MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

61. Let $[x]$ denote the greatest integer $\le x$. Consider the function $f(x) = \max\{x^2, 1+[x]\}$. Then the value of the integral $\int_0^2 f(x)dx$ is

(1) $\frac{1+5\sqrt{2}}{3}$

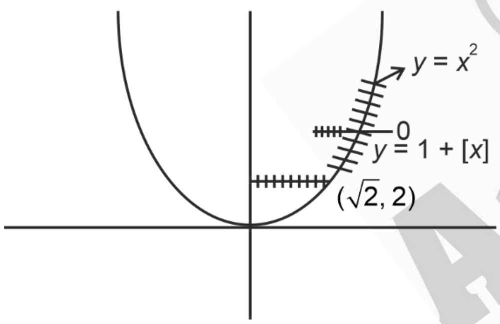
(2) $\frac{5+4\sqrt{2}}{3}$

(3) $\frac{8+4\sqrt{2}}{3}$

(4) $\frac{4+5\sqrt{2}}{3}$

Answer (2)

Sol.



$\therefore$ $f(x) = 1 \quad x \in [0, 1)$

$2 \quad x \in [1, \sqrt{2})$

$x^2 \quad x \in [\sqrt{2}, 2]$

$$\therefore \int_0^2 f(x)dx = \int_0^1 1dx + \int_1^{\sqrt{2}} 2dx + \int_{\sqrt{2}}^2 x^2 dx$$

$$= 1 + 2\sqrt{2} - 2 + \left.\frac{x^3}{3}\right|_{\sqrt{2}}^2$$

$$= 2\sqrt{2} - 1 + \frac{8}{3} - \frac{2\sqrt{2}}{3}$$

$$= \frac{5+4\sqrt{2}}{3}$$

62. Let the tangents at the points $A(4, -11)$ and $B(8, -5)$ on the circle $x^2 + y^2 - 3x + 10y - 15 = 0$, intersect at the point C. Then the radius of the circle, whose centre is C and the line joining A and B is its tangent, is equal to

(1) $\sqrt{13}$

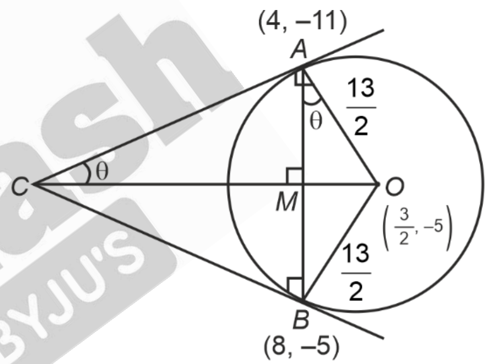
(2) $\frac{3\sqrt{3}}{4}$

(3) $\frac{2\sqrt{13}}{3}$

(4) $2\sqrt{13}$

Answer (3)

Sol.



$$OA = OB = \frac{13}{2}$$

$\because$ $AB = 2\sqrt{13}$ then $AM = \sqrt{13}$

In ΔAMO : $\angle OAM = \theta = \angle ACO$

$$\therefore \quad OC = \frac{13}{2\sin\theta}$$

$$\because \quad \sin\theta = \frac{3\sqrt{13}}{13} = \frac{3}{\sqrt{13}}$$

$$\therefore \quad OC = \frac{13\sqrt{13}}{6}$$

and $OM = \frac{3\sqrt{13}}{2}$

$$\therefore \quad CM = OC - OM = \frac{2\sqrt{13}}{3}$$

63. Let $A = \left\{(x, y) \in \mathbb{R}^2 : y \ge 0,\ 2x \le y \le \sqrt{4-(x-1)^2}\right\}$

and

$B = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \le y \le \min\left\{2x,\ \sqrt{4-(x-1)^2}\right\}\right\}$.

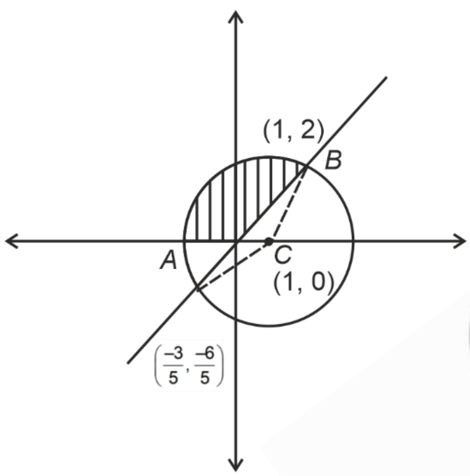
Then the ratio of the area of A to the area of B is

(1) $\frac{\pi}{\pi-1}$ (2) $\frac{\pi+1}{\pi-1}$

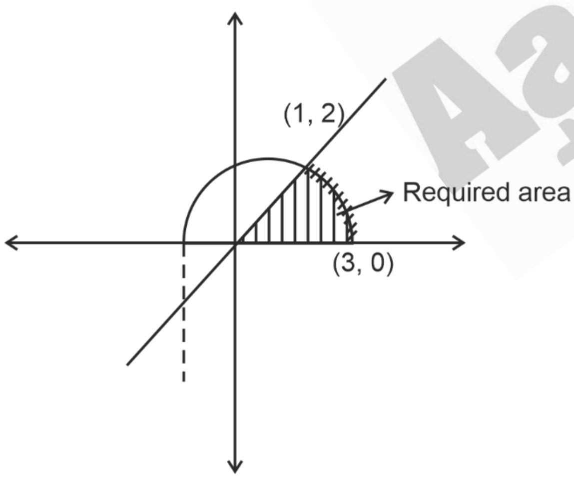
(3) $\frac{\pi-1}{\pi+1}$ (4) $\frac{\pi}{\pi+1}$

Answer (3)

Sol. For A:



For B:



Area (A) + Area (B) = 2π

$$\text{Area } B = \int_0^1 2x\,dx + \int_1^3 \sqrt{4-(x-1)^2}\,dx$$

$$= 1 + \frac{\pi 4}{4} = \pi + 1$$

Area A = $\pi - 1$

$\therefore$ Required ratio $= \frac{\pi-1}{\pi+1}$

64. Let α and β be real numbers. Consider a 3 × 3 matrix A such that $A^2 = 3A + \alpha I$. If $A^4 = 21A + \beta I$, then

(1) $\alpha = 1$ (2) $\alpha = 4$

(3) $\beta = 8$ (4) $\beta = -8$

Answer (4)

Sol. $A^4 = A^2A^2$

$= (3A + \alpha I)(3A + \alpha I)$

$= 9A^2 + 6\alpha A + \alpha^2$

$= 9(3A + \alpha I) + 6\alpha A + \alpha^2$

$= (27 + 6\alpha)A + (9\alpha + \alpha^2) = 21A + \beta I$

$\Rightarrow \alpha = -1, \beta = -8$

65. Let $f(\theta) = 3\left(\sin^4\left(\frac{3\pi}{2}-\theta\right) + \sin^4(3\pi+\theta)\right) - 2(1-\sin^2 2\theta)$

and $S = \left\{\theta \in [0, \pi] : f'(\theta) = -\frac{\sqrt{3}}{2}\right\}$. If $4\beta = \sum_{\theta \in S} \theta$,

then $f(\beta)$ is equal to

(1) $\frac{9}{8}$ (2) $\frac{3}{2}$

(3) $\frac{5}{4}$ (4) $\frac{11}{8}$

Answer (3)

Sol. $f(\theta) = 3(\cos^4\theta + \sin^4\theta) - 2(1 - \sin^2 2\theta)$

$f'(\theta) = 12(\cos^3\theta(-\sin\theta) + \sin^3\theta\cos\theta) + 4\sin 4\theta$

$= 12(\cos\theta\sin\theta(-\cos 2\theta)) + 4\sin 4\theta$

$= -6\sin 2\theta\cos 2\theta + 4\sin 4\theta$

$= -3\sin 4\theta + 4\sin 4\theta$

$= \sin 4\theta$

$$\sin 4\theta = -\frac{\sqrt{3}}{2}$$

$$4\theta = \left\{\frac{3\pi}{2}-\frac{\pi}{3}, \frac{3\pi}{2}+\frac{\pi}{3}, \frac{7\pi}{2}-\frac{\pi}{3}, \frac{7\pi}{2}+\frac{\pi}{3}\right\}$$

$$\sum\theta = \frac{5\pi}{2} \quad \Rightarrow \beta = \frac{5\pi}{8}$$

$f(\theta) = 3(1 - 2\sin^2\theta + \cos^2\theta) - 2\cos^2 2\theta$

$$= 3\left(1 - \frac{1}{2}\sin^2\frac{5\pi}{4}\right) - 2\cos^2\frac{5\pi}{4}$$

$$= 3\left(1 - \frac{1}{4}\right) - 2\cdot\frac{1}{2} = \frac{5}{4}$$

66. Consider the following system of equations

$\alpha x + 2y + z = 1$

$2\alpha x + 3y + z = 1$

$3x + ay + 2z = b$

For some $\alpha, \beta \in \mathbb{R}$. then which of the following is NOT correct?

(1) It has no solution if $\alpha = -1$ and $\beta \neq 2$

(2) It has a solution for all $\alpha \neq -1$ and $\beta = 2$

(3) It has no solution for $\alpha = 3$ and for all $\beta \neq 2$

(4) It has no solution for $\alpha = -1$ and for all $\beta \in \mathbb{R}$.

Answer (3)

Sol. $\begin{vmatrix} \alpha & 2 & 1 \\ 2\alpha & 3 & 1 \\ 3 & \alpha & 2 \end{vmatrix} = 0$

$\alpha(6 - \alpha) - 2(4\alpha - 3) + 1(2\alpha^2 - 9) = 0$

$\Rightarrow 6\alpha - \alpha^2 - 8\alpha + 6 + 2\alpha^2 - 9 = 0$

$\Rightarrow \alpha^2 - 2\alpha - 3 = 0$

OR $\alpha = 3, -1$

For $\alpha = 3$, $\beta = 2 \Rightarrow$ Infinite solution

For $\alpha = -1$, $\beta = 2 \Rightarrow$ Infinite solution

For $\alpha = -1, \beta \neq 2 \Rightarrow$ no solution

67. Let $f(x) = x + \dfrac{a}{\pi^2 - 4}\sin x + \dfrac{b}{\pi^2 - 4}\cos x, x \in \mathbb{R}$ be a function which satisfies $f(x) = x + \displaystyle\int_0^{\pi/2} \sin(x + y) f(y) dy$. Then $(a + b)$ is equal to

(1) $-\pi(\pi + 2)$

(2) $-\pi(\pi - 2)$

(3) $-2\pi(\pi + 2)$

(4) $-2\pi(\pi - 2)$

Answer (3)

Sol. $\because \quad f(x) = x + \dfrac{a}{x^2 - 4}\sin x + \dfrac{b}{x^2 - 4}\cos x, x \in R$

And $f(x) = x + \int_0^{\pi/2} \sin(x + y).f(y)dy$

$\Rightarrow \quad f(x) = x + \left(\int_0^{\pi/2} f(y).\cos y dy\right)\sin x$

$+ \left(\int_0^{\pi/2} f(y)\sin y dy\right)\cos x$

$\therefore \quad \dfrac{a}{\pi^2 - 4} = \int_0^{\pi/2} \cos y\left(y + \dfrac{a}{\pi^2 - 4}\sin y + \dfrac{b}{\pi^2 - 4}\cos y\right)dy$

$$\frac{a}{\pi^2 - 4} = \frac{\pi}{2} - 1 + \frac{a}{2(\pi^2 - 4)} + \frac{b\pi}{4(\pi^2 - 4)}$$

$\therefore \quad 2a - b\pi = 2(\pi + 2)(\pi - 2)^2 \quad \ldots$(i)

and $\dfrac{b}{\pi^2 - 4} = \int_0^{\pi/2}\left(y + \dfrac{a}{\pi^2 - 4}\sin y + \dfrac{b}{\pi^2 - 4}\cos y\right)\sin y\, dy$

$$\frac{b}{\pi^2 - 4} = 1 + \frac{a\pi}{4(\pi^2 - 4)} + \frac{b}{2(\pi^2 - 4)}$$

$\therefore \quad a\pi - 2b = -4(\pi^2 - 4) \quad \ldots$(ii)

Equation (i) – equation (ii)

$(2 - \pi)(a + b) = 2(\pi^2 - 4)(\pi - 2 + 2)$

$\therefore \quad a + b = -2\pi(\pi + 2)$

68. Fifteen football players of a club-team are given 15 T-shirts with their names written on the backside. If the players pick up the T-shirts randomly, then the probability that at least 3 players pick the correct T-shirt is

(1) $\dfrac{1}{6}$

(2) $\dfrac{5}{36}$

(3) $\dfrac{2}{15}$

(4) $\dfrac{5}{24}$

Answer (*)

69. If p, q and r are three propositions, then which of the following combination of truth values of p, q and r makes the logical expression $\{(p \vee q) \wedge ((\sim p) \vee r)\} \rightarrow ((\sim q) \vee r)$ false ?

(1) $p = F, q = T, r = F$

(2) $p = T, q = F, r = T$

(3) $p = T, q = T, r = F$

(4) $p = T, q = F, r = F$

Answer (1)

Sol. $\{(p \vee q) \wedge ((\sim p) \vee r)\} \rightarrow ((\sim q) \vee r)$

Is false when

$\{(p \vee q) \wedge ((\sim p) \vee r)\}$ T and $\sim q \vee r = F$

So, $(p \vee q) = $ T and $\sim p \vee r = T$ and

$\sim q = F$ and $r = $ F

So, $q = T$, $r = F$, and $\sim p = T$

$\therefore \quad p = F$

$\therefore \quad p = F, q = T, r = F$

70. Let $x = 2$ be a root of the equation $x^2 + px + q = 0$ and

$$f(x) - \begin{cases} \dfrac{1-\cos\left(x^2 - 4px + q^2 + 8q + 16\right)}{(x-2p)^4}, & x \neq 2p \\ 0, & x = 2p \end{cases}$$

Then $\lim\limits_{x \to 2p^+} [f(x)]$,

where [·] denotes greatest integer function, is

(1) 1 (2) 2

(3) 0 (4) –1

Answer (3)

Sol. $4 + 2p + q = 0$ …(i) $\Rightarrow 4p^2 = q^2 + 8q + 16$

For $\lim\limits_{x \to 2p^+} f(x)$ put $x = 2p + h$

$$\Rightarrow \lim_{h \to 0} \frac{\left(1-\cos\left((2p+h)^2 - 4p(2p+h) + q^2 + 8q + 16\right)\right)}{h^4}$$

$$\Rightarrow \lim_{h \to 0} \left(\frac{1-\cos\left(h^2 - 4p^2 + q^2 + 8q + 16\right)}{h^4}\right)$$

$$\Rightarrow \lim_{h \to 0} \frac{1-\cos h^2}{h^4} = \frac{1}{2}$$

$$\therefore \left[\lim_{x \to 2p^+} f(x)\right] = 0$$

71. Let B and C be the two points on the line $y + x = 0$ such that B and C are symmetric with respect to the origin. Suppose A is a point on $y - 2x = 2$ such that ΔABC is an equilateral triangle. Then, the area of the ΔABC is

(1) $3\sqrt{3}$ (2) $2\sqrt{3}$

(3) $\dfrac{10}{\sqrt{3}}$ (4) $\dfrac{8}{\sqrt{3}}$

Answer (4)

Sol. Origin (O) is mid-point of BC ($x + y = 0$).

A lies on perpendicular bisector of BC, which is $x - y = 0$

A is point of intersection of $x - y = 0$ and $y - 2x = 2$

$\therefore \quad A \equiv (-2, -2)$

$$\text{Let } h = AO = \frac{-2-2}{\sqrt{1^2 + 1^2}} = 2\sqrt{2}$$

$$\text{Area} = \frac{h^2}{\sqrt{3}} = \frac{8}{\sqrt{3}}$$

72. If the vectors $\vec{a} = \lambda\hat{i} + \mu\hat{j} + 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} + \hat{k}$ are coplanar and the projection of $\vec{a}$ on the vector $\vec{b}$ is $\sqrt{54}$ units, then the sum of all possible values of $\lambda + \mu$ is equal to

(1) 24

(2) 0

(3) 6

(4) 18

Answer (1)

Sol. $\vec{a} = \lambda\hat{i} + \mu\hat{j} + 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$, $\vec{c} = 2\hat{i} + 3\hat{j} + \hat{k}$

Now, $\vec{a} \cdot \vec{b} = \sqrt{54} \Rightarrow \dfrac{-2\lambda + 4\mu - 8}{\sqrt{24}} = \sqrt{54}$

$\Rightarrow -2\lambda + 4\mu - 8 = 36$

$\Rightarrow 2\mu - \lambda = 22$ …(i)

and $\begin{vmatrix} \lambda & \mu & 4 \\ -2 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 0$

$10\lambda - 2\mu - 56 = 0$ …(ii)

By (i) & (ii) $\lambda = \dfrac{78}{9}, \mu = \dfrac{138}{9}$

$\therefore \quad \mu + \lambda = 24$

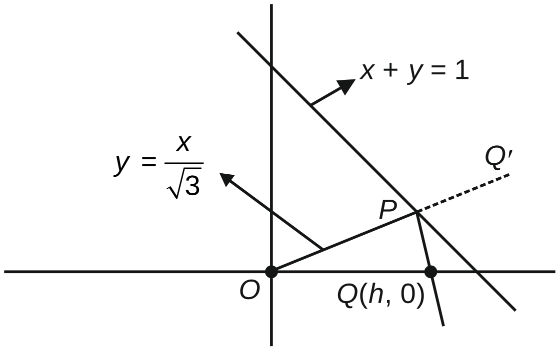
73. A light ray emits from the origin making an angle 30° with the positive x-axis. After getting reflected by the line $x + y = 1$, if this ray intersects x-axis at Q, then the abscissa of Q is

(1) $\dfrac{2}{3-\sqrt{3}}$ (2) $\dfrac{2}{3+\sqrt{3}}$

(3) $\dfrac{\sqrt{3}}{2(\sqrt{3}+1)}$ (4) $\dfrac{2}{(\sqrt{3}-1)}$

Answer (2)

Sol.



Let $Q(h, O)$

$\because$ OP reflected by $x + y = 1$.

So, image of Q lies on $y = \frac{x}{\sqrt{3}}$

$\therefore \quad \frac{x-h}{1} = \frac{y}{1} = \frac{-2(h-1)}{2}$

$\therefore \quad x = 1, y = 1 - h$

It lies on $y = \frac{x}{\sqrt{3}}$

$\therefore \quad 1 - h = \frac{1}{\sqrt{3}}$

$\therefore \quad h = 1 - \frac{1}{\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}} = \frac{2}{3+\sqrt{3}}$

Option (2) is correct.

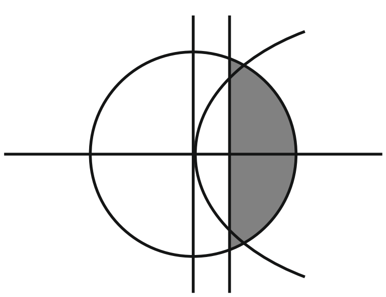
74. Let Δ be the area of the region $\left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 21,\ y^2 \le 4x,\ x \ge 1\right\}$. Then $\frac{1}{2}\left(\Delta - 21\sin^{-1}\frac{2}{\sqrt{7}}\right)$ is equal to

(1) $\sqrt{3} - \frac{4}{3}$ (2) $2\sqrt{3} - \frac{1}{3}$

(3) $\sqrt{3} - \frac{2}{3}$ (4) $2\sqrt{3} - \frac{2}{3}$

Answer (1)

Sol.



Required area $= 2\left[\int_1^3 2\sqrt{x}\,dx + \int_3^{\sqrt{21}} \sqrt{(21-x^2)}\,dx\right]$

$= 2\left(\left[2\left(\frac{x^{3/2}}{3/2}\right)\right]_1^3 + \left[\frac{x}{2}\sqrt{21-x^2} + \frac{21}{2}\sin^{-1}\left(\frac{x}{\sqrt{21}}\right)\right]_3^{\sqrt{21}}\right)$

$= 2\sqrt{3} + \frac{21\pi}{2} - \frac{8}{3} - 21\sin^{-1}\sqrt{\frac{3}{7}} = \Delta$

$\therefore \frac{1}{2}\left(\Delta - 21\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)\right) = \frac{\sqrt{3}-4}{3}$

Option (1) is correct.

75. The domain of $f(x) = \frac{\log_{(x+1)}(x-2)}{e^{2\log_e x} - (2x+3)}$, $x \in \mathbb{R}$ is

(1) $\mathbb{R} - \{-1, 3\}$ (2) $(-1, \infty) - \{3\}$

(3) $\mathbb{R} - \{3\}$ (4) $(2, \infty) - \{3\}$

Answer (4)

Sol. $f(x) = \frac{\log_{x+1}(x-2)}{e^{2\ln x} - (2x+3)}$

(i) $x - 2 > 0 \Rightarrow x > 2$

(ii) $x + 1 > 0 \Rightarrow x > -1$ and $x \ne -1$

(iii) $x > 0$

(iv) $x^2 - 2x - 3 \ne 0$

$\Rightarrow (x - 3)(x + 1) \ne 0$

$\Rightarrow x \ne -1, 3$

(i) $\cap$ (ii) $\cap$ (iii) $\cap$ (iv)

$x \in (2, \infty) - \{3\}$

76. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = \frac{x^2 + 2x + 1}{x^2 + 1}$. Then

(1) $f(x)$ is many-one in $(-\infty, -1)$

(2) $f(x)$ is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$

(3) $f(x)$ is many-one in $(1, \infty)$

(4) $f(x)$ is one-one in $(-\infty, \infty)$

Answer (2)

Sol. $f(x) = \frac{x^2 + 2x + 1}{x^2 + 1}$, where $f : \mathbb{R} \to \mathbb{R}$

$= \frac{(x+1)^2}{x^2+1} \ge 0$

$$f'(x) = \frac{(x^2+1)(2x+2)-(x^2-2x+1)(2x)}{(x^2+1)^2}$$

$$= \frac{2(x+1)(x^2+1)-(x+1)^2(2x)}{(x^2+1)^2}$$

$\Rightarrow$ $2(x + 1)(x^2 + 1 - (x + 1)x) = 0$

$\Rightarrow$ $2(x + 1)(x - 1) = 0$

$\Rightarrow$ $x = 1, -1 \Rightarrow$ points of minima and maxima

77. Let $y = f(x)$ be the solution of the differential equation $y(x + 1)dx - x^2dy = 0$, $y(1) = e$. Then $\lim\limits_{x\to 0} f(x)$ is equal to

(1) $\frac{1}{e}$ (2) 0

(3) $\frac{1}{e^2}$ (4) e^2

Answer (2)

Sol. $y(x + 1)dx = x^2dy$

$$\Rightarrow \left(\frac{x+1}{x^2}\right)dx = \frac{dy}{y}$$

$$\Rightarrow \ln x - \frac{1}{x} = \ln y + c$$

$x = 1, y = e$

$\Rightarrow$ $c = -2$

$$\Rightarrow \ln y = \ln x - \frac{1}{x} + 2$$

$$y = xe^{2-\frac{1}{x}}$$

$$\lim_{x\to 0^+} y = 0 \times e^{-\infty} = 0$$

78. For two non-zero complex numbers z_1 and z_2, if $Re(z_1z_2) = 0$ and $Re(z_1 + z_2)$, then which of the following are possible?

A. $Im(z_1) > 0$ and $Im(z_2) > 0$

B. $Im(z_1) < 0$ and $Im(z_2) > 0$

C. $Im(z_1) > 0$ and $Im(z_2) < 0$

D. $Im(z_1) < 0$ and $Im(z_2) < 0$

Choose the correct answer from the options given below

(1) B and C (2) B and D

(3) A and B (4) A and C

Answer (1)

Sol. Let $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$\Rightarrow$ $x_1x_2 - y_1y_2 = 0$ …(i)

$(x_1 + x_2) = 0$ …(ii)

$x_1^2 + y_1y_2 = 0$

$\Rightarrow$ $y_1y_2 = -x_1^2$

$\Rightarrow$ y_1 and y_2 have opposite signs.

79. Three rotten apples are mixed accidently with seven good apples and four apples are drawn one by one without replacement. Let the random variable X denote the number of rotten apples. If μ and σ^2 represent mean and variance of X, respectively, then $10(\mu^2 + \sigma^2)$ is equal to

(1) 25

(2) 250

(3) 30

(4) 20

Answer (4)

Sol.

x_i	0	1	2	3
p_i	$\frac{35}{210}=\frac{1}{6}$	$\frac{105}{210}=\frac{1}{2}$	$\frac{3\times 21}{210}=\frac{3}{10}$	$\frac{7}{210}=\frac{1}{30}$

$$\mu = \sum p_i x_i = \frac{1}{2} + \frac{6}{10} + \frac{21}{210}$$

$$= \frac{1}{2} + \frac{3}{5} + \frac{1}{10}$$

$$= \frac{6}{5}$$

$$\sigma^2 = \sum p_i x_1^2 - \mu^2$$

$$= \left(\frac{1}{2} + \frac{4.3}{10} + 9\cdot\frac{1}{30}\right) - \left(\frac{6}{5}\right)^2$$

$$= \left(\frac{1}{2} + \frac{6}{5} + \frac{3}{10}\right) - \frac{36}{25}$$

$$= \frac{14}{25}$$

Now, $10(\mu^2 + \sigma^2)$

$= 20$

80. Let $\lambda \neq 0$ be a real number. Let α, β be the roots of the equation $14x^2 - 31x + 3\lambda = 0$ and α, γ be the roots of the equation $35x^2 - 53x + 4\lambda = 0$. Then $\frac{3\alpha}{\beta}$ and $\frac{4\alpha}{\gamma}$ are the roots of the equation

(1) $7x^2 + 245x - 250 = 0$

(2) $49x^2 + 245x + 250 = 0$

(3) $7x^2 - 245x + 250 = 0$

(4) $49x^2 - 245x + 250 = 0$

Answer (4)

Sol. $35x^2 - 53x + 4\lambda = 0$ …(i)

$(14x^2 - 31x + 3\lambda = 0) \times 2.5$ …(ii)

(i) and (ii) gives

$$x = \frac{\lambda}{7} = \alpha$$

$$\alpha\beta = \frac{3\lambda}{14} \Rightarrow \beta = \frac{3\lambda}{14} \cdot \frac{7}{\lambda} = \frac{3}{2}$$

$$\alpha\gamma = \frac{4\lambda}{35} \Rightarrow \gamma = \frac{4}{35} \cdot 7 = \frac{4}{5}$$

$$\alpha + \beta = \frac{31}{14} \Rightarrow \alpha = \frac{5}{7}$$

$$\frac{3\alpha}{\beta} = \frac{10}{7}, \frac{4\alpha}{\gamma} = \frac{20}{7} \cdot \frac{5}{4} = \frac{25}{7}$$

Equation formed will be

$$x^2 - 5x + \frac{250}{49} = 0$$

$49x^2 - 245x + 250 = 0$

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

81. Let a_1, a_2, a_3, … be a *G.P* of increasing positive numbers. If the product of fourth and sixth terms is 9 and the sum of fifth and seventh terms is 24, then $a_1a_9 + a_2a_4a_9 + a_5 + a_7$ is equal to ________.

Answer (60)

Sol. Let r be the common ratio of the *G.P*

$\therefore\ a_1r^3 \times a_1r^5 = 9$

$a_1^2r^8 = 9 \Rightarrow a_1r^4 = 3$

And

$a_1(r^4 + r^6) = 24$

$\Rightarrow\ 3(1 + r^2) = 24$

$\therefore\ r^2 = 7$ and $a_1 = \frac{3}{49}$

Now

$a_1a_9 + a_2a_4a_9 + a_5 + a_7$

$= a_1^2r^8 + a_1^3r^{12} + 24$

$= 24 + \frac{9}{7^4} \times 7^4 + \frac{27}{7^6} \cdot 7^6 = 60$

82. Five digit numbers are formed using the digits 1, 2, 3, 5, 7 with repetitions and are written in descending order with serial number. For example, the number 77777 has serial number 1. Then the serial number of 35337 is ________.

Answer (1436)

Sol. Given digits 1, 2, 3, 5, 7

and number 35337

7 _ _ _ _ = 5^4 = 625

5 _ _ _ _ = 5^4 = 625

37 _ _ _ = 5^3 = 125

357 _ _ = 5^2 = 25

355 _ _ = 5^2 = 25

3537 _ = 5

3535 _ = 5

35337 = 1

$\therefore$ Serial no. = 1436

83. Let the equation of the plane P containing the line $x + 10 = \frac{8 - y}{2} = z$ be $ax + by + 3z = 2(a + b)$ and the distance of the plane P from the point (1, 27, 7) be c. Then $a^2 + b^2 + c^2$ is equal to ____________.

Answer (355)

Sol. Equation of the line:

$$x + 10 = \frac{8 - y}{2} = z$$

and plane $P : ax + by + 3z = 2(a + b)$

$\because$ line lies in P

$\therefore \quad -10a + 8b = 2a + 2b$

$12a = 6b \quad \Rightarrow \quad \boxed{b = 2a}$

and

$a - 2b + 3 = 0$

So, $\boxed{a = 1}$ $\boxed{b = 2}$

Now distance of (1, 27, 7) from $P = c$

$$\Rightarrow \frac{1 + 54 + 21 - 6}{\sqrt{14}} = c$$

$$\therefore \quad a^2 + b^2 + c^2 = 1 + 4 + \frac{4900}{14}$$

$= 355$

84. Let $f : R \to R$ be a differentiable function that satisfies the relation $f(x + y) = f(x) + f(y) - 1$, $\forall x, y \in R$. If $f'(0) = 2$, then $|f(-2)|$ is equal to ____________.

Answer (03)

Sol. $f(x + y) = f(x) + f(y) - 1 \qquad f(0) = 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) + f(h) - 1 - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(h) - 1}{h}$$

$f'(x) = f'(0)$

$f'(x) = 2$

$f(x) = 2x + c$

$\because \quad f(0) = 1$

$1 = c$

$\Rightarrow \quad c = 1$

$f(x) = 2x + 1$

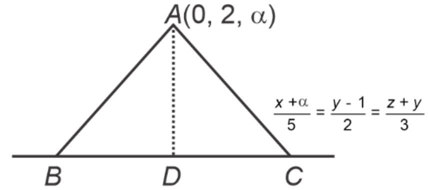
$|f(-2)| = |2(+2) - 1|$

$= 3$

85. Let the co-ordinates of one vertex of ΔABC be $A(0, 2, \alpha)$ and the other two vertices lie on the line $\frac{x+\alpha}{5} = \frac{y-1}{2} = \frac{z+4}{3}$. For $\alpha \in Z$, if the area of ΔABC is 21 sq. units and the line segment BC has length $2\sqrt{21}$ units, then α^2 is equal to ____________.

Answer (09)

Sol.



Let coordinate of $D = (5k - \alpha, 2k + 1, 3k - 4)$

$\therefore \quad D.R^s.$ of $AD = \langle 5k - \alpha, 2k - 1, 3k - 4 - \alpha \rangle$

$\therefore \quad 5(5k - \alpha) + 2(2k - 1) + 3(3k - 4 - \alpha) = 0$

$\therefore \quad 19k - 4\alpha - 7 = 0 \qquad \ldots(i)$

and $\frac{1}{2} \times 2\sqrt{21} \times AD = 21$

$\therefore \quad AD = \sqrt{21}$

$\therefore \quad (5k - \alpha)^2 + (2k - 1)^2 + (3k - 4 - \alpha)^2 = 21$

$\therefore \quad 19k^2 - 8k\alpha + \alpha^2 - 14k + 4\alpha = 2 \ldots(ii)$

from eq. (i) and (ii) : $\alpha = 3$

$\therefore \quad \alpha^2 = 9$

86. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero non-coplanar vectors. Let the position vectors of four points A, B, C and D be $\vec{a} - \vec{b} + \vec{c}, \lambda\vec{a} - 3\vec{b} + 4\vec{c}, -\vec{a} + 2\vec{b} - 3\vec{c}$ and $2\vec{a} - 4\vec{b} + 6\vec{c}$ respectively. If $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, then λ is equal to ____.

Answer (2)

Sol. $\overrightarrow{AB} = (\lambda - 1)\vec{a} + (-2)\vec{b} + 3\vec{c}$

$\overrightarrow{AC} = -2\vec{a} + 3\vec{b} - 4\vec{c}$

$\overrightarrow{AD} = -\vec{a} - 3\vec{b} + 5\vec{c}$

$\because \quad \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are co-planar

$$\begin{vmatrix} \lambda - 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}] = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 0$$

$3(\lambda - 1) - 2(6) + 3(6 - 3) = 0$

$3(\lambda - 1) - 12 + 9 = 0$

$3(\lambda - 1) = 3$

$\lambda = 2$

87. Let the coefficients of three consecutive terms in the binomial expansion of $(1 + 2x)^n$ be the ratio 2 : 5 : 8. Then the coefficient of the term, which is in the middle of these three terms, is _______.

Answer (1120)

Sol. $\dfrac{{}^nC_r\ 2^r}{{}^nC_{r+1}\ 2^{r+1}} = \dfrac{2}{5}$

$\dfrac{r+1}{n-\gamma} = \dfrac{4}{5}$ ...(i)

$\dfrac{{}^nC_{r+1}\ 2^{r+1}}{{}^nC_{r+2}\ 2^{r+2}} = \dfrac{5}{8}$

$\dfrac{r+2}{n-r-1} = \dfrac{5}{4}$ ...(ii)

Solving (i) and (ii)

$r = 3, n = 8$

Middle term = ${}^nC_{r+1}\,(2)^{r+1}$

$= {}^8C_4\ (2)^4$

$= 1120$

88. Suppose f is a function satisfying $f(x + y) + f(x) + f(y)$ for all $x, y \in \mathbb{N}$ and $f(1) = \dfrac{1}{5}$. If $\sum_{n=1}^{m} \dfrac{f(n)}{n(n+1)(n+2)} = \dfrac{1}{12}$, then m is equal to ________.

Answer (10)

Sol. $f(x + y) = f(x) + f(y)$

$\Rightarrow$ $f(x) = kx$

$f(1) = \dfrac{1}{5} \Rightarrow k = \dfrac{1}{5}$

$\therefore$ $f(x) = \dfrac{1}{5}x$

$\sum_{n=1}^{m} \dfrac{f(n)}{n(n+1)(n+2)} = \dfrac{1}{5}\sum_{n=1}^{m} \dfrac{n}{n(n+1)(n+2)} = \dfrac{1}{12}$

$\Rightarrow$ $\dfrac{1}{5}\sum_{n=1}^{m} \dfrac{1}{(n+1)(n+2)} = \dfrac{1}{12}$

$\Rightarrow$ $\dfrac{1}{5}\left[\dfrac{1}{2} - \dfrac{1}{m+2}\right] = \dfrac{1}{12}$

$\Rightarrow$ $m = 10$

89. If all the six digit numbers $x_1\ x_2\ x_3\ x_4\ x_5\ x_6$ with $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ are arranged in the increasing order, then the sum of the digits in the 72$^{\text{th}}$ number is ____________.

Answer (32)

Sol. 1 ....$\rightarrow$ ${}^8C_5 = 56$

23 ....$\rightarrow$ ${}^6C_4 = \dfrac{15}{71}$

72$^{\text{th}}$ number = 245678

Sum = 32

90. If the co-efficient of x^9 in $\left(\alpha x^3 + \dfrac{1}{\beta x}\right)^{11}$ and the co-efficient of x^{-9} in $\left(\alpha x - \dfrac{1}{\beta x^3}\right)^{11}$ are equal, then $(\alpha\beta)^2$ is equal to ________.

Answer (01)

Sol. $T_{r_1+1} = {}^{11}C_{r_1}\left(\alpha x^3\right)^{11-r_1}(\beta x)^{-r_1}$

$= {}^{11}C_{r_1}\ \alpha^{11-r_1}\ \beta^{-r_1}\ x^{33-4r_1}$

$33 - 4r_1 = 9 \Rightarrow r_1 = 6$

$T_{r_2+1} = {}^{11}C_{r_2}\ (\alpha x)^{11-r_2}\ (-1)^{r_2}\ \left(\beta x^3\right)^{-r_2}$

$= (-1)^{r_2}\ {}^{11}C_{r_2}\ \alpha^{11-r_2}\ \beta^{-r_2}\ x^{11-4r_2}$

$11 - 4r_2 = -9 \Rightarrow r_2 = 5$

Equating the coefficients

${}^{11}C_6\ \alpha^5\beta^{-6} = {}^{11}C_5\ \alpha^6\ \beta^{-5}$

$\Rightarrow$ $\alpha\beta = 1$

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