MISCELLANEOUS EXERCISE PAGE NO: 233

1. Find the values of k for which the line (k – 3) x – (4 – k²) y + k² – 7k + 6 = 0 is

(a) Parallel to the x-axis

(b) Parallel to the y-axis

(c) Passing through the origin

Solution:

It is given that

$(k-3)x-(4-k^2)y+k^2-7k+6=0$ … (1)

(a) Here, if the line is parallel to the x-axis

Slope of the line = Slope of the x-axis

It can be written as

$(4-k^2)y=(k-3)x+k^2-7k+6=0$

We get

$$y=\frac{(k-3)}{(4-k^2)}x+\frac{k^2-7k+6}{(4-k^2)}$$

Which is of the form y = mx + c

Here the slope of the given line

$$=\frac{(k-3)}{(4-k^2)}$$

Consider the slope of x-axis = 0

$$\frac{(k-3)}{(4-k^2)}=0$$

By further calculation,

$k-3=0$

$k=3$

Hence, if the given line is parallel to the x-axis, then the value of k is 3.

(b) Here, if the line is parallel to the y-axis, it is vertical, and the slope will be undefined.

So, the slope of the given line

$$= \frac{(k-3)}{(4-k^2)}$$

Here,

$\frac{(k-3)}{(4-k^2)}$ is undefined at $k^2 = 4$

$k^2 = 4$

$k = \pm 2$

Hence, if the given line is parallel to the y-axis, then the value of k is ± 2.

(c) Here, if the line is passing through (0, 0), which is the origin satisfies the given equation of the line.

$(k - 3)(0) - (4 - k^2)(0) + k^2 - 7k + 6 = 0$

By further calculation,

$k^2 - 7k + 6 = 0$

Separating the terms,

$k^2 - 6k - k + 6 = 0$

We get

$(k - 6)(k - 1) = 0$

$k = 1$ or 6

Hence, if the given line is passing through the origin, then the value of k is either 1 or 6.

2. Find the values of θ and p, if the equation x cos θ + y sin θ = p is the normal form of the line √3x + y + 2 = 0.

Solution:

It is given that

√3x + y + 2 = 0

It can be reduced as

√3x + y + 2 = 0

- √3x – y = 2

By dividing both sides by $\sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$, we get

$$-\frac{\sqrt{3}}{2}x - \frac{1}{2}y = \frac{2}{2}$$

It can be written as

$$\left(-\frac{\sqrt{3}}{2}\right)x + \left(-\frac{1}{2}\right)y = 1 \qquad ...(1)$$

By comparing equation (1) to x cos θ + y sin θ = p, we get

$$\cos\theta = -\frac{\sqrt{3}}{2},\ \sin\theta = -\frac{1}{2}, \text{ and } p = 1$$

Here the values of sin θ and cos θ are negative

$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

Hence, the respective values of θ and p are 7π/6 and 1.

3. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and –6, respectively.

Solution:

Consider the intercepts cut by the given lines on the a and b axes.

a + b = 1 …… (1)

ab = – 6 …….. (2)

By solving both equations, we get

a = 3 and b = -2 or a = – 2 and b = 3

We know that the equation of the line whose intercepts on the a and b axes is

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ or } bx + ay - ab = 0$$

Case I – a = 3 and b = – 2

So, the equation of the line is – 2x + 3y + 6 = 0, i.e. 2x – 3y = 6

Case II – a = -2 and b = 3

So, the equation of the line is 3x – 2y + 6 = 0, i.e. -3x + 2y = 6

Hence, the required equation of the lines are 2x – 3y = 6 and -3x + 2y = 6

4. What are the points on the *y*-axis whose distance from the line x/3 + y/4 = 1 is 4 units?

Solution:

Consider (0, b) as the point on the y-axis whose distance from line x/3 + y/4 = 1 is 4 units.

It can be written as 4x + 3y – 12 = 0 ……. (1)

By comparing equation (1) to the general equation of line Ax + By + C = 0, we get

A = 4, B = 3 and C = – 12

We know that the perpendicular distance (d) of a line Ax + By + C = 0 from (x_1, y_1) is written as

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

If (0, b) is the point on the y-axis whose distance from line x/3 + y/4 = 1 is 4 units, then

$$4 = \frac{|4(0) + 3(b) - 12|}{\sqrt{4^2 + 3^2}}$$

By further calculation

$$4 = \frac{|3b - 12|}{5}$$

By cross multiplication,

20 = |3b – 12|

We get

20 = ± (3b – 12)

Here, 20 = (3b – 12) or 20 = – (3b – 12)

It can be written as

3b = 20 + 12 or 3b = -20 + 12

So, we get

b = 32/3 or b = -8/3

Hence, the required points are (0, 32/3) and (0, -8/3).

5. Find the perpendicular distance from the origin to the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

Solution:

Here the equation of the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ is written as

$$y - \sin\theta = \frac{\sin\phi - \sin\theta}{\cos\phi - \cos\theta}(x - \cos\theta)$$

By cross multiplication

$$y(\cos\phi - \cos\theta) - \sin\theta(\cos\phi - \cos\theta) = x(\sin\phi - \sin\theta) - \cos\theta(\sin\phi - \sin\theta)$$

By multiplying the terms we get

$$x(\sin\theta - \sin\phi) + y(\cos\phi - \cos\theta) + \cos\theta\sin\phi - \cos\theta\sin\theta - \sin\theta\cos\phi + \sin\theta\cos\theta = 0$$

On further simplification

$$x(\sin\theta - \sin\phi) + y(\cos\phi - \cos\theta) + \sin(\phi - \theta) = 0$$

So we get

$Ax + By + C = 0$, where $A = \sin\theta - \sin\phi$, $B = \cos\phi - \cos\theta$, and $C = \sin(\phi - \theta)$

We know that the perpendicular distance (d) of a line Ax + By + C = 0 from (x_1, y_1) is written as

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

So the perpendicular distance (d) of the given line from $(x_1, y_1) = (0, 0)$ is

$$d = \frac{|(\sin\theta - \sin\phi)(0) + (\cos\phi - \cos\theta)(0) + \sin(\phi - \theta)|}{\sqrt{(\sin\theta - \sin\phi)^2 + (\cos\phi - \cos\theta)^2}}$$

By expanding using formula

$$= \frac{|\sin(\phi - \theta)|}{\sqrt{\sin^2\theta + \sin^2\phi - 2\sin\theta\sin\phi + \cos^2\phi + \cos^2\theta - 2\cos\phi\cos\theta}}$$

Grouping of terms

$$= \frac{|\sin(\phi - \theta)|}{\sqrt{(\sin^2\theta + \cos^2\theta) + (\sin^2\phi + \cos^2\phi) - 2(\sin\theta\sin\phi + \cos\theta\cos\phi)}}$$

By further simplification

$$= \frac{|\sin(\phi - \theta)|}{\sqrt{1 + 1 - 2(\cos(\phi - \theta))}}$$

Taking out 2 as common

$$= \frac{|\sin(\phi - \theta)|}{\sqrt{2(1 - \cos(\phi - \theta))}}$$

Using the formula

$$= \frac{|\sin(\phi - \theta)|}{\sqrt{2\left(2\sin^2\left(\frac{\phi - \theta}{2}\right)\right)}}$$

We get

$$= \frac{|\sin(\phi - \theta)|}{\left|2\sin\left(\frac{\phi - \theta}{2}\right)\right|}$$

6. Find the equation of the line parallel to the *y*-axis and draw through the point of intersection of the lines *x* – 7*y* + 5 = 0 and 3*x* + *y* = 0.

Solution:

Here, the equation of any line parallel to the y-axis is of the form

x = a ……. (1)

Two given lines are

x – 7y + 5 = 0 …… (2)

3x + y = 0 …… (3)

By solving equations (2) and (3), we get x = -5/22 and y = 15/22

(-5/ 22, 15/22) is the point of intersection of lines (2) and (3)

If the line x = a passes through point (-5/22, 15/22), we get a = -5/22

Hence, the required equation of the line is x = -5/22

7. Find the equation of a line drawn perpendicular to the line x/4 + y/6 = 1 through the point where it meets the y-axis.

Solution:

It is given that

x/4 + y/6 = 1

We can write it as

3x + 2y – 12 = 0

So, we get

y = -3/2 x + 6, which is of the form y = mx + c

Here, the slope of the given line = -3/2

So, the slope of line perpendicular to the given line = -1/ (-3/2) = 2/3

Consider the given line intersects, the y-axis at (0, y)

By substituting x as zero in the equation of the given line,

y/6 = 1

y = 6

Hence, the given line intersects the y-axis at (0, 6).

We know that the equation of the line that has a slope of 2/3 and passes through the point (0, 6) is

(y – 6) = 2/3 (x – 0)

By further calculation,

3y – 18 = 2x

So, we get

2x – 3y + 18 = 0

Hence, the required equation of the line is 2x – 3y + 18 = 0

8. Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - k = 0$.

Solution:

It is given that

$y - x = 0$ …… (1)

$x + y = 0$ …… (2)

$x - k = 0$ ……. (3)

Here, the point of intersection of

Lines (1) and (2) is

$x = 0$ and $y = 0$

Lines (2) and (3) is

$x = k$ and $y = -k$

Lines (3) and (1) is

$x = k$ and $y = k$

So, the vertices of the triangle formed by the three given lines are (0, 0), (k, -k) and (k, k).

Here, the area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$\frac{1}{2} |x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)|$

So, the area of the triangle formed by the three given lines is

$= \frac{1}{2} |0 (-k - k) + k (k - 0) + k (0 + k)|$ square units

By further calculation,

$= \frac{1}{2} |k^2 + k^2|$ square units

So, we get

$= \frac{1}{2} |2k^2|$

$= k^2$ square units

9. Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.

Solution:

It is given that

$3x + y - 2 = 0$ …… (1)

$px + 2y - 3 = 0$ ….. (2)

$2x - y - 3 = 0$ …… (3)

By solving equations (1) and (3), we get

$x = 1$ and $y = -1$

Here, the three lines intersect at one point, and the point of intersection of lines (1) and (3) will also satisfy line (2)

$p(1) + 2(-1) - 3 = 0$

By further calculation,

$p - 2 - 3 = 0$

So we get

$p = 5$

Hence, the required value of p is 5.

10. If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$.

Solution:

It is given that

$y = m_1x + c_1$ ….. (1)

$y = m_2x + c_2$ ….. (2)

$y = m_3x + c_3$ ….. (3)

By subtracting equation (1) from (2), we get

$0 = (m_2 - m_1)x + (c_2 - c_1)$

$(m_1 - m_2)x = c_2 - c_1$

So we get

$$x = \frac{c_2 - c_1}{m_1 - m_2}$$

By substituting this value in equation (1) we get

$$y = m_1\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_1$$

By multiplying the terms

$$y = \frac{m_1c_2 - m_1c_1}{m_1 - m_2} + c_1$$

Taking LCM

$$y = \frac{m_1c_2 - m_1c_1 + m_1c_1 - m_2c_1}{m_1 - m_2}$$

On further simplification

$$y = \frac{m_1c_2 - m_2c_1}{m_1 - m_2}$$

Here

$\left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1c_2 - m_2c_1}{m_1 - m_2}\right)$ is the point of intersection of lines (1) and (2)

Lines (1), (2) and (3) are concurrent. So the point of intersection of lines (1) and (2) will satisfy equation (3)

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = m_3\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_3$$

By multiplying the terms and taking LCM

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = \frac{m_3c_2 - m_3c_1 + c_3m_1 - c_3m_2}{m_1 - m_2}$$

By cross multiplication

$m_1 c_2 - m_2c_1 - m_3c_2 + m_3c_1 - c_3m_1 + c_3m_2 = 0$

Taking out the common terms,

$m_1 (c_2 - c_3) + m_2 (c_3 - c_1) + m_3 (c_1 - c_2) = 0$

Therefore, $m_1 (c_2 - c_3) + m_2 (c_3 - c_1) + m_3 (c_1 - c_2) = 0$

11. Find the equation of the lines through the point (3, 2), which makes an angle of 45° with the line $x - 2y = 3$.

Solution:

Consider m_1 as the slope of the required line

It can be written as

y = 1/2 x – 3/2 which is of the form y = mx + c

So, the slope of the given line m_2 = 1/2

We know that the angle between the required line and line x – 2y = 3 is 45°

If θ is the acute angle between lines l_1 and l_2 with slopes m_1 and m_2,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

We get

$$\tan 45^\circ = \frac{|m_1 - m_2|}{1 + m_1 m_2}$$

Substituting the values

$$1 = \left|\frac{\frac{1}{2} - m_1}{1 + \frac{m_1}{2}}\right|$$

By taking LCM

$$1 = \left|\frac{\left(\frac{1 - 2m_1}{2}\right)}{\frac{2 + m_1}{2}}\right|$$

On further calculation

$$1 = \left|\frac{1-2m_1}{2+m_1}\right|$$

We get

$$1 = \pm\left(\frac{1-2m_1}{2+m_1}\right)$$

Here

$$1 = \frac{1-2m_1}{2+m_1} \text{ or } 1 = -\left(\frac{1-2m_1}{2+m_1}\right)$$

It can be written as

$2 + m_1 = 1 – 2m_1$ or $2 + m_1 = – 1 + 2m_1$

$m_1 = – 1/3$ or $m_1 = 3$

Case I – $m_1 = 3$

Here, the equation of the line passing through (3, 2) and having a slope 3 is

y – 2 = 3 (x – 3)

By further calculation,

y – 2 = 3x – 9

So, we get

3x – y = 7

Case II – $m_1 = -1/3$

Here, the equation of the line passing through (3, 2) and having a slope -1/3 is

y – 2 = – 1/3 (x – 3)

By further calculation,

3y – 6 = – x + 3

So, we get

x + 3y = 9

Hence, the equations of the lines are 3x – y = 7 and x + 3y = 9

12. Find the equation of the line passing through the point of intersection of the lines 4*x* + 7*y* – 3 = 0 and 2*x* – 3*y* + 1 = 0 that has equal intercepts on the axes.

Solution:

Consider the equation of the line having equal intercepts on the axes as

x/a + y/a = 1

It can be written as

x + y = a ….. (1)

By solving equations 4x + 7y – 3 = 0 and 2x – 3y + 1 = 0, we get

x = 1/13 and y = 5/13

(1/13, 5/13) is the point of intersection of two given lines.

We know that equation (1) passes through the point (1/13, 5/13).

1/13 + 5/13 = a

a = 6/13

So, equation (1) passes through (1/13, 5/13).

1/13 + 5/13 = a

We get

a = 6/13

Her, equation (1) becomes

x + y = 6/13

13x + 13y = 6

Hence, the required equation of the line is 13x + 13y = 6

13. Show that the equation of the line passing through the origin and making an angle *θ* with the line y = mx + c is $\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$.

Solution:

Consider $y = m_1x$ as the equation of the line passing through the origin

It is given that the line makes an angle θ with line $y = mx + c$, then angle θ is written as

$$\tan\theta = \left|\frac{m_1 - m}{1 + m_1 m}\right|$$

By substituting the values

$$\tan\theta = \left|\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right|$$

We get

$$\tan\theta = \pm\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$$

Here

$$\tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m} \text{ or } \tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$$

Case I –

$$\tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}$$

We can write it as

$$\tan\theta + \frac{y}{x} m \tan\theta = \frac{y}{x} - m$$

By further simplification

$$m + \tan\theta = \frac{y}{x}(1 - m\tan\theta)$$

So we get

$$\frac{y}{x} = \frac{m + \tan\theta}{1 - m\tan\theta}$$

Case II –

$$\tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x} m}\right)$$

We can write it as

$$\tan\theta + \frac{y}{x} m \tan\theta = -\frac{y}{x} + m$$

By further simplification

$$\frac{y}{x}(1 + m\tan\theta) = m - \tan\theta$$

So we get

$$\frac{y}{x} = \frac{m - \tan\theta}{1 + m\tan\theta}$$

Hence, the required line is given by

$$\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$$

14. In what ratio, the line joining (–1, 1) and (5, 7) is divided by the line $x + y = 4$?

Solution:

We know that the equation of the line joining the points (-1, 1) and (5, 7) is given by

$$y-1=\frac{7-1}{5+1}(x+1)$$

By further calculation

$$y-1=\frac{6}{6}(x+1)$$

So we get

$x-y+2=0$ ……. (1)

So the equation of the given line is

$x+y-4=0$ ….. (2)

Here the point of intersection of lines (1) and (2) is given by

$x = 1$ and $y = 3$

Consider (1, 3) divide the line segment joining (-1, 1) and (5, 7) in the ratio 1: k.

Using the section formula

$$(1,3)=\left(\frac{k(-1)+1(5)}{1+k},\frac{k(1)+1(7)}{1+k}\right)$$

By further calculation

$$(1,3)=\left(\frac{-k+5}{1+k},\frac{k+7}{1+k}\right)$$

So we get

$$\frac{-k+5}{1+k}=1,\frac{k+7}{1+k}=3$$

We can write it as

$$\frac{-k+5}{1+k}=1$$

By cross multiplication,

$-k+5=1+k$

We get

$2k = 4$

$k = 2$

Hence, the line joining the points (-1, 1) and (5, 7) is divided by the line x + y = 4 in the ratio 1: 2.

15. Find the distance of the line 4*x* + 7*y* + 5 = 0 from the point (1, 2) along the line 2*x* – *y* = 0.

Solution:

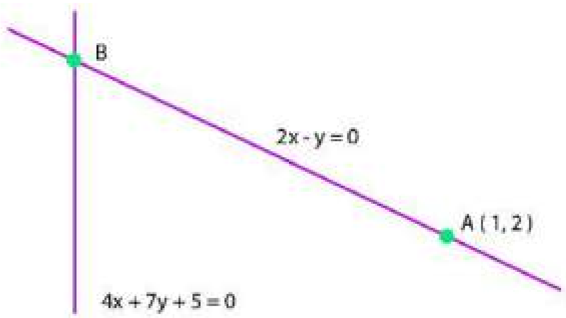
It is given that

2x – y = 0 ….. (1)

4x + 7y + 5 = 0 …… (2)

Here, A (1, 2) is a point on the line (1).

Consider B as the point of intersection of lines (1) and (2).



By solving equations (1) and (2), we get x = -5/18 and y = – 5/9

So, the coordinates of point B are (-5/18, -5/9).

From the distance formula, the distance between A and B

$$AB = \sqrt{\left(1+\frac{5}{18}\right)^2 + \left(2+\frac{5}{9}\right)^2} \text{ units}$$

By taking LCM

$$= \sqrt{\left(\frac{23}{18}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

It can be written as

$$= \sqrt{\left(\frac{23}{2\times 9}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

So we get

$$= \sqrt{\left(\frac{23}{9}\right)^2 \left(\frac{1}{2}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

By taking the common terms out

$$= \sqrt{\left(\frac{23}{9}\right)^2 \left(\frac{1}{4}+1\right)} \text{ units}$$

We get

$$= \frac{23}{9}\sqrt{\frac{5}{4}} \text{ units}$$

$$= \frac{23}{9} \times \frac{\sqrt{5}}{2} \text{ units}$$

So we get

$$= \frac{23\sqrt{5}}{18} \text{ units}$$

Hence, the required distance is

$$\frac{23\sqrt{5}}{18} \text{ units}.$$

16. Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

Solution:

Consider y = mx + c as the line passing through the point (-1, 2).

So, we get

2 = m (-1) + c

By further calculation,

2 = -m + c

c = m + 2

Substituting the value of c

y = mx + m + 2 …… (1)

So the given line is

x + y = 4 ……. (2)

By solving both equations, we get

$x=\frac{2-m}{m+1}$ and $y=\frac{5m+2}{m+1}$

$\left(\frac{2-m}{m+1},\frac{5m+2}{m+1}\right)$ is the point of intersection of lines (1) and (2)

Here the point is at a distance of 3 units from (-1, 2)

From distance formula

$$\sqrt{\left(\frac{2-m}{m+1}+1\right)^2+\left(\frac{5m+2}{m+1}-2\right)^2}=3$$

Squaring on both sides

$$\left(\frac{2-m+m+1}{m+1}\right)^2+\left(\frac{5m+2-2m-2}{m+1}\right)^2=3^2$$

By further calculation

$$\frac{9}{(m+1)^2}+\frac{9m^2}{(m+1)^2}=9$$

Dividing the equation by 9

$$\frac{1+m^2}{(m+1)^2}=1$$

By cross multiplication,

$1 + m^2 = m^2 + 1 + 2m$

So, we get

2m = 0

m = 0

Hence, the slope of the required line must be zero, i.e., the line must be parallel to the x-axis.

17. The hypotenuse of a right-angled triangle has its ends at points (1, 3) and (−4, 1). Find the equation of the legs (perpendicular sides) of the triangle.

Solution:

Consider ABC as the right angles triangle where $\angle C = 90^{\circ}$

Here, infinity such lines are present.

m is the slope of AC

So, the slope of BC = -1/m

Equation of AC –

y – 3 = m (x – 1)

By cross multiplication,

x – 1 = 1/m (y – 3)

Equation of BC –

y – 1 = – 1/m (x + 4)

By cross multiplication,

x + 4 = – m (y – 1)

By considering the values of m, we get

If m = 0,

So, we get

y – 3 = 0, x + 4 = 0

If m = ∞,

So, we get

x – 1 = 0, y – 1 = 0 we get x = 1, y = 1

18. Find the image of the point (3, 8) with respect to the line *x* + 3*y* = 7, assuming the line to be a plane mirror.

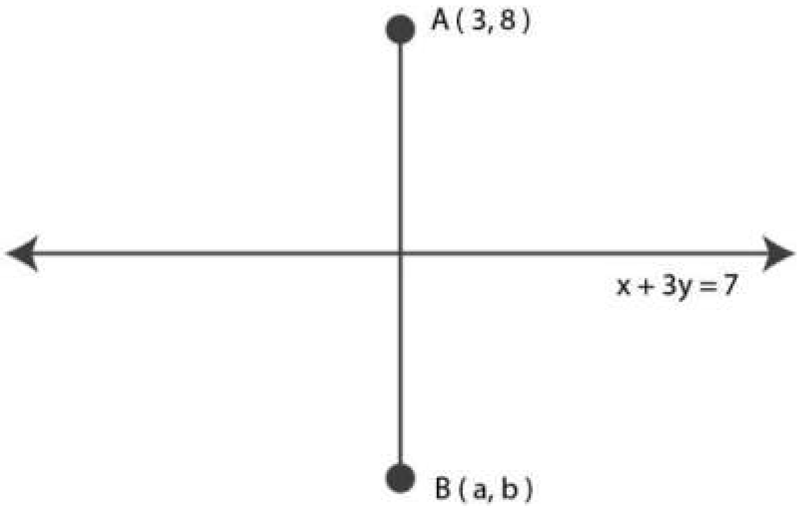
Solution:

It is given that

x + 3y = 7 ….. (1)

Consider B (a, b) as the image of point A (3, 8).

So line (1) is the perpendicular bisector of AB.



Here

$$\text{Slope of AB} = \frac{b-8}{a-3}$$

$$\text{slope of line } (1) = -\frac{1}{3}$$

Line (1) is perpendicular to AB

$$\left(\frac{b-8}{a-3}\right) \times \left(-\frac{1}{3}\right) = -1$$

By further calculation

$$\frac{b-8}{3a-9} = 1$$

By cross multiplication

b – 8 = 3a – 9

3a – b = 1 …… (2)

We know that

$$\text{Mid-point of AB} = \left(\frac{a+3}{2}, \frac{b+8}{2}\right)$$

So the mid-point of line segment AB will satisfy line (1)

From equation (1)

$$\left(\frac{a+3}{2}\right) + 3\left(\frac{b+8}{2}\right) = 7$$

By further calculation

a + 3 + 3b + 24 = 14

On further simplification,

a + 3b = – 13 ….. (3)

By solving equations (2) and (3), we get

a = – 1 and b = – 4

Hence, the image of the given point with respect to the given line is (-1, -4).

19. If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, find the value of m.

Solution:

It is given that

$y = 3x + 1$ …… (1)

$2y = x + 3$ …… (2)

$y = mx + 4$ …… (3)

Here, the slopes of

Line (1), $m_1 = 3$

Line (2), $m_2 = ½$

Line (3), $m_3 = m$

We know that lines (1) and (2) are equally inclined to line (3), which means that the angle between lines (1) and (3) equals the angle between lines (2) and (3).

$$\left|\frac{m_1 - m_3}{1 + m_1 m_3}\right| = \left|\frac{m_2 - m_3}{1 + m_2 m_3}\right|$$

Substituting the values we get

$$\left|\frac{3 - m}{1 + 3m}\right| = \left|\frac{\frac{1}{2} - m}{1 + \frac{1}{2}m}\right|$$

By taking LCM

$$\left|\frac{3 - m}{1 + 3m}\right| = \left|\frac{1 - 2m}{m + 2}\right|$$

It can be written as

$$\frac{3 - m}{1 + 3m} = \pm\left(\frac{1 - 2m}{m + 2}\right)$$

Here

$$\frac{3 - m}{1 + 3m} = \frac{1 - 2m}{m + 2} \text{ or } \frac{3 - m}{1 + 3m} = -\left(\frac{1 - 2m}{m + 2}\right)$$

If

$$\frac{3 - m}{1 + 3m} = \frac{1 - 2m}{m + 2}$$

By cross multiplication

$(3 – m) (m + 2) = (1 – 2m) (1 + 3m)$

On further calculation,

$-m^2 + m + 6 = 1 + m - 6m^2$

So, we get

$5m^2 + 5 = 0$

Dividing the equation by 5,

$m^2 + 1 = 0$

$m = \sqrt{-1}$, which is not real.

Therefore, this case is not possible.

If

$$\frac{3-m}{1+3m} = -\left(\frac{1-2m}{m+2}\right)$$

By cross multiplication

$(3 - m)(m + 2) = -(1 - 2m)(1 + 3m)$

On further calculation

$-m^2 + m + 6 = -(1 + m - 6m^2)$

So we get

$7m^2 - 2m - 7 = 0$

Here we get

$$m = \frac{2 \pm \sqrt{4 - 4(7)(-7)}}{2(7)}$$

By further simplification

$$m = \frac{2 \pm 2\sqrt{1+49}}{14}$$

We can write it as

$$m = \frac{1 \pm 5\sqrt{2}}{7}$$

Hence, the required value of m is

$$\frac{1 \pm 5\sqrt{2}}{7}$$

20. If the sum of the perpendicular distances of a variable point P (*x, y*) from the lines *x* + *y* – 5 = 0 and 3*x* – 2*y* + 7 = 0 is always 10. Show that P must move on a line.

Solution:

It is given that

x + y – 5 = 0 …. (1)

3x – 2y + 7 = 0 ….. (2)

Here the perpendicular distances of P (x, y) from lines (1) and (2) are written as

$$d_1 = \frac{|x+y-5|}{\sqrt{(1)^2+(1)^2}} \text{ and } d_2 = \frac{|3x-2y+7|}{\sqrt{(3)^2+(-2)^2}}$$

So we get

$$d_1 = \frac{|x+y-5|}{\sqrt{2}} \text{ and } d_2 = \frac{|3x-2y+7|}{\sqrt{13}}$$

We know that $d_1 + d_2 = 10$

Substituting the values

$$\frac{|x+y-5|}{\sqrt{2}} + \frac{|3x-2y+7|}{\sqrt{13}} = 10$$

By further calculation

$$\sqrt{13}\,|x+y-5| + \sqrt{2}\,|3x-2y+7| - 10\sqrt{26} = 0$$

It can be written as

$$\sqrt{13}\,(x+y-5) + \sqrt{2}\,(3x-2y+7) - 10\sqrt{26} = 0$$

Now by assuming (x + y – 5) and (3x – 2y + 7) are positive

$$\sqrt{13}x + \sqrt{13}y - 5\sqrt{13} + 3\sqrt{2}x - 2\sqrt{2}y + 7\sqrt{2} - 10\sqrt{26} = 0$$

Taking out the common terms

$x\left(\sqrt{13}+3\sqrt{2}\right) + y\left(\sqrt{13}-2\sqrt{2}\right) + \left(7\sqrt{2}-5\sqrt{13}-10\sqrt{26}\right) = 0$, which is the equation of a line.

In the same way, we can find the equation of the line for any signs of (x + y – 5) and (3x – 2y + 7)

Hence, point P must move on a line.

21. Find the equation of the line which is equidistant from parallel lines 9*x* + 6*y* – 7 = 0 and 3*x* + 2*y* + 6 = 0.

Solution:

It is given that

9x + 6y – 7 = 0 …… (1)

3x + 2y + 6 = 0 …… (2)

Consider P (h, k) be the arbitrary point that is equidistant from lines (1) and (2)

Here the perpendicular distance of P (h, k) from line (1) is written as

$$d_1 = \frac{|9h+6k-7|}{(9)^2+(6)^2} = \frac{|9h+6k-7|}{\sqrt{117}} = \frac{|9h+6k-7|}{3\sqrt{13}}$$

Similarly the perpendicular distance of P (h, k) from line (2) is written as

$$d_2 = \frac{|3h+2k+6|}{\sqrt{(3)^2+(2)^2}} = \frac{|3h+2k+6|}{\sqrt{13}}$$

We know that P (h, k) is equidistant from lines (1) and (2) $d_1 = d_2$

Substituting the values

$$\frac{|9h+6k-7|}{3\sqrt{13}} = \frac{|3h+2k+6|}{\sqrt{13}}$$

By further calculation

$$|9h+6k-7| = 3|3h+2k+6|$$

It can be written as

$$|9h+6k-7| = \pm 3(3h+2k+6)$$

Here,

9h + 6k – 7 = 3 (3h + 2k + 6) or 9h + 6k – 7 = – 3 (3h + 2k + 6)

9h + 6k – 7 = 3 (3h + 2k + 6) is not possible as

9h + 6k – 7 = 3 (3h + 2k + 6)

By further calculation,

– 7 = 18 (which is wrong)

We know that

9h + 6k – 7 = -3 (3h + 2k + 6)

By multiplication,

9h + 6k – 7 = -9h – 6k – 18

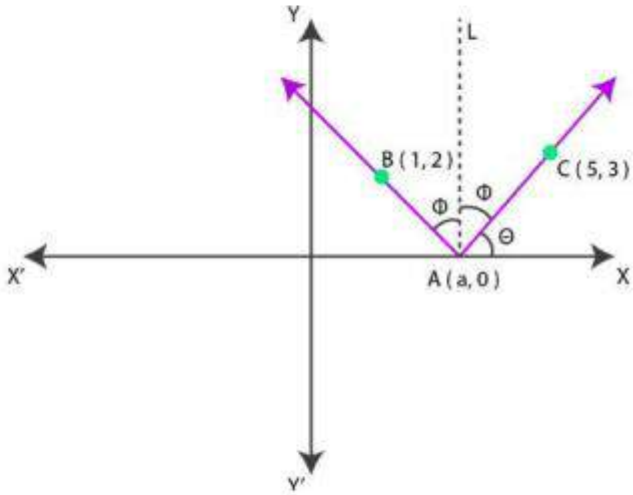
We get

18h + 12k + 11 = 0

Hence, the required equation of the line is 18x + 12y + 11 = 0

22. A ray of light passing through the point (1, 2) reflects on the *x*-axis at point A, and the reflected ray passes through the point (5, 3). Find the coordinates of A.

Solution:



Consider the coordinates of point A as (a, 0).

Construct a line (AL) which is perpendicular to the x-axis.

Here, the angle of incidence is equal to the angle of reflection

$\angle BAL = \angle CAL = \Phi$

$\angle CAX = \theta$

It can be written as

$\angle OAB = 180° – (\theta + 2\Phi) = 180° – [\theta + 2(90° – \theta)]$

On further calculation,

$= 180° – \theta – 180° + 2\theta$

$= \theta$

So, we get

$\angle BAX = 180° – \theta$

$$\text{slope of line AC} = \frac{3-0}{5-a}$$

$$\tan\theta = \frac{3}{5-a} \qquad ...(1)$$

$$\text{Slope of line AB} = \frac{2-0}{1-a}$$

We get

$$\tan(180° - \theta) = \frac{2}{1-a}$$

By further calculation

$$-\tan\theta = \frac{2}{1-a}$$

$$\tan\theta = \frac{2}{a-1} \qquad ...(2)$$

From equations (1) and (2) we get

$$\frac{3}{5-a} = \frac{2}{a-1}$$

By cross multiplication,

3a – 3 = 10 – 2a

We get

a = 13/5

Hence, the coordinates of point A are (13/5, 0).

23. Prove that the product of the lengths of the perpendiculars drawn from points $\left(\sqrt{a^2-b^2}, 0\right)$ and $\left(-\sqrt{a^2-b^2}, 0\right)$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

Solution:

It is given that

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

We can write it as

bx cos θ + ay sin θ – ab = 0 ….. (1)

Here the length of the perpendicular from point $\left(\sqrt{a^2-b^2},0\right)$ to line (1)

$$p_1=\frac{\left|b\cos\theta\left(\sqrt{a^2-b^2}\right)+a\sin\theta(0)-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{\left|b\cos\theta\sqrt{a^2-b^2}-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}} \qquad ...(2)$$

Similarly the length of the perpendicular from point $\left(-\sqrt{a^2-b^2},0\right)$ to line (2)

$$p_2=\frac{\left|b\cos\theta\left(-\sqrt{a^2-b^2}\right)+a\sin\theta(0)-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{\left|b\cos\theta\sqrt{a^2-b^2}+ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}} \qquad ...(3)$$

By multiplying equations (2) and (3) we get

$$p_1p_2=\frac{\left|b\cos\theta\sqrt{a^2-b^2}-ab\right|\left|\left(b\cos\theta\sqrt{a^2-b^2}+ab\right)\right|}{\left(\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}\right)^2}$$

We get

$$=\frac{\left|\left(b\cos\theta\sqrt{a^2-b^2}-ab\right)\left(b\cos\theta\sqrt{a^2-b^2}+ab\right)\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

From the formula

$$=\frac{\left|\left(b\cos\theta\sqrt{a^2-b^2}\right)^2-(ab)^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

By squaring the numerator we get

$$=\frac{\left|b^2\cos^2\theta\left(a^2-b^2\right)-a^2b^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

By expanding using formula

$$=\frac{\left|a^2b^2\cos^2\theta-b^4\cos^2\theta-a^2b^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

Taking out the common terms

$$=\frac{b^2\left|a^2\cos^2\theta-b^2\cos^2\theta-a^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

We get

$$= \frac{b^2 \left| a^2 \cos^2\theta - b^2 \cos^2\theta - a^2 \sin^2\theta - a^2 \cos^2\theta \right|}{b^2 \cos^2\theta + a^2 \sin^2\theta}$$

Here $\sin^2 \theta + \cos^2 \theta = 1$

$$= \frac{b^2 \left| -\left(b^2 \cos^2\theta + a^2 \sin^2\theta \right) \right|}{b^2 \cos^2\theta + a^2 \sin^2\theta}$$

So we get

$$= \frac{b^2 \left(b^2 \cos^2\theta + a^2 \sin^2\theta \right)}{\left(b^2 \cos^2\theta + a^2 \sin^2\theta \right)}$$

$= b^2$

Therefore, it is proved.

24. A person standing at the junction (crossing) of two straight paths represented by the equations 2*x* – 3*y* + 4 = 0 and 3*x* + 4*y* – 5 = 0 wants to reach the path whose equation is 6*x* – 7*y* + 8 = 0 in the least time. Find the equation of the path that he should follow.

Solution:

It is given that

2x – 3y + 4 = 0 …… (1)

3x + 4y – 5 = 0 ……. (2)

6x – 7y + 8 = 0 …… (3)

Here, the person is standing at the junction of the paths represented by lines (1) and (2).

By solving equations (1) and (2), we get

x = – 1/17 and y = 22/17

Hence, the person is standing at point (-1/17, 22/17).

We know that the person can reach path (3) in the least time if they walk along the perpendicular line to (3) from point (-1/17, 22/17)

Here, the slope of line (3) = 6/7

We get the slope of the line perpendicular to the line (3) = -1/ (6/7) = – 7/6

So, the equation of the line passing through (-1/17, 22/17) and having a slope of -7/6 is written as

$$\left(y-\frac{22}{17}\right)=-\frac{7}{6}\left(x+\frac{1}{17}\right)$$

By further calculation,

6 (17y – 22) = – 7 (17x + 1)

By multiplication,

102y – 132 = – 119x – 7

We get

1119x + 102y = 125

Therefore, the path that the person should follow is 119x + 102y = 125