

MISCELLANEOUS EXERCISE

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Prove that:

1.

$$2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0$$

Solution:

$$\text{L.H.S.} = 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

Using the formula

$$\cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$$

So we get

$$= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \left[\frac{\frac{3\pi}{13} + \frac{5\pi}{13}}{2} \right] \cos \left[\frac{\frac{3\pi}{13} - \frac{5\pi}{13}}{2} \right]$$

By further calculation

$$= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \left(\frac{-\pi}{13} \right)$$

We get

$$= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \frac{\pi}{13}$$

Taking out the common terms

$$= 2 \cos \frac{\pi}{13} \left[\cos \frac{9\pi}{13} + \cos \frac{4\pi}{13} \right]$$

It can be written as

$$= 2 \cos \frac{\pi}{13} \left[2 \cos \left(\frac{\frac{9\pi}{13} + \frac{4\pi}{13}}{2} \right) \cos \left(\frac{\frac{9\pi}{13} - \frac{4\pi}{13}}{2} \right) \right]$$

On further calculation

$$= 2 \cos \frac{\pi}{13} \left[2 \cos \frac{\pi}{2} \cos \frac{5\pi}{26} \right]$$

We get

$$= 2 \cos \frac{\pi}{13} \times 2 \times 0 \times \cos \frac{5\pi}{26}$$

$$= 0$$

= RHS

$$2. (\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0$$

Solution:

Consider

$$\text{LHS} = (\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x$$

By further calculation

$$= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x - \cos^2 x$$

Taking out the common terms

$$= \cos 3x \cos x + \sin 3x \sin x - (\cos^2 x - \sin^2 x)$$

Using the formula

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$= \cos(3x - x) - \cos 2x$$

So we get

$$= \cos 2x - \cos 2x$$

$$= 0$$

= RHS

3.

$$(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4 \cos^2 \frac{x+y}{2}$$

Solution:

Consider

$$\text{LHS} = (\cos x + \cos y)^2 + (\sin x - \sin y)^2$$

By expanding using formula we get

$$= \cos^2 x + \cos^2 y + 2 \cos x \cos y + \sin^2 x + \sin^2 y - 2 \sin x \sin y$$

Grouping the terms

$$= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) + 2(\cos x \cos y - \sin x \sin y)$$

Using the formula $\cos(A + B) = (\cos A \cos B - \sin A \sin B)$

$$= 1 + 1 + 2 \cos(x + y)$$

By further calculation

$$= 2 + 2 \cos(x + y)$$

Taking 2 as common

$$= 2 [1 + \cos(x + y)]$$

From the formula $\cos 2A = 2 \cos^2 A - 1$

$$= 2 \left[1 + 2 \cos^2 \left(\frac{x+y}{2} \right) - 1 \right]$$

We get

$$= 4 \cos^2 \left(\frac{x+y}{2} \right)$$

= RHS

4.

$$(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \frac{x-y}{2}$$

Solution:

$$\text{LHS} = (\cos x - \cos y)^2 + (\sin x - \sin y)^2$$

By expanding using formula

$$= \cos^2 x + \cos^2 y - 2 \cos x \cos y + \sin^2 x + \sin^2 y - 2 \sin x \sin y$$

Grouping the terms

$$= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2 (\cos x \cos y + \sin x \sin y)$$

Using the formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$= 1 + 1 - 2 [\cos(x - y)]$$

By further calculation

$$= 2 [1 - \cos(x - y)]$$

From formula $\cos 2A = 1 - 2 \sin^2 A$

$$= 2 \left[1 - \left\{ 1 - 2 \sin^2 \left(\frac{x-y}{2} \right) \right\} \right]$$

We get

$$= 4 \sin^2 \left(\frac{x-y}{2} \right)$$

= RHS

$$5. \sin x + \sin 3x + \sin 5x + \sin 7x = 4 \cos x \cos 2x \sin 4x$$

Solution:

Consider

$$\text{LHS} = \sin x + \sin 3x + \sin 5x + \sin 7x$$

Grouping the terms

$$= (\sin x + \sin 5x) + (\sin 3x + \sin 7x)$$

Using the formula

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)$$

So we get

$$= 2 \sin\left(\frac{x+5x}{2}\right) \cdot \cos\left(\frac{x-5x}{2}\right) + 2 \sin\left(\frac{3x+7x}{2}\right) \cos\left(\frac{3x-7x}{2}\right)$$

By further calculation

$$= 2 \sin 3x \cos(-2x) + 2 \sin 5x \cos(-2x)$$

We get

$$= 2 \sin 3x \cos 2x + 2 \sin 5x \cos 2x$$

Taking out the common terms

$$= 2 \cos 2x [\sin 3x + \sin 5x]$$

Using the formula we can write it as

$$= 2 \cos 2x \left[2 \sin\left(\frac{3x+5x}{2}\right) \cdot \cos\left(\frac{3x-5x}{2}\right) \right]$$

We get

$$= 2 \cos 2x [2 \sin 4x \cos(-x)]$$

$$= 4 \cos 2x \sin 4x \cos x$$

$$= \text{RHS}$$

6.

$$\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$$

Solution:

$$\text{L.H.S.} = \frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)}$$

Using the formula

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right), \quad \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)$$

$$= \frac{\left[2 \sin\left(\frac{7x+5x}{2}\right) \cdot \cos\left(\frac{7x-5x}{2}\right) \right] + \left[2 \sin\left(\frac{9x+3x}{2}\right) \cdot \cos\left(\frac{9x-3x}{2}\right) \right]}{\left[2 \cos\left(\frac{7x+5x}{2}\right) \cdot \cos\left(\frac{7x-5x}{2}\right) \right] + \left[2 \cos\left(\frac{9x+3x}{2}\right) \cdot \cos\left(\frac{9x-3x}{2}\right) \right]}$$

By further calculation

$$= \frac{[2 \sin 6x \cdot \cos x] + [2 \sin 6x \cdot \cos 3x]}{[2 \cos 6x \cdot \cos x] + [2 \cos 6x \cdot \cos 3x]}$$

Taking out the common terms

$$= \frac{2 \sin 6x [\cos x + \cos 3x]}{2 \cos 6x [\cos x + \cos 3x]}$$

We get

$$= \tan 6x$$

$$= \text{RHS}$$

7.

$$\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$$

Solution:

$$\text{LHS} = \sin 3x + \sin 2x - \sin x$$

It can be written as

$$= \sin 3x + (\sin 2x - \sin x)$$

Using the formula

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$= \sin 3x + \left[2 \cos\left(\frac{2x+x}{2}\right) \sin\left(\frac{2x-x}{2}\right) \right]$$

By further simplification

$$= \sin 3x + \left[2 \cos\left(\frac{3x}{2}\right) \sin\left(\frac{x}{2}\right) \right]$$

$$= \sin 3x + 2 \cos \frac{3x}{2} \sin \frac{x}{2}$$

Using formula $\sin 2A = 2 \sin A \cos B$

$$= 2 \sin \frac{3x}{2} \cdot \cos \frac{3x}{2} + 2 \cos \frac{3x}{2} \sin \frac{x}{2}$$

Taking out the common terms

$$= 2 \cos\left(\frac{3x}{2}\right) \left[\sin\left(\frac{3x}{2}\right) + \sin\left(\frac{x}{2}\right) \right]$$

From the formula

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$= 2 \cos\left(\frac{3x}{2}\right) \left[2 \sin\left\{\frac{\left(\frac{3x}{2}\right) + \left(\frac{x}{2}\right)}{2}\right\} \cos\left\{\frac{\left(\frac{3x}{2}\right) - \left(\frac{x}{2}\right)}{2}\right\} \right]$$

By further simplification

$$= 2 \cos\left(\frac{3x}{2}\right) \cdot 2 \sin x \cos\left(\frac{x}{2}\right)$$

We get

$$= 4 \sin x \cos\left(\frac{x}{2}\right) \cos\left(\frac{3x}{2}\right)$$

= RHS

8. Find $\sin x/2$, $\cos x/2$ and $\tan x/2$ in each of the following:

$$\tan x = -\frac{4}{3}, x \text{ in quadrant II}$$

Solution:

It is given that

x is in quadrant II

$$\frac{\pi}{2} < x < \pi$$

Dividing by 2

$$\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Hence, $\sin x/2$, $\cos x/2$ and $\tan x/2$ are all positive.

$$\tan x = -\frac{4}{3}$$

From the formula $\sec^2 x = 1 + \tan^2 x$

Substituting the values

$$\sec^2 x = 1 + (-4/3)^2$$

We get

$$= 1 + 16/9 = 25/9$$

Here

$$\cos^2 x = \frac{9}{25}$$

$$\cos x = \pm \frac{3}{5}$$

Here x is in quadrant II, $\cos x$ is negative.

$$\cos x = -3/5$$

From the formula

$$\cos x = 2 \cos^2 \frac{x}{2} - 1$$

Substituting the values

$$\frac{-3}{5} = 2 \cos^2 \frac{x}{2} - 1$$

By further calculation

$$2 \cos^2 \frac{x}{2} = 1 - \frac{3}{5}$$

$$2 \cos^2 \frac{x}{2} = \frac{2}{5}$$

$$\cos^2 \frac{x}{2} = \frac{1}{5}$$

We get

$$\cos \frac{x}{2} = \frac{1}{\sqrt{5}}$$

$$\cos \frac{x}{2} = \frac{\sqrt{5}}{5}$$

From the formula

$$\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1$$

Substituting the value

$$\sin^2 \frac{x}{2} + \left(\frac{1}{\sqrt{5}} \right)^2 = 1$$

By further calculation

$$\sin^2 \frac{x}{2} = 1 - \frac{1}{5} = \frac{4}{5}$$

We get

$$\sin \frac{x}{2} = \frac{2}{\sqrt{5}}$$

$$\sin \frac{x}{2} = \frac{2\sqrt{5}}{5}$$

Here

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{2}{\sqrt{5}}\right)}{\left(\frac{1}{\sqrt{5}}\right)} = 2$$

Hence, the respective values of $\sin x/2$, $\cos x/2$ and $\tan x/2$ are

$$\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, \text{ and } 2$$

9. $\cos x = -1/3$, x in quadrant III

Solution:

It is given that

x is in quadrant III

$$\pi < x < \frac{3\pi}{2}$$

Dividing by 2

$$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$

Hence, $\cos x/2$ and $\tan x/2$ are negative where $\sin x/2$ is positive.

$$\cos x = -\frac{1}{3}$$

From the formula $\cos x = 1 - 2 \sin^2 x/2$

We get

$$\sin^2 x/2 = (1 - \cos x)/2$$

Substituting the values

$$\sin^2 \frac{x}{2} = \frac{1 - \left(-\frac{1}{3}\right)}{2} = \frac{\left(1 + \frac{1}{3}\right)}{2}$$

We get

$$= \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

Here

$$\sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

Using the formula

$$\cos x = 2 \cos^2 \frac{x}{2} - 1$$

We get

$$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$$

Substituting the values

$$\begin{aligned} &= \frac{1 + \left(-\frac{1}{3}\right)}{2} = \frac{\left(\frac{3-1}{3}\right)}{2} \\ &= \frac{\left(\frac{2}{3}\right)}{2} = \frac{1}{3} \end{aligned}$$

We get

$$\cos \frac{x}{2} = -\frac{1}{\sqrt{3}}$$

By further calculation

$$\cos \frac{x}{2} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$$

Here

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{\left(\frac{-1}{\sqrt{3}}\right)} = -\sqrt{2}$$

Therefore, the respective values of $\sin x/2$, $\cos x/2$ and $\tan x/2$ are

$$\frac{\sqrt{6}}{3}, \frac{-\sqrt{3}}{3}, \text{ and } -\sqrt{2}$$

10. $\sin x = 1/4$, x in quadrant II

Solution:

It is given that

x is in quadrant II

$$\frac{\pi}{2} < x < \pi$$

Dividing by 2

$$\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Hence, $\sin x/2$, $\cos x/2$ and $\tan x/2$ are positive.

$$\sin x = \frac{1}{4}$$

From the formula $\cos^2 x = 1 - \sin^2 x$

We get

$$\cos^2 x = 1 - (1/4)^2$$

Substituting the values

$$\cos^2 x = 1 - 1/16 = 15/16$$

We get

$$\cos x = -\frac{\sqrt{15}}{4}$$

Here

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$$

Substituting the values

$$= \frac{1 - \left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4 + \sqrt{15}}{8}$$

We get

$$\sin \frac{x}{2} = \sqrt{\frac{4 + \sqrt{15}}{8}}$$

Multiplying and dividing by 2

$$= \sqrt{\frac{4+\sqrt{15}}{8} \times \frac{2}{2}}$$

By further calculation

$$= \sqrt{\frac{8+2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8+2\sqrt{15}}}{4}$$

Here

$$\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$$

By substituting the values

$$= \frac{1 + \left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4 - \sqrt{15}}{8}$$

We get

$$\cos \frac{x}{2} = \sqrt{\frac{4-\sqrt{15}}{8}}$$

By multiplying and dividing by 2

$$= \sqrt{\frac{4-\sqrt{15}}{8} \times \frac{2}{2}}$$

It can be written as

$$= \sqrt{\frac{8-2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8-2\sqrt{15}}}{4}$$

We know that

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}$$

Substituting the values

$$= \frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4} \right)}{\left(\frac{\sqrt{8-2\sqrt{15}}}{4} \right)} = \frac{\sqrt{8+2\sqrt{15}}}{\sqrt{8-2\sqrt{15}}}$$

By multiplying and dividing the terms

$$= \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}}} \times \frac{8+2\sqrt{15}}{8+2\sqrt{15}}$$

We get

$$= \sqrt{\frac{(8+2\sqrt{15})^2}{64-60}} = \frac{8+2\sqrt{15}}{2}$$

$$= 4 + \sqrt{15}$$

Therefore, the respective values of $\sin x/2$, $\cos x/2$ and $\tan x/2$ are

$$\frac{\sqrt{8+2\sqrt{15}}}{4}, \frac{\sqrt{8-2\sqrt{15}}}{4} \text{ and } 4 + \sqrt{15}$$