MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

61. Let the mean and standard deviation of marks of class A of 100 students be respectively 40 and α (>0), and the mean and standard deviation of marks of class B of n students be respectively 55 and $30 - \alpha$. If the mean and variance of the marks of the combined class of 100 + n students are respectively 50 and 350, then the sum of variances of classes A and B is :

(1) 450 (2) 650

(3) 900 (4) 500

Answer (4)

Sol. Let mean of class $A = \bar{x}_A$

& Mean of class $B = \bar{x}_B$

$\therefore \quad \frac{\sum x_A}{100} = 40 \quad \ldots(i)$

$\& \quad \frac{\sum x_A^2}{100} - 40^2 = \alpha^2 \quad \ldots(ii)$

also $\frac{\sum x_B}{n} = 55 \quad \ldots(iii)$

$\& \quad \frac{\sum x_B^2}{n} - 55^2 = (30-\alpha)^2 \quad \ldots(iv)$

and $\frac{\sum x_A + \sum x_B}{100+n} = 50 \quad \ldots(v)$

$\frac{\sum x_A^2 + \sum x_B^2}{100+n} - 50^2 = 350 \quad \ldots(vi)$

By equation (i), (ii) & (iii)

$4000 + 55n = 50(100 + n)$

$\Rightarrow 5n = 1000 \qquad \Rightarrow n = 200$

also by (ii), (iii) & (iv)

$(\alpha^2 + 40^2)100 + (55^2 + (30 - \alpha)^2)200 = (50^2 + 350)300$

$\Rightarrow 3\alpha^2 - 120\alpha + (40^2 + 2 \times 55^2) - 3(50^2 + 350) = 0$

$\Rightarrow 3\alpha^2 - 120\alpha + 900 = 0$

$\Rightarrow \alpha^2 - 40\alpha + 300 = 0$

$\Rightarrow \alpha = 10$ or 30 (rejected)

Sum of variances $= 10^2 + 20^2 = 500$

62. The foot of perpendicular form the origin O to a plane P which meets the co-ordinate axes at the points A, B, C is (2, a, 4), $a \in N$. If the volume of the tetrahedron OABC is 144 unit^3, then which of the following points is **NOT** on P?

(1) (2, 2, 4) (2) (3, 0, 4)

(3) (0, 6, 3) (4) (0, 4, 4)

Answer (2)

Sol. As (2, *a*, 4) is foot of perpendicular

equation of plane

$2(x - 2) + a(y - a) + 4(z - 4) = 0$

Clearly for (3, 0, 4) $a \notin N$.

63. Let $y = y(x)$ be the solution of the differential equation $(3y^2 - 5x^2)y\,dx + 2x(x^2 - y^2)dy = 0$ such that $y(1) = 1$. Then $|(y(2))^3 - 12y(2)|$ is equal to :

(1) $16\sqrt{2}$ (2) 32

(3) $32\sqrt{2}$ (4) 64

Answer (3)

Sol. $\frac{dy}{dx} = \frac{(5x^2-3y^2)y}{(x^2-y^2)2x} \qquad y(1) = 1$

$\Rightarrow v = 1$

Put $y = vx$

$v + x\frac{dv}{dx} = \frac{v(5-3v^2)}{2(1-v^2)}$

$\Rightarrow x\frac{dv}{dx} = \frac{5v - 3v^3 - 2v + 2v^3}{2(1-v^2)}$

$\Rightarrow \frac{2(v^2-1)dv}{v^3-3v} = \frac{dx}{x}$

$\Rightarrow \frac{2}{3}\ln\left|v^3 - 3v\right| = \ln x + c$

$\downarrow y(1) = 1$

$\Rightarrow \quad \frac{2}{3}\ln 2 = c$

$\therefore \quad \frac{2}{3}\ln\left(\frac{\left(\left(\frac{y}{x}\right)^{-3} - \frac{3y}{x}\right)}{2}\right) = \ln x$

$\downarrow y(2)$

$\Rightarrow \quad \left(\frac{y^3}{8} - \frac{3y}{2}\right) = 2.2^{\frac{3}{2}}$

$\Rightarrow \quad y^3(2) - 12y(2) = 32\sqrt{2}$

64. Let (a, b) ⊂ (0, 2π) be the largest interval for which $\sin^{-1}(\sin\theta) - \cos^{-1}(\sin\theta) > 0$, θ ∈ (0, 2π), holds. If $\alpha x^2 + \beta x + \sin^{-1}(x^2 - 6x + 10) + \cos^{-1}(x^2 - 6x + 10) = 0$ and α – β = b – a, then α is equal to :

(1) $\frac{\pi}{16}$ (2) $\frac{\pi}{8}$

(3) $\frac{\pi}{48}$ (4) $\frac{\pi}{12}$

Answer (4)

Sol. $\sin^{-1}(\sin\theta) > \cos^{-1}\sin\theta$

For $Q \to \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

$\therefore \quad \alpha - \beta = \frac{\pi}{2}$ …(i)

also $\alpha x^2 + \beta x + \frac{\pi}{2} = 0$ & $x = 3$ only

$\therefore \quad 9\alpha + 3\beta = -\frac{\pi}{2}$ …(ii)

$12\alpha = \pi \Rightarrow \alpha = \frac{\pi}{12}$

65. $\lim_{x\to\infty} \frac{\left(\sqrt{3x+1} + \sqrt{3x-1}\right)^6 + \left(\sqrt{3x+1} - \sqrt{3x-1}\right)^6}{\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6} x^3$

(1) is equal to 9

(2) does not exist

(3) is equal to $\frac{27}{2}$

(4) is equal to 27

Answer (3)

Sol. $\lim_{x\to\infty} \frac{2\left({}^6C_0\left(\sqrt{3x+1}\right)^6 + {}^6C_2 \times \left(\sqrt{3x+1}\right)^4 + {}^6C_4\left(\sqrt{3x+1}\right)^2 + {}^6C_6\right)x^3}{2\left({}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6\right)}$

$= \frac{3^3}{1} = 27$

66. The equation $e^{4x} + 8e^{3x} + 13e^{2x} - 8e^x + 1 = 0$, $x \in R$ has :

(1) two solutions and only one of them is negative

(2) two solutions and both are negative

(3) four solutions two of which are negative

(4) no solution

Answer (2)

Sol. $\left(e^{2x} + e^{-2x}\right) + 8\left(e^x - e^{-x}\right) + 13 = 0$

$e^x - e^{-x} = t$

$\left(t^2 + 2\right) + 8t + 13 = 0$

$t = -5, \ -3$

$e^x - e^{-x} = -5$	$e^x - e^{-x} = -3$
One negative Root	One negative Root

67. Let P be the plane, passing through the point (1, –1, –5) and perpendicular to the line joining the points (4, 1, –3) and (2, 4, 3). Then the distance of P from the point (3, –2, 2) is

(1) 4

(2) 7

(3) 5

(4) 6

Answer (3)

Sol. $\vec{n} = 2\hat{i} - 3\hat{j} - 6\hat{k}$

Equation of plane is

$2x - 3y - 6z = 35$

Or $2x - 3y - 6z - 35 = 0$

Distance from (3, –2, 2)

$= \left|\frac{6 + 6 - 12 - 35}{7}\right|$

= 5 units

68. The complex number $z = \dfrac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ is equal to :

(1) $\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

(2) $\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}$

(3) $\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

(4) $\sqrt{2}i\left(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right)$

Answer (1)

Sol. $z = \dfrac{\sqrt{2}e^{\left(3\frac{\pi}{4}\right)i}}{e^{i\frac{\pi}{3}}}$

$= \sqrt{2}e^{i\left(\frac{5\pi}{12}\right)}$

69. Let the plane $P: 8x + a_1y + a_2z + 12 = 0$ be parallel to the line $L: \dfrac{x+2}{2} = \dfrac{y-3}{3} = \dfrac{z+4}{5}$. If the intercept of P on the y-axis is 1, then the distance between P and L is:

(1) $\sqrt{\frac{7}{2}}$

(2) $\frac{6}{\sqrt{14}}$

(3) $\sqrt{\frac{2}{7}}$

(4) $\sqrt{14}$

Answer (4)

Sol. $16 + 3\alpha_1 + 5\alpha_2 = 0$

At *y*-axis $x = z = 0$

$\alpha_1 y + 12 = 0$

$\boxed{\alpha_1 = -12}$

$\boxed{\alpha_2 = 4}$

Equation of plane is

$8x - 12y + 4z + 12 = 0$

Or $2x - 3y + z + 3 = 0$

Distance from (–2, 3, –4)

$= \left|\dfrac{-4-9-4+3}{\sqrt{14}}\right| = \sqrt{14}$

70. Let H be the hyperbola, whose foci are $(1 \pm \sqrt{2}, 0)$ and eccentricity is $\sqrt{2}$. Then the length of its latus rectum is ________.

(1) 3

(2) $\frac{3}{2}$

(3) $\frac{5}{2}$

(4) 2

Answer (4)

Sol. $2ae = \left(1+\sqrt{2}\right) - \left(1-\sqrt{2}\right) = 2\sqrt{2}$

$2 \times a \times \sqrt{2} = 2\sqrt{2}$

$a = 1$

$b^2 = a^2\left(e^2 - 1\right) = i(2-1) = 1$

$\text{LR} = \dfrac{2b^2}{a} = 2$

71. If a point $P(\alpha, \beta, \gamma)$ satisfying

$$(\alpha\beta\gamma)\begin{pmatrix} 2 & 10 & 8 \\ 9 & 3 & 8 \\ 8 & 4 & 8 \end{pmatrix} = (0\ 0\ 0)$$

lies on the plane $2x + 4y + 3z = 5$, then $6\alpha + 9\beta + 7\gamma$ is equal to

(1) $\frac{11}{5}$

(2) 11

(3) $\frac{5}{4}$

(4) –1

Answer (2)

Sol. $[\alpha\quad \beta\quad \gamma]\begin{bmatrix} 2 & 10 & 8 \\ 9 & 3 & 8 \\ 8 & 4 & 8 \end{bmatrix} = [0\quad 0\quad 0]$

$2\alpha + 9\beta + 8\gamma = 0$ …(i)

$10\alpha + 3\beta + 4\gamma = 0$ …(ii)

$\alpha + \beta + \gamma = 0$ …(iii)

$\begin{vmatrix} 2 & 10 & 8 \\ 9 & 3 & 8 \\ 8 & 4 & 8 \end{vmatrix} = 0$

$\therefore$ Above system of equations has infinitely many Solutions

(ii) – 4(iii) $\Rightarrow \beta = 6\alpha$ …(iv)

(iii) $\Rightarrow \gamma = -7\alpha$ …(v)

(α, β, γ) lies on $2x + 4y + 3z = 5$

$\therefore \quad 2\alpha + 4\beta + 3\gamma = 5 \qquad \ldots$(vi)

Using (iv) and (v) in (vi);

$\alpha = 1, \beta = 6, \gamma = -7$

$\therefore \quad 6\alpha + 9\beta + 7\gamma = 11$

72. Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = 5\hat{i} - 3\hat{j} + 3\hat{k}$ be three vectors if $\vec{r}$ is a vector such that, $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$, then $25|\vec{r}|^2$ is equal to

(1) 560 (2) 449

(3) 336 (4) 339

Answer (4)

Sol. $\vec{r} - \vec{c} = \lambda\vec{b}$

$\vec{r} = \vec{c} + \lambda\vec{b}$

$\vec{r} \cdot \vec{a} = 0$

$\vec{c} \cdot \vec{a} + \lambda(\vec{b} \cdot \vec{a}) = 0$

$8 + \lambda(5) = 0$

$\lambda = \frac{-8}{5}$

$\vec{r} = \vec{c} - \frac{8}{5}\vec{b}$

$5\vec{r} = 5\vec{c} - 8\vec{b}$

$= 17\hat{i} - 7\hat{j} - \hat{k}$

$25|\vec{r}|^2 = 339$

73. The set of all values of a^2 for which the line $x + y = 0$ bisects two distinct chords drawn from a point $P\left(\frac{1+a}{2}, \frac{1-a}{2}\right)$ on the circle $2x^2 + 2y^2 - (1 + a)x - (1 - a)y = 0$ is equal to

(1) $(8, \infty)$ (2) $(4, \infty)$

(3) $(0, 4]$ (4) $(2, 12]$

Answer (1)

Sol. If $(k, -k)$ is mid-point

Equation of chord :

$$2xk + 2y(-k) - \frac{1+a}{2}(x+k) - \frac{(1-a)}{2}(y-k)$$

$$= 4k^2 - (1+a)k - (1-a)(-k)$$

or $$x\left(2k - \frac{1+a}{2}\right) + y\left(-2k - \frac{1-a}{2}\right)$$

$$= 4k^2 - \left(\frac{1+a}{2}\right)k - (-k)\left(\frac{1-a}{2}\right)$$

As it passes through $\left(\frac{1+a}{2}, \frac{1-a}{2}\right)$

$$\frac{1+a}{2}\left(2k - \frac{1+a}{2}\right) + \frac{1-a}{2}\left(-2k - \frac{1-a}{2}\right)$$

$$= 4k^2 - \frac{(1+a)}{2}k + k\frac{(1-a)}{2}$$

So, quadratic in k should have $D > 0$.

$\boxed{a > 8}$

74. Among the relations

$S = \left\{(a,b) : a, b \in \mathbb{R} - \{0\}, 2 + \frac{a}{b} > 0\right\}$ and

$T = \left\{(a,b) : a, b \in \mathbb{R}, a^2 - b^2 \in \mathbb{Z}\right\}$,

(1) S is transitive but T is not

(2) neither S nor T is transitive

(3) both S and T are symmetric

(4) T is symmetric but S is not

Answer (4)

Sol. For S

If $2 + \frac{a}{b} > 0$ or $\frac{a}{b} > -2$

$\Rightarrow \frac{b}{a} > -2 \qquad \therefore$ not symmetric

For T

$a^2 - b^2 \in I \Rightarrow b^2 - a^2 \in I \ \forall\, a, b \in \mathbb{R}$

$\therefore$ T is symmetric but S is not.

75. The number of values of $r \in \{p, q, \sim p, \sim q\}$ for which $((p \wedge q) \Rightarrow (r \vee q)) \wedge ((p \wedge r) \Rightarrow q)$ is a tautology, is

(1) 4 (2) 1

(3) 2 (4) 3

Answer (3)

Sol. $((p \wedge q) \Rightarrow (r \vee q)) \wedge ((p \wedge r) \Rightarrow q) \equiv T$ (given)

$\equiv ((\sim p \vee \sim q) \vee (r \vee q)) \wedge (\sim p \vee \sim r \vee q)$

$\equiv ((\sim p \vee r) \vee (\sim q \vee q)) \wedge (\sim p \vee \sim r \vee q)$

$\equiv \sim p \vee \sim r \vee q$

For above statement to be tautology

r can be $\sim p$ or q

$\therefore$ Two values of r are possible.

76. If $\phi(x) = \frac{1}{\sqrt{x}}\int_{\frac{\pi}{4}}^{x}(4\sqrt{2}\sin t - 3\phi'(t)dt, x > 0$, then $\phi'\left(\frac{\pi}{4}\right)$ is equal to

(1) $\frac{8}{6+\sqrt{\pi}}$ (2) $\frac{4}{6+\sqrt{\pi}}$

(3) $\frac{4}{6-\sqrt{\pi}}$ (4) $\frac{8}{\sqrt{\pi}}$

Answer (1)

Sol. $\phi(x) = \frac{1}{\sqrt{x}}\int_{\pi/4}^{x}\left(4\sqrt{2}\sin t - 3\phi'(t)\right)dt$

$\Rightarrow \phi'(x) = \frac{-1}{2x^{3/2}}\int_{\pi/4}^{x}\left(4\sqrt{2}\sin t - 3\phi'(t)\right)dt + \frac{1}{\sqrt{x}}\left(4\sqrt{2}\sin(x) - 3\phi'(x)\right)$

$x = \frac{\pi}{4}$

$\phi'\left(\frac{\pi}{4}\right) = \frac{-1}{2\left(\frac{\pi}{4}\right)^{3/2}} \times 0 + \sqrt{\frac{4}{\pi}}\left(4\sqrt{2} \times \frac{1}{\sqrt{2}} - 3\phi'\left(\frac{\pi}{4}\right)\right)$

$\Rightarrow \phi'\left(\frac{\pi}{4}\right)\left[1 + \frac{6}{\sqrt{\pi}}\right] = \frac{2}{\sqrt{\pi}} \times 4$

$\Rightarrow \phi'\left(\frac{\pi}{4}\right) = \frac{8}{6+\sqrt{x}}$

Option (1) is correct.

77. Let a_1, a_2, a_3, …. be an A.P. If a_7 = 3, the product a_1a_4 is minimum and the sum of its first n terms is zero, then $n! - 4a_n(n + 2)$ is equal to :

(1) $\frac{381}{4}$ (2) 24

(3) 9 (4) $\frac{33}{4}$

Answer (2)

Sol. $a_7 = 3 \Rightarrow a + 6d = 3 \Rightarrow a = 3 - 6d$

$a_1 \cdot a_4 = a(a + 3d)$

$\Rightarrow$ $(3 - 6d)(3 - 6d + 3d)$

$\Rightarrow$ $3(1 - 2d)3(1 - d)$

$\Rightarrow$ $9(2d^2 - 3d + 1)$

Let $f(d) = 2d^2 - 3d + 1$

$f'(d) = 4d - 3 \Rightarrow d = \frac{3}{4}$

$\therefore$ $a = 3 - 6 \cdot \frac{3}{4} = 3 - \frac{9}{2} = -\frac{3}{2}$

$S_n = 0$

$\frac{n}{2}(29 + (n-1)d) = 0$

$\Rightarrow$ $2 \cdot \left(-\frac{3}{2}\right) + (n-1)\left(\frac{3}{4}\right) = 0$

$\Rightarrow$ $3 = \frac{3}{4}(n-1)$

$\Rightarrow$ $n = 5$

Now, $n! - 4.a_{n(n+2)}$

$= 5! - 4 \cdot a_{35}$

$= 120 - 4\left(-\frac{3}{2} + 34 . \frac{3}{4}\right)$

$= 120 - (-6 + 102)$

$= 120 - (96)$

$= 24$

78. The absolute minimum value, of the function $f(x) = |x^2 - x + 1| + [x^2 - x + 1]$, where $[t]$ denotes the greatest integer function, in the interval [–1. 2], is

(1) $\frac{3}{4}$ (2) $\frac{3}{2}$

(3) $\frac{1}{4}$ (4) $\frac{5}{4}$

Answer (1)

Sol. $x^2 - x + 1 > 0$

$\Rightarrow$ $f(x) = (x^2 - x + 1) + [x^2 - x + 1]$

Now,

$x^2 - x + 1$ attains its minimum value at $x = \frac{1}{2}$

and $\min\left[x^2 - x + 1\right] = 0$ as $x^2 - x + 1 > 0$

$\Rightarrow$ $f(x)$ attains it min at $x = \frac{1}{2}$

$\therefore$ $f\left(\frac{1}{2}\right) = \frac{3}{4} + 0 = \frac{3}{4}$

Option (1) is correct.

79. Let $\alpha > 0$. If $\int_0^{\alpha} \frac{x}{\sqrt{x+\alpha}-\sqrt{x}}dx = \frac{16+20\sqrt{2}}{15}$, then α is equal to:

(1) 2 (2) 4

(3) $\sqrt{2}$ (4) $2\sqrt{2}$

Answer (3)

Sol. $I = \int_0^{\alpha} \frac{x}{\sqrt{x+\alpha}-\sqrt{x}}dx, \alpha > 0$

$$= \frac{1}{\alpha}\int_0^{\alpha} x\left(\sqrt{x+\alpha}+\sqrt{x}\right)dx$$

$$= \frac{1}{\alpha}\left\{\int_0^{\alpha} x\sqrt{x+\alpha}\,dx + \int_0^{\alpha} x^{3/2}\,dx\right\}$$

$$= \frac{2\alpha^{3/2}}{15}\left(2^{3/2}+2\right)+\frac{2\alpha^{3/2}}{5}$$

$$= \frac{2\alpha^{3/2}\left(2^{3/2}+5\right)}{15}$$

When $\alpha = \sqrt{2}$ then $\int_0^{\alpha} \frac{x}{\sqrt{x+\alpha}-\sqrt{x}}dx = \frac{16+20\sqrt{2}}{15}$

$\therefore \quad \alpha = \sqrt{2}$

80. Let $f : \mathbb{R}-\{2,6\} \to \mathbb{R}$ be real valued function defined as $f(x) = \frac{x^2+2x+1}{x^2-8x+12}$. Then range of f is

(1) $\left(-\infty,-\frac{21}{4}\right] \cup [0,\infty)$ (2) $\left(-\infty,-\frac{21}{4}\right) \cup (0,\infty)$

(3) $\left(-\infty,-\frac{21}{4}\right] \cup [1,\infty)$ (4) $\left(-\infty,-\frac{21}{4}\right] \cup \left[\frac{21}{4},\infty\right)$

Answer (1)

Sol. $y = \frac{x^2+2x+1}{x^2-8x+12}$

$\Rightarrow \quad (y-1)x^2-(8y+2)x+12y-1=0$

Let $y \neq 1$, then $D \geq 0$

$4(4y+1)^2-4(y-1)(12y-1) \geq 0$

$\Rightarrow \quad 16y^2+1+8y-(12y^2-13y+1) \geq 0$

$\Rightarrow \quad 4y^2+21y \geq 0$

$\Rightarrow \quad y \in \left(-\infty,-\frac{21}{4}\right] \cup [0,\infty)-\{1\}$

for $y = 1$,

$-8x + 12 = 2x + 1$

$x = \frac{11}{10} \quad \therefore\ I \in R$

$\therefore \quad$ Range $= \left(-\infty,-\frac{21}{4}\right] \cup [0,\infty)$

$\therefore \quad$ option (1) is correct.

SECTION - B

Numerical Value Type Questions: This section contains 10 questions. In Section B, attempt any five questions out of 10. The answer to each question is a **NUMERICAL VALUE.** For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the second decimal place; e.g. 06.25, 07.00, –00.33, –00.30, 30.27, –27.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

81. If the constant term in the binomial expansion of $\left(\frac{x^{\frac{5}{2}}}{2}-\frac{4}{x^l}\right)^9$ is –84 and the coefficient of x^{-3l} is $2^{\alpha}\beta$, where $\beta < 0$ is an odd number, then $|\alpha l - \beta|$ is equal to ________.

Answer (98)

Sol. Given binomial expansion of $\left(\frac{x^{\frac{5}{2}}}{2}-\frac{4}{x^l}\right)^9$

$$T_{r+1} = {}^9C_r\left(\frac{x^{\frac{5}{2}}}{2}\right)^{9-r}\left(\frac{-4}{x^l}\right)^r$$

$$= {}^9C_r\, x^{\frac{45-5r}{2}-lr}\cdot 2^{r-9}\cdot r^r\cdot(-1)^r$$

Now constant term = –84

So, $\frac{45-5r}{2} = lr \Rightarrow 2lr+5r = 45$

and ${}^9C_r\cdot 2^{3r-9}(-1)^r = -84$

So, $\boxed{r=3}$ and $l = 5$

Now for x^{-15} $\frac{45-5r}{2} - 5r = -15$

$45 - 15r = -30$

$\boxed{r = 5}$

$\therefore$ Coefficient $= -{}^9C_5 2^6 = -63.2^7$

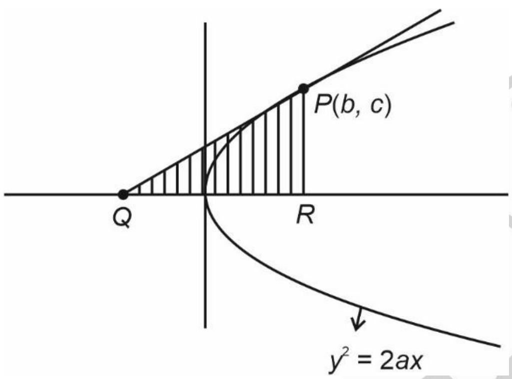
$\therefore$ $\alpha = 7, \beta = -63$

and $|\alpha l - \beta| = |7 \times 5 + 63| = 98$

82. Let S be the set of all $a \in \mathbb{N}$ such that the area of the triangle formed by the tangent at the point $P(b, c)$, $b, c \in \mathbb{N}$, on the parabola $y^2 = 2ax$ the lines $x = b$, $y = 0$ is 16 unit^2, then $\sum_{a \in S} a$ is equal to_______.

Answer (146)

Sol.



Tangent at $P(b, c)$:

$yc = a(x + b)$

for Q : $y = 0$ $\boxed{x = -b}$

$\therefore$ Area of shaded region = 16

$\frac{1}{2} \times 2b \times c = 16$

$bc = 16$ and $c^2 = 2ab$

$\therefore$ $2a = \frac{c^2.c}{16}$ $\frac{16 \times 16 \times 16}{32}$

$a = \frac{c^3}{32}, 1 \le C \le 16$ and divisor of 16

$\therefore$ $a = 2, 16, 128$

$\therefore$ $\Sigma a = 146$

83. The sum $1^2 - 2\cdot3^2 + 3\cdot5^2 - 4\cdot7^2 + 5\cdot9^2 - \ldots + 15\cdot29^2$ is _______.

Answer (6592)

Sol. $S = 1^2 - 2.3^2 + 3.5^2 - 4.7^2 + \ldots\ldots + 15.29^2$

$= \sum_{r=1}^{8}(2r-1)(4r-3)^2 - \sum_{r=1}^{7} 2r(4r-1)^2$

$= \sum_{r=1}^{8} 32r^3 - 64r^2 + 32r - 9 - 2 - \sum_{r=1}^{7} 16r^3 - 8r^2 + r$

$= 32 \times 36^2 - 64 \times 204 + 1152 - 72$

$- 2(16 \times 28^2 - 1120 + 28)$

$= 6592$

84. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $|\vec{a}| = \sqrt{31}$, $4|\vec{b}| = |\vec{c}| = 2$ and $2(\vec{a} \times \vec{b}) = 2(\vec{c} \times \vec{a})$. If the angle between $\vec{b}$ and $\vec{c}$ is $\frac{2\pi}{3}$, then $\left(\frac{\vec{a} \times \vec{c}}{\vec{a} \cdot \vec{b}}\right)^2$ is equal to _______.

Answer (03)

Sol. $\vec{a} \times (2\vec{b} + 3\vec{c}) = 0$

$\vec{a} = \lambda(2\vec{b} + 3\vec{c})$

$|\vec{a}|^2 = \lambda^2(4|b|^2 + 9|c|^2 + 12\vec{b} \cdot \vec{c})$

$31 = 31\lambda^2$

$\lambda = \pm 1$

$\vec{a} = \pm(2\vec{b} + 3\vec{c})$

$\frac{|\vec{a} \times \vec{c}|}{|\vec{a} \cdot \vec{b}|} = \frac{2|\vec{b} \times \vec{c}|}{2\vec{b} \cdot \vec{b} + 3\vec{c} \cdot \vec{b}}$

$|\vec{b} \times \vec{c}|^2 = \frac{1}{4} \cdot 4 - \left(1 - \frac{1}{2}\right)^2$

$= \frac{3}{4}$

$\therefore$ $\frac{|\vec{a} \times \vec{c}|}{|\vec{a} \cdot \vec{b}|} = \frac{\sqrt{3}}{2 \cdot \frac{1}{4} - \frac{3}{2}} = \frac{\sqrt{3}}{-1}$

$\left(\frac{|\vec{a} \times \vec{c}|}{|\vec{a} \cdot \vec{b}|}\right)^2 = 3$

85. Let A be a $n \times n$ matrix such that $|A| = 2$. If the determinant of the matrix $\text{Adj}(2 \cdot \text{Adj}(2A^{-1}))\cdot$ is 2^{84}, then n is equal to _______.

Answer (05)

Sol. $\because$ $\left|adj\left(2\cdot adj(2A^{-1})\right)\right| = 2^{84}$

$\Rightarrow$ $2^{n\cdot(n-1)}\left|adj\left(2A^{-1}\right)\right|^{(n-1)} = 2^{84}$

$\Rightarrow$ $2^{n(n-1)}\left|2A^{-1}\right|^{(n-1)^2} = 2^{84}$

$\Rightarrow$ $2^{n(n-1)}\cdot 2^{n(n-1)^2}\cdot\dfrac{1}{|A|^{(n-1)^2}} = 2^{84}$

$\Rightarrow$ $2^{n(n-1)+n(n-1)^2-(n-1)^2} = 2^{84}$ $\{\because |4| = 2\}$

$\therefore$ $n(n-1) + (n-1)^3 = 84$

$\therefore$ $n = 5$

86. The coefficient of x^{-6}, in the expansion of $\left(\dfrac{4x}{5}+\dfrac{5}{2x^2}\right)^9$, is ________.

Answer (5040)

Sol. Coeff of x^{-6} in $\left(\dfrac{4x}{5}+\dfrac{5}{2x^2}\right)^9$

$T_{r+1} = {}^9C_r\left(\dfrac{4x}{5}\right)^{9-r}\left(\dfrac{5}{2x^2}\right)^r$

$9 - 3r = -6$

$r = 5$

Coeff of $x^{-6} = {}^9C_5\left(\dfrac{4}{5}\right)^4\left(\dfrac{5}{2}\right)^5$

$= 5040$

87. Let $A = \left[a_{ij}\right], a_{ij} \in \mathbb{Z}\cap[0,4], 1\le i,j\le 2$. The number of matrices A such that the sum of all entries is a prime number $p\in(2,13)$ is ______.

Answer (204)

Sol. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{13} & a_{14}\end{bmatrix}$

Such that Σa_{ii} = 3, 5, 7 or 11

Then for sum 3, the possible entries are (0, 0, 0, 3), (0, 0, 1, 2), (0, 1, 1, 1).

Then total number of possible matrices

$= 4 + 12 + 4$

$= 20$

For sum 5 the possible entries are (0, 0, 1, 4), (0, 0, 2, 3), (0, 1, 2, 2), (0, 1, 1, 3) and (1, 1, 1, 2).

$\therefore$ Total possible matrices = 12+12+12+12+4 = 52

For sum 7 the possible entries are (0, 0, 3, 4), (0, 2, 2, 3), (0, 1, 2, 4), (0, 1, 3, 3), (1, 2, 2, 2), (1, 1, 2, 3) and (1, 1, 1, 4).

$\therefore$ Total possible matrices = 80

For sum 11 the possible entries are (0, 3, 4, 4), (1, 2, 4, 4), (2, 3, 3, 3), (2, 2, 3, 4).

$\therefore$ Total number of matrices = 52

$\therefore$ Total required matrices = 20 + 52 + 80 + 52

= 204

88. Let the area of the region $\left\{(x,y): |2x-1| \le y \le \left|x^2-x\right|, 0\le x\le 1\right\}$ be A. Then $(6A + 11)^2$ is equal to ______.

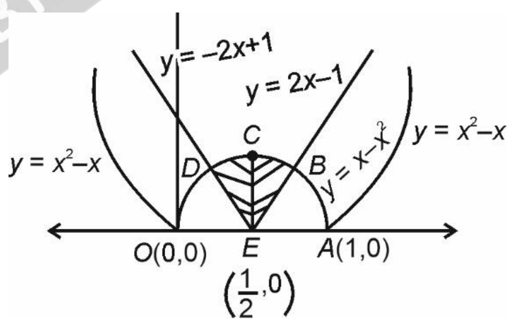
Answer (125)

Sol. For B,

$x - x^2 = 2x - 1$

$x^2 + x - 1 = 0$

$x = \dfrac{-1+\sqrt{5}}{2}$



Area = 2(area of BCE)

$A = 2\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{5}-1}{2}}\left(x-x^2\right)-\left(2x-1\right)dx$

$= 2\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{5}-1}{2}} 1 - x - x^2\, dx = 2\left(x - \dfrac{x^2}{2} - \dfrac{x^3}{3}\right)\Bigg]_{\frac{1}{2}}^{\frac{\sqrt{5}-1}{2}}$

$$= 2\left\{\left(\frac{\sqrt{5}-1}{2}-\frac{1}{2}\right)-\left\{\left(\frac{\sqrt{5}-1}{2}\right)^2-\frac{1}{4}\right\}\frac{1}{2}\right.$$

$$\left.-\left[\left(\frac{\sqrt{5}-1}{2}\right)^3-\frac{1}{8}\right]\frac{1}{3}\right\}$$

$$A=\frac{-11+5\sqrt{5}}{6}$$

$\Rightarrow$ $(6A + 11)^2 = 125$

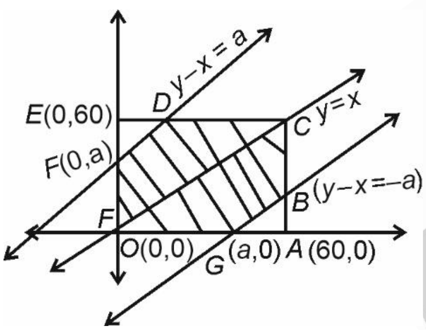
89. Let A be the event that the absolute difference between two randomly chosen real numbers in the sample space [0, 60] is less than or equal to a. If $P(A)=\frac{11}{36}$, then a is equal to _______.

Answer (10)

Sol. Let two numbers be x and y

$|y - x| < a$ (where $a > 0$)

$-a < y - x < a$



$$\text{Required probability}=\frac{\text{Area of shaded region}}{\text{Total area}}$$

$$\frac{[ABG]+[DEF]}{[OGAB\ CDEF]}=1-\frac{11}{36}$$

$$\frac{2[ABG]}{3600}=\frac{25}{36}$$

$[ABG] = 1250$

$$\frac{1}{2}(60-a)^2=1250$$

$(60 - a)^2 = 2500$

$a = 10, 110$ (Rejected)

$a = 10$

90. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 11 : 21$, then $n^2 + n + 15$ is equal to

Answer (45)

Sol. $$\frac{\dfrac{(2n+1)!}{(n+2)!}}{\dfrac{(2n-1)!}{(n-1)!}}=\frac{11}{21}$$

$$\frac{(2n+1)2n}{(n+2)(n+1)n}=\frac{11}{21}$$

$84n + 42 = 11(n^2 + 3n + 2)$

$11n^2 - 51n - 20 = 0$

$(n - 5)(11n + 4) = 0$

$n = 5, \frac{-4}{11}$ (Rejected)

$n^2 + n + 15 = 45$

❑ ❑ ❑