \times \frac{6!}{3!2!} = 50400$

90. Let $\vec{v} = a\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{w} = 2\alpha\hat{i} + \hat{j} - \hat{k}$ and $\vec{u}$ be a vector such that $|\vec{u}| = \alpha > 0$. If the minimum value of the scalar triple product $[\vec{u}\,\vec{v}\,\vec{w}]$ is $-\alpha\sqrt{3401}$, and $|\vec{u}\cdot\hat{i}|^2 = \frac{m}{n}$ where m and n are coprime natural numbers, then $m + n$ is equal to ______.

Answer (3501)

Sol. $\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 2 & -3 \\ 2\alpha & 1 & -1 \end{vmatrix} = \hat{i} - 5\alpha\hat{j} - 3\alpha\hat{k}$

$$[\vec{u}\,\vec{v}\;\;\vec{w}] = \vec{u}\cdot(\vec{v}\times\vec{w})$$

$$= |\vec{u}||\vec{v}\times\vec{w}| \times \cos\theta$$

$$= \alpha\sqrt{34\alpha^2 + 1}\cos\theta$$

$$[\vec{u}\,\vec{v}\,\vec{w}]_{\min} = -\alpha\sqrt{3401}$$

$$\alpha\sqrt{34\alpha^2 + 1} \times (-1) = -\alpha\sqrt{3401}$$

(taking $\cos\theta = 1$)

$\Rightarrow \quad \alpha = 10$

$$\vec{v} \times \vec{w} = \hat{i} - 50\hat{j} - 30\hat{k}$$

$\cos\theta = -1 \Rightarrow \vec{u}$ is antiparallel to $\vec{v} \times \vec{w}$

$$\vec{u} = -|\vec{u}| \cdot \frac{\vec{v}\times\vec{w}}{|\vec{v}\times\vec{w}|} = \frac{-10\left(\hat{i} - 50\hat{j} - 30\hat{k}\right)}{\sqrt{3401}}$$

$$|\vec{u}\cdot\hat{i}|^2 = \left|\frac{-10}{\sqrt{3401}}\right|^2 = \frac{100}{3401} = \frac{m}{n}$$

$m + n = 3501$

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