## EXERCISE 13.1

1. Evaluate the given limit: $\lim _{x \rightarrow 3} x+3$

## Solution:

Given,
$\lim _{x \rightarrow 3} x+3$
Substituting $x=3$, we get
$=3+3$
$=6$
2. Evaluate the given limit: $\lim _{x \rightarrow \pi}\left(x-\frac{22}{7}\right)$

## Solution:

Given limit,
$\lim _{x \rightarrow \pi}\left(x-\frac{22}{7}\right)$
Substituting $x=\pi$, we get

$$
\lim _{x \rightarrow \pi}\left(x-\frac{22}{7}\right)=(\pi-22 / 7)
$$

3. Evaluate the given limit: $\lim _{r \rightarrow 1} \pi r^{2}$

## Solution:

Given limit, $\lim _{r \rightarrow 1} \pi r^{2}$
Substituting $r=1$, we get

$$
\begin{aligned}
& \lim _{r \rightarrow 1} \pi r^{2}=\pi(1)^{2} \\
& =\pi
\end{aligned}
$$

4. Evaluate the given limit: $\lim _{x \rightarrow 4} \frac{4 x+3}{x-2}$

## Solution:

Given limit,
$\lim _{x \rightarrow 4} \frac{4 x+3}{x-2}$

Substituting $x=4$, we get
$\lim _{x \rightarrow 4} \frac{4 x+3}{x-2}=[4(4)+3] /(4-2)$
$=(16+3) / 2$
$=19 / 2$
5. Evaluate the given limit: $\lim _{x \rightarrow-1} \frac{x^{10}+x^{5}+1}{x-1}$

## Solution:

Given limit,
$\lim _{x \rightarrow-1} \frac{x^{10}+x^{5}+1}{x-1}$
Substituting $x=-1$, we get
$\lim _{x \rightarrow-1} \frac{x^{10}+x^{5}+1}{x-1}$
$=\left[(-1)^{10}+(-1)^{5}+1\right] /(-1-1)$
$=(1-1+1) /-2$
$=-1 / 2$
6. Evaluate the given limit:

$$
\lim _{x \rightarrow 0} \frac{(x+1)^{5}-1}{x}
$$

## Solution:

Given limit,
$\lim _{x \rightarrow 0} \frac{(x+1)^{5}-1}{x}$
$=\left[(0+1)^{5}-1\right] / 0$
$=0$
Since this limit is undefined,
Substitute $x+1=y$, then $x=y-1$

$$
\lim _{y \rightarrow 1} \frac{(y)^{5}-1}{y-1}
$$

$$
=\lim _{y \rightarrow 1} \frac{(y)^{5}-1^{5}}{y-1}
$$

We know that,

$$
\lim _{x \rightarrow a} \frac{x^{n}-a^{n}}{x-a}=n a^{n-1}
$$

## Hence,

$$
\begin{aligned}
& \lim _{y \rightarrow 1} \frac{(y)^{5}-1^{5}}{y-1} \\
= & 5(1)^{5-1} \\
= & 5(1)^{4} \\
= & 5
\end{aligned}
$$

7. Evaluate the given limit: $\lim _{x \rightarrow 2} \frac{3 x^{2}-x-10}{x^{2}-4}$

## Solution:

By evaluating the limit at $x=2$, we get

$$
\begin{aligned}
& \lim _{x \rightarrow 2} \frac{3 x^{2}-x-10}{x^{2}-4}=\left[3(2)^{2}-x-10\right] / 4-4 \\
& =0
\end{aligned}
$$

Now, by factorising numerator, we get

$$
\lim _{x \rightarrow 2} \frac{3 x^{2}-x-10}{x^{2}-4}=\lim _{x \rightarrow 2} \frac{3 x^{2}-6 x+5 x-10}{x^{2}-2^{2}}
$$

We know that,
$a^{2}-b^{2}=(a-b)(a+b)$

$$
\begin{aligned}
& =\lim _{x \rightarrow 2} \frac{(x-2)(3 x+5)}{(x-2)(x+2)} \\
& =\lim _{x \rightarrow 2} \frac{(3 x+5)}{(x+2)}
\end{aligned}
$$

By substituting $x=2$, we get,
$=[3(2)+5] /(2+2)$
$=11 / 4$
8. Evaluate the given limit: $\lim _{x \rightarrow 3} \frac{x^{4}-81}{2 x^{2}-5 x-3}$

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## Solution:

First substitute $\mathrm{x}=3$ in the given limit, we get

$$
\begin{aligned}
& \lim _{x \rightarrow 3} \frac{(3)^{4}-81}{2(3)^{2}-5 \times 3-3} \\
= & (81-81) /(18-18) \\
= & 0 / 0
\end{aligned}
$$

Since the limit is of the form $0 / 0$, we need to factorise the numerator and denominator

$$
\lim _{x \rightarrow 3} \frac{\left(x^{2}-9\right)\left(x^{2}+9\right)}{2 x^{2}-6 x+x-3} \lim _{x \rightarrow 3} \frac{(x-3)(x+3)\left(x^{2}+9\right)}{(2 x+1)(x-3)}
$$

$$
\lim _{x \rightarrow 3} \frac{x^{4}-81}{2 x^{2}-5 x-3}=\lim _{x \rightarrow 3} \frac{(x+3)\left(x^{2}+9\right)}{(2 x+1)}
$$

Now substituting $x=3$, we get

$$
=\frac{(3+3)\left(3^{2}+9\right)}{(2 \times 3+1)}
$$

$$
=108 / 7
$$

Hence,

$$
\lim _{x \rightarrow 3} \frac{x^{4}-81}{2 x^{2}-5 x-3}=108 / 7
$$

9. Evaluate the given limit: $\lim _{x \rightarrow 0} \frac{a x+b}{c x+1}$

Solution:

$$
\begin{aligned}
& \lim _{x \rightarrow 0} \frac{a x+b}{c x+1} \\
= & {[a(0)+b] / c(0)+1 } \\
= & b / 1 \\
= & b
\end{aligned}
$$

10. Evaluate the given limit: $\lim _{z \rightarrow 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

## Solution:

$$
\begin{aligned}
& \lim _{z \rightarrow 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}-1}}=(1-1) /(1-1) \\
& =0
\end{aligned}
$$

Let the value of $z^{1 / 6}$ be $x$
$\left(z^{1 / 6}\right)^{2}=x^{2}$
$z^{1 / 3}=x^{2}$
Now, substituting $z^{1 / 3}=x^{2}$ we get
$\lim _{x \rightarrow 1} \frac{x^{2}-1}{x-1}=\frac{x^{2}-1^{2}}{x-1}$
We know that,

$$
\begin{aligned}
& \lim _{x \rightarrow a} \frac{x^{n}-a^{n}}{x-a}=n a^{n-1} \\
& \lim _{x \rightarrow 1} \frac{x^{2}-1^{2}}{x-1}=2(1)^{2-1} \\
& =2
\end{aligned}
$$

11. Evaluate the given limit: $\lim _{x \rightarrow 1} \frac{a x^{2}+b x+c}{c x^{2}+b x+a}, a+b+c \neq 0$

## Solution:

Given limit,
$\lim _{x \rightarrow 1} \frac{a x^{2}+b x+c}{c x^{2}+b x+a}, a+b+c \neq 0$
Substituting $x=1$,
$\lim _{x \rightarrow 1} \frac{a x^{2}+b x+c}{c x^{2}+b x+a}$
$=\left[a(1)^{2}+b(1)+c\right] /\left[c(1)^{2}+b(1)+a\right]$
$=(a+b+c) /(a+b+c)$
Given,
$[a+b+c \neq 0]$
$=1$

$$
\lim _{x \rightarrow-2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}
$$

## Solution:

By substituting $x=-2$, we get

$$
\lim _{x \rightarrow-2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}=0 / 0
$$

Now,

$$
\lim _{x \rightarrow-2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\frac{\frac{2+x}{2 x}}{x+2}
$$

$=1 / 2 \mathrm{x}$
$=1 / 2(-2)$
$=-1 / 4$
13. Evaluate the given limit: $\lim _{x \rightarrow 0} \frac{\sin a x}{b x}$

Solution:
Given ${ }^{\lim _{x \rightarrow 0} \frac{\sin a x}{b x}}$

Formula used here
$\lim _{x \rightarrow 0} \frac{\sin x}{x}=1$
By applying the limits in the given expression
$\lim _{x \rightarrow 0} \frac{\sin \mathrm{ax}}{\mathrm{bx}}=\frac{0}{0}$
By multiplying and dividing by ' $a$ ' in the given expression, we get
$\lim _{x \rightarrow 0} \frac{\sin a x}{b x} \times \frac{a}{a}$
We get,
$\lim _{x \rightarrow 0} \frac{\sin a x}{a x} \times \frac{a}{b}$
We know that,

$$
\lim _{x \rightarrow 0} \frac{\sin x}{x}=1
$$

$$
=\frac{\mathrm{a}}{\mathrm{~b}} \lim _{\mathrm{ax} \rightarrow 0} \frac{\sin \mathrm{ax}}{\mathrm{ax}}=\frac{\mathrm{a}}{\mathrm{~b}} \times 1
$$

$$
=a / b
$$

14. Evaluate the given limit: $\lim _{x \rightarrow 0} \frac{\sin a x}{\sin b x}, a, b \neq 0$

Solution:

$$
\lim _{x \rightarrow 0} \frac{\sin a x}{\sin b x}=0 / 0
$$

By multiplying ax and bx in numerator and denominator, we get

$$
\lim _{x \rightarrow 0} \frac{\sin a x}{\sin b x}=\lim _{x \rightarrow 0} \frac{\frac{\sin a x}{\sin b x} \times a x}{b x} \times b x
$$

Now, we get

- $\frac{\lim _{\mathrm{ax} \rightarrow 0} \frac{\sin \mathrm{ax}}{\mathrm{ax}}}{\lim _{\mathrm{bx} \rightarrow 0} \frac{\sin b \mathrm{x}}{\mathrm{bx}}}$

We know that,
$\lim _{x \rightarrow 0} \frac{\sin x}{x}=1$
Hence, $\mathrm{a} / \mathrm{b} \times 1$
$=a / b$
15. Evaluate the given limit:
$\lim _{x \rightarrow \pi} \frac{\sin (\pi-x)}{\pi(\pi-x)}$

## Solution:

$$
\lim _{x \rightarrow \pi} \frac{\sin (\pi-x)}{\pi(\pi-x)}
$$

$$
\lim _{x \rightarrow \pi} \frac{\sin (\pi-x)}{\pi(\pi-x)}=\lim _{\pi-x \rightarrow 0} \frac{\sin (\pi-x)}{(\pi-x)} \times \frac{1}{\pi}
$$

$$
=\frac{1}{\pi} \lim _{\pi-x \rightarrow 0} \frac{\sin (\pi-x)}{(\pi-x)}
$$

We know that

$$
\begin{aligned}
& \lim _{x \rightarrow 0} \frac{\sin x}{x}=1 \\
& \frac{1}{\pi} \lim _{\pi-x \rightarrow 0} \frac{\sin (\pi-x)}{(\pi-x)}=\frac{1}{\pi} \times 1 \\
& =1 / \pi
\end{aligned}
$$

16. Evaluate the given limit:
$\lim _{x \rightarrow 0} \frac{\cos x}{\pi-x}$
Solution:

$$
\begin{aligned}
& \lim _{x \rightarrow 0} \frac{\cos x}{\pi-x}=\frac{\cos 0}{\pi-0} \\
& =1 / \pi
\end{aligned}
$$

17. Evaluate the given limit:

$$
\lim _{x \rightarrow 0} \frac{\cos 2 x-1}{\cos x-1}
$$

Solution:

$$
\lim _{x \rightarrow 0} \frac{\cos 2 x-1}{\cos x-1}=\frac{0}{0}
$$

Hence,

$$
\lim _{x \rightarrow 0} \frac{\cos 2 x-1}{\cos x-1}=\lim _{x \rightarrow 0} \frac{1-2 \sin ^{2} x-1}{1-2 \sin ^{2} \frac{x}{2}-1}
$$

$$
\left(\cos 2 x=1-2 \sin ^{2} x\right)
$$

$$
\lim _{x \rightarrow 0} \frac{\sin ^{2} x}{\sin ^{2} \frac{x}{2}}=\lim _{x \rightarrow 0} \frac{\frac{\sin ^{2} x \times x^{2}}{x^{2}}}{\frac{\sin 2 \frac{x}{2} \times \frac{x^{2}}{4}}{\left(\frac{x}{2}\right)^{2}}}
$$

$$
=4^{\frac{\lim _{x \rightarrow 0} \frac{\sin ^{2} x}{x^{2}}}{\left.\lim _{x \rightarrow 0} \frac{\left(i^{2 x} \frac{x}{2}\right.}{2}\right)^{2}}}
$$

$$
=4^{\frac{\lim _{x \rightarrow 0}\left(\frac{\sin ^{2} x}{x^{2}}\right)^{2}}{\lim _{x \rightarrow 0}\left(\frac{\sin 2 \frac{x}{2}}{\left(\frac{x}{2}\right)^{2}}\right)^{2}}}
$$

We know that,

$$
\lim _{x \rightarrow 0} \frac{\sin x}{x}=1
$$

$=4 \times 1^{2} / 1^{2}$
$=4$
18. Evaluate the given limit:

$$
\lim _{x \rightarrow 0} \frac{a x+x \cos x}{b \sin x}
$$

Solution:

$$
\lim _{x \rightarrow 0} \frac{a x+x \cos x}{b \sin x}=\frac{0}{0}
$$

Hence,

$$
\begin{aligned}
& \lim _{x \rightarrow 0} \frac{a x+x \cos x}{b \sin x}=\frac{1}{b} \lim _{x \rightarrow 0} \frac{x(a+\cos x)}{\sin x} \\
& =\frac{1}{b} \lim _{x \rightarrow 0} \times \lim _{x \rightarrow 0}(a+\cos x) \\
& =\frac{1}{b} \times \frac{1}{\lim _{x \rightarrow 0} \frac{\sin x}{x}} \times \lim _{x \rightarrow 0}(a+\cos x)
\end{aligned}
$$

We know that,

$$
\begin{aligned}
& \lim _{x \rightarrow 0} \frac{\sin x}{x}=1 \\
& =\frac{1}{b} \times(a+\cos 0) \\
& =(a+1) / b
\end{aligned}
$$

19. Evaluate the given limit:
$\lim _{x \rightarrow 0} x \sec x$

## Solution:

$$
\lim _{x \rightarrow 0} x \sec x=\lim _{x \rightarrow 0} \frac{x}{\cos x}
$$

$$
\begin{aligned}
& =\lim _{x \rightarrow 0} \frac{0}{\cos 0}=\frac{0}{1} \\
& =0
\end{aligned}
$$

20. Evaluate the given limit:

$$
\lim _{x \rightarrow 0} \frac{\sin a x+b x}{a x+\sin b x} a, b, a+b \neq 0
$$

## Solution:

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$$
\lim _{x \rightarrow 0} \frac{\sin a x+b x}{a x+\sin b x}=\frac{0}{0}
$$

Hence,
$\lim _{x \rightarrow 0} \frac{\sin a x+b x}{a x+\sin b x}=\lim _{x \rightarrow 0} \frac{\left(\sin \frac{a x}{a x}\right) a x+b x}{a x+\left(\sin \frac{b x}{b x}\right)}$
$=\frac{\left(\lim _{a x \rightarrow 0} \sin \frac{a x}{a x}\right) \times \lim _{x \rightarrow 0} a x+\lim _{x \rightarrow 0} b x}{\lim _{x \rightarrow 0} a x+\lim _{x \rightarrow 0} b x \times\left(\lim _{b x \rightarrow 0} \sin \frac{b x}{b x}\right)}$
We know that,
$\lim _{x \rightarrow 0} \frac{\sin x}{x}=1$
$=\frac{\lim _{X \rightarrow 0} a x+\lim _{X \rightarrow 0} b x}{\lim _{x \rightarrow 0} a x+\lim _{x \rightarrow 0} b x}$
We get,
$\lim _{x \rightarrow 0}(a x+b x)$
$\lim _{x \rightarrow 0}(a x+b x)$
$=1$
21. Evaluate the given limit:
$\lim _{x \rightarrow 0}(\operatorname{cosec} x-\cot x)$
Solution:

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$$
\lim _{x \rightarrow 0}(\operatorname{cosec} x-\cot x)
$$

Applying the formulas for $\operatorname{cosec} x$ and $\cot x$, we get

$$
\begin{aligned}
& \operatorname{cosec} x=\frac{1}{\sin x} \text { and } \cot x=\frac{\cos x}{\sin x} \\
& \lim _{x \rightarrow 0}(\operatorname{cosec} x-\cot x)=\lim _{x \rightarrow 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right) \\
& \lim _{x \rightarrow 0}(\operatorname{cosec} x-\cot x)=\lim _{x \rightarrow 0} \frac{1-\cos x}{\sin x}
\end{aligned}
$$

Now, by applying the formula we get,

$$
\begin{aligned}
& 1-\cos x=2 \sin ^{2} \frac{x}{2} \text { and } \sin x=2 \sin \frac{x}{2} \cos \frac{x}{2} \\
& \lim _{x \rightarrow 0}(\operatorname{cosec} x-\cot x)=\lim _{x \rightarrow 0} \frac{2 \sin ^{2} \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \\
& \lim _{x \rightarrow 0}(\operatorname{cosec} x-\cot x)=\lim _{x \rightarrow 0} \tan \frac{x}{2} \\
& =0
\end{aligned}
$$

22. Evaluate the given limit:

$$
\lim _{x \rightarrow \frac{\pi}{2}} \frac{\tan 2 x}{x-\frac{\pi}{2}}
$$

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$\lim _{x \rightarrow \frac{\pi}{2}} \frac{\tan 2 x}{x-\frac{\pi}{2}}=\frac{0}{0}$
Let $\mathrm{x}-(\pi / 2)=\mathrm{y}$
Then, $x \rightarrow(\pi / 2)=y \rightarrow 0$
Now, we get
$\lim _{x \rightarrow \frac{\pi}{2}} \frac{\tan 2 x}{x-\frac{\pi}{2}}=\lim _{y \rightarrow 0} \frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$
$=\lim _{\mathrm{y} \rightarrow 0} \frac{\tan (2 \mathrm{y}+\pi)}{\mathrm{y}}$
$=\lim _{\mathrm{y} \rightarrow 0} \frac{\tan (2 \mathrm{y})}{\mathrm{y}}$
We know that,
$\tan x=\sin x / \cos x$
$=\lim _{\mathrm{y} \rightarrow 0 \mathrm{y} \cos 2 \mathrm{y}} \frac{\sin 2 \mathrm{y}}{}$
By multiplying and dividing by 2 , we get

$$
\begin{aligned}
& =\lim _{y \rightarrow 0} \frac{\sin 2 y}{2 y} \times \frac{2}{\cos 2 y} \\
& =\lim _{2 y \rightarrow 0} \frac{\sin 2 y}{2 y} \times \lim _{y \rightarrow 0} \frac{2}{\cos 2 y} \\
& =1 \times 2 / \cos 0 \\
& =1 \times 2 / 1 \\
& =2
\end{aligned}
$$

23. 

Find $\lim _{x \rightarrow 0} f(x)$ and $\lim _{x \rightarrow 1} f(x)$, where $f(x)=\left\{\begin{array}{r}2 x+3 x \leq 0 \\ 3(x+1) x>0\end{array}\right.$

## Solution:

Given function is $f(x)=\left\{\begin{array}{r}2 x+3 x \leq 0 \\ 3(x+1) x>0\end{array}\right.$
$\lim _{x \rightarrow 0} f(x):$

$$
\begin{aligned}
& \lim _{x \rightarrow 0^{-}} f(x)=\lim _{x \rightarrow 0}(2 x+3) \\
&= 2(0)+3 \\
&= 0+3 \\
&= 3 \\
& \lim _{x \rightarrow 0^{+}} f(x)=\lim _{x \rightarrow 0} 3(x+1): \\
&=3(0+1) \\
&=3(1) \\
&=3
\end{aligned}
$$

Hence, $\lim _{x \rightarrow 0^{-}} f(x)=\lim _{x \rightarrow 0^{+}} f(x)=\lim _{x \rightarrow 0} f(x)=3$

Now, for $\lim _{x \rightarrow 1} f(x)$ :

$$
\begin{aligned}
& \lim _{x \rightarrow 1^{-}} f(x)=\lim _{x \rightarrow 1} 3(x+1) \\
& =3(1+1) \\
& =3(2) \\
& =6
\end{aligned}
$$

$\lim _{x \rightarrow 1^{+}} f(x)=\lim _{x \rightarrow 1} 3(x+1)$
$=3(1+1)$
$=3(2)$
$=6$
Hence, $\lim _{x \rightarrow 1^{-}} f(x)=\lim _{x \rightarrow 1^{+}} f(x)=\lim _{x \rightarrow 1} f(x)=6$
$\lim _{x \rightarrow 0} f(x)=3 \underset{\text { and }}{ } \lim _{x \rightarrow 1} f(x)=6$
24. Find
$\lim _{x \rightarrow 1} f(x)$, where
$f(x)=\left\{\begin{array}{c}x^{2}-1 x \leq 1 \\ -x^{2}-1 x>1\end{array}\right.$
Solution:

Given function is:
$f(x)=\left\{\begin{array}{c}x^{2}-1 x \leq 1 \\ -x^{2}-1 x>1\end{array}\right.$
$\lim _{x \rightarrow 1} f(x):$
$\lim _{x \rightarrow 1^{-}} f(x)=\lim _{x \rightarrow 1} x^{2}-1$
$=1^{2}-1$
$=1-1$
$=0$
$\lim _{x \rightarrow 1^{+}} f(x)=\lim _{x \rightarrow 1}\left(-x^{2}-1\right)$
$=\left(-1^{2}-1\right)$
$=-1-1$
$=-2$
We find,
$\lim _{x \rightarrow 1^{-}} f(x) \neq \lim _{x \rightarrow 1^{+}} f(x)$

Hence, $\lim _{x \rightarrow 1} f(x)$ does not exist
25. Evaluate
$\lim _{x \rightarrow 0} f(x)$ , where $f(x)=$
$\left\{\begin{array}{l}\frac{|x|}{x}, x \neq 0 \\ x \\ 0, x=0\end{array}\right.$

## Solution:

Given function is $f(x)=\left\{\begin{array}{l}\frac{|x|}{x}, x \neq 0 \\ x \\ 0, x=0\end{array}\right.$
We know that,
$\lim _{x \rightarrow a} f(x)$
exists only when $\lim _{x \rightarrow a} f(x)=\lim _{x \rightarrow a^{+}} f(x)$
$\lim _{x \rightarrow 0^{-}} f(x)=\lim _{x \rightarrow 0^{+}} f(x)$
Now, we need to prove that: $x \rightarrow 0 \quad x \rightarrow 0^{+}$

We know,
$|x|=x$, if $x>=-x$, if $x<0$
Hence,

$$
\begin{aligned}
& \lim _{x \rightarrow 0^{-}} f(x)=\lim _{x \rightarrow 0^{-}} \frac{|x|}{x} \\
& =\lim _{x \rightarrow 0} \frac{-x}{x}=\lim _{x \rightarrow 0}(-1) \\
& =-1
\end{aligned}
$$

$$
\lim _{x \rightarrow 0^{+}} f(x)=\lim _{x \rightarrow 0^{+}} \frac{|x|}{x}
$$

$$
=\lim _{x \rightarrow 0} \frac{x}{x}=\lim _{x \rightarrow 0}(1)
$$

$$
=1
$$

We find here,

$$
\lim _{x \rightarrow 0^{-}} f(x) \neq \lim _{x \rightarrow 0^{+}} f(x)
$$

Hence, ${ }^{\lim _{x \rightarrow 0} f(x)}$ does not exist.
26. Find
$\lim _{x \rightarrow 0} f(x)$
$x \rightarrow 0$, where $f(x)=$
$\left\{\begin{array}{c}\frac{x}{|x|}, \\ x \neq 0 \\ 0,\end{array}\right.$
Solution:
Given function is:
$\mathrm{f}(\mathrm{x})=\left\{\begin{array}{c}\frac{\mathrm{x}}{|\mathrm{x}|}, \\ \mathrm{x}=0 \\ 0,\end{array}\right.$
$\lim _{x \rightarrow 0} f(x):$
$\lim _{x \rightarrow 0^{-}} f(x)=\lim _{x \rightarrow 0^{-}} \frac{x}{|x|}$
$=\lim _{x \rightarrow 0} \frac{x}{-x}=\lim _{x \rightarrow 0} \frac{1}{-1}$
$=-1$
$\lim _{x \rightarrow 0^{+}} f(x)=\lim _{x \rightarrow 0^{+}} \frac{x}{|x|}$
$=\lim _{x \rightarrow 0} \frac{x}{x}=\lim _{x \rightarrow 0}(1)$
$=1$
We find here,

$$
\lim _{x \rightarrow 0^{-}} f(x) \neq \lim _{x \rightarrow 0^{+}} f(x)
$$

Hence, ${ }^{\lim _{\mathrm{x} \rightarrow 0} \mathrm{f}(\mathrm{x})}$ does not exist.

## 27. Find

$\lim _{x \rightarrow 5} f(x)$
$f(x)=|x|-5$

## Solution:

Given function is:

$$
\begin{aligned}
& f(x)=|x|-5 \\
& \lim _{x \rightarrow 5} f(x): \\
& \lim _{x \rightarrow 5^{-}} f(x)=\lim _{x \rightarrow 5^{-}}|x|-5 \\
& =\lim _{x \rightarrow 5}(x-5)=5-5 \\
& =0
\end{aligned}
$$

$$
\lim _{x \rightarrow 5^{+}} f(x)=\lim _{x \rightarrow 5^{+}}|x|-5
$$

$$
=\lim _{x \rightarrow 5}(x-5)
$$

$=5-5$
$=0$

Hence, $\lim _{x \rightarrow 5^{-}} f(x)=\lim _{x \rightarrow 5^{+}} f(x)=\lim _{x \rightarrow 5} f(x)=0$
28. Suppose
$f(x)=\left\{\begin{array}{c}a+b x, x<1 \\ 4, \quad x=1 \\ b-a x x>1\end{array}\right.$ and if
$\lim _{x \rightarrow 1} f(x)=f(1)$ what are the possible values of $a$ and $b ?$
Solution:
Given function is:

$$
f(x)=\left\{\begin{array}{l}
a+b x, x<1 \\
4, x=1 \\
b-a x, x>1
\end{array}\right. \text { and }
$$

$\lim _{x \rightarrow 1} f(x)=f(1)$
$\lim _{x \rightarrow 1^{-}} f(x)=\lim _{x \rightarrow 1} a+b x$
$=\mathrm{a}+\mathrm{b}(1)$
$=\mathrm{a}+\mathrm{b}$
$\lim _{x \rightarrow 1^{+}} f(x)=\lim _{x \rightarrow 1} b-a x$
$=\mathrm{b}-\mathrm{a}(1)$
$=\mathrm{b}-\mathrm{a}$

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Here,
$\mathrm{f}(1)=4$

Hence, $\lim _{x \rightarrow 1^{-}} f(x)=\lim _{x \rightarrow 1^{+}} f(x)=\lim _{x \rightarrow 1} f(x)=f(1)$
Then, $\mathrm{a}+\mathrm{b}=4$ and $\mathrm{b}-\mathrm{a}=4$
By solving the above two equations, we get,
$\mathrm{a}=0$ and $\mathrm{b}=4$
Therefore, the possible values of a and b is 0 and 4 respectively
29. Let $a_{1}, a_{2}, \ldots \ldots . . a_{n}$ be fixed real numbers and define a function
$f(x)=\left(x-a_{1}\right)\left(x-a_{2}\right) \ldots \ldots\left(x-a_{n}\right)$.
What is
$\lim _{x \rightarrow a_{1}} f(x)$ ?
For some $\mathbf{a} \neq \mathbf{a}_{1}, a_{2}, \ldots \ldots . a_{n}$, compute
$\lim _{x \rightarrow \mathrm{a}} f(x)$
Solution:

Given function is:
$f(x)=\left(x-a_{1}\right)\left(x-a_{2}\right) \ldots\left(x-a_{n}\right)$
$\lim _{x \rightarrow a_{1}} f(x):$

$$
\begin{aligned}
& \lim _{x \rightarrow a_{1}} f(x)=\lim _{x \rightarrow a_{1}}\left[\left(x-a_{1}\right)\left(x-a_{2}\right) \ldots\left(x-a_{n}\right)\right] \\
& =\left[\lim _{x \rightarrow a_{1}}\left(x-a_{1}\right)\right]\left[[ \operatorname { l i m } _ { x \rightarrow a _ { 1 } } ( x - a _ { 2 } ) ] \ldots \left[\left[\lim _{x \rightarrow a_{1}}\left(x-a_{n}\right)\right]\right.\right.
\end{aligned}
$$

We get,

$$
=\left(a_{1}-a_{1}\right)\left(a_{1}-a_{2}\right) \ldots\left(a_{1}-a_{n}\right)=0
$$

$$
\text { Hence, }{ }^{\lim _{x \rightarrow a_{1}} f(x)}=0
$$

$\lim _{x \rightarrow a} f(x):$
$\lim _{x \rightarrow a} f(x)=\lim _{x \rightarrow a}\left[\left(x-a_{1}\right)\left(x-a_{2}\right) \ldots\left(x-a_{n}\right)\right]$
$=\left[\lim _{x \rightarrow a}\left(x-a_{1}\right)\right]\left[\lim _{x \rightarrow a}\left(x-a_{2}\right)\right] \ldots\left[\lim _{x \rightarrow a}\left(x-a_{n}\right)\right]$
We get,
$=\left(a-a_{1}\right)\left(a-a_{2}\right) \ldots . .\left(a-a_{n}\right)$
Hence, $\lim _{x \rightarrow a} f(x)=\left(a-a_{1}\right)\left(a-a_{2}\right) \ldots\left(a-a_{n}\right)$
Therefore, $\lim _{x \rightarrow a_{1}} f(x)=0$ and $\lim _{x \rightarrow a} f(x)=\left(a-a_{1}\right)\left(a-a_{2}\right) \ldots\left(a-a_{n}\right)$

$$
\mathrm{f}(\mathrm{x})=\left\{\begin{array}{rl}
|\mathrm{x}|+1, & \mathrm{x}
\end{array}<000 \mathrm{x}=0\right.
$$

30. If

$$
\text { For what value (s) of a does } \lim _{x \rightarrow a} f(x) \text { exist? }
$$

Solution:

Given function is:

There are three cases.
Case 1:
When $\mathrm{a}=0$
$\lim _{x \rightarrow 0} f(x):$
$\lim _{x \rightarrow 0^{-}} f(x)=\lim _{x \rightarrow 0^{-}}(|x|+1)$
$=\lim _{x \rightarrow 0}(-x+1)=-0+1$
$=1$

$$
\lim _{x \rightarrow 0^{+}} f(x)=\lim _{x \rightarrow 0^{+}}(|x|-1)
$$

$=\lim _{x \rightarrow 0}(x-1)=0-1$
$=-1$
Here, we find

$$
\lim _{x \rightarrow 0^{-}} f(x) \neq \lim _{x \rightarrow 0^{+}} f(x)
$$

Hence, $\lim _{x \rightarrow 0} f(x)$ does not exit.
Case 2:
When a $<0$

$$
\lim _{x \rightarrow a} f(x):
$$

$$
\lim _{x \rightarrow a^{-}} f(x)=\lim _{x \rightarrow a^{-}}(|x|+1)
$$

$$
\lim _{x \rightarrow a}(-x+1)=-a+1
$$

$$
\lim _{\mathrm{x} \rightarrow \mathrm{a}^{+}} \mathrm{f}(\mathrm{x})=\lim _{\mathrm{x} \rightarrow \mathrm{a}^{+}}(|\mathrm{x}|+1)
$$

$$
=\lim _{x \rightarrow a}(-x+1)=-a+1
$$

Hence, $\lim _{x \rightarrow a^{-}} f(x)=\lim _{x \rightarrow a^{+}} f(x)=\lim _{x \rightarrow a} f(x)=-a+1$
Therefore, $\varliminf_{\lim }(\mathrm{f}(\mathrm{x}))$ exists at $\mathrm{x}=\mathrm{a}$ and $\mathrm{a}<0$

Case 3:
When $\mathrm{a}>0$
$\lim _{x \rightarrow a} f(x):$
$\lim _{x \rightarrow a^{-}} f(x)=\lim _{x \rightarrow a^{-}}(|x|-1)$
$=\lim _{x \rightarrow a}(x-1)=a-1$
$\lim _{x \rightarrow \mathrm{a}^{+}} \mathrm{f}(\mathrm{x})=\lim _{\mathrm{x} \rightarrow \mathrm{a}^{+}}(|\mathrm{x}|-1)$
$=\lim _{x \rightarrow a}(x-1)=a-1$
Hence, $\lim _{x \rightarrow a^{-}} f(x)=\lim _{x \rightarrow a^{+}} f(x)=\lim _{x \rightarrow a} f(x)=a-1$
Therefore, $\left.\left.\lim _{(\mathrm{f}}^{\mathrm{f}} \mathrm{x}\right)\right)$ exists at $\mathrm{x}=\mathrm{a}$ when $\mathrm{a}>0$
31. If the function $f(x)$ satisfies $\lim _{x \rightarrow 1} \frac{f(x)-2}{x^{2}-1}=\pi$, evaluate $\lim _{x \rightarrow 1} f(x)$

Solution:

Given function that $\mathrm{f}(\mathrm{x})$ satisfies

$$
\lim _{x \rightarrow 1} \frac{f(x)-2}{x^{2}-1}=\pi
$$

$$
\begin{aligned}
& \frac{\lim _{x \rightarrow 1} f(x)-2}{\lim _{x \rightarrow 1} x^{2}-1}=\pi \\
& \lim _{x \rightarrow 1}(f(x)-2)=\pi\left(\lim _{x \rightarrow 1}\left(x^{2}-1\right)\right)
\end{aligned}
$$

Substituting $\mathrm{x}=1$, we get,

$$
\begin{aligned}
& \lim _{x \rightarrow 1}(f(x)-2)=\pi\left(1^{2}-1\right) \\
& \lim _{x \rightarrow 1}(f(x)-2)=\pi(1-1) \\
& \lim _{x \rightarrow 1}(f(x)-2)=0 \\
& \lim _{x \rightarrow 1} f(x)-\lim _{x \rightarrow 1} 2=0 \\
& \lim _{x \rightarrow 1} f(x)-2=0 \\
& =2
\end{aligned}
$$

32. If

$$
\mathrm{f}(\mathrm{x})=\left\{\begin{array}{l}
\mathrm{mx} \mathrm{x}^{2}+\mathrm{n}, \quad \mathrm{x}<0 \\
\mathrm{nx}+\mathrm{m}, \quad 0 \leq \mathrm{x} \leq 1 \\
\mathrm{nx}+\mathrm{m}, \quad \mathrm{x}>1
\end{array}\right.
$$ For what integers $m$ and $n$ does both $\lim _{x \rightarrow 0} f(x)$ and $\lim _{x \rightarrow 1} f(x)$ exist?

Solution:

Given function is

$$
f(x)=\left\{\begin{array}{l}
\mathrm{mx}^{2}+\mathrm{n}, \quad \mathrm{x}<0 \\
\mathrm{nx}+\mathrm{m}, \quad 0 \leq \mathrm{x} \leq 1 \\
\mathrm{nx}+\mathrm{m} . \\
\mathrm{n}
\end{array} \mathrm{x}>1\right.
$$

$\lim _{x \rightarrow 0} f(x):$
$\lim _{x \rightarrow 0^{-}} f(x)=\lim _{x \rightarrow 0}\left(m x^{2}+n\right)$
$=m(0)+n$
$=0+\mathrm{n}$
$=n$
$\lim _{x \rightarrow 0^{+}} f(x)=\lim _{x \rightarrow 0}(n x+m)$
$=n(0)+\mathrm{m}$
$=0+\mathrm{m}$
$=\mathrm{m}$

Hence,
$\lim _{x \rightarrow 0} f(x)$ exists if $n=m$.
Now,

$$
\lim _{x \rightarrow 1} f(x):
$$

$$
\begin{aligned}
& \lim _{x \rightarrow 1^{-}} f(x)=\lim _{x \rightarrow 1}(n x+m) \\
& =n(1)+m \\
& =n+m \\
& \lim _{x \rightarrow 1^{+}} f(x)=\lim _{x \rightarrow 1}\left(n x^{3}+m\right) \\
& =n(1)^{3}+m \\
& =n(1)+m \\
& =n+m
\end{aligned}
$$

Therefore $\lim _{x \rightarrow 1^{-}} f(x)=\lim _{x \rightarrow 1^{+}} f(x)=\lim _{x \rightarrow 1} f(x)$
Hence, for any integral value of $m$ and $n^{\lim _{x \rightarrow 1} f(x)}$ exists.

## EXERCISE 13.2

1. Find the derivative of $x^{2}-2$ at $x=10$.

## Solution:

Let $f(x)=x^{2}-2$

## From first principle

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

Put $x=10$, we get

$$
\begin{aligned}
& f^{\prime}(10)=\lim _{h \rightarrow 0} \frac{f(10+h)-f(10)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\left[(10+h)^{2}-2\right]-\left(10^{2}-2\right)}{h}
\end{aligned}
$$

$$
=\lim _{\mathrm{h} \rightarrow 0} \frac{10^{2}+2 \times 10 \times \mathrm{h}+\mathrm{h}^{2}-2-10^{2}+2}{\mathrm{~h}}
$$

$$
=\lim _{h \rightarrow 0} \frac{20 h+h^{2}}{h}
$$

$$
=\lim _{h \rightarrow 0}(20+h)
$$

$$
=20+0
$$

$$
=20
$$

2. Find the derivative of $x$ at $x=1$.

Solution:
Let $\mathrm{f}(\mathrm{x})=\mathrm{x}$
Then,

From first principle

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

Let $f(x)=x$
From first principle

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(10)}{h}
$$

Put $\mathrm{x}=1$, we get
$f^{\prime}(1)=\lim _{h \rightarrow 0} \frac{f(1+h)-f(1)}{h}$
$=\lim _{h \rightarrow 0} \frac{(1+h)-1}{h}$
$=\lim _{h \rightarrow 0} \frac{1+h-1}{h}$
$=\lim _{h \rightarrow 0} \frac{h}{h}$
$=\lim _{h \rightarrow 0} 1$
$=1$
3. Find the derivative of 99 x at $\mathrm{x}=100$.

## Solution:

Let $\mathrm{f}(\mathrm{x})=99 \mathrm{x}$,
From the first principle,

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

Put $\mathrm{x}=100$, we get

$$
\begin{aligned}
& f^{\prime}(100)=\lim _{h \rightarrow 0} \frac{f(100+h)-f(100)}{h} \\
& =\lim _{h \rightarrow 0} \frac{99(100+h)-99 \times 100}{h}
\end{aligned}
$$

$$
=\lim _{h \rightarrow 0} \frac{99 \times 100+99 h-99 \times 100}{h}
$$

$$
=\lim _{h \rightarrow 0} \frac{99 \times h}{h}
$$

$$
=\lim _{h \rightarrow 0} 99
$$

$$
=99
$$

4. Find the derivative of the following functions from the first principle.
(i) $x^{3}-27$
(ii) $(x-1)(x-2)$
(iii) $1 / x^{2}$
(iv) $x+1 / x-1$

Solution:
(i) Let $\mathrm{f}(\mathrm{x})=\mathrm{x}^{3}-27$

From the first principle,

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\left[(x+h)^{3}-27\right]-\left(x^{3}-27\right)}{h} \\
& =\lim _{h \rightarrow 0} \frac{x^{3}+h^{3}+3 x^{2} h+3 x^{2}-x^{3}}{h} \\
& =\lim _{h \rightarrow 0} \frac{h^{3}+3 x^{2} h+3 x^{2}}{h} \\
& =\lim _{h \rightarrow 0}\left(h^{2}+3 x^{2}+3 x h\right) \\
& =0+3 x^{2} \\
& =3 x^{2} \\
& \text { (ii) Let } f(x)=(x-1)(x-2)
\end{aligned}
$$

From the first principle,

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

$$
=\lim _{h \rightarrow 0} \frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}
$$

$$
\lim _{h \rightarrow 0} \frac{\left(x^{2}+h x-2 x+h x+h^{2}-2 h-x-h+2\right)-\left(x^{2}-2 x-x+2\right)}{h}
$$

$$
=\lim _{h \rightarrow 0} \frac{h x+h x+h^{2}-2 h-h}{h}
$$

$$
=\lim _{\mathrm{h} \rightarrow 0}(\mathrm{~h}+2 \mathrm{x}-3)
$$

$$
\begin{aligned}
& =0+2 x-3 \\
& =2 x-3
\end{aligned}
$$

(iii) Let $f(x)=1 / x^{2}$

From the first principle, we get

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\frac{1}{(x+h)^{2}}-\frac{1}{x^{2}}}{h}
\end{aligned}
$$

$$
=\lim _{h \rightarrow 0} \frac{x^{2}-(x+h)^{2}}{h x^{2}(x+h)^{2}}
$$

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{x^{2}-x^{2}-h^{2}-2 h x}{x^{2}(x+h)^{2}}\right]
$$

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-h^{2}-2 h x}{x^{2}(x+h)^{2}}\right]
$$

$$
=\lim _{h \rightarrow 0}\left[\frac{-h-2 x}{x^{2}(x+h)^{2}}\right]
$$

$$
=(0-2 x) /\left[x^{2}(x+0)^{2}\right]
$$

$$
=\left(-2 / x^{3}\right)
$$

(iv) Let $\mathrm{f}(\mathrm{x})=\mathrm{x}+1 / \mathrm{x}-1$

From the first principle, we get

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}}{h} \\
& =\lim _{h \rightarrow 0} \frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{h(x-1)(x+h-1)} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\left(x^{2}+h x+x-x-h-1\right)-\left(x^{2}+h x+x-x+h-1\right)}{(x-1)(x+h-1)}\right] \\
& =\lim _{h \rightarrow 0} \frac{-2 h}{h(x-1)(x+h-1)} \quad \text { Activate Windo } \\
& =\lim _{h \rightarrow 0} \frac{-2}{(x-1)(x+h-1)} \\
& =-\frac{2}{(x-1)(x-1)} \\
& =-\frac{2}{(x-1)^{2}}
\end{aligned}
$$

5. For the function $f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots \frac{x^{2}}{2}+x+1$, prove that $f^{\prime}(1)=100 f^{\prime}(0)$.

## Solution:

Given function is:
$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots \frac{x^{2}}{2}+x+1$
By differentiating both sides, we get
$\frac{d}{d x} f(x)=\frac{d}{d x}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\cdots+\frac{x^{2}}{2}+x+1\right]$
$=\frac{d}{d x}\left(\frac{x^{100}}{100}\right)+\frac{d}{d x}\left(\frac{x^{99}}{99}\right)+\cdots+\frac{d}{d x}\left(\frac{x^{2}}{2}\right)+\frac{d}{d x}(x)+\frac{d}{d x}(1)$
We know that,
$\frac{\mathrm{d}}{\mathrm{dx}}\left(\mathrm{x}^{\mathrm{n}}\right)=\mathrm{nx} \mathrm{x}^{\mathrm{n}-1}$
$\therefore \frac{\mathrm{d}}{\mathrm{dx}} \mathrm{f}(\mathrm{x})=\frac{100 \mathrm{x}^{99}}{100}+\frac{99 \mathrm{x}^{98}}{99}+\cdots+\frac{2 \mathrm{x}}{2}+1+0$

$$
f^{\prime}(x)=x^{99}+x^{98}+\cdots+x+1
$$

At $x=0$, we get

$$
f^{\prime}(0)=0+0+\ldots+0+1
$$

$f^{\prime}(0)=1$
At $x=1$, we get

$$
f^{\prime}(1)=1^{99}+1^{98}+\ldots+1+1=[1+1 \ldots .+1] 100 \text { times }=1 \times 100=100
$$

Hence, $f^{\prime}(1)=100 f^{\prime}(0)$
6. Find the derivative of $x^{n}+a x^{n-1}+a^{2} x^{n-2}+\ldots+a^{n-1} x+a^{n}$ for some fixed real number $a$. Solution:

The Learning App
Given function is:

$$
f(x)=x^{n}+a x^{n-1}+a^{2} x^{n-2}+\ldots+a^{n-1} x+a^{n}
$$

By differentiating both sides, we get

$$
\begin{aligned}
& f^{\prime}(x)=\frac{d}{d x}\left(x^{n}+a x^{n-1}+a^{2} x^{n-2}+\ldots+a^{n-1} x+a^{n}\right) \\
& =\frac{d}{d x}\left(x^{n}\right)+a \frac{d}{d x}\left(x^{n-1}\right)+a^{2} \frac{d}{d x}\left(x^{n-2}\right)+\cdots+a^{n-1} \frac{d}{d x}(x)+a^{n} \frac{d}{d x}(1)
\end{aligned}
$$

We know that,

$$
\frac{\mathrm{d}}{\mathrm{dx}}\left(\mathrm{x}^{\mathrm{n}}\right)=\mathrm{n} \mathrm{x}^{\mathrm{n}-1}
$$

$$
f^{\prime}(x)=n x^{n-1}+a(n-1) x^{n-2}+a^{2}(n-2) x^{n-3}+\ldots+a^{n-1}+a^{n}(0)
$$

$$
f^{\prime}(x)=n x^{n-1}+a(n-1) x^{n-2}+a^{2}(n-2) x^{n-3}+\ldots+a^{n-1}
$$

7. For some constants $a \operatorname{and} b$, find the derivative of
(i) $(x-a)(x-b)$
(ii) $\left(a x^{2}+b\right)^{2}$
(iii) $\mathrm{x}-\mathrm{a} / \mathrm{x}-\mathrm{b}$

## Solution:

(i) $(x-a)(x-b)$

Let $\mathrm{f}(\mathrm{x})=(\mathrm{x}-\mathrm{a})(\mathrm{x}-\mathrm{b})$
$f(x)=x^{2}-(a+b) x+a b$
Now, by differentiating both sides, we get

$$
\begin{aligned}
& f^{\prime}(x)=\frac{d}{d x}\left(x^{2}-(a+b) x+a b\right) \\
& =\frac{d}{d x}\left(x^{2}\right)-(a+b) \frac{d}{d x}(x)+\frac{d}{d x}(a b)
\end{aligned}
$$

We know that,
$\frac{\mathrm{d}}{\mathrm{dx}}\left(\mathrm{x}^{\mathrm{n}}\right)=\mathrm{nx} \mathrm{n}^{\mathrm{n}-1}$
$f^{\prime}(x)=2 x-(a+b)+0$
$=2 \mathrm{x}-\mathrm{a}-\mathrm{b}$
(ii) $\left(a x^{2}+b\right)^{2}$

Let $\mathrm{f}(\mathrm{x})=\left(\mathrm{ax}{ }^{2}+\mathrm{b}\right)_{\sim}^{2}$
$f(x)=a^{2} x^{4}+2 a b x^{2}+b^{2}$
By differentiating both sides, we get
$f^{\prime}(x)=\frac{d}{d x}\left(a^{2} x^{4}+2 a b x^{2}+b^{2}\right)$
$f^{\prime}(x)=\frac{d}{d x}\left(x^{4}\right)+(2 a b) \frac{d}{d x}\left(x^{2}\right)+\frac{d}{d x}\left(b^{2}\right)$

We know that,
$\frac{\mathrm{d}}{\mathrm{dx}}\left(\mathrm{X}^{\mathrm{n}}\right)=\mathrm{n} \mathrm{x}^{\mathrm{n}-1}$
$f^{\prime}(x)=a^{2} \times 4 x^{3}+2 a b \times 2 x+0$
$=4 a^{2} x^{3}+4 a b x$
$=4 \mathrm{ax}\left(\mathrm{ax}^{2}+\mathrm{b}\right)$
(iii) $x-a / x-b$

Let $f(x)=\frac{(x-a)}{(x-b)}$

By differentiating both sides and using quotient rule, we get
$f^{\prime}(x)=\frac{d}{d x}\left(\frac{x-a}{x-b}\right)$
$f^{\prime}(x)=\frac{(x-b) \frac{d}{d x}(x-a)-(x-a) \frac{d}{d x}(x-b)}{(x-b)^{2}}$
$=\frac{(x-b)(1)-(x-a)(1)}{(x-b)^{2}}$

By further calculation, we get

$$
\begin{aligned}
& =\frac{x-b-x+a}{(x-b)^{2}} \\
& =\frac{a-b}{(x-b)^{2}}
\end{aligned}
$$

$$
x^{n}-a^{n}
$$

8. Find the derivative of $\mathrm{x}-\mathrm{a}$ for some constant a .

## Solution:

$$
\text { Let } f(x)=\frac{x^{n}-a^{n}}{x-a}
$$

By differentiating both sides and using quotient rule, we get

$$
\begin{aligned}
& f^{\prime}(x)=\frac{d}{d x}\left(\frac{x^{n}-a^{n}}{x-a}\right) \\
& f^{\prime}(x)=\frac{(x-a) \frac{d}{d x}\left(x^{n}-a^{n}\right)-\left(x^{n}-a^{n}\right) \frac{d}{d x}(x-a)}{(x-a)^{2}}
\end{aligned}
$$

By further calculation, we get

$$
\begin{aligned}
& =\frac{(x-a)\left(n x^{n-1}-0\right)-\left(x^{n}-a^{n}\right)}{(x-a)^{2}} \\
& =\frac{n x^{n}-a n x^{n-1}-x^{n}+a^{n}}{(x-a)^{2}}
\end{aligned}
$$

9. Find the derivative of
(i) $2 x-3 / 4$
(ii) $\left(5 x^{3}+3 x-1\right)(x-1)$
(iii) $x^{-3}(5+3 x)$
(iv) $x^{5}\left(3-6 x^{-9}\right)$
(v) $x^{-4}\left(3-4 x^{-5}\right)$
(vi) $(2 / x+1)-x^{2} / 3 x-1$

## Solution:

(i)

Let $\mathrm{f}(\mathrm{x})=2 \mathrm{x}-3 / 4$
By differentiating both sides, we get

$$
\begin{aligned}
& \mathrm{f}^{\prime}(\mathrm{x})=\frac{\mathrm{d}}{\mathrm{dx}}\left(2 \mathrm{x}-\frac{3}{4}\right) \\
& =2 \frac{\mathrm{~d}}{\mathrm{dx}}(\mathrm{x})-\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{3}{4}\right) \\
& =2-0 \\
& =2
\end{aligned}
$$

(ii)

Let $f(x)=\left(5 x^{3}+3 x-1\right)(x-1)$
By differentiating both sides and using the product rule, we get

$$
\begin{aligned}
& f^{\prime}(x)=\left(5 x^{3}+3 x-1\right) \frac{d}{d x}(x-1)+(x-1) \frac{d}{d x}\left(5 x^{3}+3 x+1\right) \\
& =\left(5 x^{3}+3 x-1\right) \times 1+(x-1) \times\left(15 x^{2}+3\right) \\
& =\left(5 x^{3}+3 x-1\right)+(x-1)\left(15 x^{2}+3\right) \\
& =5 x^{3}+3 x-1+15 x^{3}+3 x-15 x^{2}-3 \\
& =20 x^{3}-15 x^{2}+6 x-4 \\
& \text { (iii) }
\end{aligned}
$$

Let $\mathrm{f}(\mathrm{x})=\mathrm{x}^{-3}(5+3 \mathrm{x})$
By differentiating both sides and using Leibnitz product rule, we get

$$
\begin{aligned}
& f^{\prime}(x)=x^{-3} \frac{d}{d x}(5+3 x)+(5+3 x) \frac{d}{d x}\left(x^{-3}\right) \\
& =x^{-3}(0+3)+(5+3 x)\left(-3 x^{-3-1}\right)
\end{aligned}
$$

By further calculation, we get

$$
\begin{aligned}
& =x^{-3}(3)+(5+3 x)\left(-3 x^{-4}\right) \\
& =3 x^{-3}-15 x^{-4}-9 x^{-3} \\
& =-6 x^{-3}-15 x^{-4} \\
& =-3 x^{-3}\left(2+\frac{5}{x}\right) \\
& =\frac{-3 x^{-3}}{x}(2 x+5) \\
& =\frac{-3}{x^{4}}(5+2 x)
\end{aligned}
$$

(iv)

Let $f(x)=x^{5}\left(3-6 x^{-9}\right)$
By differentiating both sides and using Leibnitz product rule, we get

$$
\begin{aligned}
& f^{\prime}(x)=x^{5} \frac{d}{d x}\left(3-6 x^{-9}\right)+\left(3-6 x^{-9}\right) \frac{d}{d x}\left(x^{5}\right) \\
& =x^{5}\left\{0-6(-9) x^{-9-1}\right\}+\left(3-6 x^{-9}\right)\left(5 x^{4}\right)
\end{aligned}
$$

By further calculation, we get

$$
\begin{aligned}
& =x^{5}\left(54 x^{-10}\right)+15 x^{4}-30 x^{-5} \\
& =54 x^{-5}+15 x^{4}-30 x^{-5} \\
& =24 x^{-5}+15 x^{4} \\
& =15 x^{4}+\frac{24}{x^{5}}
\end{aligned}
$$

(v)

Let $f(x)=x^{-4}\left(3-4 x^{-5}\right)$
By differentiating both sides and using Leibnitz product rule, we get

$$
\begin{aligned}
& f^{\prime}(x)=x^{-4} \frac{d}{d x}\left(3-4 x^{-5}\right)+\left(3-4 x^{-5}\right) \frac{d}{d x}\left(x^{-4}\right) \\
& =x^{-4}\left\{0-4(-5) x^{-5-1}\right\}+\left(3-4 x^{-5}\right)(-4) x^{-4-1}
\end{aligned}
$$

By further calculation, we get

$$
=x^{-4}\left(20 x^{-6}\right)+\left(3-4 x^{-5}\right)\left(-4 x^{-5}\right)
$$

$$
\begin{aligned}
& =20 x^{-10}-12 x^{-5}+16 x^{-10} \\
& =36 x^{-10}-12 x^{-5} \\
& =-\frac{12}{x^{5}}+\frac{36}{x^{10}}
\end{aligned}
$$

(vi)

Let

$$
f(x)=\frac{2}{x+1}-\frac{x^{2}}{3 x-1}
$$

By differentiating both sides we get,

$$
f^{\prime}(x)=\frac{d}{d x}\left(\frac{2}{x+1}-\frac{x^{2}}{3 x-1}\right)
$$

Using quotient rule we get,

$$
\begin{aligned}
& f^{\prime}(x)=\left[\frac{(x+1) \frac{d}{d x}(2)-2 \frac{d}{d x}(x+1)}{(x+1)^{2}}\right]-\left[\frac{(3 x-1) \frac{d}{d x}\left(x^{2}\right)-x^{2} \frac{d}{d x}(3 x-1)}{(3 x-1)^{2}}\right] \\
& =\left[\frac{(x+1)(0)-2(1)}{(x+1)^{2}}\right]-\left[\frac{(3 x-1)(2 x)-\left(x^{2}\right) \times 3}{(3 x-1)^{2}}\right] \\
& =-\frac{2}{(x+1)^{2}}-\left[\frac{6 x^{2}-2 x-3 x^{2}}{(3 x-1)^{2}}\right] \\
& =-\frac{2}{(x+1)^{2}}-\frac{x(3 x-2)}{(3 x-1)^{2}}
\end{aligned}
$$

10. Find the derivative of $\cos x$ from the first principle.

Solution:

Let $f(x)=\cos x$
Accordingly, $\mathrm{f}(\mathrm{x}+\mathrm{h})=\cos (\mathrm{x}+\mathrm{h})$
By first principle, we get

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

So, we get

$$
=\lim _{h \rightarrow 0} \frac{1}{h}[\cos (x+h)-\cos (x)]
$$

$=\lim _{\mathrm{h} \rightarrow 0} \frac{1}{\mathrm{~h}}\left[-2 \sin \left(\frac{\mathrm{x}+\mathrm{h}+\mathrm{x}}{2}\right) \sin \left(\frac{\mathrm{x}+\mathrm{h}-\mathrm{x}}{2}\right)\right]$
By further calculation, we get
$=\lim _{\mathrm{h} \rightarrow 0} \frac{1}{\mathrm{~h}}\left[-2 \sin \left(\frac{2 \mathrm{x}+\mathrm{h}}{2}\right) \sin \left(\frac{\mathrm{h}}{2}\right)\right]$
$=\lim _{h \rightarrow 0}-\sin \left(\frac{2 x+h}{2}\right) \times \lim _{h \rightarrow 0} \frac{\sin \left(\frac{h}{2}\right)}{\frac{h}{2}}$
$=-\sin \left(\frac{2 \mathrm{x}+0}{2}\right) \times 1$
$=-\sin (2 x / 2)$
$=-\sin (\mathrm{x})$
11. Find the derivative of the following functions.
(i) $\sin x \cos x$
(ii) $\sec x$
(iii) $5 \sec x+4 \cos x$
(iv) $\operatorname{cosec} x$
(v) $3 \cot x+5 \operatorname{cosec} x$
(vi) $5 \sin x-6 \cos x+7$
(vii) $2 \boldsymbol{\operatorname { t a n }} \mathrm{x}-7 \boldsymbol{\operatorname { s e c }} \mathrm{x}$

## Solution:

(i) $\sin x \cos x$

Let $\mathrm{f}(\mathrm{x})=\sin \mathrm{x} \cos \mathrm{x}$
Accordingly, from the first principle,

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sin (x+h) \cos (x+h)-\sin x \cos x}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{2 h}[2 \sin (x+h) \cos (x+h)-2 \sin x \cos x] \\
& =\lim _{h \rightarrow 0} \frac{1}{2 h}[\sin 2(x+h)-\sin 2 x] \\
& =\lim _{h \rightarrow 0} \frac{1}{2 h}\left[2 \cos \frac{2 x+2 h+2 x}{2} \cdot \sin \frac{2 x+2 h-2 x}{2}\right]
\end{aligned}
$$

By further calculation, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\cos \frac{4 x+2 h}{2} \sin \frac{2 h}{2}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}[\cos (2 x+h) \sin h] \\
& =\lim _{h \rightarrow 0} \cos (2 x+h) \cdot \lim _{h \rightarrow 0} \frac{\sin h}{h} \\
& =\cos (2 x+0) \cdot 1 \\
& =\cos 2 x
\end{aligned}
$$

(ii) $\sec x$

Let $\mathrm{f}(\mathrm{x})=\sec \mathrm{x}$
$=1 / \cos x$
By differentiating both sides, we get
$f^{\prime}(x)=\frac{d}{d x}\left(\frac{1}{\cos x}\right)$

Using quotient rule, we get
$f^{\prime}(x)=\frac{\cos x \frac{d}{d x}(1)-1 \frac{d}{d x}(\cos x)}{\cos ^{2} x}$
$=\frac{\cos x \times 0-(-\sin x)}{\cos ^{2} x}$
We get
$=\frac{\sin x}{\cos ^{2} x}$
$=\frac{\sin x}{\cos x} \times \frac{1}{\cos x}$
$=\tan \mathrm{x} \sec \mathrm{x}$
(iii) $5 \sec x+4 \cos x$

Let $f(x)=5 \sec x+4 \cos x$
By differentiating both sides, we get
$f^{\prime}(x)=\frac{d}{d x}(5 \sec x+4 \cos x)$
By further calculation, we get
$=5 \frac{d}{d x}(\sec x)+4 \frac{d}{d x}(\cos x)$
$=5 \sec x \tan x+4 \times(-\sin x)$
$=5 \sec x \tan x-4 \sin x$
(iv) $\operatorname{cosec} x$

Let $\mathrm{f}(\mathrm{x})=\operatorname{cosec} \mathrm{x}$
Accordingly $f(x+h)=\operatorname{cosec}(x+h)$
By first principle, we get
$f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$
$=\lim _{h \rightarrow 0} \frac{\operatorname{cosec}(x+h)-\operatorname{cosec} x}{h}$
$=\lim _{h \rightarrow 0} \frac{1}{h}\left(\frac{1}{\sin (x+h)}-\frac{1}{\sin x}\right)$

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin x-\sin (x+h)}{\sin x \sin (x+h)}\right] \\
& =\frac{1}{\sin x} \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{2 \cos \left(\frac{x+x+h}{2}\right) \sin \left(\frac{x-x-h}{2}\right)}{\sin (x+h)}\right] \\
& =\frac{1}{\sin x} \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{2 \cos \left(\frac{2 x+h}{2}\right) \sin \left(\frac{-h}{2}\right)}{\sin (x+h)}\right]
\end{aligned}
$$

By further calculation, we get

$$
=\frac{1}{\sin x} \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-\sin \left(\frac{h}{2}\right) \cos \left(\frac{2 x+h}{2}\right)}{\left(\frac{h}{2}\right) \sin (x+h)}\right]
$$

$$
=-\frac{1}{\sin x} \lim _{h \rightarrow 0} \frac{\sin \left(\frac{h}{2}\right)}{\frac{h}{2}} \times \lim _{h \rightarrow 0} \frac{\cos \left(\frac{2 x+h}{2}\right)}{\sin (x+h)}
$$

$$
=-\frac{1}{\sin x} \times 1 \times \frac{\cos \left(\frac{2 x+0}{2}\right)}{\sin (x+0)}
$$

$$
=-\frac{1}{\sin x} \times \frac{\cos x}{\sin x}
$$

$$
=-\operatorname{cosec} x \cot x
$$

(v) $3 \cot x+5 \operatorname{cosec} x$

Let $f(x)=3 \cot x+5 \operatorname{cosec} x$

$$
f^{\prime}(x)=3(\cot x)^{\prime}+5(\operatorname{cosec} x)^{\prime}
$$

Let $f_{1}(x)=\cot x$,
Accordingly $f_{1}(x+h)=\cot (x+h)$
By using first principle, we get

$$
\begin{aligned}
& f_{1}^{\prime}(x)=\lim _{x \rightarrow 0} \frac{f_{1}(x+h)-f_{1}(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\cot (x+h)-\cot x}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left(\frac{\cos (x+h)}{\sin (x+h)}-\frac{\cos x}{\sin x}\right)
\end{aligned}
$$

By further calculation, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left(\frac{\sin x \cos (x+h)-\cos x \sin (x+h)}{\sin x \sin (x+h)}\right) \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left(\frac{\sin (x-x-h)}{\sin x \sin (x+h)}\right)
\end{aligned}
$$

$$
=1 / \sin x^{\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (-h)}{\sin (x+h)}\right]}
$$

$$
=-\frac{1}{\sin x}\left(\lim _{h \rightarrow 0} \frac{\sin h}{h}\right)\left(\lim _{h \rightarrow 0} \frac{1}{\sin (x+h)}\right)
$$

$$
=-\frac{1}{\sin x} \times 1 \times \frac{1}{\sin (x+0)}
$$

$$
=-\frac{1}{\sin ^{2} x}
$$

$$
=-\operatorname{cosec}^{2} x
$$

Let $f_{2}(x)=\operatorname{cosec} x$,
Accordingly $f_{2}(x+h)=\operatorname{cosec}(x+h)$
By using first principle, we get

$$
\begin{aligned}
& f_{2}^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f_{2}(x+h)-f_{2}(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\operatorname{cosed}(x+h)-\operatorname{cosec} x}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left(\frac{1}{\sin (x+h)}-\frac{1}{\sin x}\right)
\end{aligned}
$$

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin x-\sin (x+h)}{\sin x \sin (x+h)}\right]
$$

By further calculation, we get
$=\frac{1}{\sin x} \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{2 \cos \left(\frac{x+x+h}{2}\right) \sin \left(\frac{x-x-h}{2}\right)}{\sin (x+h)}\right]$
$=\frac{1}{\sin x} \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{2 \cos \left(\frac{2 x+h}{2}\right) \sin \left(\frac{-h}{2}\right)}{\sin (x+h)}\right]$
$=\frac{1}{\sin x} \lim _{h \rightarrow 0}\left[\frac{-\sin \left(\frac{h}{2}\right) \cos \left(\frac{2 x+h}{2}\right)}{\left(\frac{h}{2}\right) \sin (x+h)}\right]$
$=-\frac{1}{\sin x} \lim _{h \rightarrow 0} \frac{\sin \left(\frac{h}{2}\right)}{\frac{h}{2}} \times \lim _{h \rightarrow 0} \frac{\cos \left(\frac{2 x+h}{2}\right)}{\sin (x+h)}$
$=-\frac{1}{\sin x} \times 1 \times \frac{\cos \left(\frac{2 \mathrm{x}+0}{2}\right)}{\sin (\mathrm{x}+0)}$
$=-\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$
$=-\operatorname{cosec} x \cot x$
Now, substitute the value of $(\cot x)^{\prime}$ and $(\operatorname{cosec} x)^{\prime}$ in $f^{\prime}(x)$, we get

$$
\begin{aligned}
& f^{\prime}(x)=3(\cot x)^{\prime}+5(\operatorname{cosec} x)^{\prime} \\
& f^{\prime}(x)=3 x\left(-\operatorname{cosec}^{2} x\right)+5 x(-\operatorname{cosec} x \cot x) \\
& f^{\prime}(x)=-3 \operatorname{cosec}^{2} x-5 \operatorname{cosec} x \cot x \\
& (v i) 5 \sin x-6 \cos x+7
\end{aligned}
$$

Let $f(x)=5 \sin x-6 \cos x+7$
Accordingly, from the first principle,

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}[5 \sin (x+h)-6 \cos (x+h)+7-5 \sin x+6 \cos x-7] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}[5\{\sin (x+h)-\sin x\}-6\{\cos (x+h)-\cos x\}] \\
& =5 \lim _{h \rightarrow 0} \frac{1}{h}[\sin (x+h)-\sin x]-6 \lim _{h \rightarrow 0} \frac{1}{h}[\cos (x+h)-\cos x]
\end{aligned}
$$

By further calculation, we get

$$
\begin{aligned}
& =5 \lim _{h \rightarrow 0} \frac{1}{h}\left[2 \cos \left(\frac{x+h+x}{2}\right) \sin \left(\frac{x+h-x}{2}\right)\right]-6 \lim _{h \rightarrow 0} \frac{\cos x \cos h-\sin x \sin h-\cos x}{h} \\
& =5 \lim _{h \rightarrow 0} \frac{1}{h}\left[2 \cos \left(\frac{2 x+h}{2}\right) \sin \frac{h}{2}\right]-6 \lim _{h \rightarrow 0}\left[\frac{-\cos x(1-\cos h)-\sin x \sin h}{h}\right]
\end{aligned}
$$

Now, we get

$$
\begin{aligned}
& =5 \lim _{h \rightarrow 0}\left(\cos \left(\frac{2 x+h}{2}\right) \frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)-6 \lim _{h \rightarrow 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x \sin h}{h}\right] \\
& =5\left[\lim _{h \rightarrow 0} \cos \left(\frac{2 x+h}{2}\right)\right]\left[\lim _{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}}\right]-6\left[(-\cos x)\left(\lim _{h \rightarrow 0} \frac{1-\cos h}{h}\right)-\sin x \lim _{h \rightarrow 0}\left(\frac{\sin h}{h}\right)\right] \\
& =5 \cos x \cdot 1-6[(-\cos x) \cdot(0)-\sin x \cdot 1] \\
& =5 \cos x+6 \sin x
\end{aligned}
$$

(vii) $2 \tan x-7 \sec x$

Let $f(x)=2 \tan x-7 \sec x$
Accordingly, from the first principle,

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}[2 \tan (x+h)-7 \sec (x+h)-2 \tan x+7 \sec x] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}[2\{\tan (x+h)-\tan x\}-7\{\sec (x+h)-\sec x\}] \\
& =2 \lim _{h \rightarrow 0} \frac{1}{h}[\tan (x+h)-\tan x]-7 \lim _{h \rightarrow 0} \frac{1}{h}[\sec (x+h)-\sec x]
\end{aligned}
$$

By further calculation, we get
$=2 \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h)}{\cos (x+h)}-\frac{\sin x}{\cos x}\right]-7 \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{1}{\cos (x+h)}-\frac{1}{\cos x}\right]$

$$
\begin{aligned}
& =2 \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h) \cos x-\sin x \cos (x+h)}{\cos x \cos (x+h)}\right]-7 \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\cos x-\cos (x+h)}{\cos x \cos (x+h)}\right] \\
& =2 \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h-x)}{\cos x \cos (x+h)}\right]-7 \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-2 \sin \left(\frac{x+x+h}{2}\right) \sin \left(\frac{x-x-h}{2}\right)}{\cos x \cos (x+h)}\right]
\end{aligned}
$$

Now, we get

$$
\begin{aligned}
& =2 \lim _{h \rightarrow 0}\left[\left(\frac{\sin h}{h}\right) \frac{1}{\cos x \cos (x+h)}\right]-7 \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-2 \sin \left(\frac{2 x+h}{2}\right) \sin \left(-\frac{h}{2}\right)}{\cos x \cos (x+h)}\right] \\
& =2\left(\lim _{h \rightarrow 0} \frac{\sin h}{h}\right)\left(\lim _{h \rightarrow 0} \frac{1}{\cos x \cos (x+h)}\right)-7\left(\lim _{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)\left(\lim _{h \rightarrow 0} \frac{\sin \left(\frac{2 x+h}{2}\right)}{\cos x \cos (x+h)}\right)
\end{aligned}
$$

$$
=2.1 \cdot \frac{1}{\cos x \cos x}-7.1\left(\frac{\sin x}{\cos x \cos x}\right)
$$

$$
=2 \sec ^{2} x-7 \sec x \tan x
$$

## MISCELLNNEOUS EXERCISE

1. Find the derivative of the following functions from the first principle.
(i) $-x$
(ii) $(-x)^{-1}$
(iii) $\sin (x+1)$
(iv) $\cos \left(x-\frac{\pi}{8}\right)$

Solution:
(i) -X

Let $\mathrm{f}(\mathrm{x})=-\mathrm{x}$
Accordingly, $\mathrm{f}(\mathrm{x}+\mathrm{h})=-(\mathrm{x}+\mathrm{h})$
Using first principle, we get

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{-(x+h)-(-x)}{h}
\end{aligned}
$$

Now, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{-x-h+x}{h} \\
& =\lim _{h \rightarrow 0} \frac{-h}{h} \\
& =\lim _{h \rightarrow 0}(-1)=-1
\end{aligned}
$$

(ii) $(-x)^{-1}$

Let $f(x)=(-x)^{-1}=\frac{1}{-x}=\frac{-1}{x}$
Accordingly, $f(x+h)=\frac{-1}{(x+h)}$

Using first principle, we get

$$
\begin{aligned}
& f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-1}{x+h}-\left(\frac{-1}{x}\right)\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-1}{x+h}+\frac{1}{x}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right]
\end{aligned}
$$

By further calculation, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-x+x+h}{x(x+h)}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{h}{x(x+h)}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{x(x+h)} \\
& =\frac{1}{x \cdot x} \\
& =1 / x^{2} \\
& \text { (iii) } \sin (x+1)
\end{aligned}
$$

Let $f(x)=\sin (x+1)$
Accordingly, $\mathrm{f}(\mathrm{x}+\mathrm{h})=\sin (\mathrm{x}+\mathrm{h}+1)$
By using first principle, we get

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

$$
\begin{aligned}
& =\lim _{\mathrm{h} \rightarrow 0} \frac{1}{h}[\sin (x+h+1)-\sin (x+1)] \\
& =\lim _{\mathrm{h} \rightarrow 0} \frac{1}{\mathrm{~h}}\left[2 \cos \left(\frac{\mathrm{x}+\mathrm{h}+1+\mathrm{x}+1}{2}\right) \sin \left(\frac{\mathrm{x}+\mathrm{h}+1-\mathrm{x}-1}{2}\right)\right] \\
& =\lim _{\mathrm{h} \rightarrow 0} \frac{1}{\mathrm{~h}}\left[2 \cos \left(\frac{2 \mathrm{x}+\mathrm{h}+2}{2}\right) \sin \left(\frac{\mathrm{h}}{2}\right)\right] \\
& =\lim _{\mathrm{h} \rightarrow 0}\left[\cos \left(\frac{2 \mathrm{x}+\mathrm{h}+2}{2}\right) \cdot \frac{\sin \left(\frac{\mathrm{h}}{2}\right)}{\left(\frac{\mathrm{h}}{2}\right)}\right]
\end{aligned}
$$

We get,

$$
=\lim _{h \rightarrow 0} \cos \left(\frac{2 x+h+2}{2}\right) \cdot \lim _{\substack{h \\ 2}} \frac{\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}
$$

We know that,

$$
\mathrm{h} \rightarrow 0 \Rightarrow \frac{\mathrm{~h}}{2} \rightarrow 0
$$

$$
=\cos \left(\frac{2 x+0+2}{2}\right) \cdot 1
$$

$$
=\cos (x+1)
$$

(iv) $\cos \left(x-\frac{\pi}{8}\right)$

Let $f(x)=\cos \left(x-\frac{\pi}{8}\right)$

$$
\text { Accordingly, } \mathrm{f}(\mathrm{x}+\mathrm{h})=\cos \left(\mathrm{x}+\mathrm{h}-\frac{\pi}{8}\right)
$$

By using first principle, we get

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\cos \left(x+h-\frac{\pi}{8}\right)-\cos \left(x-\frac{\pi}{8}\right)\right]
\end{aligned}
$$

We get,

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left[-2 \sin \frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2} \sin \left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]
$$

## Further we get,

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left[-2 \sin \left(\frac{2 x+h-\frac{\pi}{4}}{2}\right) \sin \frac{h}{2}\right]
$$

So,

$$
\begin{aligned}
& =\lim _{h \rightarrow 0}\left[-\sin \left(\frac{2 x+h-\frac{\pi}{4}}{2}\right) \frac{\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right] \\
& =\lim _{h \rightarrow 0}\left[-\sin \left(\frac{2 x+h-\frac{\pi}{4}}{2}\right)\right] \cdot \lim _{\frac{h}{2} \rightarrow 0} \frac{\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}
\end{aligned}
$$

$$
\begin{aligned}
& {\left[\text { As h } \rightarrow 0 \Rightarrow \frac{\mathrm{~h}}{2} \rightarrow 0\right] } \\
= & -\sin \left(\frac{2 \mathrm{x}+0-\frac{\pi}{4}}{2}\right) \cdot 1
\end{aligned}
$$

Hence, we get

$$
=-\sin \left(x-\frac{\pi}{8}\right)
$$

Find the derivative of the following functions. (It is to be understood that $a, b, c, d, p, q, r$ and $s$ are fixed non-zero constants, and $m$ and $n$ are integers.)
2. $(x+a)$

Solution:
Let $\mathrm{f}(\mathrm{x})=\mathrm{x}+\mathrm{a}$
Accordingly, $\mathrm{f}(\mathrm{x}+\mathrm{h})=\mathrm{x}+\mathrm{h}+\mathrm{a}$
Using first principle, we get

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

So, now we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{x+h+a-x-a}{h} \\
& =\lim _{h \rightarrow 0}\left(\frac{h}{h}\right) \\
& =\lim _{h \rightarrow 0}(1) \\
& =1
\end{aligned}
$$

3. $(p x+q)(r / x+s)$

## Solution:

Let $f(x)=(p x+q)\left(\frac{r}{x}+s\right)$
Using Leibnitz product rule, we get

$$
f^{\prime}(x)=(p x+q)\left(\frac{r}{x}+s\right)^{\prime}+\left(\frac{r}{x}+s\right)(p x+q)^{\prime}
$$

We get,

$$
=(p x+q)\left(r x^{-1}+s\right)^{\prime}+\left(\frac{r}{x}+s\right)(p)
$$

By further calculation, we get

$$
\begin{aligned}
& =(p x+q)\left(-r x^{-2}\right)+\left(\frac{r}{x}+s\right) p \\
& =(p x+q)\left(\frac{-r}{x^{2}}\right)+\left(\frac{r}{x}+s\right) p
\end{aligned}
$$

Now, we get

$$
\begin{aligned}
& =\frac{-p r}{x}-\frac{q r}{x^{2}}+\frac{p r}{x}+p s \\
& =p s-\frac{q r}{x^{2}}
\end{aligned}
$$

4. $(a x+b)(c x+d)^{2}$

## Solution:

Let $f(x)=(a x+b)(c x+d)^{2}$
By using Leibnitz product rule, we get

$$
f^{\prime}(x)=(a x+b) \frac{d}{d x}(c x+d)^{2}+(c x+d)^{2} \frac{d}{d x}(a x+b)
$$

We get,

$$
=(a x+b) \frac{d}{d x}\left(c^{2} x^{2}+2 c d x+d^{2}\right)+(c x+d)^{2} \frac{d}{d x}(a x+b)
$$

By differentiating separately, we get

$$
=(a x+b)\left[\frac{d}{d x}\left(c^{2} x^{2}\right)+\frac{d}{d x}(2 c d x)+\frac{d}{d x} d^{2}\right]+(c x+d)^{2}\left[\frac{d}{d x} a x+\frac{d}{d x} b\right]
$$

So,

$$
\begin{aligned}
& =(a x+b)\left(2 c^{2} x+2 c d\right)+\left(c x+d^{2}\right) a \\
& =2 c(a x+b)(c x+d)+a(c x+d)^{2}
\end{aligned}
$$

5. $(a x+b) /(c x+d)$

Solution:
Let $f(x)=\frac{a x+b}{c x+d}$
Using quotient rule, we get

$$
f^{\prime}(x)=\frac{(c x+d) \frac{d}{d x}(a x+b)-(a x+b) \frac{d}{d x}(c x+d)}{(c x+d)^{2}}
$$

## Further we get

$$
=\frac{(c x+d)(a)-(a x+b)(c)}{(c x+d)^{2}}
$$

So, now we get

$$
=\frac{a c x+a d-a c x-b c}{(c x+d)^{2}}
$$

## Hence,

$=\frac{a d-b c}{(c x+d)^{2}}$
6. $(1+1 / x) /(1-1 / x)$

## Solution:

Let $f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1}$, where $x \neq 0$
Using quotient rule, we get

$$
f^{\prime}(x)=\frac{(x-1) \frac{d}{d x}(x+1)-(x+1) \frac{d}{d x}(x-1)}{(x-1)^{2}}, x \neq 0,1
$$

Further, we get

$$
=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^{2}}, x \neq 0,1
$$

So,

$$
\begin{aligned}
& =\frac{x-1-x-1}{(x-1)^{2}}, x \neq 0,1 \\
& =\frac{-2}{(x-1)^{2}}, x \neq 0,1
\end{aligned}
$$

7. 1 / $\left(a x^{2}+b x+c\right)$

## Solution:

$$
\text { Let } f(x)=\frac{1}{a x^{2}+b x+c}
$$

Using quotient rule, we get

$$
f^{\prime}(x)=\frac{\left(a x^{2}+b x+c\right) \frac{d}{d x}(1)-\frac{d}{d x}\left(a x^{2}+b x+c\right)}{\left(a x^{2}+b x+c\right)^{2}}
$$

By further calculation, we get

$$
\begin{aligned}
& =\frac{\left(a x^{2}+b x+c\right)(0)-(2 a x+b)}{\left(a x^{2}+b x+c\right)^{2}} \\
& =\frac{-(2 a x+b)}{\left(a x^{2}+b x+c\right)^{2}}
\end{aligned}
$$

$$
\text { 8. }(a x+b) / p x^{2}+q x+r
$$

## Solution:

$$
\text { Let } f(x)=\frac{a x+b}{p x^{2}+q x+r}
$$

Using quotient rule, we get

$$
f^{\prime}(x)=\frac{\left(p x^{2}+q x+r\right) \frac{d}{d x}(a x+b)-(a x+b) \frac{d}{d x}\left(p x^{2}+q x+r\right)}{\left(p x^{2}+q x+r\right)^{2}}
$$

Further we get,

$$
=\frac{\left(p x^{2}+q x+r\right)(a)-(a x+b)(2 p x+q)}{\left(p x^{2}+q x+r\right)^{2}}
$$

Again by further calculation, we get

$$
\begin{aligned}
& =\frac{a p x^{2}+a q x+a r-2 a p x^{2}-a q x-2 b p x-b q}{\left(p x^{2}+q x+r\right)^{2}} \\
& =\frac{-a p x^{2}-2 b p x+a r-b q}{\left(p x^{2}+q x+r\right)^{2}}
\end{aligned}
$$

9. $\left(p x^{2}+q x+r\right) / a x+b$

## Solution:

Let $f(x)=\frac{p x^{2}+q x+r}{a x+b}$

Using quotient rule, we get

$$
f^{\prime}(x)=\frac{(a x+b) \frac{d}{d x}\left(p x^{2}+q x+r\right)-\left(p x^{2}+q x+r\right) \frac{d}{d x}(a x+b)}{(a x+b)^{2}}
$$

By further calculation, we get

$$
=\frac{(a x+b)(2 p x+q)-\left(p x^{2}+q x+r\right)(a)}{(a x+b)^{2}}
$$

So, we get

$$
\begin{aligned}
& =\frac{2 a p x^{2}+a q x+2 b p x+b q-a p x^{2}-a q x-a r}{(a x+b)^{2}} \\
& =\frac{a p x^{2}+2 b p x+b q-a r}{(a x+b)^{2}}
\end{aligned}
$$

10. $\left(a / x^{4}\right)-\left(b / x^{2}\right)+\cos x$

Solution:
Let $f(x)=\frac{a}{x^{4}}-\frac{b}{x^{2}}+\cos x$

By differentiating we get,

$$
f^{\prime}(x)=\frac{d}{d x}\left(\frac{a}{x^{4}}\right)-\frac{d}{d x}\left(\frac{b}{x^{2}}\right)+\frac{d}{d x}(\cos x)
$$

## On further calculation, we get

$$
=a \frac{d}{d x}\left(x^{-4}\right)-b \frac{d}{d x}\left(x^{-2}\right)+\frac{d}{d x}(\cos x)
$$

We know that,

$$
\left[\frac{d}{d x}\left(x^{n}\right)=n x^{n-1} \text { and } \frac{d}{d x}(\cos x)=-\sin x\right]
$$

So,

$$
\begin{aligned}
& =a\left(-4 x^{-5}\right)-b\left(-2 x^{-3}\right)+(-\sin x) \\
& =\frac{-4 a}{x^{3}}+\frac{2 b}{x^{3}}-\sin x
\end{aligned}
$$

11. $4 \sqrt{x}-2$

## Solution:

$$
\text { Let } f(x)=4 \sqrt{x}-2
$$

By differentiating we get,

$$
f^{\prime}(x)=\frac{d}{d x}(4 \sqrt{x}-2)=\frac{d}{d x}(4 \sqrt{x})-\frac{d}{d x}(2)
$$

Further, we get
$=4 \frac{d}{d x}\left(x^{\frac{1}{2}}\right)-0$
$=4\left(\frac{1}{2} x^{\frac{1}{2}-1}\right)$
$=\left(2 x^{-\frac{1}{2}}\right)$
$=\frac{2}{\sqrt{x}}$
12. $(a x+b)^{n}$

## Solution:

Let $f(x)=(a x+b)^{n}$

Accordingly, $f(x+h)=\{a(x+h)+b\}^{n}=(a x+a h+b)^{n}$
Using first principle, we get

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{(a x+a h+b)^{n}-(a x+b)^{n}}{h}
\end{aligned}
$$

Further we get,

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{(a x+b)^{n}\left(1+\frac{a h}{a x+b}\right)^{n}-(a x+b)^{n}}{h} \\
& =(a x+b)^{n} \lim _{h \rightarrow 0} \frac{\left(1+\frac{a h}{a x+b}\right)^{n}-1}{h}
\end{aligned}
$$

By using binomial theorem, we get

$$
=(a x+b)^{n} \lim _{b \rightarrow 0} \frac{1}{n}\left[\left\{1+n\left(\frac{a h}{a x+b}\right)+\frac{n(n-1)}{\underline{2}}\left(\frac{a h}{a x+b}\right)^{2}+\ldots\right\}-1\right]
$$

Now, we get

$$
=(a x+b)^{n} \lim _{b \rightarrow 0} \frac{1}{h}\left[n\left(\frac{a h}{a x+b}\right)+\frac{n(n-1) a^{2} h^{2}}{\left\lfloor 2(a x+b)^{2}\right.}+\ldots(\text { Terms containing higher degrees of } h)\right]
$$

So, we get

$$
=(a x+b)^{n} \lim _{h \rightarrow 0}\left[\frac{n a}{(a x+b)}+\frac{n(n-1) a^{2} h}{12(a x+b)^{2}}+\ldots\right]
$$

On further calculation, we get

$$
\begin{aligned}
& =(a x+b)^{n}\left[\frac{n a}{(a x+b)}+0\right] \\
& =n a \frac{(a x+b)^{n}}{(a x+b)} \\
& =n a(a x+b)^{n-1}
\end{aligned}
$$

13. $(a x+b)^{n}(c x+d)^{m}$

## Solution:

$$
\text { Let } f(x)=(a x+b)^{n}(c x+d)^{n}
$$

By using Leibnitz product rule, we get

$$
f^{\prime}(x)=(a x+b)^{n} \frac{d}{d x}(c x+d)^{m \prime}+(c x+d)^{m} \frac{d}{d x}(a x+b)^{n}
$$

let $f_{1}(x)=(c x+d)^{m}$

$$
f_{1}(x+h)=(c x+c h+d)^{m}
$$

Then,

$$
\begin{aligned}
f_{1}^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f_{1}(x+h)-f_{1}(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{(c x+c h+d)^{m}-(c x+d)^{m}}{h}
\end{aligned}
$$

By taking ( $c x+d$ ) ${ }_{m}^{m}$ as common, we get

$$
=(c x+d)^{m} \lim _{h \rightarrow 0} \frac{1}{h}\left[\left(1+\frac{c h}{c x+d}\right)^{m}-1\right]
$$

On further calculation, we get

$$
=(c x+d)^{m} \lim _{h \rightarrow 0} \frac{1}{h}\left[\left(1+\frac{m c h}{(c x+d)}+\frac{m(m-1)}{2} \frac{\left(c^{2} h^{2}\right)}{(c x+d)^{2}}+\ldots\right)-1\right]
$$

Now, we get

$$
=(c x+d)^{m} \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{m c h}{(c x+d)}+\frac{m(m-1) c^{2} h^{2}}{2(c x+d)^{2}}+\ldots(\text { Terms containing higher degrees of } h)\right]
$$

We know that,

$$
\frac{d}{d x}(c x+d)^{\prime \prime \prime}=m c(c x+d)^{m-1}
$$

Similarly, $\frac{d}{d x}(a x+b)^{n}=n a(a x+b)^{n-1}$

$$
=(c x+d)^{m} \lim _{h \rightarrow 0}\left[\frac{m c}{(c x+d)}+\frac{m(m-1) c^{2} h}{2(c x+d)^{2}}+\ldots\right]
$$

Now, we get

$$
\begin{aligned}
& =(c x+d)^{m \prime}\left[\frac{m c}{c x+d}+0\right] \\
& =\frac{m c(c x+d)^{m}}{(c x+d)} \\
& =m c(c x+d)^{m-1}
\end{aligned}
$$

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Hence, we get

$$
\begin{aligned}
f^{\prime}(x) & =(a x+b)^{n}\left\{m c(c x+d)^{m-1}\right\}+(c x+d)^{m}\left\{n a(a x+b)^{n-1}\right\} \\
& =(a x+b)^{n-1}(c x+d)^{m-1}[m c(a x+b)+n a(c x+d)]
\end{aligned}
$$

14. $\sin (x+a)$

## Solution:

$$
\begin{aligned}
& \text { Let } f(x)=\sin (x+a) \\
& f(x+h)=\sin (x+h+a)
\end{aligned}
$$

## By using first principle, we get

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sin (x+h+a)-\sin (x+a)}{h}
\end{aligned}
$$

On further calculation, we get

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left[2 \cos \left(\frac{x+h+a+x+a}{2}\right) \sin \left(\frac{x+h+a-x-a}{2}\right)\right]
$$

So, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[2 \cos \left(\frac{2 x+2 a+h}{2}\right) \sin \left(\frac{h}{2}\right)\right] \\
& =\lim _{h \rightarrow 0}\left[\cos \left(\frac{2 x+2 a+h}{2}\right)\left\{\frac{\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]
\end{aligned}
$$

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## By taking limits, we get

$$
=\lim _{h \rightarrow 0} \cos \left(\frac{2 x+2 a+h}{2}\right) \lim _{\frac{h}{2} \rightarrow 0}\left\{\frac{\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}
$$

Hence, we get

$$
\begin{aligned}
& =\cos \left(\frac{2 x+2 a}{2}\right) \times 1 \\
& =\cos (x+a)
\end{aligned}
$$

15. $\operatorname{cosec} x \cot x$

## Solution:

Let $f(x)=\operatorname{cosec} x \cot x$
By using Leibnitz product rule, we get

$$
\begin{equation*}
f^{\prime}(x)=\operatorname{cosec} x(\cot x)^{\prime}+\cot x(\operatorname{cosec} x)^{\prime} \tag{1}
\end{equation*}
$$

Let $f_{1}(x)=\cot x$.

Accordingly, $f_{1}(x+h)=\cot (x+h)$
By using first principle, we get

$$
\begin{aligned}
f_{1}^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f_{1}(x+h)-f_{1}(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{\cot (x+h)-\cot x}{h}
\end{aligned}
$$

On further calculation, we get

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left(\frac{\cos (x+h)}{\sin (x+h)}-\frac{\cos x}{\sin x}\right)
$$

## Now, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin x \cos (x+h)-\cos x \sin (x+h)}{\sin x \sin (x+h)}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x-x-h)}{\sin x \sin (x+h)}\right]
\end{aligned}
$$

We get

$$
\begin{aligned}
& =\frac{1}{\sin x} \cdot \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (-h)}{\sin (x+h)}\right] \\
& =\frac{-1}{\sin x} \cdot\left(\lim _{h \rightarrow 0} \frac{\sin h}{h}\right)\left(\lim _{h \rightarrow 0} \frac{1}{\sin (x+h)}\right)
\end{aligned}
$$

So, we get
$=\frac{-1}{\sin x} \cdot 1 \cdot\left(\frac{1}{\sin (x+0)}\right)$
$=\frac{-1}{\sin ^{2} x}$
$=-\operatorname{cosec}^{2} x$
Hence, we get
$(\cot x)^{\prime}=-\operatorname{cosec}^{2} x$
Now, let $f_{2}(x)=\operatorname{cosec} x$. Accordingly, $f_{2}(x+h)=\operatorname{cosec}(x+h)$
By using first principle, we get

$$
\begin{aligned}
f_{2}^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f_{2}(x+h)-f_{2}(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}[\operatorname{cosec}(x+h)-\operatorname{cosec} x]
\end{aligned}
$$

## By calculating further, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{1}{\sin (x+h)}-\frac{1}{\sin x}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin x-\sin (x+h)}{\sin x \sin (x+h)}\right]
\end{aligned}
$$

So,

$$
\begin{aligned}
& =\frac{1}{\sin x} \cdot \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{2 \cos \left(\frac{x+x+h}{2}\right) \sin \left(\frac{x-x-h}{2}\right)}{\sin (x+h)}\right] \\
& =\frac{1}{\sin x} \cdot \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{2 \cos \left(\frac{2 x+h}{2}\right) \sin \left(\frac{-h}{2}\right)}{\sin (x+h)}\right]
\end{aligned}
$$

$$
=\frac{1}{\sin x} \cdot \lim _{h \rightarrow 0}\left[\frac{-\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos \left(\frac{2 x+h}{2}\right)}{\sin (x+h)}\right]
$$

We get,

$$
\begin{aligned}
& =\frac{-1}{\sin x} \cdot \lim _{h \rightarrow 0} \frac{\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim _{h \rightarrow 0} \frac{\cos \left(\frac{2 x+h}{2}\right)}{\sin (x+h)} \\
& =\frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos \left(\frac{2 x+0}{2}\right)}{\sin (x+0)} \\
& =\frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} \\
& =-\cos \mathrm{ec} x \cdot \cot x
\end{aligned}
$$

## Hence,



From equations (1) (2) and (3) we get,

$$
\begin{aligned}
f^{\prime}(x) & =\operatorname{cosec} x\left(-\operatorname{cosec}^{2} x\right)+\cot x(-\operatorname{cosec} x \cot x) \\
& =-\operatorname{cosec}^{3} x-\cot ^{2} x \operatorname{cosec} x
\end{aligned}
$$

16. $\frac{\cos x}{1+\sin x}$

## Solution:

Let $f(x)=\frac{\cos x}{1+\sin x}$
By using quotient rule, we get

$$
\begin{aligned}
f^{\prime}(x) & =\frac{(1+\sin x) \frac{d}{d x}(\cos x)-(\cos x) \frac{d}{d x}(1+\sin x)}{(1+\sin x)^{2}} \\
& =\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^{2}}
\end{aligned}
$$

We get,

$$
\begin{aligned}
& =\frac{-\sin x-\sin ^{2} x-\cos ^{2} x}{(1+\sin x)^{2}} \\
& =\frac{-\sin x-\left(\sin ^{2} x+\cos ^{2} x\right)}{(1+\sin x)^{2}}
\end{aligned}
$$

Now, we get

$$
\begin{aligned}
& =\frac{-\sin x-1}{(1+\sin x)^{2}} \\
& =\frac{-(1+\sin x)}{(1+\sin x)^{2}} \\
& =\frac{-1}{(1+\sin x)}
\end{aligned}
$$

17. 

$\frac{\sin x+\cos x}{\sin x-\cos x}$

## Solution:

$$
\text { Let } f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}
$$

By differentiating and using quotient rule, we get

$$
f^{\prime}(x)=\frac{(\sin x-\cos x) \frac{d}{d x}(\sin x+\cos x)-(\sin x+\cos x) \frac{d}{d x}(\sin x-\cos x)}{(\sin x-\cos x)^{2}}
$$

On further calculation, we get

$$
\begin{aligned}
& =\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^{2}} \\
& =\frac{-(\sin x-\cos x)^{2}-(\sin x+\cos x)^{2}}{(\sin x-\cos x)^{2}}
\end{aligned}
$$

By expanding the terms, we get

$$
=\frac{-\left[\sin ^{2} x+\cos ^{2} x-2 \sin x \cos x+\sin ^{2} x+\cos ^{2} x+2 \sin x \cos x\right]}{(\sin x-\cos x)^{2}}
$$

We get

$$
\begin{aligned}
& =\frac{-[1+1]}{(\sin x-\cos x)^{2}} \\
& =\frac{-2}{(\sin x-\cos x)^{2}}
\end{aligned}
$$

18. 

$\frac{\sec x-1}{\sec x+1}$

## Solution:

$$
\text { Let } f(x)=\frac{\sec x-1}{\sec x+1}
$$

Now, this can be written as

$$
f(x)=\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}
$$

By differentiating and using quotient rule, we get

$$
\begin{aligned}
f^{\prime}(x) & =\frac{(1+\cos x) \frac{d}{d x}(1-\cos x)-(1-\cos x) \frac{d}{d x}(1+\cos x)}{(1+\cos x)^{2}} \\
& =\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^{2}}
\end{aligned}
$$

On multiplying we get

$$
\begin{aligned}
& =\frac{\sin x+\cos x \sin x+\sin x-\sin x \cos x}{(1+\cos x)^{2}} \\
& =\frac{2 \sin x}{(1+\cos x)^{2}}
\end{aligned}
$$

This can be written as

$$
=\frac{2 \sin x}{\left(1+\frac{1}{\sec x}\right)^{2}}
$$

On taking L.C.M we get

$$
=\frac{2 \sin x}{\frac{(\sec x+1)^{2}}{\sec ^{2} x}}
$$

On further calculation, we get

$$
\begin{aligned}
& =\frac{2 \sin x \sec ^{2} x}{(\sec x+1)^{2}} \\
& =\frac{\frac{2 \sin x}{\cos x} \sec x}{(\sec x+1)^{2}} \\
& =\frac{2 \sec x \tan x}{(\sec x+1)^{2}}
\end{aligned}
$$

19. $\sin ^{n} x$

## Solution:

Let $y=\sin ^{n} x$.
Accordingly, for $n=1, y=\sin x$.
We know that,
$\frac{d y}{d x}=\cos x$, i.e., $\frac{d}{d x} \sin x=\cos x$
For $n=2, y=\sin ^{2} x$.
So, $\frac{d y}{d x}=\frac{d}{d x}(\sin x \sin x)$
By Leibnitz product rule, we get
$=(\sin x)^{\prime} \sin x+\sin x(\sin x)^{\prime}$
$=\cos x \sin x+\sin x \cos x$
$=2 \sin x \cos x$

For $n=3, y=\sin ^{3} x$.
So, $\frac{d y}{d x}=\frac{d}{d x}\left(\sin x \sin ^{2} x\right)$
By Leibnitz product rule, we get

$$
=(\sin x)^{\prime} \sin ^{2} x+\sin x\left(\sin ^{2} x\right)^{\prime}
$$

From equation (1) we get

$$
=\cos x \sin ^{2} x+\sin x(2 \sin x \cos x)
$$

$=\cos x \sin ^{2} x+2 \sin ^{2} x \cos x$
$=3 \sin ^{2} x \cos x$
We state that, $\frac{d}{d x}\left(\sin ^{n} x\right)=n \sin ^{(n-1)} x \cos x$
For $\mathrm{n}=\mathrm{k}$, let our assertion be true

$$
\begin{equation*}
\text { i.e., } \frac{d}{d x}\left(\sin ^{k} x\right)=k \sin ^{(k-1)} x \cos x \tag{2}
\end{equation*}
$$

Now, consider

$$
\frac{d}{d x}\left(\sin ^{k+1} x\right)=\frac{d}{d x}\left(\sin x \sin ^{k} x\right)
$$

By using Leibnitz product rule, we get

$$
=(\sin x)^{\prime} \sin ^{k} x+\sin x\left(\sin ^{k} x\right)^{\prime}
$$

From equation (2) we get

$$
\begin{aligned}
& =\cos x \sin ^{k} x+\sin x\left(k \sin ^{(k-1)} x \cos x\right) \\
& =\cos x \sin ^{k} x+k \sin ^{k} x \cos x \\
& =(k+1) \sin ^{k} x \cos x
\end{aligned}
$$

Hence, our assertion is true for $\mathrm{n}=\mathrm{k}+1$

Therefore,
by mathematical induction, $\frac{d}{d x}\left(\sin ^{n} x\right)=n \sin ^{(n-1)} x \cos x$
20. $\frac{a+b \sin x}{c+d \cos x}$

## Solution:

Let $f(x)=\frac{a+b \sin x}{c+d \cos x}$
By differentiating and using quotient rule, we get

$$
\begin{aligned}
f^{\prime}(x) & =\frac{(c+d \cos x) \frac{d}{d x}(a+b \sin x)-(a+b \sin x) \frac{d}{d x}(c+d \cos x)}{(c+d \cos x)^{2}} \\
& =\frac{(c+d \cos x)(b \cos x)-(a+b \sin x)(-d \sin x)}{(c+d \cos x)^{2}}
\end{aligned}
$$

On multiplying we get

$$
=\frac{c b \cos x+b d \cos ^{2} x+a d \sin x+b d \sin ^{2} x}{(c+d \cos x)^{2}}
$$

Now, taking bd as common we get

$$
\begin{aligned}
& =\frac{b c \cos x+a d \sin x+b d\left(\cos ^{2} x+\sin ^{2} x\right)}{(c+d \cos x)^{2}} \\
& =\frac{b c \cos x+a d \sin x+b d}{(c+d \cos x)^{2}}
\end{aligned}
$$

21. 

$$
\sin (x+a)
$$

$\cos x$

## Solution:

Let $f(x)=\frac{\sin (x+a)}{\cos x}$
By differentiating and using quotient rule, we get
$f^{\prime}(x)=\frac{\cos x \frac{d}{d x}[\sin (x+a)]-\sin (x+a) \frac{d}{d x} \cos x}{\cos ^{2} x}$
$f^{\prime}(x)=\frac{\cos x \frac{d}{d x}[\sin (x+a)]-\sin (x+a)(-\sin x)}{\cos ^{2} x}$
Let $g(x)=\sin (x+a)$. Accordingly, $g(x+h)=\sin (x+h+a)$
By using first principle, we get

$$
\begin{aligned}
g^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h} \\
& =\lim _{h \rightarrow 0} \frac{1}{h}[\sin (x+h+a)-\sin (x+a)]
\end{aligned}
$$

On further calculation, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[2 \cos \left(\frac{x+h+a+x+a}{2}\right) \sin \left(\frac{x+h+a-x-a}{2}\right)\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[2 \cos \left(\frac{2 x+2 a+h}{2}\right) \sin \left(\frac{h}{2}\right)\right] \\
& =\lim _{h \rightarrow 0}\left[\cos \left(\frac{2 x+2 a+h}{2}\right)\left\{\frac{\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]
\end{aligned}
$$

Now, taking limits we get

$$
=\lim _{h \rightarrow 0} \cos \left(\frac{2 x+2 a+h}{2}\right) \cdot \lim _{\frac{h}{2} \rightarrow 0}\left\{\frac{\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \quad\left[\text { As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0\right]
$$

We know that,

$$
\begin{align*}
& {\left[\lim _{h \rightarrow 0} \frac{\sin h}{h}=1\right]} \\
& =\left(\cos \frac{2 x+2 a}{2}\right) \times 1 \\
& =\cos (x+a) \tag{ii}
\end{align*}
$$

From equation (i) and (ii) we get

$$
\begin{aligned}
f^{\prime}(x) & =\frac{\cos x \cdot \cos (x+a)+\sin x \sin (x+a)}{\cos ^{2} x} \\
& =\frac{\cos (x+a-x)}{\cos ^{2} x} \\
& =\frac{\cos a}{\cos ^{2} x}
\end{aligned}
$$

22. $x^{4}(5 \sin x-3 \cos x)$

## Solution:

$$
\text { Let } f(x)=x^{4}(5 \sin x-3 \cos x)
$$

By differentiating and using product rule, we get

$$
f^{\prime}(x)=x^{4} \frac{d}{d x}(5 \sin x-3 \cos x)+(5 \sin x-3 \cos x) \frac{d}{d x}\left(x^{4}\right)
$$

On further calculation, we get

$$
=x^{+}\left[5 \frac{d}{d x}(\sin x)-3 \frac{d}{d x}(\cos x)\right]+(5 \sin x-3 \cos x) \frac{d}{d x}\left(x^{+}\right)
$$

So, we get

$$
=x^{+}[5 \cos x-3(-\sin x)]+(5 \sin x-3 \cos x)\left(4 x^{3}\right)
$$

By taking $\mathrm{x}^{3}$ as common, we get

$$
=x^{3}[5 x \cos x+3 x \sin x+20 \sin x-12 \cos x]
$$

23. $\left(x^{2}+1\right) \cos x$

Solution:
Let $f(x)=\left(x^{2}+1\right) \cos x$
By differentiating and using product rule, we get

$$
f^{\prime}(x)=\left(x^{2}+1\right) \frac{d}{d x}(\cos x)+\cos x \frac{d}{d x}\left(x^{2}+1\right)
$$

On further calcualtion, we get

$$
=\left(x^{2}+1\right)(-\sin x)+\cos x(2 x)
$$

## By multiplying we get

$$
=-x^{2} \sin x-\sin x+2 x \cos x
$$

24. $\left(a x^{2}+\sin x\right)(p+q \cos x)$

Solution:
Let $f(x)=\left(a x^{2}+\sin x\right)(p+q \cos x)$
By differentiating and using product rule, we get

$$
f^{\prime}(x)=\left(a x^{2}+\sin x\right) \frac{d}{d x}(p+q \cos x)+(p+q \cos x) \frac{d}{d x}\left(a x^{2}+\sin x\right)
$$

On further calculation, we get

$$
\begin{aligned}
& =\left(a x^{2}+\sin x\right)(-q \sin x)+(p+q \cos x)(2 a x+\cos x) \\
& =-q \sin x\left(a x^{2}+\sin x\right)+(p+q \cos x)(2 a x+\cos x)
\end{aligned}
$$

25. $(x+\cos x)(x-\tan x)$

## Solution:

$$
\text { Let } f(x)=(x+\cos x)(x-\tan x)
$$

## By differentiating and using product rule, we get

$$
\begin{aligned}
& f^{\prime}(x)=(x+\cos x) \frac{d}{d x}(x-\tan x)+(x-\tan x) \frac{d}{d x}(x+\cos x) \\
& =(x+\cos x)\left[\frac{d}{d x}(x)-\frac{d}{d x}(\tan x)\right]+(x-\tan x)(1-\sin x)
\end{aligned}
$$

Now, we get

$$
\begin{equation*}
=(x+\cos x)\left[1-\frac{d}{d x} \tan x\right]+(x-\tan x)(1-\sin x) \tag{i}
\end{equation*}
$$

Let $g(x)=\tan x$. Accordingly, $g(x+h)=\tan (x+h)$
By using first principle, we get

$$
\begin{aligned}
g^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h} \\
& =\lim _{h \rightarrow 0}\left(\frac{\tan (x+h)-\tan x}{h}\right)
\end{aligned}
$$

On further calculation, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h)}{\cos (x+h)}-\frac{\sin x}{\cos x}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h) \cos x-\sin x \cos (x+h)}{\cos (x+h) \cos x}\right]
\end{aligned}
$$

Now, we get

$$
\begin{aligned}
& =\frac{1}{\cos x} \cdot \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h-x)}{\cos (x+h)}\right] \\
& =\frac{1}{\cos x} \cdot \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin h}{\cos (x+h)}\right]
\end{aligned}
$$

So, we get

$$
=\frac{1}{\cos x} \cdot\left(\lim _{h \rightarrow 0} \frac{\sin h}{h}\right) \cdot\left(\lim _{h \rightarrow 0} \frac{1}{\cos (x+h)}\right)
$$

We get

$$
\begin{align*}
& =\frac{1}{\cos x} \cdot 1 \cdot \frac{1}{\cos (x+0)} \\
& =\frac{1}{\cos ^{2} x} \\
& =\sec ^{2} x \tag{ii}
\end{align*}
$$

Hence, from equation (i) and (ii) we get

$$
\begin{aligned}
f^{\prime}(x) & =(x+\cos x)\left(1-\sec ^{2} x\right)+(x-\tan x)(1-\sin x) \\
& =(x+\cos x)\left(-\tan ^{2} x\right)+(x-\tan x)(1-\sin x) \\
& =-\tan ^{2} x(x+\cos x)+(x-\tan x)(1-\sin x)
\end{aligned}
$$

26. $\frac{4 x+5 \sin x}{3 x+7 \cos x}$

## Solution:

$$
\text { Let } f(x)=\frac{4 x+5 \sin x}{3 x+7 \cos x}
$$

By differentiating and using quotient rule, we get

$$
f^{\prime}(x)=\frac{(3 x+7 \cos x) \frac{d}{d x}(4 x+5 \sin x)-(4 x+5 \sin x) \frac{d}{d x}(3 x+7 \cos x)}{(3 x+7 \cos x)^{2}}
$$

On further calculation, we get

$$
\begin{aligned}
& =\frac{(3 x+7 \cos x)\left[4 \frac{d}{d x}(x)+5 \frac{d}{d x}(\sin x)\right]-(4 x+5 \sin x)\left[3 \frac{d}{d x} x+7 \frac{d}{d x} \cos x\right]}{(3 x+7 \cos x)^{2}} \\
& =\frac{(3 x+7 \cos x)(4+5 \cos x)-(4 x+5 \sin x)(3-7 \sin x)}{(3 x+7 \cos x)^{2}}
\end{aligned}
$$

On multiplying we get

$$
=\frac{12 x+15 x \cos x+28 \cos x+35 \cos ^{2} x-12 x+28 x \sin x-15 \sin x+35 \sin ^{2} x}{(3 x+7 \cos x)^{2}}
$$

We get
$=\frac{15 x \cos x+28 \cos x+28 x \sin x-15 \sin x+35\left(\cos ^{2} x+\sin ^{2} x\right)}{(3 x+7 \cos x)^{2}}$
$=\frac{35+15 x \cos x+28 \cos x+28 x \sin x-15 \sin x}{(3 x+7 \cos x)^{2}}$
27. $\frac{x^{2} \cos \left(\frac{\pi}{4}\right)}{\sin x}$

Solution:

$$
\text { Let } f(x)=\frac{x^{2} \cos \left(\frac{\pi}{4}\right)}{\sin x}
$$

By differentiating and using quotient rule, we get

$$
f^{\prime}(x)=\cos \frac{\pi}{4} \cdot\left[\frac{\sin x \frac{d}{d x}\left(x^{2}\right)-x^{2} \frac{d}{d x}(\sin x)}{\sin ^{2} x}\right]
$$

By further calculation, we get

$$
=\cos \frac{\pi}{4} \cdot\left[\frac{\sin x \cdot 2 x-x^{2} \cos x}{\sin ^{2} x}\right]
$$

By taking x as common, we get

$$
=\frac{x \cos \frac{\pi}{4}[2 \sin x-x \cos x]}{\sin ^{2} x}
$$

28. $\frac{x}{1+\tan x}$

## Solution:

$$
\text { Let } f(x)=\frac{x}{1+\tan x}
$$

By differentiating and using quotient rule, we get

$$
\begin{align*}
& f^{\prime}(x)=\frac{(1+\tan x) \frac{d}{d x}(x)-x \frac{d}{d x}(1+\tan x)}{(1+\tan x)^{2}} \\
& f^{\prime}(x)=\frac{(1+\tan x)-x \cdot \frac{d}{d x}(1+\tan x)}{(1+\tan x)^{2}} \tag{i}
\end{align*}
$$

Let $g(x)=1+\tan x$. Accordingly, $g(x+h)=1+\tan (x+h)$.
Using first principle, we get

$$
\begin{aligned}
g^{\prime}(x)= & \lim _{h \rightarrow 0} \frac{g(x+h)-g(x)}{h} \\
& =\lim _{h \rightarrow 0}\left[\frac{1+\tan (x+h)-1-\tan x}{h}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h)}{\cos (x+h)}-\frac{\sin x}{\cos x}\right]
\end{aligned}
$$

By taking L.C.M we get

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h) \cos x-\sin x \cos (x+h)}{\cos (x+h) \cos x}\right]
$$

We get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h-x)}{\cos (x+h) \cos x}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin h}{\cos (x+h) \cos x}\right]
\end{aligned}
$$

So, we get

$$
\begin{aligned}
& =\left(\lim _{h \rightarrow 0} \frac{\sin h}{h}\right) \cdot\left(\lim _{h \rightarrow 0} \frac{1}{\cos (x+h) \cos x}\right) \\
& =1 \times \frac{1}{\cos ^{2} x}=\sec ^{2} x
\end{aligned}
$$

$$
\begin{equation*}
\frac{d}{d x}(1+\tan x)=\sec ^{2} x \tag{ii}
\end{equation*}
$$

From equation (i) and (ii) we get
$f^{\prime}(x)=\frac{1+\tan x-x \sec ^{2} x}{(1+\tan x)^{2}}$
29. $(x+\sec x)(x-\tan x)$

Solution:

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Let $f(x)=(x+\sec x)(x-\tan x)$
By differentiating and using product rule, we get

$$
f^{\prime}(x)=(x+\sec x) \frac{d}{d x}(x-\tan x)+(x-\tan x) \frac{d}{d x}(x+\sec x)
$$

## So, we get

$$
=(x+\sec x)\left[\frac{d}{d x}(x)-\frac{d}{d x} \tan x\right]+(x-\tan x)\left[\frac{d}{d x}(x)+\frac{d}{d x} \sec x\right]
$$

$=(x+\sec x)\left[1-\frac{d}{d x} \tan x\right]+(x-\tan x)\left[1+\frac{d}{d x} \sec x\right]$
Let $f_{1}(x)=\tan x, f_{2}(x)=\sec x$
Accordingly, $f_{1}(x+h)=\tan (x+h)$ and $f_{2}(x+h)=\sec (x+h)$

$$
\begin{aligned}
f_{1}^{\prime}(x) & =\lim _{h \rightarrow 0}\left(\frac{f_{1}(x+h)-f_{1}(x)}{h}\right) \\
& =\lim _{h \rightarrow 0}\left(\frac{\tan (x+h)-\tan x}{h}\right)
\end{aligned}
$$

By further calculation, we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0}\left[\frac{\tan (x+h)-\tan x}{h}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h)}{\cos (x+h)}-\frac{\sin x}{\cos x}\right]
\end{aligned}
$$

Now, by taking L.C.M we get

$$
\begin{aligned}
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h) \cos x-\sin x \cos (x+h)}{\cos (x+h) \cos x}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin (x+h-x)}{\cos (x+h) \cos x}\right] \\
& =\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\sin h}{\cos (x+h) \cos x}\right]
\end{aligned}
$$

$$
\begin{aligned}
& =\left(\lim _{h \rightarrow 0} \frac{\sin h}{h}\right) \cdot\left(\lim _{h \rightarrow 0} \frac{1}{\cos (x+h) \cos x}\right) \\
& =1 \times \frac{1}{\cos ^{2} x}=\sec ^{2} x
\end{aligned}
$$

Hence we get

$$
\begin{equation*}
\frac{d}{d x} \tan x=\sec ^{2} x \tag{ii}
\end{equation*}
$$

Now, take

$$
\begin{aligned}
f_{2}^{\prime}(x)= & \lim _{h \rightarrow 0}\left(\frac{f_{2}(x+h)-f_{2}(x)}{h}\right) \\
& =\lim _{h \rightarrow 0}\left(\frac{\sec (x+h)-\sec x}{h}\right)
\end{aligned}
$$

This can be written as

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{1}{\cos (x+h)}-\frac{1}{\cos x}\right]
$$

By taking L.C.M we get

$$
=\lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{\cos x-\cos (x+h)}{\cos (x+h) \cos x}\right]
$$

On further calculation, we get

$$
\begin{aligned}
& =\frac{1}{\cos x} \cdot \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-2 \sin \left(\frac{x+x+h}{2}\right) \cdot \sin \left(\frac{x-x-h}{2}\right)}{\cos (x+h)}\right] \\
& =\frac{1}{\cos x} \cdot \lim _{h \rightarrow 0} \frac{1}{h}\left[\frac{-2 \sin \left(\frac{2 x+h}{2}\right) \cdot \sin \left(\frac{-h}{2}\right)}{\cos (x+h)}\right]
\end{aligned}
$$

## We get

$$
=\frac{1}{\cos x} \cdot \lim _{h \rightarrow 0}\left[\frac{\sin \left(\frac{2 x+h}{2}\right)\left\{\frac{\sin \left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\cos (x+h)}\right]
$$

By taking limits, we get


We get

$$
\begin{gather*}
=\sec x \cdot \frac{\sin x \cdot 1}{\cos x} \\
\frac{d}{d x} \sec x=\sec x \tan x \tag{iii}
\end{gather*}
$$

From equation (i) (ii) and (iii) we get

$$
f^{\prime}(x)=(x+\sec x)\left(1-\sec ^{2} x\right)+(x-\tan x)(1+\sec x \tan x)
$$

30. $\frac{x}{\sin ^{n} x}$

## Solution:

$$
\text { Let } f(x)=\frac{x}{\sin ^{n} x}
$$

By differentiating and using quotient rule, we get

$$
f^{\prime}(x)=\frac{\sin ^{n} x \frac{d}{d x} x-x \frac{d}{d x} \sin ^{n} x}{\sin ^{2 n} x}
$$

Easily, it can be shown that,
$\frac{d}{d x} \sin ^{n} x=n \sin ^{n-1} x \cos x$

Hence,

$$
f^{\prime}(x)=\frac{\sin ^{n} x \frac{d}{d x} x-x \frac{d}{d x} \sin ^{n} x}{\sin ^{2 n} x}
$$

By further calculation, we get

$$
=\frac{\sin ^{n} x \cdot 1-x\left(n \sin ^{n-1} x \cos x\right)}{\sin ^{2 n} x}
$$

By taking common terms, we get

$$
=\frac{\sin ^{n-1} x(\sin x-n x \cos x)}{\sin ^{2 n} x}
$$

Hence, we get
$=\frac{\sin x-n x \cos x}{\sin ^{n+1} x}$


[^0]:    Solution:

