EXERCISE 4.1 PAGE: 94

Prove the following by using the principle of mathematical induction for all n ∈ N:

1.

$$1+3+3^2+\ldots+3^{n-1}=\frac{(3^n-1)}{2}$$

Solution:

We can write the given statement as

$$P(n): 1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{(3^n-1)}{2}$$

If n = 1 we get

$$P(1): 1 = \frac{(3^1-1)}{2}=\frac{3-1}{2}=\frac{2}{2}=1$$

Which is true.

Consider P (k) be true for some positive integer k

$$1+3+3^2+\ldots+3^{k-1}=\frac{(3^k-1)}{2} \quad \ldots(i)$$

Now let us prove that P (k + 1) is true.

Here

$$1 + 3 + 3^2 + \ldots + 3^{k-1} + 3^{(k+1)-1} = (1 + 3 + 3^2 + \ldots + 3^{k-1}) + 3^k$$

By using equation (i)

$$=\frac{(3^k-1)}{2}+3^k$$

Taking LCM

$$=\frac{(3^k-1)+2.3^k}{2}$$

Taking the common terms out

$$= \frac{(1+2)3^k - 1}{2}$$

We get

$$= \frac{3.3^k - 1}{2}$$

$$= \frac{3^{k+1} - 1}{2}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

2.

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Solution:

We can write the given statement as

$$P(n):\ 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

If n = 1 we get

$$P(1):\ 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{1.2}{2}\right)^2 = 1^2 = 1$$

Which is true.

Consider P (k) be true for some positive integer k

$$1^3 + 2^3 + 3^3 + \ldots. + k^3 = \left(\frac{k(k+1)}{2}\right)^2 \qquad \ldots (i)$$

Now let us prove that P (k + 1) is true.

Here

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 = (1^3 + 2^3 + 3^3 + \ldots\ldots.. + k^3) + (k + 1)^3$

By using equation (i)

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

So we get

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

Taking LCM

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

Taking the common terms out

$$=\frac{(k+1)^2\left\{k^2+4(k+1)\right\}}{4}$$

We get

$$=\frac{(k+1)^2\left\{k^2+4k+4\right\}}{4}$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

By expanding using formula

$$=\frac{(k+1)^2(k+1+1)^2}{4}$$

$$=\left(\frac{(k+1)(k+1+1)}{2}\right)^2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

3.

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+...+\frac{1}{(1+2+3+...n)}=\frac{2n}{(n+1)}$$

Solution:

We can write the given statement as

$$P(n):\ 1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots n}=\frac{2n}{n+1}$$

If n = 1 we get

$$P(1):\ 1=\frac{2.1}{1+1}=\frac{2}{2}=1$$

Which is true.

Consider P (k) be true for some positive integer k

$$1+\frac{1}{1+2}+\ldots+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}=\frac{2k}{k+1} \qquad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+k}+\frac{1}{1+2+3+...+k+(k+1)}$$
$$=\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...k}\right)+\frac{1}{1+2+3+...+k+(k+1)}$$

By using equation (i)

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+...+k+(k+1)}$$

We know that

$$1+2+3+...+n=\frac{n(n+1)}{2}$$

So we get

$$=\frac{2k}{k+1}+\frac{1}{\left(\frac{(k+1)(k+1+1)}{2}\right)}$$

It can be written as

$$=\frac{2k}{(k+1)}+\frac{2}{(k+1)(k+2)}$$

Taking the common terms out

$$=\frac{2}{(k+1)}\left(k+\frac{1}{k+2}\right)$$

By taking LCM

$$=\frac{2}{k+1}\left(\frac{k(k+2)+1}{k+2}\right)$$

We get

$$=\frac{2}{(k+1)}\left(\frac{k^2+2k+1}{k+2}\right)$$
$$=\frac{2\cdot(k+1)^2}{(k+1)(k+2)}$$
$$=\frac{2(k+1)}{(k+2)}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

4.

$$1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

Solution:

We can write the given statement as

$$P(n): 1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

If n = 1 we get

$$P(1): 1.2.3 = 6 = \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1.2.3.4}{4} = 6$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.2.3 + 2.3.4 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4} \quad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$1.2.3 + 2.3.4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \{1.2.3 + 2.3.4 + \ldots + k(k+1)(k+2)\} + (k+1)(k+2)(k+3)$

By using equation (i)

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

So we get

$$= (k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$

It can be written as

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

By further simplification

$$= \frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

5.

$$1.3 + 2.3^2 + 3.3^3 + ... + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

Solution:

We can write the given statement as

$$P(n): 1.3+2.3^2+3.3^3+...+n3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

If n = 1 we get

$$P(1): 1.3 = 3=\frac{(2.1-1)3^{1+1}+3}{4}=\frac{3^2+3}{4}=\frac{12}{4}=3$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.3+2.3^2+3.3^3+...+k3^k=\frac{(2k-1)3^{k+1}+3}{4} \qquad ... (i)$$

Now let us prove that P (k + 1) is true.

Here

$1.3 + 2.3^2 + 3.3^3 + \ldots + k3^k + (k+1)\,3^{k+1} = = (1.3 + 2.3^2 + 3.3^3 + \ldots + k.3^k) + (k+1)\,3^{k+1}$

By using equation (i)

$$=\frac{(2k-1)3^{k+1}+3}{4}+(k+1)3^{k+1}$$

By taking LCM

$$=\frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

Taking the common terms out

$$=\frac{3^{k+1}\{2k-1+4(k+1)\}+3}{4}$$

By further simplification

$$=\frac{3^{k+1}\{6k+3\}+3}{4}$$

Taking 3 as common

$$=\frac{3^{k+1}.3\{2k+1\}+3}{4}$$

$$=\frac{3^{(k+1)+1}\{2k+1\}+3}{4}$$

$$=\frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

6.

$$1.2+2.3+3.4+...+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

Solution:

We can write the given statement as

$$P(n):\ 1.2+2.3+3.4+\ldots+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

If n = 1 we get

$$P(1):\ 1.2=2=\frac{1(1+1)(1+2)}{3}=\frac{1.2.3}{3}=2$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.2+2.3+3.4+\ldots..+k.(k+1)=\left[\frac{k(k+1)(k+2)}{3}\right] \quad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$$1.2+2.3+3.4+\ldots+k.(k+1)+(k+1).(k+2)=[1.2+2.3+3.4+\ldots+k.(k+1)]+(k+1).(k+2)$$

By using equation (i)

$$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)$$

We can write it as

$$=(k+1)(k+2)\left(\frac{k}{3}+1\right)$$

We get

$$=\frac{(k+1)(k+2)(k+3)}{3}$$

By further simplification

$$=\frac{(k+1)(k+1+1)(k+1+2)}{3}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

7.

$$1.3 + 3.5 + 5.7 + ... + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$$

Solution:

We can write the given statement as

P(n): $1.3 + 3.5 + 5.7 + ... + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$

If n = 1 we get

$$P(1): 1.3 = 3 = \frac{1(4.1^2+6.1-1)}{3} = \frac{4+6-1}{3} = \frac{9}{3} = 3$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.3 + 3.5 + 5.7 + ... + (2k-1)(2k+1) = \frac{k(4k^2+6k-1)}{3} \qquad ... (i)$$

Now let us prove that P (k + 1) is true.

Here

$(1.3 + 3.5 + 5.7 + ... + (2k-1)(2k+1) + \{2(k+1)-1\} \{2(k+1)+1\}$

By using equation (i)

$$= \frac{k(4k^2+6k-1)}{3} + (2k+2-1)(2k+2+1)$$

$$= \frac{k\left(4k^2+6k-1\right)}{3} + (2k+2-1)(2k+2+1)$$

On further calculation

$$= \frac{k\left(4k^2+6k-1\right)}{3} + (2k+1)(2k+3)$$

By multiplying the terms

$$= \frac{k\left(4k^2+6k-1\right)}{3} + \left(4k^2+8k+3\right)$$

Taking LCM

$$= \frac{k\left(4k^2+6k-1\right)+3\left(4k^2+8k+3\right)}{3}$$

By further simplification

$$= \frac{4k^3+6k^2-k+12k^2+24k+9}{3}$$

So we get

$$= \frac{4k^3+18k^2+23k+9}{3}$$

It can be written as

$$= \frac{4k^3+14k^2+9k+4k^2+14k+9}{3}$$

$$= \frac{k\left(4k^2+14k+9\right)+1\left(4k^2+14k+9\right)}{3}$$

Separating the terms

$$= \frac{(k+1)\left\{4k^2+8k+4+6k+6-1\right\}}{3}$$

Taking the common terms out

$$= \frac{(k+1)\left\{4\left(k^2+2k+1\right)+6(k+1)-1\right\}}{3}$$

Using the formula

$$= \frac{(k+1)\left\{4(k+1)^2 + 6(k+1) - 1\right\}}{3}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

8. $1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\, 2^{n+1} + 2$

Solution:

We can write the given statement as

P (n): $1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\, 2^{n+1} + 2$

If n = 1 we get

P (1): $1.2 = 2 = (1 - 1)\, 2^{1+1} + 2 = 0 + 2 = 2$

Which is true.

Consider P (k) be true for some positive integer k

$1.2 + 2.2^2 + 3.2^2 + \ldots + k.2^k = (k - 1)\, 2^{k+1} + 2$ … (i)

Now let us prove that P (k + 1) is true.

Here

$$\left\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k\right\} + (k+1) \cdot 2^{k+1}$$

By using equation (i)

$$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$

Taking the common terms out

$$= 2^{k+1}\left\{(k-1) + (k+1)\right\} + 2$$

So we get

$$= 2^{k+1}.2k + 2$$

It can be written as

$$= k.2^{(k+1)+1} + 2$$

$$= \left\{(k+1) - 1\right\} 2^{(k+1)+1} + 2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

9.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

If n = 1 we get

$$P(1): \frac{1}{2}=1-\frac{1}{2^1}=\frac{1}{2}$$

Which is true.

Consider P (k) be true for some positive integer k

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots.+\frac{1}{2^k}=1-\frac{1}{2^k} \qquad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots.+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

By using equation (i)

$$=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

We can write it as

$$=1-\frac{1}{2^k}+\frac{1}{2.2^k}$$

Taking the common terms out

$$=1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$$

So we get

$$=1-\frac{1}{2^k}\left(\frac{1}{2}\right)$$

It can be written as

$$=1-\frac{1}{2^{k+1}}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

10.

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

If n = 1 we get

$$P(1)=\frac{1}{2.5}=\frac{1}{10}=\frac{1}{6.1+4}=\frac{1}{10}$$

Which is true.

Consider P (k) be true for some positive integer k

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4} \qquad \ldots \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots\ldots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

By using equation (i)

$$=\frac{k}{6k+4}+\frac{1}{(3k+3-1)(3k+3+2)}$$

By simplification of terms

$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$

Taking 2 as common

$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

Taking the common terms out

$$=\frac{1}{(3k+2)}\left(\frac{k}{2}+\frac{1}{3k+5}\right)$$

Taking LCM

$$= \frac{1}{(3k+2)}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$$

By multiplication

$$= \frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$$

Separating the terms

$$= \frac{1}{(3k+2)}\left(\frac{(3k+2)(k+1)}{2(3k+5)}\right)$$

By further calculation

$$= \frac{(k+1)}{6k+10}$$

So we get

$$= \frac{(k+1)}{6(k+1)+4}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

11.

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+...+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

If n = 1 we get

$$P(1): \frac{1}{1\cdot2\cdot3}=\frac{1\cdot(1+3)}{4(1+1)(1+2)}=\frac{1\cdot4}{4\cdot2\cdot3}=\frac{1}{1\cdot2\cdot3}$$

Which is true.

Consider P (k) be true for some positive integer k

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+...+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)} \quad ... \text{(i)}$$

Now let us prove that P (k + 1) is true.

Here

$$\left[\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+.....+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

By using equation (i)

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

Taking out the common terms

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4}+\frac{1}{k+3}\right\}$$

Taking LCM

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2+4}{4(k+3)}\right\}$$

Expanding using formula

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k^2+6k+9)+4}{4(k+3)}\right\}$$

By further calculation

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+6k^2+9k+4}{4(k+3)}\right\}$$

We can write it as

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^2+k+4k^2+8k+4}{4(k+3)}\right\}$$

Taking the common terms

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\left(k^2+2k+1\right)+4\left(k^2+2k+1\right)}{4(k+3)}\right\}$$

We get

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+1)^2+4(k+1)^2}{4(k+3)}\right\}$$

Here

$$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

12.

$$a+ar+ar^2+\ldots+ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

Solution:

We can write the given statement as

$$P(n): a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

If n = 1 we get

$$P(1): a = \frac{a(r^1 - 1)}{(r - 1)} = a$$

Which is true.

Consider P (k) be true for some positive integer k

$$a + ar + ar^2 + ... + ar^{k-1} = \frac{a(r^k - 1)}{r - 1} \quad ... (i)$$

Now let us prove that P (k + 1) is true.

Here

$$\left\{a + ar + ar^2 + ... + ar^{k-1}\right\} + ar^{(k+1)-1}$$

By using equation (i)

$$= \frac{a(r^k - 1)}{r - 1} + ar^k$$

Taking LCM

$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$$

Multiplying the terms

$$= \frac{a\left(r^k - 1\right) + ar^{k+1} - ar^k}{r - 1}$$

So we get

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r - 1}$$

By further simplification

$$= \frac{ar^{k+1} - a}{r - 1}$$

Taking the common terms out

$$= \frac{a\left(r^{k+1} - 1\right)}{r - 1}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

13.

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2$$

Solution:

We can write the given statement as

$$P(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

If n = 1 we get

$$P(1):\left(1+\frac{3}{1}\right)=4=(1+1)^2=2^2=4,$$

Which is true.

Consider P (k) be true for some positive integer k

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)=(k+1)^2 \qquad \ldots (1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{\{2(k+1)+1\}}{(k+1)^2}\right\}$$

By using equation (i)

$$=(k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right)$$

Taking LCM

$$=(k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$$

So we get

$$=(k+1)^2+2(k+1)+1$$

By further simplification

$$=\{(k+1)+1\}^2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

14.

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

Solution:

We can write the given statement as

$$P(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

If n = 1 we get

$$P(1):\left(1+\frac{1}{1}\right)=2=(1+1)$$

Which is true.

Consider P (k) be true for some positive integer k

$$P(k):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)=(k+1) \qquad \ldots (1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

By using equation (i)

$$=(k+1)\left(1+\frac{1}{k+1}\right)$$

Taking LCM

$$=(k+1)\left(\frac{(k+1)+1}{(k+1)}\right)$$

By further simplification

= (k + 1) + 1

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

15.

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Solution:

We can write the given statement as

$$P(n) = 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

If n = 1 we get

$$P(1) = 1^2 = 1 = \frac{1(2.1-1)(2.1+1)}{3} = \frac{1.1.3}{3} = 1.$$

Which is true.

Consider P (k) be true for some positive integer k

$$P(k) = 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \ldots (1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left\{1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2\right\} + \left\{2(k+1)-1\right\}^2$$

By using equation (i)

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+2-1)^2$$

So we get

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

Taking LCM

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

Taking the common terms out

$$= \frac{(2k+1)\{k(2k-1) + 3(2k+1)\}}{3}$$

By further simplification

$$= \frac{(2k+1)\{2k^2 - k + 6k + 3\}}{3}$$

So we get

$$= \frac{(2k+1)\{2k^2 + 5k + 3\}}{3}$$

We can write it as

$$= \frac{(2k+1)\{2k^2 + 2k + 3k + 3\}}{3}$$

Splitting the terms

$$= \frac{(2k+1)\{2k(k+1) + 3(k+1)\}}{3}$$

We get

$$= \frac{(2k+1)(k+1)(2k+3)}{3}$$

$$= \frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

16.

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

If n = 1 we get

$$P(1)=\frac{1}{1.4}=\frac{1}{3.1+1}=\frac{1}{4}=\frac{1}{1.4}$$

Which is true.

Consider P (k) be true for some positive integer k

$$P(k)=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \qquad \ldots (1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

By using equation (i)

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

So we get

$$=\frac{1}{(3k+1)}\left\{k+\frac{1}{(3k+4)}\right\}$$

Taking LCM

$$= \frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$$

Multiplying the terms

$$= \frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$$

It can be written as

$$= \frac{1}{(3k+1)}\left\{\frac{3k^2+3k+k+1}{(3k+4)}\right\}$$

Separating the terms

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

By further calculation

$$= \frac{(k+1)}{3(k+1)+1}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

17.

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

If n = 1 we get

$$P(1): \frac{1}{3.5}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}$$

Which is true.

Consider P (k) be true for some positive integer k

$$P(k): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)} \quad ...(1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

By using equation (i)

$$= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)}$$

So we get

$$= \frac{1}{(2k+3)}\left[\frac{k}{3} + \frac{1}{(2k+5)}\right]$$

Taking LCM

$$= \frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{3(2k+5)}\right]$$

Multiplying the terms

$$= \frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$$

It can be written as

$$= \frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{3(2k+5)}\right]$$

Separating the terms

$$= \frac{1}{(2k+3)}\left[\frac{2k(k+1)+3(k+1)}{3(2k+5)}\right]$$

By further calculation

$$= \frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$= \frac{(k+1)}{3\{2(k+1)+3\}}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

18.

$$1+2+3+\ldots+n < \frac{1}{8}(2n+1)^2$$

Solution:

We can write the given statement as

$$P(n): 1+2+3+...+n < \frac{1}{8}(2n+1)^2$$

If n = 1 we get

$$1 < \frac{1}{8}(2.1+1)^2 = \frac{9}{8}$$

Which is true.

Consider P (k) be true for some positive integer k

$$1+2+...+k < \frac{1}{8}(2k+1)^2 \qquad ... (1)$$

Now let us prove that P (k + 1) is true.

Here

$$(1+2+...+k)+(k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$$

By using equation (i)

$$< \frac{1}{8}\{(2k+1)^2 + 8(k+1)\}$$

Expanding terms using formula

$$< \frac{1}{8}\{4k^2 + 4k + 1 + 8k + 8\}$$

By further calculation

$$< \frac{1}{8}\{4k^2 + 12k + 9\}$$

So we get

$$< \frac{1}{8}(2k+3)^2$$

$$< \frac{1}{8}\{2(k+1)+1\}^2$$

$$(1+2+3+...+k)+(k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

19. $n\,(n + 1)\,(n + 5)$ **is a multiple of 3**

Solution:

We can write the given statement as

P (n): n (n + 1) (n + 5), which is a multiple of 3

If n = 1 we get

1 (1 + 1) (1 + 5) = 12, which is a multiple of 3

Which is true.

Consider P (k) be true for some positive integer k

k (k + 1) (k + 5) is a multiple of 3

k (k + 1) (k + 5) = 3m, where m ∈ N …… (1)

Now let us prove that P (k + 1) is true.

Here

(k + 1) {(k + 1) + 1} {(k + 1) + 5}

We can write it as

= (k + 1) (k + 2) {(k + 5) + 1}

By multiplying the terms

= (k + 1) (k + 2) (k + 5) + (k + 1) (k + 2)

So we get

= {k (k + 1) (k + 5) + 2 (k + 1) (k + 5)} + (k + 1) (k + 2)

Substituting equation (1)

= 3m + (k + 1) {2 (k + 5) + (k + 2)}

By multiplication

= 3m + (k + 1) {2k + 10 + k + 2}

On further calculation

= 3m + (k + 1) (3k + 12)

Taking 3 as common

= 3m + 3 (k + 1) (k + 4)

We get

= 3 {m + (k + 1) (k + 4)}

= 3 × q where q = {m + (k + 1) (k + 4)} is some natural number

(k + 1) {(k + 1) + 1} {(k + 1) + 5} is a multiple of 3

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

20. $10^{2n-1} + 1$ is divisible by 11

Solution:

We can write the given statement as

P (*n*): $10^{2n-1} + 1$ is divisible by 11

If n = 1 we get

P (1) = $10^{2.1-1} + 1 = 11$, which is divisible by 11

Which is true.

Consider P (k) be true for some positive integer k

$10^{2k-1} + 1$ is divisible by 11

$10^{2k-1} + 1 = 11m$, where $m \in N$ …… (1)

Now let us prove that P (k + 1) is true.

Here

$10^{2(k+1)-1} + 1$

We can write it as

$= 10^{2k+2-1} + 1$

$= 10^{2k+1} + 1$

By addition and subtraction of 1

$= 10^2 (10^{2k-1} + 1 - 1) + 1$

We get

$= 10^2 (10^{2k-1} + 1) - 10^2 + 1$

Using equation 1 we get

$= 10^2. 11m – 100 + 1$

$= 100 \times 11m – 99$

Taking out the common terms

$= 11 (100m – 9)$

$= 11 r$, where $r = (100m – 9)$ is some natural number

$10^{2(k + 1) – 1} + 1$ is divisible by 11

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

21. $x^{2n} – y^{2n}$ is divisible by $x + y$

Solution:

We can write the given statement as

P (*n*): $x^{2n} – y^{2n}$ is divisible by $x + y$

If n = 1 we get

$P (1) = x^{2 \times 1} – y^{2 \times 1} = x^2 – y^2 = (x + y) (x – y)$, which is divisible by $(x + y)$

Which is true.

Consider P (k) be true for some positive integer k

$x^{2k} – y^{2k}$ is divisible by $x + y$

$x^{2k} – y^{2k} = m (x + y)$, where $m \in N$ …… (1)

Now let us prove that P (k + 1) is true.

Here

$x^{2(k + 1)} – y^{2(k + 1)}$

We can write it as

$= x^{2k} . x^2 – y^{2k} . y^2$

By adding and subtracting y^{2k} we get

$= x^2 (x^{2k} – y^{2k} + y^{2k}) – y^{2k}. y^2$

From equation (1) we get

$= x^2 \{m (x + y) + y^{2k}\} – y^{2k}. y^2$

By multiplying the terms

$= m (x + y) x^2 + y^{2k}. x^2 – y^{2k}. y^2$

Taking out the common terms

$= m (x + y) x^2 + y^{2k} (x^2 – y^2)$

Expanding using formula

$= m (x + y) x^2 + y^{2k} (x + y) (x – y)$

So we get

$= (x + y) \{mx^2 + y^{2k} (x – y)\}$, which is a factor of $(x + y)$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

22. $3^{2n+2} – 8n – 9$ is divisible by 8

Solution:

We can write the given statement as

P (n): $3^{2n+2} – 8n – 9$ is divisible by 8

If n = 1 we get

P (1) = $3^{2 \times 1 + 2} – 8 \times 1 – 9 = 64$, which is divisible by 8

Which is true.

Consider P (k) be true for some positive integer k

$3^{2k+2} – 8k – 9$ is divisible by 8

$3^{2k+2} – 8k – 9 = 8m$, where $m \in N$ …… (1)

Now let us prove that P (k + 1) is true.

Here

$3^{2(k+1)+2} – 8 (k + 1) – 9$

We can write it as

$= 3^{2k+2} . 3^2 – 8k – 8 – 9$

By adding and subtracting 8k and 9 we get

$= 3^2 (3^{2k+2} – 8k – 9 + 8k + 9) – 8k – 17$

On further simplification

$= 3^2 (3^{2k+2} – 8k – 9) + 3^2 (8k + 9) – 8k – 17$

From equation (1) we get

$= 9. 8m + 9 (8k + 9) – 8k – 17$

By multiplying the terms

$= 9. 8m + 72k + 81 – 8k – 17$

So we get

$= 9. 8m + 64k + 64$

By taking out the common terms

$= 8 (9m + 8k + 8)$

= 8r, where r = (9m + 8k + 8) is a natural number

So $3^{2(k+1)+2} – 8 (k + 1) – 9$ is divisible by 8

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

23. $41^n – 14^n$ is a multiple of 27

Solution:

We can write the given statement as

P (n):$41^n – 14^n$is a multiple of 27

If n = 1 we get

P (1) = $41^1 – 14^1 = 27$, which is a multiple by 27

Which is true.

Consider P (k) be true for some positive integer k

$41^k – 14^k$is a multiple of 27

$41^k – 14^k = 27m$, where $m \in N$ …… (1)

Now let us prove that P (k + 1) is true.

Here

$41^{k+1} – 14^{k+1}$

We can write it as

$= 41^k. 41 – 14^k. 14$

By adding and subtracting 14^k we get

$= 41 (41^k – 14^k + 14^k) – 14^k. 14$

On further simplification

$= 41 (41^k – 14^k) + 41. 14^k – 14^k. 14$

From equation (1) we get

$= 41. 27m + 14^k (41 – 14)$

By multiplying the terms

$= 41. 27m + 27. 14^k$

By taking out the common terms

$= 27 (41m – 14^k)$

$= 27r$, where $r = (41m – 14^k)$ is a natural number

So $41^{k+1} – 14^{k+1}$ is a multiple of 27

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.

24. $(2n +7) < (n + 3)^2$

Solution:

We can write the given statement as

P(n): $(2n +7) < (n + 3)2$

If n = 1 we get

$2.1 + 7 = 9 < (1 + 3)^2 = 16$

Which is true.

Consider P (k) be true for some positive integer k

$(2k + 7) < (k + 3)^2$ … (1)

Now let us prove that P (k + 1) is true.

Here

{2 (k + 1) + 7} = (2k + 7) + 2

We can write it as

= {2 (k + 1) + 7}

From equation (1) we get

$(2k + 7) + 2 < (k + 3)^2 + 2$

By expanding the terms

$2 (k + 1) + 7 < k^2 + 6k + 9 + 2$

On further calculation

$2 (k + 1) + 7 < k^2 + 6k + 11$

Here $k^2 + 6k + 11 < k^2 + 8k + 16$

We can write it as

$2 (k + 1) + 7 < (k + 4)^2$

$2 (k + 1) + 7 < \{(k + 1) + 3\}^2$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers, i.e., n.