1. Find the derivative of the following functions from the first principle.

(i) –*x*

(ii) (–*x*)$^{-1}$

(iii) sin (*x* + 1)

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Solution:

(i) - x

Let f (x) = - x

Accordingly, f (x + h) = - (x + h)

Using first principle, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{-(x+h)-(-x)}{h}$$

Now, we get

$$=\lim_{h\to 0}\frac{-x-h+x}{h}$$

$$=\lim_{h\to 0}\frac{-h}{h}$$

$$=\lim_{h\to 0}(-1)=-1$$

(ii) (-x)$^{-1}$

Let $f(x)=(-x)^{-1}=\frac{1}{-x}=\frac{-1}{x}$

Accordingly, $f(x+h)=\frac{-1}{(x+h)}$

Using first principle, we get

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-1}{x+h} - \left(\frac{-1}{x}\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-1}{x+h} + \frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-x + (x+h)}{x(x+h)}\right]$$

By further calculation, we get

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-x + x + h}{x(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h}{x(x+h)}\right]$$

$$= \lim_{h\to 0} \frac{1}{x(x+h)}$$

$$= \frac{1}{x \cdot x}$$

$= 1 / x^2$

(iii) sin (x + 1)

Let f (x) = sin (x + 1)

Accordingly, f (x + h) = sin (x + h + 1)

By using first principle, we get

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+1)-\sin(x+1)\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

We get,

$$= \lim_{h\to 0}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

We know that,

$$h\to 0 \Rightarrow \frac{h}{2}\to 0$$

$$= \cos\left(\frac{2x+0+2}{2}\right)\cdot 1$$
$$= \cos(x+1)$$

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Let $f(x)=\cos\left(x-\frac{\pi}{8}\right)$

Accordingly, $f(x+h)=\cos\left(x+h-\frac{\pi}{8}\right)$

By using first principle, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]$$

We get,

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

Further we get,

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}\right]$$

So,

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right].\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$\left[\text{As } h \to 0 \Rightarrow \frac{h}{2} \to 0\right]$$

$$= -\sin\left(\frac{2x + 0 - \frac{\pi}{4}}{2}\right).1$$

Hence, we get

$$= -\sin\left(x - \frac{\pi}{8}\right)$$

Find the derivative of the following functions. (It is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants, and *m* and *n* are integers.)

2. (*x* + *a*)

Solution:

Let $f(x) = x + a$

Accordingly, $f(x + h) = x + h + a$

Using first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So, now we get

$$= \lim_{h \to 0} \frac{x + h + a - x - a}{h}$$

$$= \lim_{h \to 0} \left(\frac{h}{h}\right)$$

$$= \lim_{h \to 0} (1)$$

$$= 1$$

3. (px + q) (r / x + s)

Solution:

Let $f(x)=(px+q)\left(\frac{r}{x}+s\right)$

Using Leibnitz product rule, we get

$$f'(x)=(px+q)\left(\frac{r}{x}+s\right)'+\left(\frac{r}{x}+s\right)(px+q)'$$

We get,

$$=(px+q)\left(rx^{-1}+s\right)'+\left(\frac{r}{x}+s\right)(p)$$

By further calculation, we get

$$=(px+q)\left(-rx^{-2}\right)+\left(\frac{r}{x}+s\right)p$$

$$=(px+q)\left(\frac{-r}{x^2}\right)+\left(\frac{r}{x}+s\right)p$$

Now, we get

$$=\frac{-pr}{x}-\frac{qr}{x^2}+\frac{pr}{x}+ps$$
$$=ps-\frac{qr}{x^2}$$

4. (*ax* + *b*) (*cx* + *d*)^2

Solution:

Let $f(x)=(ax+b)(cx+d)^2$

By using Leibnitz product rule, we get

$$f'(x)=(ax+b)\frac{d}{dx}(cx+d)^2+(cx+d)^2\frac{d}{dx}(ax+b)$$

We get,

$$=(ax+b)\frac{d}{dx}\left(c^2x^2+2cdx+d^2\right)+(cx+d)^2\frac{d}{dx}(ax+b)$$

By differentiating separately, we get

$$=(ax+b)\left[\frac{d}{dx}\left(c^2x^2\right)+\frac{d}{dx}(2cdx)+\frac{d}{dx}d^2\right]+(cx+d)^2\left[\frac{d}{dx}ax+\frac{d}{dx}b\right]$$

So,

$$=(ax+b)\left(2c^2x+2cd\right)+\left(cx+d^2\right)a$$
$$=2c(ax+b)(cx+d)+a(cx+d)^2$$

5. (ax + b) / (cx + d)

Solution:

Let $f(x)=\dfrac{ax+b}{cx+d}$

Using quotient rule, we get

$$f'(x)=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

Further we get

$$=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$

So, now we get

$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$

Hence,

$$=\frac{ad-bc}{(cx+d)^2}$$

6. (1 + 1 / x) / (1 – 1 / x)

Solution:

Let $f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1}$, where $x\neq 0$

Using quotient rule, we get

$$f'(x)=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2},\ x\neq 0,\ 1$$

Further, we get

$$=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2},\ x\neq 0,\ 1$$

So,

$$= \frac{x-1-x-1}{(x-1)^2},\ x \neq 0,\ 1$$

$$= \frac{-2}{(x-1)^2},\ x \neq 0,\ 1$$

7. 1 / (ax² + bx + c)

Solution:

Let $f(x) = \frac{1}{ax^2+bx+c}$

Using quotient rule, we get

$$f'(x) = \frac{(ax^2+bx+c)\frac{d}{dx}(1) - \frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

By further calculation, we get

$$= \frac{(ax^2+bx+c)(0)-(2ax+b)}{(ax^2+bx+c)^2}$$

$$= \frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

8. (ax + b) / px² + qx + r

Solution:

Let $f(x) = \frac{ax+b}{px^2+qx+r}$

Using quotient rule, we get

$$f'(x) = \frac{(px^2+qx+r)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(px^2+qx+r)}{(px^2+qx+r)^2}$$

Further we get,

$$=\frac{(px^2+qx+r)(a)-(ax+b)(2px+q)}{(px^2+qx+r)^2}$$

Again by further calculation, we get

$$=\frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{(px^2+qx+r)^2}$$

$$=\frac{-apx^2-2bpx+ar-bq}{(px^2+qx+r)^2}$$

9. (px^2 + qx + r) / ax + b

Solution:

$$\text{Let } f(x)=\frac{px^2+qx+r}{ax+b}$$

Using quotient rule, we get

$$f'(x)=\frac{(ax+b)\frac{d}{dx}(px^2+qx+r)-(px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

By further calculation, we get

$$=\frac{(ax+b)(2px+q)-(px^2+qx+r)(a)}{(ax+b)^2}$$

So, we get

$$=\frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$=\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

10. (a / x^4) – (b / x^2) + cos x

Solution:

$$\text{Let } f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$$

By differentiating we get,

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{b}{x^2}\right)+\frac{d}{dx}(\cos x)$$

On further calculation, we get

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

We know that,

$$\left[\frac{d}{dx}\left(x^n\right)=nx^{n-1}\text{and}\frac{d}{dx}(\cos x)=-\sin x\right]$$

So,

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x)$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

11. $4\sqrt{x}-2$

Solution:

Let $f(x)=4\sqrt{x}-2$

By differentiating we get,

$$f'(x)=\frac{d}{dx}\left(4\sqrt{x}-2\right)=\frac{d}{dx}\left(4\sqrt{x}\right)-\frac{d}{dx}(2)$$

Further, we get

$$=4\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-0$$

$$= 4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$=\left(2x^{-\frac{1}{2}}\right)$$

$$=\frac{2}{\sqrt{x}}$$

12. (ax + b)^n

Solution:

$$\text{Let } f(x) = (ax+b)^n$$

$$\text{Accordingly, } f(x+h) = \{a(x+h)+b\}^n = (ax+ah+b)^n$$

Using first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(ax+ah+b)^n-(ax+b)^n}{h}$$

Further we get,

$$= \lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$
$$= (ax+b)^n\lim_{h\to 0}\frac{\left(1+\frac{ah}{ax+b}\right)^n-1}{h}$$

By using binomial theorem, we get

$$= (ax+b)^n\lim_{h\to 0}\frac{1}{n}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{\underline{|2}}\left(\frac{ah}{ax+b}\right)^2+\ldots\right\}-1\right]$$

Now, we get

$$= (ax+b)^n \lim_{h \to 0} \frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right) + \frac{n(n-1)a^2h^2}{\underline{|2}(ax+b)^2} + \ldots(\text{Terms containing higher degrees of } h)\right]$$

So, we get

$$= (ax+b)^n \lim_{h \to 0}\left[\frac{na}{(ax+b)} + \frac{n(n-1)a^2h}{\underline{|2}(ax+b)^2} + \ldots\right]$$

On further calculation, we get

$$= (ax+b)^n\left[\frac{na}{(ax+b)} + 0\right]$$

$$= na\frac{(ax+b)^n}{(ax+b)}$$

$$= na(ax+b)^{n-1}$$

13. (*ax* + *b*)^n (*cx* + *d*)^m

Solution:

Let $f(x) = (ax+b)^n (cx+d)^m$

By using Leibnitz product rule, we get

$$f'(x) = (ax+b)^n \frac{d}{dx}(cx+d)^m + (cx+d)^m \frac{d}{dx}(ax+b)^n$$

let $f_1(x) = (cx+d)^m$

Then, $f_1(x+h) = (cx+ch+d)^m$

$$f_1'(x) = \lim_{h \to 0}\frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h \to 0}\frac{(cx+ch+d)^m - (cx+d)^m}{h}$$

By taking $(cx + d)^m$ as common, we get

$$=(cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m-1\right]$$

On further calculation, we get

$$=(cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{(c^2h^2)}{(cx+d)^2}+\ldots\right)-1\right]$$

Now, we get

$$=(cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+\ldots(\text{Terms containing higher degrees of } h)\right]$$

We know that,

$$\frac{d}{dx}(cx+d)^m = mc(cx+d)^{m-1}$$

Similarly, $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

$$=(cx+d)^m \lim_{h\to 0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h}{2(cx+d)^2}+\ldots\right]$$

Now, we get

$$=(cx+d)^m\left[\frac{mc}{cx+d}+0\right]$$

$$=\frac{mc(cx+d)^m}{(cx+d)}$$

$$=mc(cx+d)^{m-1}$$

Hence, we get

$$f'(x) = (ax+b)^n \left\{mc(cx+d)^{m-1}\right\} + (cx+d)^m \left\{na(ax+b)^{n-1}\right\}$$
$$= (ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b) + na(cx+d)\right]$$

14. sin (*x* + *a*)

Solution:

Let $f(x) = \sin(x+a)$

$f(x+h) = \sin(x+h+a)$

By using first principle, we get

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sin(x+h+a) - \sin(x+a)}{h}$$

On further calculation, we get

$$= \lim_{h\to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

So, we get

$$= \lim_{h\to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$
$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

By taking limits, we get

$$= \lim_{h\to 0} \cos\left(\frac{2x+2a+h}{2}\right) \lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}$$

Hence, we get

$$= \cos\left(\frac{2x+2a}{2}\right)\times 1$$
$$= \cos(x+a)$$

15. cosec *x* cot *x*

Solution:

Let $f(x) = \operatorname{cosec} x \cot x$

By using Leibnitz product rule, we get

$$f'(x) = \operatorname{cosec} x(\cot x)' + \cot x(\operatorname{cosec} x)' \qquad ...(1)$$

Let $f_1(x) = \cot x.$

Accordingly, $f_1(x+h) = \cot(x+h)$

By using first principle, we get

$$f_1'(x) = \lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$
$$= \lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}$$

On further calculation, we get

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right)$$

Now, we get

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)}\right]$$

We get

$$= \frac{1}{\sin x} . \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$= \frac{-1}{\sin x} . \left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \frac{1}{\sin(x+h)}\right)$$

So, we get

$$= \frac{-1}{\sin x}.1.\left(\frac{1}{\sin(x+0)}\right)$$
$$= \frac{-1}{\sin^2 x}$$
$$= -\text{cosec}^2 x$$

Hence, we get

$$(\cot x)' = -\text{cosec}^2 x \qquad ...(2)$$

Now, let $f_2(x)$ = cosec x. Accordingly, $f_2(x+h) = \text{cosec}(x+h)$

By using first principle, we get

$$f_2'(x) = \lim_{h\to 0}\frac{f_2(x+h) - f_2(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\text{cosec}(x+h) - \text{cosec}\, x\right]$$

By calculating further, we get

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)}\right]$$

So,

$$= \frac{1}{\sin x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$
$$= \frac{1}{\sin x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \right]$$

We get,

$$= \frac{-1}{\sin x} \cdot \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cdot \cot x$$

Hence,

$$(\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x \qquad \ldots(3)$$

From equations (1) (2) and (3) we get,

$$f'(x) = \operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right) + \cot x\left(-\operatorname{cosec} x \cot x\right)$$
$$= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

16. $\frac{\cos x}{1+\sin x}$

Solution:

Let $f(x)=\frac{\cos x}{1+\sin x}$

By using quotient rule, we get

$$f'(x)=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$
$$=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

We get,

$$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$
$$=\frac{-\sin x-(\sin^2 x+\cos^2 x)}{(1+\sin x)^2}$$

Now, we get

$$=\frac{-\sin x-1}{(1+\sin x)^2}$$
$$=\frac{-(1+\sin x)}{(1+\sin x)^2}$$
$$=\frac{-1}{(1+\sin x)}$$

17.

$\frac{\sin x+\cos x}{\sin x-\cos x}$

Solution:

Let $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$

By differentiating and using quotient rule, we get

$$f'(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

On further calculation, we get

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

By expanding the terms, we get

$$= \frac{-\left[\sin^2 x + \cos^2 x - 2\sin x\cos x + \sin^2 x + \cos^2 x + 2\sin x\cos x\right]}{(\sin x - \cos x)^2}$$

We get

$$= \frac{-[1+1]}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

18.

$$\frac{\sec x - 1}{\sec x + 1}$$

Solution:

Let $f(x) = \frac{\sec x - 1}{\sec x + 1}$

Now, this can be written as

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(1+\cos x)\frac{d}{dx}(1-\cos x) - (1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

On multiplying we get

$$= \frac{\sin x + \cos x\sin x + \sin x - \sin x\cos x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

This can be written as

$$= \frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2}$$

On taking L.C.M we get

$$= \frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

On further calculation, we get

$$= \frac{2\sin x\sec^2 x}{(\sec x+1)^2}$$

$$= \frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$= \frac{2\sec x\tan x}{(\sec x+1)^2}$$

19. $\sin^n x$

Solution:

Let $y = \sin^n x$.

Accordingly, for $n = 1$, $y = \sin x$.

We know that,

$$\frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx}\sin x = \cos x$$

For $n = 2$, $y = \sin^2 x$.

So, $\frac{dy}{dx} = \frac{d}{dx}(\sin x \sin x)$

By Leibnitz product rule, we get

$$= (\sin x)' \sin x + \sin x(\sin x)'$$

$$= \cos x\sin x + \sin x\cos x$$

$$= 2\sin x\cos x \qquad ...(1)$$

For $n = 3$, $y = \sin^3 x$.

So, $\frac{dy}{dx} = \frac{d}{dx}\left(\sin x \sin^2 x\right)$

By Leibnitz product rule, we get

$= (\sin x)' \sin^2 x + \sin x\left(\sin^2 x\right)'$

From equation (1) we get

$= \cos x \sin^2 x + \sin x\left(2\sin x \cos x\right)$

$= \cos x \sin^2 x + 2\sin^2 x \cos x$

$= 3\sin^2 x \cos x$

We state that, $\frac{d}{dx}\left(\sin^n x\right) = n\sin^{(n-1)} x \cos x$

For n = k, let our assertion be true

i.e., $\frac{d}{dx}\left(\sin^k x\right) = k\sin^{(k-1)} x \cos x$ ...(2)

Now, consider

$\frac{d}{dx}\left(\sin^{k+1} x\right) = \frac{d}{dx}\left(\sin x \sin^k x\right)$

By using Leibnitz product rule, we get

$= (\sin x)' \sin^k x + \sin x\left(\sin^k x\right)'$

From equation (2) we get

$= \cos x \sin^k x + \sin x\left(k\sin^{(k-1)} x \cos x\right)$

$= \cos x \sin^k x + k\sin^k x \cos x$

$= (k+1)\sin^k x \cos x$

Hence, our assertion is true for n = k + 1

Therefore, by mathematical induction, $\frac{d}{dx}\left(\sin^n x\right) = n\sin^{(n-1)} x \cos x$

20. $\frac{a + b\sin x}{c + d\cos x}$

Solution:

Let $f(x)=\frac{a+b\sin x}{c+d\cos x}$

By differentiating and using quotient rule, we get

$$f'(x)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

On multiplying we get

$$=\frac{cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x}{(c+d\cos x)^2}$$

Now, taking bd as common we get

$$=\frac{bc\cos x+ad\sin x+bd(\cos^2 x+\sin^2 x)}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

21.

$\frac{\sin(x+a)}{\cos x}$

Solution:

Let $f(x)=\frac{\sin(x+a)}{\cos x}$

By differentiating and using quotient rule, we get

$$f'(x)=\frac{\cos x\frac{d}{dx}[\sin(x+a)]-\sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x)=\frac{\cos x\frac{d}{dx}[\sin(x+a)]-\sin(x+a)(-\sin x)}{\cos^2 x} \quad \ldots \text{(i)}$$

Let $g(x)=\sin(x+a)$. Accordingly, $g(x+h)=\sin(x+h+a)$

By using first principle, we get

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}[\sin(x+h+a)-\sin(x+a)]$$

On further calculation, we get

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h \to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

Now, taking limits we get

$$= \lim_{h \to 0} \cos\left(\frac{2x+2a+h}{2}\right) . \lim_{\frac{h}{2} \to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \qquad \left[\text{As } h \to 0 \Rightarrow \frac{h}{2} \to 0\right]$$

We know that,

$$\left[\lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$= \left(\cos\frac{2x+2a}{2}\right) \times 1$$

$$= \cos(x+a) \qquad \text{... (ii)}$$

From equation (i) and (ii) we get

$$f'(x) = \frac{\cos x \cdot \cos(x+a) + \sin x \sin(x+a)}{\cos^2 x}$$

$$= \frac{\cos(x+a-x)}{\cos^2 x}$$

$$= \frac{\cos a}{\cos^2 x}$$

22. x^4 (5 sin x – 3 cos x)

Solution:

Let $f(x) = x^4(5\sin x - 3\cos x)$

By differentiating and using product rule, we get

$$f'(x) = x^4 \frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$

On further calculation, we get

$$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$

So, we get

$$= x^4\left[5\cos x - 3(-\sin x)\right] + (5\sin x - 3\cos x)(4x^3)$$

By taking x^3 as common, we get

$$= x^3\left[5x\cos x + 3x\sin x + 20\sin x - 12\cos x\right]$$

23. (x^2 + 1) cos *x*

Solution:

Let $f(x) = (x^2+1)\cos x$

By differentiating and using product rule, we get

$$f'(x) = (x^2+1)\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x^2+1)$$

On further calcualtion, we get

$$= (x^2+1)(-\sin x) + \cos x(2x)$$

By multiplying we get

$$= -x^2\sin x - \sin x + 2x\cos x$$

24. (ax^2 + sin *x*) (*p* + *q* cos *x*)

Solution:

Let $f(x) = (ax^2 + \sin x)(p + q\cos x)$

By differentiating and using product rule, we get

$$f'(x) = (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x)$$

On further calculation, we get

$$= (ax^2 + \sin x)(-q\sin x) + (p + q\cos x)(2ax + \cos x)$$

$$= -q\sin x(ax^2 + \sin x) + (p + q\cos x)(2ax + \cos x)$$

25. $(x + \cos x)(x - \tan x)$

Solution:

Let $f(x)=(x+\cos x)(x-\tan x)$

By differentiating and using product rule, we get

$$f'(x)=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)$$

$$=(x+\cos x)\left[\frac{d}{dx}(x)-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)$$

Now, we get

$$=(x+\cos x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)(1-\sin x) \qquad \text{... (i)}$$

Let $g(x)=\tan x$. Accordingly, $g(x+h)=\tan(x+h)$

By using first principle, we get

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

On further calculation, we get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

Now, we get

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$$

So, we get

$$=\frac{1}{\cos x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right)$$

We get

$$= \frac{1}{\cos x}.1.\frac{1}{\cos(x+0)}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x \qquad \text{... (ii)}$$

Hence, from equation (i) and (ii) we get

$$f'(x) = (x+\cos x)(1-\sec^2 x) + (x-\tan x)(1-\sin x)$$

$$= (x+\cos x)(-\tan^2 x) + (x-\tan x)(1-\sin x)$$

$$= -\tan^2 x(x+\cos x) + (x-\tan x)(1-\sin x)$$

26. $\frac{4x+5\sin x}{3x+7\cos x}$

Solution:

Let $f(x) = \frac{4x+5\sin x}{3x+7\cos x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x) - (4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

On further calculation, we get

$$= \frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right] - (4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$= \frac{(3x+7\cos x)(4+5\cos x) - (4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

On multiplying we get

$$= \frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

We get

$$= \frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$= \frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

27. $\dfrac{x^2 \cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

Solution:

Let $f(x) = \dfrac{x^2 \cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

By differentiating and using quotient rule, we get

$$f'(x) = \cos\frac{\pi}{4} \cdot \left[\frac{\sin x \dfrac{d}{dx}\left(x^2\right) - x^2 \dfrac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

By further calculation, we get

$$= \cos\frac{\pi}{4} \cdot \left[\frac{\sin x \cdot 2x - x^2 \cos x}{\sin^2 x}\right]$$

By taking x as common, we get

$$= \frac{x \cos\dfrac{\pi}{4}\left[2\sin x - x\cos x\right]}{\sin^2 x}$$

28. $\dfrac{x}{1+\tan x}$

Solution:

Let $f(x) = \dfrac{x}{1+\tan x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(1+\tan x)\dfrac{d}{dx}(x) - x\dfrac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x) = \frac{(1+\tan x) - x \cdot \dfrac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \qquad \text{... (i)}$$

Let $g(x)=1+\tan x$. Accordingly, $g(x+h)=1+\tan(x+h)$.

Using first principle, we get

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\left[\frac{1+\tan(x+h)-1-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

By taking L.C.M we get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

We get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

So, we get

$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$

$$\frac{d}{dx}(1+\tan x)=\sec^2 x \qquad \text{... (ii)}$$

From equation (i) and (ii) we get

$$f'(x)=\frac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$$

29. (*x* + sec *x*) (*x* – tan *x*)

Solution:

Let $f(x)=(x+\sec x)(x-\tan x)$

By differentiating and using product rule, we get

$$f'(x)=(x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$

So, we get

$$=(x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)+\frac{d}{dx}\sec x\right]$$

$$=(x+\sec x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right] \qquad \text{... (i)}$$

Let $f_1(x)=\tan x,\ f_2(x)=\sec x$

Accordingly, $f_1(x+h)=\tan(x+h)$ and $f_2(x+h)=\sec(x+h)$

$$f_1'(x)=\lim_{h\to 0}\left(\frac{f_1(x+h)-f_1(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

By further calculation, we get

$$=\lim_{h\to 0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

Now, by taking L.C.M we get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$= \left(\lim_{h \to 0} \frac{\sin h}{h} \right) . \left(\lim_{h \to 0} \frac{1}{\cos(x+h)\cos x} \right)$$

$$= 1 \times \frac{1}{\cos^2 x} = \sec^2 x$$

Hence we get

$$\frac{d}{dx} \tan x = \sec^2 x \qquad \text{... (ii)}$$

Now, take

$$f_2'(x) = \lim_{h \to 0} \left(\frac{f_2(x+h) - f_2(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\sec(x+h) - \sec x}{h} \right)$$

This can be written as

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]$$

By taking L.C.M we get

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x} \right]$$

On further calculation, we get

$$= \frac{1}{\cos x} . \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{x+x+h}{2} \right) \cdot \sin\left(\frac{x-x-h}{2} \right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} . \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{2x+h}{2} \right) \cdot \sin\left(\frac{-h}{2} \right)}{\cos(x+h)} \right]$$

We get

$$=\frac{1}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\cos(x+h)}\right]$$

By taking limits, we get

$$=\sec x.\frac{\left\{\lim_{h\to 0}\sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\lim_{h\to 0}\cos(x+h)}$$

We get

$$=\sec x.\frac{\sin x.1}{\cos x}$$

$$\frac{d}{dx}\sec x=\sec x\tan x \quad \text{.. (iii)}$$

From equation (i) (ii) and (iii) we get

$$f'(x)=(x+\sec x)(1-\sec^2 x)+(x-\tan x)(1+\sec x\tan x)$$

30. $\frac{x}{\sin^n x}$

Solution:

Let $f(x)=\frac{x}{\sin^n x}$

By differentiating and using quotient rule, we get

$$f'(x)=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

Easily, it can be shown that,

$$\frac{d}{dx}\sin^n x=n\sin^{n-1} x\cos x$$

Hence,

$$f'(x)=\frac{\sin^n x\frac{d}{dx}x-x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

By further calculation, we get

$$=\frac{\sin^n x.1-x\left(n\sin^{n-1} x\cos x\right)}{\sin^{2n} x}$$

By taking common terms, we get

$$=\frac{\sin^{n-1} x\left(\sin x-nx\cos x\right)}{\sin^{2n} x}$$

Hence, we get

$$=\frac{\sin x-nx\cos x}{\sin^{n+1} x}$$