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1. Express the trigonometric ratios sin A, sec A and tan A in terms of cot A.

Solution:

To convert the given trigonometric ratios in terms of cot functions, use trigonometric formulas

We know that,

$\text{cosec}^2A - \cot^2A = 1$

$\text{cosec}^2A = 1 + \cot^2A$

Since cosec function is the inverse of sin function, it is written as

$1/\sin^2A = 1 + \cot^2A$

Now, rearrange the terms, it becomes

$\sin^2A = 1/(1+\cot^2A)$

Now, take square roots on both sides, we get

$\sin A = \pm 1/(\sqrt{(1+\cot^2A)}$

The above equation defines the sin function in terms of cot function

Now, to express sec function in terms of cot function, use this formula

$\sin^2A = 1/ (1+\cot^2A)$

Now, represent the sin function as cos function

$1 - \cos^2A = 1/ (1+\cot^2A)$

Rearrange the terms,

$\cos^2A = 1 - 1/(1+\cot^2A)$

$\Rightarrow \cos^2A = (1-1+\cot^2A)/(1+\cot^2A)$

Since sec function is the inverse of cos function,

$\Rightarrow 1/\sec^2A = \cot^2A/(1+\cot^2A)$

Take the reciprocal and square roots on both sides, we get

$\Rightarrow \sec A = \pm\sqrt{(1+\cot^2A)}/\cot A$

Now, to express tan function in terms of cot function

tan A = sin A/cos A and cot A = cos A/sin A

Since cot function is the inverse of tan function, it is rewritten as

tan A = 1/cot A

2. Write all the other trigonometric ratios of ∠A in terms of sec A.

Solution:

Cos A function in terms of sec A:

sec A = 1/cos A

⇒ cos A = 1/sec A

sec A function in terms of sec A:

$\cos^2 A + \sin^2 A = 1$

Rearrange the terms

$\sin^2 A = 1 – \cos^2 A$

$\sin^2 A = 1 – (1/\sec^2 A)$

$\sin^2 A = (\sec^2 A-1)/\sec^2 A$

$\sin A = \pm \sqrt{(\sec^2 A-1)}/\sec A$

cosec A function in terms of sec A:

sin A = 1/cosec A

⇒cosec A = 1/sin A

$\text{cosec } A = \pm \sec A/\sqrt{(\sec^2 A-1)}$

Now, tan A function in terms of sec A:

$\sec^2 A – \tan^2 A = 1$

Rearrange the terms

$\Rightarrow \tan^2 A = \sec^2 A – 1$

$\tan A = \sqrt{(\sec^2 A – 1)}$

cot A function in terms of sec A:

tan A = 1/cot A

⇒ cot A = 1/tan A

$\cot A = \pm 1/\sqrt{(\sec^2 A – 1)}$

3. Evaluate:

(i) $(\sin^2 63° + \sin^2 27°)/(\cos^2 17° + \cos^2 73°)$
(ii) sin 25° cos 65° + cos 25° sin 65°

Solution:

(i) $(\sin^2 63° + \sin^2 27°)/(\cos^2 17° + \cos^2 73°)$

To simplify this, convert some of the sin functions into cos functions and cos function into sin function and it becomes,

$= [\sin^2(90°-27°) + \sin^2 27°] / [\cos^2(90°-73°) + \cos^2 73°)]$

$= (\cos^2 27° + \sin^2 27°)/(\sin^2 27° + \cos^2 73°)$

$= 1/1 =1$ (since $\sin^2 A + \cos^2 A = 1$)

Therefore, $(\sin^2 63° + \sin^2 27°)/(\cos^2 17° + \cos^2 73°) = 1$

(ii) sin 25° cos 65° + cos 25° sin 65°

To simplify this, convert some of the sin functions into cos functions and cos function into sin function and it becomes,

= sin(90°-25°) cos 65° + cos (90°-65°) sin 65°

= cos 65° cos 65° + sin 65° sin 65°

= $\cos^2 65° + \sin^2 65°$ = 1 (since $\sin^2 A + \cos^2 A = 1$)

Therefore, sin 25° cos 65° + cos 25° sin 65° = 1

4. Choose the correct option. Justify your choice.
(i) 9 $\sec^2 A$ – 9 $\tan^2 A$ =
(A) 1 (B) 9 (C) 8 (D) 0
(ii) (1 + tan θ + sec θ) (1 + cot θ – cosec θ)
(A) 0 (B) 1 (C) 2 (D) – 1
(iii) (sec A + tan A) (1 – sin A) =
(A) sec A (B) sin A (C) cosec A (D) cos A

(iv) 1+$\tan^2 A$/1+$\cot^2 A$ =

(A) $\sec^2 A$ (B) -1 (C) $\cot^2 A$ (D) $\tan^2 A$

Solution:

(i) (B) is correct.

Justification:

Take 9 outside, and it becomes

9 $\sec^2 A$ – 9 $\tan^2 A$

= 9 ($\sec^2 A$ – $\tan^2 A$)

= 9×1 = 9 (∵ sec2 A – tan2 A = 1)

Therefore, 9 $\sec^2 A$ – 9 $\tan^2 A$ = 9

(ii) (C) is correct

Justification:

(1 + tan θ + sec θ) (1 + cot θ – cosec θ)

We know that, tan θ = sin θ/cos θ

sec θ = 1/ cos θ

cot θ = cos θ/sin θ

cosec θ = 1/sin θ

Now, substitute the above values in the given problem, we get

= (1 + sin θ/cos θ + 1/ cos θ) (1 + cos θ/sin θ – 1/sin θ)

Simplify the above equation,

= (cos θ +sin θ+1)/cos θ × (sin θ+cos θ-1)/sin θ

$= (\cos\theta+\sin\theta)^2-1^2/(\cos\theta \sin\theta)$

$= (\cos^2\theta + \sin^2\theta + 2\cos\theta \sin\theta -1)/(\cos\theta \sin\theta)$

$= (1+ 2\cos\theta \sin\theta -1)/(\cos\theta \sin\theta)$ (Since $\cos^2\theta + \sin^2\theta = 1$)

$= (2\cos\theta \sin\theta)/(\cos\theta \sin\theta) = 2$

Therefore, $(1 + \tan\theta + \sec\theta) (1 + \cot\theta - \operatorname{cosec}\theta) =2$

(iii) (D) is correct.

Justification:

We know that,

Sec A= 1/cos A

Tan A = sin A / cos A

Now, substitute the above values in the given problem, we get

(secA + tanA) (1 – sinA)

= (1/cos A + sin A/cos A) (1 – sinA)

= (1+sin A/cos A) (1 – sinA)

= $(1 - \sin^2 A)/\cos A$

= $\cos^2 A/\cos A = \cos A$

Therefore, (secA + tanA) (1 – sinA) = cos A

(iv) (D) is correct.

Justification:

We know that,

$\tan^2 A = 1/\cot^2 A$

Now, substitute this in the given problem, we get

$1+\tan^2 A/1+\cot^2 A$

$= (1+1/\cot^2 A)/1+\cot^2 A$

$= (\cot^2 A+1/\cot^2 A)\times(1/1+\cot^2 A)$

$= 1/\cot^2 A = \tan^2 A$

So, $1+\tan^2 A/1+\cot^2 A = \tan^2 A$

5. Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

(i) $(\operatorname{cosec}\theta - \cot\theta)^2 = (1-\cos\theta)/(1+\cos\theta)$

(ii) cos A/(1+sin A) + (1+sin A)/cos A = 2 sec A

(iii) $\tan\theta/(1-\cot\theta) + \cot\theta/(1-\tan\theta) = 1 + \sec\theta \operatorname{cosec}\theta$

[Hint : Write the expression in terms of $\sin\theta$ and $\cos\theta$]

(iv) (1 + sec A)/sec A = $\sin^2$A/(1-cos A)

[Hint : Simplify LHS and RHS separately]

(v) (cos A–sin A+1)/(cos A +sin A–1) = cosec A + cot A, using the identity cosec^2A = 1+$\cot^2$A.

(vi) $\sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$

(vii) (sin θ – 2$\sin^3$θ)/(2$\cos^3$θ-cos θ) = tan θ
(viii) (sin A + cosec A)^2 + (cos A + sec A)^2 = 7+$\tan^2$A+$\cot^2$A
(ix) (cosec A – sin A)(sec A – cos A) = 1/(tan A+cotA)
[Hint : Simplify LHS and RHS separately] (x) (1+$\tan^2$A/1+$\cot^2$A) = (1-tan A/1-cot A)^2 = $\tan^2$A

Solution:

(i) (cosec θ – cot θ)^2 = (1-cos θ)/(1+cos θ)

To prove this, first take the Left-Hand side (L.H.S) of the given equation, to prove the Right Hand Side (R.H.S)

L.H.S. = (cosec θ – cot θ)^2

The above equation is in the form of (a-b)^2, and expand it

Since (a-b)^2 = a^2 + b^2 – 2ab

Here a = cosec θ and b = cot θ

= (cosec^2θ + $\cot^2$θ – 2cosec θ cot θ)

Now, apply the corresponding inverse functions and equivalent ratios to simplify

= (1/$\sin^2$θ + $\cos^2$θ/$\sin^2$θ – 2cos θ/$\sin^2$θ)

= (1 + $\cos^2$θ – 2cos θ)/(1 – $\cos^2$θ)

= (1-cos θ)^2/(1 – cosθ)(1+cos θ)

= (1-cos θ)/(1+cos θ) = R.H.S.

Therefore, (cosec θ – cot θ)^2 = (1-cos θ)/(1+cos θ)

Hence proved.

(ii) (cos A/(1+sin A)) + ((1+sin A)/cos A) = 2 sec A

Now, take the L.H.S of the given equation.

L.H.S. = (cos A/(1+sin A)) + ((1+sin A)/cos A)

= [$\cos^2$A + (1+sin A)^2]/(1+sin A)cos A

= ($\cos^2$A + $\sin^2$A + 1 + 2sin A)/(1+sin A) cos A

Since $\cos^2$A + $\sin^2$A = 1, we can write it as

= (1 + 1 + 2sin A)/(1+sin A) cos A

= (2+ 2sin A)/(1+sin A)cos A

= 2(1+sin A)/(1+sin A)cos A

= 2/cos A = 2 sec A = R.H.S.

L.H.S. = R.H.S.

(cos A/(1+sin A)) + ((1+sin A)/cos A) = 2 sec A

Hence proved.

(iii) tan θ/(1-cot θ) + cot θ/(1-tan θ) = 1 + sec θ cosec θ

L.H.S. = tan θ/(1-cot θ) + cot θ/(1-tan θ)

We know that tan θ =sin θ/cos θ

cot θ = cos θ/sin θ

Now, substitute it in the given equation, to convert it in a simplified form

= [(sin θ/cos θ)/1-(cos θ/sin θ)] + [(cos θ/sin θ)/1-(sin θ/cos θ)]

= [(sin θ/cos θ)/(sin θ-cos θ)/sin θ] + [(cos θ/sin θ)/(cos θ-sin θ)/cos θ]

= $\sin^2\theta$/[cos θ(sin θ-cos θ)] + $\cos^2\theta$/[sin θ(cos θ-sin θ)]

= $\sin^2\theta$/[cos θ(sin θ-cos θ)] – $\cos^2\theta$/[sin θ(sin θ-cos θ)]

= 1/(sin θ-cos θ) [($\sin^2\theta$/cos θ) – ($\cos^2\theta$/sin θ)]

= 1/(sin θ-cos θ) × [($\sin^3\theta$ – $\cos^3\theta$)/sin θ cos θ]

= [(sin θ-cos θ)($\sin^2\theta$+$\cos^2\theta$+sin θ cos θ)]/[(sin θ-cos θ)sin θ cos θ]

= (1 + sin θ cos θ)/sin θ cos θ

= 1/sin θ cos θ + 1

= 1 + sec θ cosec θ = R.H.S.

Therefore, L.H.S. = R.H.S.

Hence proved

(iv) (1 + sec A)/sec A = $\sin^2$A/(1-cos A)

First find the simplified form of L.H.S

L.H.S. = (1 + sec A)/sec A

Since secant function is the inverse function of cos function and it is written as

= (1 + 1/cos A)/1/cos A

= (cos A + 1)/cos A/1/cos A

Therefore, (1 + sec A)/sec A = cos A + 1

R.H.S. = $\sin^2$A/(1-cos A)

We know that $\sin^2$A = (1 – $\cos^2$A), we get

= (1 – $\cos^2$A)/(1-cos A)

= (1-cos A)(1+cos A)/(1-cos A)

Therefore, $\sin^2$A/(1-cos A)= cos A + 1

L.H.S. = R.H.S.

Hence proved

(v) (cos A–sin A+1)/(cos A+sin A–1) = cosec A + cot A, using the identity cosec^{2}A = 1+cot^{2}A.

With the help of identity function, cosec^{2}A = 1+cot^{2}A, let us prove the above equation.

L.H.S. = (cos A–sin A+1)/(cos A+sin A–1)

Divide the numerator and denominator by sin A, we get

= (cos A–sin A+1)/sin A/(cos A+sin A–1)/sin A

We know that cos A/sin A = cot A and 1/sin A = cosec A

= (cot A – 1 + cosec A)/(cot A+ 1 – cosec A)

= (cot A – cosec^{2}A + cot^{2}A + cosec A)/(cot A+ 1 – cosec A) (using cosec^{2}A – cot^{2}A = 1

= [(cot A + cosec A) – (cosec^{2}A – cot^{2}A)]/(cot A+ 1 – cosec A)

= [(cot A + cosec A) – (cosec A + cot A)(cosec A – cot A)]/(1 – cosec A + cot A)

= (cot A + cosec A)(1 – cosec A + cot A)/(1 – cosec A + cot A)

= cot A + cosec A = R.H.S.

Therefore, (cos A–sin A+1)/(cos A+sin A–1) = cosec A + cot A

Hence Proved

(vi) $\sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$

$$\text{L.H.S} = \sqrt{\frac{1+\sin A}{1-\sin A}}$$

First divide the numerator and denominator of L.H.S. by cos A,

$$= \sqrt{\frac{\frac{1}{\cos A}+\frac{\sin A}{\cos A}}{\frac{1}{\cos A}-\frac{\sin A}{\cos A}}}$$

We know that 1/cos A = sec A and sin A/ cos A = tan A and it becomes,

= √(sec A+ tan A)/(sec A-tan A)

Now using rationalization, we get

$= \sqrt{\frac{sec\,A + tan\,A}{Sec\,A - tan\,A}} \times \sqrt{\frac{sec\,A + tan\,A}{Sec\,A + tan\,A}}$

$= \sqrt{\frac{(sec\,A + tan\,A)^2}{sec^2 A - tan^2 A}}$

= (sec A + tan A)/1

= sec A + tan A = R.H.S

Hence proved

(vii) (sin θ – $2\sin^3θ$)/($2\cos^3θ$-cos θ) = tan θ

L.H.S. = (sin θ – $2\sin^3θ$)/($2\cos^3θ$ – cos θ)

Take sin θ as in numerator and cos θ in denominator as outside, it becomes

= [sin θ(1 – $2\sin^2θ$)]/[cos θ($2\cos^2θ$- 1)]

We know that $\sin^2θ$ = 1-$\cos^2θ$

= sin θ[1 – 2(1-$\cos^2θ$)]/[cos θ($2\cos^2θ$ -1)]

= [sin θ($2\cos^2θ$ -1)]/[cos θ($2\cos^2θ$ -1)]

= tan θ = R.H.S.

Hence proved

(viii) $(\sin A + \text{cosec } A)^2 + (\cos A + \sec A)^2 = 7+\tan^2A+\cot^2A$

L.H.S. = $(\sin A + \text{cosec } A)^2 + (\cos A + \sec A)^2$

It is of the form $(a+b)^2$, expand it

$(a+b)^2 = a^2 + b^2 + 2ab$

= ($\sin^2A$ + cosec^2A + 2 sin A cosec A) + ($\cos^2A$ + $\sec^2A$ + 2 cos A sec A)

= ($\sin^2A$ + $\cos^2A$) + 2 sin A(1/sin A) + 2 cos A(1/cos A) + 1 + $\tan^2A$ + 1 + $\cot^2A$

= 1 + 2 + 2 + 2 + $\tan^2A$ + $\cot^2A$

= 7+$\tan^2A$+$\cot^2A$ = R.H.S.

Therefore, $(\sin A + \text{cosec } A)^2 + (\cos A + \sec A)^2 = 7+\tan^2A+\cot^2A$

Hence proved.

(ix) (cosec A – sin A)(sec A – cos A) = 1/(tan A+cotA)

First, find the simplified form of L.H.S

L.H.S. = (cosec A – sin A)(sec A – cos A)

Now, substitute the inverse and equivalent trigonometric ratio forms

= (1/sin A – sin A)(1/cos A – cos A)

$= [(1-\sin^2 A)/\sin A][(1-\cos^2 A)/\cos A]$

$= (\cos^2 A/\sin A)\times(\sin^2 A/\cos A)$

$= \cos A \sin A$

Now, simplify the R.H.S

R.H.S. $= 1/(\tan A+\cot A)$

$= 1/(\sin A/\cos A + \cos A/\sin A)$

$= 1/[(\sin^2 A+\cos^2 A)/\sin A \cos A]$

$= \cos A \sin A$

L.H.S. = R.H.S.

$(\text{cosec } A - \sin A)(\sec A - \cos A) = 1/(\tan A+\cot A)$

Hence proved

(x) $(1+\tan^2 A/1+\cot^2 A) = (1-\tan A/1-\cot A)^2 = \tan^2 A$

L.H.S. $= (1+\tan^2 A/1+\cot^2 A)$

Since cot function is the inverse of tan function,

$= (1+\tan^2 A/1+1/\tan^2 A)$

$= 1+\tan^2 A/[(1+\tan^2 A)/\tan^2 A]$

Now cancel the $1+\tan^2 A$ terms, we get

$= \tan^2 A$

$(1+\tan^2 A/1+\cot^2 A) = \tan^2 A$

Similarly,

$(1-\tan A/1-\cot A)^2 = \tan^2 A$

Hence proved