EXERCISE 8.1

Expand each of the expressions in Exercises 1 to 5.

1. $(1 - 2x)^5$

Solution:

From binomial theorem expansion, we can write as

$(1 - 2x)^5$

$= {}^5C_0 (1)^5 - {}^5C_1 (1)^4 (2x) + {}^5C_2 (1)^3 (2x)^2 - {}^5C_3 (1)^2 (2x)^3 + {}^5C_4 (1)^1 (2x)^4 - {}^5C_5 (2x)^5$

$= 1 - 5 (2x) + 10 (4x)^2 - 10 (8x^3) + 5 (16 x^4) - (32 x^5)$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

2. $\left(\frac{2}{x} - \frac{x}{2}\right)^5$

Solution:

From the binomial theorem, the given equation can be expanded as

$$\left(\frac{2}{x} - \frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2$$
$$- {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$$
$$= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32}$$
$$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$$

3. $(2x - 3)^6$

Solution:

From the binomial theorem, the given equation can be expanded as

$$(2x-3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_1(2x)^4(3)^2 - {}^4C_3(2x)^3(3)^3$$
$$= 64x^6 - 6\left(32x^5\right)(3) + 15\left(16x^4\right)(9) - 20\left(8x^3\right)(27)$$
$$+ 15\left(4x^2\right)(81) - 6(2x)(243) + 729$$
$$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$$

4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

Solution:

From the binomial theorem, the given equation can be expanded as

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 + {}^3C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^3C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2$$
$$= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$
$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^3}$$

5. $\left(x+\frac{1}{x}\right)^6$

Solution:

From the binomial theorem, the given equation can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)'\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2$$
$$+ {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_3(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$
$$= x^4 + 6(x)^3\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6}$$
$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

6. Using the binomial theorem, find $(96)^3$.

Solution:

Given $(96)^3$

96 can be expressed as the sum or difference of two numbers, and then the binomial theorem can be applied.

The given question can be written as 96 = 100 – 4

$(96)^3 = (100 - 4)^3$

$= {}^3C_0\,(100)^3 - {}^3C_1\,(100)^2\,(4) - {}^3C_2\,(100)\,(4)^2 - {}^3C_3\,(4)^3$

$= (100)^3 - 3\,(100)^2\,(4) + 3\,(100)\,(4)^2 - (4)^3$

= 1000000 – 120000 + 4800 – 64

= 884736

7. Using the binomial theorem, find $(102)^5$.

Solution:

Given $(102)^5$

102 can be expressed as the sum or difference of two numbers, and then the binomial theorem can be applied.

The given question can be written as 102 = 100 + 2

$(102)^5 = (100 + 2)^5$

$= {}^5C_0\,(100)^5 + {}^5C_1\,(100)^4\,(2) + {}^5C_2\,(100)^3\,(2)^2 + {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 + {}^5C_5\,(2)^5$

$= (100)^5 + 5\,(100)^4\,(2) + 10\,(100)^3\,(2)^2 + 5\,(100)\,(2)^3 + 5\,(100)\,(2)^4 + (2)^5$

= 1000000000 + 1000000000 + 40000000 + 80000 + 8000 + 32

= 11040808032

8. Using the binomial theorem, find $(101)^4$.

Solution:

Given $(101)^4$

101 can be expressed as the sum or difference of two numbers, and then the binomial theorem can be applied.

The given question can be written as 101 = 100 + 1

$(101)^4 = (100 + 1)^4$

$= {}^4C_0\,(100)^4 + {}^4C_1\,(100)^3\,(1) + {}^4C_2\,(100)^2\,(1)^2 + {}^4C_3\,(100)\,(1)^3 + {}^4C_4\,(1)^4$

$= (100)^4 + 4\,(100)^3 + 6\,(100)^2 + 4\,(100) + (1)^4$

= 100000000 + 4000000 + 60000 + 400 + 1

= 104060401

9. Using the binomial theorem, find $(99)^5$m.

Solution:

Given $(99)^5$

99 can be written as the sum or difference of two numbers then the binomial theorem can be applied.

The given question can be written as 99 = 100 -1

$(99)^5 = (100 – 1)^5$

$= {}^5C_0\ (100)^5 – {}^5C_1\ (100)^4\ (1) + {}^5C_2\ (100)^3\ (1)^2 – {}^5C_3\ (100)^2\ (1)^3 + {}^5C_4\ (100)\ (1)^4 – {}^5C_5\ (1)^5$

$= (100)^5 – 5\ (100)^4 + 10\ (100)^3 – 10\ (100)^2 + 5\ (100) – 1$

= 1000000000 – 5000000000 + 10000000 – 100000 + 500 – 1

= 9509900499

10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution:

By splitting the given 1.1 and then applying the binomial theorem, the first few terms of $(1.1)^{10000}$ can be obtained as

$(1.1)^{10000} = (1 + 0.1)^{10000}$

$= (1 + 0.1)^{10000}\ C_1\ (1.1)$ + other positive terms

$= 1 + 10000 \times 1.1$ + other positive terms

= 1 + 11000 + other positive terms

> 1000

$(1.1)^{10000} > 1000$

11. Find $(a + b)^4 – (a – b)^4$. Hence, evaluate

$$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4 .$$

Solution:

Using the binomial theorem, the expression $(a + b)^4$ and $(a – b)^4$ can be expanded

$(a + b)^4 = {}^4C_0\ a^4 + {}^4C_1\ a^3\ b + {}^4C_2\ a^2\ b^2 + {}^4C_3\ a\ b^3 + {}^4C_4\ b^4$

$(a – b)^4 = {}^4C_0\ a^4 – {}^4C_1\ a^3\ b + {}^4C_2\ a^2\ b^2 – {}^4C_3\ a\ b^3 + {}^4C_4\ b^4$

Now $(a + b)^4 – (a – b)^4 = {}^4C_0\ a^4 + {}^4C_1\ a^3\ b + {}^4C_2\ a^2\ b^2 + {}^4C_3\ a\ b^3 + {}^4C_4\ b^4 – [{}^4C_0\ a^4 – {}^4C_1\ a^3\ b + {}^4C_2\ a^2\ b^2 – {}^4C_3\ a\ b^3 + {}^4C_4\ b^4]$

$= 2\ ({}^4C_1\ a^3\ b + {}^4C_3\ a\ b^3)$

$= 2\ (4a^3 b + 4ab^3)$

$= 8ab\ (a^2 + b^2)$

Now by substituting $a = \sqrt{3}$ and $b = \sqrt{2}$, we get

$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8\ (\sqrt{3})\ (\sqrt{2})\ \{(\sqrt{3})^2 + (\sqrt{2})^2\}$

$= 8\ (\sqrt{6})\ (3 + 2)$

$= 40\ \sqrt{6}$

12. Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate

$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$$

Solution:

Using binomial theorem, the expressions $(x + 1)^6$ and $(x - 1)^6$ can be expressed as

$(x + 1)^6 = {}^6C_0\ x^6 + {}^6C_1\ x^5 + {}^6C_2\ x^4 + {}^6C_3\ x^3 + {}^6C_4\ x^2 + {}^6C_5\ x + {}^6C_6$

$(x - 1)^6 = {}^6C_0\ x^6 - {}^6C_1\ x^5 + {}^6C_2\ x^4 - {}^6C_3\ x^3 + {}^6C_4\ x^2 - {}^6C_5\ x + {}^6C_6$

Now, $(x + 1)^6 - (x - 1)^6 = {}^6C_0\ x^6 + {}^6C_1\ x^5 + {}^6C_2\ x^4 + {}^6C_3\ x^3 + {}^6C_4\ x^2 + {}^6C_5\ x + {}^6C_6 - [{}^6C_0\ x^6 - {}^6C_1\ x^5 + {}^6C_2\ x^4 - {}^6C_3\ x^3 + {}^6C_4\ x^2 - {}^6C_5\ x + {}^6C_6]$

$= 2\ [{}^6C_0\ x^6 + {}^6C_2\ x^4 + {}^6C_4\ x^2 + {}^6C_6]$

$= 2\ [x^6 + 15x^4 + 15x^2 + 1]$

Now by substituting $x = \sqrt{2}$, we get

$(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6 = 2\ [(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1]$

$= 2\ (8 + 15 \times 4 + 15 \times 2 + 1)$

$= 2\ (8 + 60 + 30 + 1)$

$= 2\ (99)$

$= 198$

13. Show that $9^{n+1} - 8n - 9$ is divisible by 64 whenever n is a positive integer.

Solution:

In order to show that $9^{n+1} - 8n - 9$ is divisible by 64, it has to be shown that $9^{n+1} - 8n - 9 = 64\ k$, where k is some natural number.

Using the binomial theorem,

$(1 + a)^m = {}^mC_0 + {}^mC_1\, a + {}^mC_2\, a^2 + \ldots + {}^mC_m\, a^m$

For a = 8 and m = n + 1 we get

$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\,(8) + {}^{n+1}C_2\,(8)^2 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n+1}$

$9^{n+1} = 1 + (n + 1)\,8 + 8^2\,[{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + \ldots + {}^{n+1}C_{n+1}\,(8)^{n-1}]$

$9^{n+1} = 9 + 8n + 64\,[{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + \ldots + {}^{n+1}C_{n+1}\,(8)^{n-1}]$

$9^{n+1} - 8n - 9 = 64\,k$

Where $k = [{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + \ldots + {}^{n+1}C_{n+1}\,(8)^{n-1}]$ is a natural number

Thus, $9^{n+1} - 8n - 9$ is divisible by 64 whenever n is a positive integer.

Hence proved.

14. Prove that

$$\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$$

Solution:

By Binomial Theorem

$$\sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r = (a+b)^n$$

On right side we need 4^n so we will put the values as,
Putting b = 3 & a = 1 in the above equation, we get

$$\sum_{r=0}^{n} \binom{n}{r} (1)^{n-r} (3)^r = (1+3)^n$$

$$\sum_{r=0}^{n} \binom{n}{r} (1)(3)^r = (4)^n$$

$$\sum_{r=0}^{n} \binom{n}{r} (3)^r = (4)^n$$

Hence Proved.