

## MATHEMATICS

### SECTION - A

**Multiple Choice Questions:** This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

**Choose the correct answer :**

61. The area of the region

$$A = \left\{ (x, y) : |\cos x - \sin x| \leq y \leq \sin x, 0 \leq x \leq \frac{\pi}{2} \right\}$$

is

$$(1) \sqrt{5} - 2\sqrt{2} + 1$$

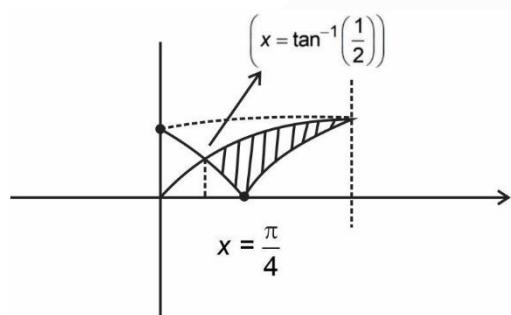
$$(2) \frac{3}{\sqrt{5}} - \frac{3}{\sqrt{2}} + 1$$

$$(3) \sqrt{5} + 2\sqrt{2} - 4.5$$

$$(4) 1 - \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{5}}$$

**Answer (1)**

**Sol.**



$$\text{Area} = \int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} (\sin x - (\cos x - \sin x)) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - (\sin x - \cos x)) dx$$

$$= \int_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} (2\sin x - \cos x) dx + (\sin x)_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= 2\cos x - \sin x \Big|_{\tan^{-1}\frac{1}{2}}^{\frac{\pi}{4}} + \left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} + 1 - \frac{1}{\sqrt{2}}$$

$$= \sqrt{5} - 2\sqrt{2} + 1$$

62. Let  $K$  be the sum of the coefficients of the odd powers of  $x$  in the expansion of  $(1+x)^{99}$ . Let  $a$  be the middle term in the expansion of  $\left(2 + \frac{1}{\sqrt{2}}\right)^{200}$ . If

$$\frac{^{200}C_{99}K}{a} = \frac{2^l m}{n}, \text{ where } m \text{ and } n \text{ are odd numbers,}$$

then the ordered pair  $(l, n)$  is equal to

$$(1) (51, 101) \quad (2) (51, 99)$$

$$(3) (50, 101) \quad (4) (50, 51)$$

**Answer (3)**

$$\text{Sol. } K = 2^{98}$$

$$a = ^{200}C_{100} 2^{50}$$

$$\therefore \frac{^{200}C_{99} \cdot 2^{98}}{^{200}C_{100} \cdot 2^{50}} = \frac{2^l m}{n}$$

$$\Rightarrow \frac{100}{101} \cdot 2^{48} = \frac{2^l m}{n}$$

$$\Rightarrow \frac{25}{101} \cdot 2^{50} = \frac{2^l m}{n}$$

$$\therefore l = 50, m = 25, n = 101$$

63. If the lines  $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+3}{1}$  and  $\frac{x-a}{2} = \frac{y+2}{3} = \frac{z-3}{1}$  intersect at the point  $P$ , then

the distance of the point  $P$  from the plane  $z = a$  is

$$(1) 28 \quad (2) 16$$

$$(3) 10 \quad (4) 22$$

**Answer (1)**

$$\text{Sol. } \frac{x-1}{1} = \frac{y-2}{2} = \frac{z+3}{1} = \lambda \text{ (say)}$$

$$\& \frac{x-a}{2} = \frac{y+2}{3} = \frac{z-3}{1} = \mu \text{ (say)}$$

$$\therefore \lambda + 1 = 2\mu + a \quad \dots(i)$$

$$2\lambda + 2 = 3\mu + 2 \quad \dots(ii)$$

$$\lambda - 3 = \mu + 3 \quad \dots(iii)$$

By (i) & (ii)

$$\Rightarrow 3\mu - 2 = 4\mu + 2a + 2$$

$$\Rightarrow \mu = -2(1+a) \text{ & } \lambda = 5 - 3a$$

Put  $\lambda$  &  $\mu$  in (iii) we get

$$a = -9$$

$$\mu = 16$$

$$\lambda = 22$$

$$\therefore \text{Point of intersection} \equiv (23, 46, 19)$$

Distance from  $z = -9$  is 28

64. If the tangent at a point  $P$  on the parabola  $y^2 = 3x$  is parallel to the line  $x + 2y = 1$  and the tangents at the points  $Q$  and  $R$  on the ellipse  $\frac{x^2}{4} + \frac{y^2}{1} = 1$  are perpendicular to the line  $x - y = 2$ , then the area of the triangle  $PQR$  is

$$(1) 5\sqrt{3}$$

$$(2) 3\sqrt{5}$$

$$(3) \frac{9}{\sqrt{5}}$$

$$(4) \frac{3}{2}\sqrt{5}$$

### Answer (2)

**Sol.**  $P \equiv \left( \frac{A}{m^2}, \frac{2A}{m} \right)$  where  $\left( A = \frac{3}{4}, m = \frac{-1}{2} \right)$

$$\& Q, R = \left( \mp \frac{a^2 m_1}{a^2 m_1^2 + b^2}, \frac{\mp \cdot b^2}{\sqrt{a^2 m_1^2 + b^2}} \right)$$

Where  $a^2 = 4$ ,  $b^2 = 1$  and  $m_1 = 1$

$$\therefore P \equiv (3, -3)$$

$$Q \equiv \left( \frac{-4}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right) \& R \left( \frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$$

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 3 & -3 & 1 \\ -4 & -1 & \sqrt{5} \\ \frac{4}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 \end{vmatrix} = \frac{1}{10} \begin{vmatrix} 3 & -3 & 1 \\ -4 & -1 & \sqrt{5} \\ 0 & 0 & 2\sqrt{5} \end{vmatrix}$$

$$= \frac{2\sqrt{5}}{10}(-15) = 3\sqrt{5}$$

65. The statement  $B \Rightarrow ((\sim A) \vee B)$  is equivalent to

- (1)  $A \Rightarrow (A \Leftrightarrow B)$
- (2)  $B \Rightarrow ((\sim A) \Rightarrow B)$
- (3)  $B \Rightarrow (A \Rightarrow B)$
- (4)  $A \Rightarrow ((\sim A) \Rightarrow B)$

### Answer (2, 3, 4)

**Sol.**  $B \Rightarrow ((\sim A) \vee B) \equiv \sim B \vee (\sim A \vee B)$

$$\equiv (\sim B \vee B) \vee (\sim A) \equiv T$$

$$\text{Option (2)} B \Rightarrow ((\sim A) \Rightarrow B) \equiv (\sim B) \vee (\sim A \Rightarrow B)$$

$$\equiv (\sim B) \vee (A \vee B) \equiv T$$

$$\text{Option (3)} B \Rightarrow (A \Rightarrow B) \equiv B \Rightarrow ((\sim A) \vee B)$$

(same as given)

$$\text{Option (4)} A \Rightarrow ((\sim A) \Rightarrow B) \equiv (\sim A) \vee (A \vee B) \equiv T$$

66. The set of all value of  $t \in \mathbb{R}$ , for which the matrix

$$\begin{bmatrix} e^t & e^{-t}(\sin t - 2\cos t) & e^{-1}(-2\sin t - \cos t) \\ e^t & e^{-t}(2\sin t + \cos t) & e^{-1}(\sin t - 2\cos t) \\ e^t & e^{-1}\cos t & e^{-1}\sin t \end{bmatrix} \quad \text{is}$$

invertible, is

$$(1) \left\{ (2k+1)\frac{\pi}{2}, k \in \mathbb{Z} \right\} \quad (2) \mathbb{R}$$

$$(3) \left\{ k\pi + \frac{\pi}{4}, k \in \mathbb{Z} \right\} \quad (4) \{k\pi, k \in \mathbb{Z}\}$$

### Answer (2)

$$\text{Sol.} \begin{vmatrix} 0 & e^{-t}(-\sin t - 3\cos t) & e^{-t}(-3\sin t + \cos t) \\ 8 & e^{-t}(2\sin t) & e^{-t}(-2\cos t) \\ e^t & e^{-t}\cos t & e^{-t}\sin t \end{vmatrix} = 0$$

(If matrix is non invertible)

$$-2\cos t(-\sin t - 3\cos t) - 2\sin t(\cos t - 3\sin t) = 0$$

$$\Rightarrow 6\cos^2 t + 6\sin^2 t = 0$$

$$t \in 0$$

67. Let  $S = \{w_1, w_2, \dots\}$  be the sample space associated to a random experiment. Let  $P(w_n) = \frac{P(w_{n-1})}{2}$ ,  $n \geq 2$ . Let  $A = \{2k+3l : k, l \in \mathbb{N}\}$  and  $B = \{w_n : n \in A\}$ . Then  $P(B)$  is equal to

$$(1) \frac{1}{16} \quad (2) \frac{1}{32}$$

$$(3) \frac{3}{64} \quad (4) \frac{3}{32}$$

### Answer (3)

$$\text{Sol. } P(w_1) + \frac{P(w_1)}{2} + \frac{P(w_1)}{2^2} + \dots = 1$$

$$\therefore P(w_1) = \frac{1}{2}$$

$$\text{Hence, } P(w_n) = \frac{1}{2^n}$$

Every number except 1, 2, 3, 4, 6 is representable in the form

$2k + 3l$  where  $k, l \in N$ .

$$\begin{aligned}\therefore P(B) &= 1 - P(w_1) - P(w_2) \\ &\quad - P(w_3) - P(w_4) - P(w_6) \\ &= \frac{3}{64}\end{aligned}$$

68. The value of the integral  $\int_1^2 \left( \frac{t^4 + 1}{t^6 + 1} \right) dt$  is

- (1)  $\tan^{-1} 2 - \frac{1}{3} \tan^{-1} 8 + \frac{\pi}{3}$
  - (2)  $\tan^{-1} 2 + \frac{1}{3} \tan^{-1} 8 - \frac{\pi}{3}$
  - (3)  $\tan^{-1} \frac{1}{2} + \frac{1}{3} \tan^{-1} 8 - \frac{\pi}{3}$
  - (4)  $\tan^{-1} \frac{1}{2} - \frac{1}{3} \tan^{-1} 8 + \frac{\pi}{3}$

## Answer (2)

$$\begin{aligned}
 \text{Sol. } & \int_1^2 \frac{t^4 + 1}{t^6 + 1} dt \\
 &= \int_1^2 \frac{(t^2 + 1)^2}{t^6 + 1} dt - 2 \int_1^2 \frac{t^2}{t^6 + 1} dt \\
 &= \int_1^2 \frac{t^2 + 1}{t^4 - t^2 + 1} dt - 2 \int_1^2 \frac{t^2}{(t^3)^2 + 1} dt \\
 &= \tan^{-1}(2t + \sqrt{3}) + \tan^{-1}(2t - \sqrt{3}) \Big|_1^2 \\
 &\quad - \frac{2}{3} \tan^{-1}(t^3) \Big|_1^2 \\
 &= \tan^{-1}(4 + \sqrt{3}) + \tan^{-1}(4 - \sqrt{3}) - \tan^{-1}(2 + \sqrt{3}) \\
 &\quad - \tan^{-1}(2 - \sqrt{3}) \\
 &= \tan^{-1} 2 + \frac{1}{3} \tan^{-1} 8 - \frac{\pi}{3}
 \end{aligned}$$

69. Consider a function  $f : \mathbb{N} \rightarrow \mathbb{R}$ , satisfying

$$f(1) + 2f(2) + 3f(3) + \dots + xf(x) = x(x+1)f(x); x \geq 2$$

with  $f(1) = 1$ . Then  $\frac{1}{f(2022)} + \frac{1}{f(2028)}$  is equal to



### **Answer (3)**

**Sol.**  $f(1) + 2f(2) + 3f(3) + \dots + nf(n) = n(n+1) + (n)$  ... (i)

$$n \rightarrow n + 1$$

$$f(1) + 2f(2) + \dots + (n+1)f(n+1) = (n+1)(n+2)$$

$f(n+1)$  ... (ii)

(i) and (ii) gives

$$3f(3) - 2f(2) = 0$$

$$4f(4) - 3f(3) = 0$$

•  
•  
•

$$(n+1)f(n+1) - nf(n) = 0$$

$$\Rightarrow f(n+1) = \frac{2f(2)}{n+1}$$

$$f(n) = \frac{1}{2n}$$

$$\frac{1}{f(2022)} + \frac{1}{f(2028)} = 8100$$

70. Let  $y = y(x)$  be the solution of the differential equation  $x \log_e x \frac{dy}{dx} + y = x^2 \log_e x$ , ( $x > 1$ ). If  $y(2) = 2$ , then  $y(e)$  is equal to

- (1)  $\frac{1+e^2}{2}$       (2)  $\frac{1+e^2}{4}$   
 (3)  $\frac{4+e^2}{4}$       (4)  $\frac{2+e^2}{2}$

### **Answer (3)**

**Sol.**  $x \ln x \frac{dy}{dx} + y = x^2 \ln x$

$$\frac{dy}{dx} + \frac{1}{x \ln x} \cdot y = x$$

$$\text{If } I = e^{\int \frac{1}{x \ln x} dx} = e^{\int \frac{1}{t} dt}, \text{ where } t = \ln x$$

$$= e^{\ln t} = t = \ln x$$

$$y \cdot \ln x = \int x \ln x = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}$$

$$y \ln x = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C \quad \dots(i)$$

$$y(2) = 2 \Rightarrow C = 1$$

Putting  $x = e$  in (i),

$$y = \frac{e^2}{4} + 1 = \frac{4 + e^2}{4}$$

71. The shortest distance between the lines  $\frac{x-1}{2} = \frac{y+8}{-7} = \frac{z-4}{5}$  and  $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-6}{-3}$  is

- (1)  $4\sqrt{3}$       (2)  $3\sqrt{3}$   
 (3)  $5\sqrt{3}$       (4)  $2\sqrt{3}$

**Answer (1)**

**Sol.**  $\vec{r}_1 = \hat{i} - 8\hat{j} + 4\hat{k}$

$$\vec{r}_2 = \hat{i} + 2\hat{j} + 6\hat{k}$$

$$\vec{a} = 2\hat{i} - 7\hat{j} + 5\hat{k}$$

$$\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$$

$$\text{S.D.} = \frac{|\vec{r}_1 - \vec{r}_2 \quad \vec{a} \quad \vec{b}|}{|\vec{a} \times \vec{b}|}$$

$$\begin{bmatrix} \vec{r}_1 - \vec{r}_2 & \vec{a} & \vec{b} \end{bmatrix} = \begin{vmatrix} 0 & -10 & -2 \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix}$$

$$\therefore 10(-16) - 2(16) = -192$$

$$|\vec{r}_1 - \vec{r}_2 \quad \vec{a} \quad \vec{b}| = 192$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix} = 16\hat{i} + 16\hat{j} + 16\hat{k}$$

$$\vec{a} \times \vec{b} = 16\sqrt{3}$$

$$\text{S.D.} = \frac{192}{16\sqrt{3}} = 4\sqrt{3}$$

72. The letters of the word OUGHT are written in all possible ways and these words are arranged as in a dictionary, in a series. Then the serial number of the word TOUGH is

- (1) 84      (2) 86  
 (3) 89      (4) 79

**Answer (3)**

<b>Sol.</b> G	.....	24
H	.....	24
O	.....	24
T G	.....	6
T H	.....	6
T O G	.....	2
T O H	.....	2
T O U G H	.....	1
		89

73. The value of the integral  $\int_{\frac{1}{2}}^2 \frac{\tan^{-1} x}{x} dx$  is equal to

- (1)  $\frac{\pi}{4} \log_e 2$       (2)  $\pi \log_e 2$   
 (3)  $\frac{\pi}{2} \log_e 2$       (4)  $\frac{1}{2} \log_e 2$

**Answer (3)**

**Sol.**  $I = \int_{\frac{1}{2}}^2 \frac{\tan^{-1} x}{x} dx \quad \dots(i)$

$$x \rightarrow \frac{1}{x}$$

$$I = \int_{\frac{1}{2}}^2 \frac{1}{x} \tan^{-1} \frac{1}{x} dx \quad \dots(ii)$$

$$2I = \int_{\frac{1}{2}}^2 \frac{1}{x} \cdot \frac{\pi}{2} dx$$

$$= \frac{\pi}{2} \ln x \Big|_{\frac{1}{2}}^2 = \pi \ln 2$$

$$\Rightarrow I = \frac{\pi}{2} \ln 2$$

74. Let  $f$  and  $g$  be twice differentiable functions on  $\mathbb{R}$  such that

$$f''(x) = g''(x) + 6x$$

$$f'(1) = 4g'(1) - 3 = 9$$

$$f(2) = 3g(2) = 12.$$

Then which of the following is NOT true?

- (1) There exists  $x_0 \in (1, 3/2)$  such that  $f(x_0) = g(x_0)$   
 (2)  $|f'(x) - g'(x)| < 6 \Rightarrow -1 < x < 1$   
 (3)  $g(-2) - f(-2) = 20$   
 (4) If  $-1 < x < 2$ , then  $|f(x) - g(x)| < 8$

**Answer (4)**

**Sol.**  $f''(x) = g''(x) + 6x$

$$\Rightarrow f'(x) = g'(x) + 3x^2 + C$$

$$f'(1) = g'(1) + 3 + C$$

$$\Rightarrow g = 3 + 3 + C \Rightarrow C = 3$$

$$\Rightarrow f'(x) = g'(x) + 3x^2 + 3$$

$$\Rightarrow f(x) = g(x) + x^2 + 3x + C'$$

$$x = 2$$

$$f(2) = g(2) + 14 + C'$$

$$12 = 4 + 14 + C'$$

$$\Rightarrow C' = -6$$

$$\Rightarrow f(x) = g(2) + x^3 + 3x - 6$$

$$f(-2) = g(-2) - 8 - 6 - 6$$

$$g(-2) - f(-2) = 20$$

$$f'(x) - g'(x) = 3x^2 + 3$$

$$x \in (-1, 1)$$

$$3x^2 + 3 \in (0, 6)$$

$$\Rightarrow f'(x) - g'(x) \in (0, 6)$$

$$f(x) - g(x) = x^3 + 3x - 6$$

At  $x = -1$

$$|f(-1) - g(-1)| = 10$$

∴ Option (4) is false.

75. If  $\vec{a} = \hat{i} + 2\hat{k}$ ,  $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ ,  $\vec{c} = 7\hat{i} - 3\hat{j} + 4\hat{k}$ ,  $\vec{r} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$  and  $\vec{r} \cdot \vec{a} = 0$ . Then  $\vec{r} \cdot \vec{c}$  is equal to

(1) 32

(2) 30

(3) 34

(4) 36

**Answer (3)**

**Sol.**  $(\vec{r} - \vec{c}) \times \vec{b} = 0$

$$\vec{r} = \lambda \vec{b} + \vec{c}$$

$$\vec{r} \cdot \vec{a} = 0$$

$$\Rightarrow \lambda \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a} = 0$$

$$\Rightarrow \lambda(3) + (7 + 8) = 0$$

$$\Rightarrow \lambda = -5$$

$$\vec{r} = 5\vec{b} + \vec{c}$$

$$= -5\hat{i} - 5\hat{j} - 5\hat{k} + (7\hat{i} + 3\hat{j} + 4\hat{k})$$

$$= 2\hat{i} - 8\hat{j} - \hat{k}$$

$$\therefore \vec{r} \cdot \vec{c} = 17 + 24 - 4 = 34$$

76. The plane  $2x - y + z = 4$  intersects the line segment joining the points  $A(a, -2, 4)$  and  $B(2, b, -3)$  at the point  $C$  in the ratio  $2 : 1$  and the distance of the point  $C$  from the origin is  $\sqrt{5}$ . If  $ab < 0$  and  $P$  is the point  $(a - b, b, 2b - a)$  then  $CP^2$  is equal to

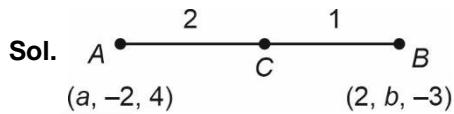
(1)  $\frac{73}{3}$

(2)  $\frac{16}{3}$

(3)  $\frac{97}{3}$

(4)  $\frac{17}{3}$

**Answer (4)**

**Sol.** 

$$C: \left( \frac{a+4}{3}, \frac{-2+2b}{3}, \frac{4-6}{3} \right)$$

$$= \left( \frac{a+4}{3}, \frac{-2+2b}{3}, \frac{-2}{3} \right)$$

$C$  lies on plane  $2x - y + z = 4$

$$2\left(\frac{a+4}{3}\right) - \left(\frac{2b-2}{3}\right) - \frac{2}{3} = 4$$

$$\Rightarrow 2a + 8 - 2b + 2 - 2 = 12$$

$$\Rightarrow a - b = 2 \quad \dots(i)$$

Now,  $OP = \sqrt{5}$

$$\left(\frac{a+4}{3}\right)^2 + \left(\frac{2b-2}{3}\right)^2 + \frac{4}{9} = 5 \text{ and using (i)}$$

$$a = \frac{11}{5}, 1$$

$$\Rightarrow b = \frac{1}{5}, -1$$

as also  $\Rightarrow a = 1, b = -1$

$$\therefore P(2, -1, -3), C\left(\frac{5}{3}, \frac{-4}{3}, \frac{-2}{3}\right)$$

$$CP^2 = \frac{1}{9} + \frac{1}{9} + \frac{49}{9}$$

$$= \frac{17}{3}$$

∴ Option (4) is correct.



- $$3a + 2b + 2a + 3b = 5n$$
- $$\therefore 3a + 2b = 5(n - m)$$
- $$\therefore (b, a) \in R$$
- $$\therefore R \text{ is symmetric}$$
- (3) If  $(a, b) \in R$  and  $(b, c) \in R$  then
- $$2a + 3b = 5m, 2b + 3c = 5n$$
- $$\Rightarrow 2a + 5b + 3c = 5(m + n)$$
- $$\Rightarrow 2a + 3c = 5(m + n - b)$$
- $$\therefore (a, c) \in R$$
- $$\therefore R \text{ is transitive}$$
- Hence  $R$  is equivalence relation.
- option (1) is correct.

### SECTION - B

**Numerical Value Type Questions:** This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE**. For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, -00.33, -00.30, 30.27, -27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

81. Let  $\{a_k\}$  and  $\{b_k\}$ ,  $k \in \mathbb{N}$ , be two G.P.s with common ratios  $r_1$  and  $r_2$  respectively such that  $a_1 = b_1 = 4$  and  $r_1 < r_2$ . Let  $c_k = a_k + b_k$ ,  $k \in \mathbb{N}$ . If  $c_2 = 5$  and  $c_3 = \frac{13}{4}$  then  $\sum_{k=1}^{\infty} c_k - (12a_6 + 8b_4)$  is equal to \_\_\_\_\_.

### Answer (09)

**Sol.**  $\{a_k\}$  be a G.P. with  $a_1 = 4$ ,  $r = r_1$

And

$\{b_k\}$  be G.P. with  $b_1 = 4$ ,  $r = r_2$  ( $r_1 < r_2$ )

Now

$$C_k = a_k + b_k$$

$$c_1 = 4 + 4 = 8 \text{ and } c_2 = 5$$

$$a_2 + b_2 = 5$$

$$\therefore r_1 + r_2 = \frac{5}{4}$$

$$\text{and } c_3 = \frac{13}{4} \Rightarrow r_4^2 + r_2^2 = \frac{13}{16}$$

$$\therefore \frac{25}{16} - 2r_1r_2 = \frac{13}{16} \Rightarrow 2r_1r_2 = \frac{3}{4}$$

$$\therefore r_2 - r_1 = \sqrt{\frac{25}{16} - \frac{3}{2}} = \frac{1}{4}$$

$$\therefore r_2 = \frac{3}{4}, r_1 = \frac{1}{2}$$

$$\therefore a_6 = 4 \times \frac{1}{2^5} = \frac{1}{8}, b_4 = 4 \times \frac{27}{64} = \frac{27}{16}$$

$$\text{and } \sum_{K=1}^{\infty} C_K = 4 \left[ \frac{1}{1 - \frac{1}{2}} + \frac{1}{1 - \frac{3}{4}} \right] = 24$$

$$\therefore \sum_{K=1}^{\infty} C_K - (12a_6 + 8b_4) = 09$$

82. the total number of 4-digit numbers whose greatest common divisor with 54 is 2, is \_\_\_\_\_.

### Answer (3000)

**Sol.**  $\gcd(a, 54) = 2$  when  $a$  is a 4 digit no.

$$\text{And } 54 = 3 \times 3 \times 2$$

So,  $a = \text{all even no. of 4 digits}$

– Even multiple of 3 (4 digits)

$$= 4500 - 1500$$

$$= 3000$$

83. If the equation of the normal to the curve  $y = \frac{x-a}{(x+b)(x-2)}$  at the point  $(1, -3)$  is  $x - 4y = 13$ , then the value of  $a + b$  is equal to \_\_\_\_\_.

### Answer (04)

**Sol.** Given curve :  $y = \frac{x-a}{(x+b)(x-2)}$  at  $(1, -3)$

$$\therefore -3 = \frac{1-a}{(1+b)(-1)} \Rightarrow 3 + 3b = 1 - a$$

$$\beta \Rightarrow a + 3b + 2 = 0$$

$$y = \frac{x-a}{(x+b)(x-2)}$$

$$\frac{dy}{dx} = \frac{(x+b)(x-2) - (x-a)[(x+b) + (x-2)]}{[(x+b)(x-2)]^2}$$

$$\text{at } (1, -3) \quad m_T = \frac{-(1+b) - (1-a)(b)}{(1+b)^2} = -4$$

$$\begin{aligned} \therefore 1 + b + b - ab &= 4(1 + b)^2 \\ \Rightarrow 1 + 2b + b(3b + 2) &= 4b^2 + 4 + 8b \\ \Rightarrow b^2 + 4b + 3 &= 0 \\ (b+1)(b+3) &= 0 \\ b = -1, a = 1 &\quad \text{but } 1 + b \neq 0 \\ b = -3, a = 7 &\quad \therefore b \neq -1 \\ \therefore a + b &= 04 \end{aligned}$$

84. A triangle is formed by the tangents at the point  $(2, 2)$  on the curves  $y^2 = 2x$  and  $x^2 + y^2 = 4x$ , and the line  $x + y + 2 = 0$ . If  $r$  is the radius of its circumcircle, then  $r^2$  is equal to \_\_\_\_\_.

**Answer (10)**

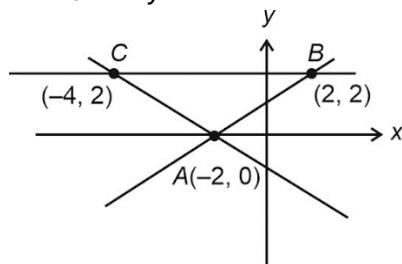
**Sol.** Tangent for  $y^2 = 2x$  at  $(2, 2)$  is

$$L_1 : 2y = x + 2$$

Tangent for  $x^2 + y^2 = 4x$  at  $(2, 2)$  is

$$L_2 : y = 2$$

$$L_3 : x + y = 2 = 0$$



$$\text{Radius of circumcircle} = \frac{abc}{4\Delta}$$

$$= \frac{(\sqrt{20})(6)(\sqrt{8})}{4 \times \frac{1}{2} \times 6 \times 2}$$

$$R = \sqrt{10}$$

$$R^2 = 10$$

85. Let  $X = \{11, 12, 13, \dots, 40, 41\}$  and  $Y = \{61, 62, 63, \dots, 90, 91\}$  be the two sets of observations. If  $\bar{x}$  and  $\bar{y}$  are their respective means and  $\sigma^2$  is the variance of all the observations in  $X \cup Y$ , then  $|\bar{x} + \bar{y} - \sigma^2|$  is equal to \_\_\_\_\_.

**Answer (603)**

**Sol.**  $x = \{11, 12, 13, \dots, 40, 41\}$

$y = \{61, 62, 63, \dots, 90, 91\}$

$$\bar{x} = \frac{\frac{31}{2}(11+41)}{31} = \frac{1}{2} \times 52 = 26$$

$$\bar{y} = \frac{\frac{31}{2}(61+91)}{31} = \frac{1}{2} \times 152 = 76$$

$$\begin{aligned} \sigma^2 &= \frac{\sum x_i^2 + \sum y_i^2}{62} - \left( \frac{\sum x + \sum y}{62} \right)^2 \\ &= 705 \end{aligned}$$

Now

$$\begin{aligned} &|\bar{x} + \bar{y} - \sigma^2| \\ &= |26 + 76 - 705| \\ &= 603 \end{aligned}$$

86. Let  $\alpha = 8 - 14t$ ,  $A = \left\{ z \in \mathbb{C} : \frac{\alpha z - \bar{\alpha} \bar{z}}{z^2 - (\bar{z})^2 - 112i} \right\}$  and

$B = \{z \in \mathbb{C} : |z + 3i| = 4\}$ . Then  $\sum_{z \in A \cap B} (\operatorname{Re} z - \operatorname{Im} z)$  is equal to \_\_\_\_\_.

**Answer (07)**

**Sol.** Let  $z = x + iy$

and  $\alpha = 8 - 14i$

$$\frac{\alpha z - \bar{\alpha} \bar{z}}{z^2 - \bar{z}^2 - 112i} = 1$$

$$\therefore \frac{(16y - 28x)i}{4xy - 112i} = 1$$

$$(16y - 28x + 112)i = 4xy$$

$$\therefore z = -7i \text{ or } 4$$

Now,  $z = -7i$  satisfy  $B$

$$B : x^2 + (y + 3)^2 = 16$$

$$A \cap B = (0, -7)$$

$$\operatorname{Re} z - \operatorname{Im} z = 7$$

87. Let  $A$  be a symmetric matrix such that

$$|A| = 2 \text{ and } \begin{bmatrix} 2 & 1 \\ 3 & \frac{3}{2} \end{bmatrix} A - \begin{bmatrix} 1 & 2 \\ \alpha & \beta \end{bmatrix}$$

diagonal elements of  $A$  is  $s$ , then  $\frac{\beta s}{\alpha^2}$  is equal to \_\_\_\_\_.

**Answer (05)**

**Sol.**  $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

$$|A| = ab - c^2 = 2 \quad \dots(1)$$

$$\begin{pmatrix} 2 & 1 \\ 3 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ \alpha & \beta \end{pmatrix}$$

$$2a + c = 1 \quad \dots(2)$$

$$2c + b = 2 \quad \dots(3)$$

$$3a + \frac{3}{2}c = \alpha \quad \dots(4)$$

$$3c + \frac{3}{2}b = \beta \quad \dots(5)$$

From (1), (2) and (3)

$$a = \frac{3}{4}, b = 3, c = -\frac{1}{2}$$

$$\Rightarrow \text{Now, } \alpha = \frac{6}{4}$$

$$\beta = 3$$

$$s = \frac{15}{4}$$

$$\frac{\beta s}{\alpha^2} = \frac{3 \times \frac{15}{4}}{\left(\frac{6}{4}\right)^2} = \frac{\frac{45}{4}}{\frac{9}{4}} = 5$$

88. Let  $a_1 = b_1 = 1$  and  $a_n = a_{n-1} + (n-1)$ ,  $b_n = b_{n-1} + a_{n-1}$ ,

$$\forall n \geq 2. \text{ If } S = \sum_{n=1}^{10} \frac{b_n}{2^n} \text{ and } T = \sum_{n=1}^8 \frac{n}{2^{n-1}}, \text{ then }$$

$$2^7(2S-T) \text{ is equal to } \dots$$

**Answer (461)**

**Sol.**  $\because a_n = a_{n-1} + (n-1)$  and  $a_1 = b_1 = 1$

$$b_n = b_{n-1} + a_{n-1}$$

$$\therefore b_{n+1} = 2b_n - b_{n-1} + n - 1$$

<b>n</b>	<b>b<sub>n</sub></b>	<b>b<sub>n</sub> - n</b>
1	1	0
2	2	0
3	4	1
4	8	4
5	15	10
6	26	20
7	42	35
8	64	56
9	93	84
10	130	120

$$\begin{aligned} \therefore 2S - T &= \left( \sum_{n=1}^8 \frac{b_n - n}{2^{n-1}} \right) + \frac{b_9}{2^8} + \frac{b_{10}}{2^9} \\ &= \frac{461}{128} \end{aligned}$$

$$\therefore 2^7(2S - T) = 461$$

89. A circle with centre  $(2, 3)$  and radius 4 intersects the line  $x + y = 3$  at the points  $P$  and  $Q$ . If the tangents at  $P$  and  $Q$  intersect at the point  $S(\alpha, \beta)$ , then  $4\alpha - 7\beta$  is equal to \_\_\_\_\_.

**Answer (11)**

**Sol.** The line  $x + y = 3$  ... (i)

is polar of  $S(\alpha, \beta)$  w.r.t. circle

$$(x - 2)^2 + (y - 3)^2 = 16$$

$$\Rightarrow x^2 + y^2 - 4x - 6y - 3 = 0$$

Equation of polar is

$$\alpha x + \beta y - 2(x + \alpha) - 3(4 + \beta) - 3 = 0$$

$$(\alpha - 2)x + (\beta - 3)y - (2\alpha + 3\beta + 3) = 0 \quad \dots \text{(ii)}$$

(i) and (ii) represent the same.

$$\therefore \frac{\alpha - 2}{1} = \frac{\beta - 3}{1} = \frac{2\alpha + 3\beta + 3}{3}$$

$$\alpha - \beta + 1 = 0$$

$$\alpha - 3\beta - 9 = 0$$

$$\Rightarrow \alpha = -6, \beta = -5$$

$$4\alpha - 7\beta = 11$$

90. Let  $\alpha_1, \alpha_2, \dots, \alpha_7$  be the roots of the equation  $x^7 + 3x^5 - 13x^3 - 15x = 0$  and  $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_7|$ . Then  $\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$  is equal to \_\_\_\_\_.

**Answer (09)**

**Sol.**  $x^7 + 3x^5 - 13x^3 - 15x = 0$

$$x(x^6 + 3x^4 - 13x^2 - 15) = 0$$

$$x = 0 = \alpha_7$$

$$\text{Let } x^2 = t$$

$$t^3 + 3t^2 - 13t - 15 = 0$$

$$(t+1)(t+5)(t-3) = 0$$

$$t = x^2 = -1, -5, 3$$

$$x = \pm i, \pm \sqrt{5}i, \pm \sqrt{3}$$

$$\alpha_1, \alpha_2 = \pm \sqrt{5}i, \alpha_3, \alpha_4 = \pm \sqrt{3}, \alpha_5, \alpha_6 = \pm i$$

$$\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6 = 5 + 3 + 1 = 9$$