EXERCISE 8.2

PAGE NO: 171

Find the coefficient of

1. x^5 in $(x + 3)^8$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

Here x^5 is the T_{r+1} term so a= x, b = 3 and n =8

$T_{r+1} = {}^8C_r\, x^{8-r}\, 3^r$…………… (i)

To find out x^5

We have to equate $x^5 = x^{8-r}$

⇒ r= 3

Putting the value of r in (I), we get

$$T_{3+1} = {}^8C_3\, x^{8-3}\, 3^3$$

$$T_4 = \frac{8!}{3!\,5!} \times x^5 \times 27$$

= 1512 x^5

Hence the coefficient of x^5= 1512.

2. a^5b^7 in $(a – 2b)^{12}$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

Here a = a, b = -2b & n =12

Substituting the values, we get

$T_{r+1} = {}^{12}C_r\, a^{12-r}\, (-2b)^r$………. (i)

To find a^5

We equate $a^{12-r} = a^5$

r = 7

Putting r = 7 in (i)

$T_8 = {}^{12}C_7\, a^5\, (-2b)^7$

$$T_8 = \frac{12!}{7!\,5!} \times a^5 \times (-2)^7\, b^7$$

$= -101376\, a^5\, b^7$

Hence, the coefficient of a^5b^7= -101376.

Write the general term in the expansion of

3. $(x^2 – y)^6$

Solution:

The general term T_{r+1} in the binomial expansion is given by

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$…….. (i)

Here, $a = x^2$, $n = 6$ and $b = -y$

Putting values in (i)

$T_{r+1} = {}^6C_r\, x^{2(6-r)}\, (-1)^r\, y^r$

$$= \frac{6!}{r!\,(6-r)!} \times x^{12-2r} \times (-1)^r \times y^r$$

$$= -1^r \frac{6!}{r!\,(6-r)!} \times x^{12-2r} \times y^r$$

$= -1^r\, {}^6c_r\, .x^{12-2r}.\, y^r$

4. $(x^2 – y\, x)^{12}$, $x \neq 0$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here $n = 12$, $a= x^2$ and $b = -y\, x$

Substituting the values, we get

$T_{n+1} = {}^{12}C_r \times x^{2(12-r)}\, (-1)^r\, y^r\, x^r$

$= \frac{12!}{r!(12-r)!} \times x^{24-2r} -1^r y^r x^r$

$= -1^r \frac{12!}{r!(12-r)!} x^{24-r} y^r$

$= -1^r\ {}^{12}C_r .x^{24-2r}. y^r$

5. Find the 4th term in the expansion of $(x – 2y)^{12}$.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

Here, a= x, n =12, r= 3 and b = -2y

By substituting the values, we get

$T_4 = {}^{12}C_3\, x^9 (-2y)^3$

$= \frac{12!}{3!9!} \times x^9 \times -8 \times y^3$

$= -\frac{12 \times 11 \times 10 \times 8}{3 \times 2 \times 1} \times x^9 y^3$

$= -1760\, x^9 y^3$

6. Find the 13th term in the expansion of

$$\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here a=9x, $b = -\frac{1}{3\sqrt{x}}$ n =18 and r = 12

Putting values

$$T_{13} = \frac{18!}{12!\,6!}\ 9x^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$= \frac{(18\times17\times16\times15\times14\times13\times12!)}{12!\times6\times5\times4\times3\times2\times1}\times 3^{12}\times x^6\times\frac{1}{x^6}\times\frac{1}{3^{12}}$$

= 18564

Find the middle terms in the expansions of

7. $\left(3-\frac{x^3}{6}\right)^7$

Solution:

Here n = 7 so there would be two middle terms given by

$$\left(\frac{n+1}{2}^{th}\right)\text{term} = 4^{\text{ th}} \text{ and } \left(\frac{n+1}{2}+1\right)^{\text{ th}}\text{ term} = 5^{th}$$

We have

$a = 3, n = 7 \text{ and } b = -\frac{x^3}{6}$

For T_4, r= 3

The term will be

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$T_4 = \frac{7!}{3!}\, 3^4 \left(-\frac{x^3}{6}\right)^3$$

$$= -\frac{7\times 6\times 5\times 4}{3\times 2\times 1} \times 3^4 \times \frac{x^9}{2^3 3^3}$$

$$= -\frac{105}{8}\, x^9$$

For T_5 term, r = 4

The term T_{r+1} in the binomial expansion is given by

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$T_5 = \frac{7!}{4!\,3!}\, 3^3 \left(-\frac{x^3}{6}\right)^4$$

$$= \frac{7\times 6\times 5\times 4!}{4!\,3!} \times \frac{3^3}{2^4 3^4} \times x^3 = \frac{35\, x^{12}}{48}$$

8. $\left(\dfrac{x}{3} + 9y\right)^{10}$

Solution:

Here n is even so the middle term will be given by $\left(\frac{n+1}{2}\right)^{th}$ term $= 6^{th}$ term

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Now $a = \frac{x}{3}$, $b = 9y$, $n = 10$ and $r = 5$

Substituting the values

$$T_6 = \frac{10!}{5!\,5!} \times \left(\frac{x}{3}\right)^5 \times (9y)^5$$

$$T_6 = \frac{10!}{5!\,5!} \times \left(\frac{x}{3}\right)^5 \times (9y)^5$$

$$= \frac{10\times9\times8\times7\times6\times5!}{5!\times5\times4\times3\times2\times1} \times \frac{x^5}{3^5} \times 3^{10} \times y^5$$

$= 61236\, x^5y^5$

9. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Solution:

We know that the general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here n= m+n, a = 1 and b= a

Substituting the values in the general form

$T_{r+1} = {}^{m+n}C_r\, 1^{m+n-r}\, a^r$

$= {}^{m+n}C_r\, a^r$…………. (i)

Now, we have that the general term for the expression is,

$T_{r+1} = {}^{m+n}C_r\, a^r$

Now, for coefficient of a^m

$T_{m+1} = {}^{m+n}C_m\, a^m$

Hence, for the coefficient of a^m, the value of r = m

So, the coefficient is ${}^{m+n}C_m$

Similarly, the coefficient of a^n is ${}^{m+n}C_n$

$${}^{m+n}C_m = \frac{(m+n)!}{m!n!}$$

And also, ${}^{m+n}C_n = \frac{(m+n)!}{m!n!}$

The coefficient of a^m and a^n are same that is $\frac{(m+n)!}{m!n!}$

10. The coefficients of the $(r – 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1:3:5. Find n and r.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here, the binomial is $(1+x)^n$ with a = 1 , b = x and n = n

The $(r+1)^{th}$ term is given by

$T_{(r+1)} = {}^nC_r\, 1^{n-r}\, x^r$

$T_{(r+1)} = {}^nC_r\, x^r$

The coefficient of $(r+1)^{th}$ term is ${}^nC_r$

The r^{th} term is given by $(r-1)^{th}$ term

$T_{(r+1-1)} = {}^nC_{r-1}\, x^{r-1}$

$T_r = {}^nC_{r-1}\, x^{r-1}$

∴ the coefficient of r^{th} term is ${}^nC_{r-1}$

For $(r-1)^{th}$ term, we will take $(r-2)^{th}$ term

$T_{r-2+1} = {}^nC_{r-2}\, x^{r-2}$

$T_{r-1} = {}^nC_{r-2}\, x^{r-2}$

∴ the coefficient of $(r-1)^{th}$ term is ${}^nC_{r-2}$

Given that the coefficient of $(r-1)^{th}$, r^{th} and $r+1^{th}$ term are in ratio 1:3:5

Therefore,

$$\frac{\text{the coefficient of } r-1^{\text{th}} \text{ term}}{\text{coefficient of } r^{\text{th}} \text{ term}} = \frac{1}{3}$$

$$\frac{{}^{n}C_{r-2}}{{}^{n}C_{r-1}} = \frac{1}{3}$$

$$\Rightarrow \frac{\frac{n!}{(r-2)!(n-r+2)!}}{\frac{n!}{(r-1)!(n-r+1)!}} = \frac{1}{3}$$

On rearranging we get

$$\frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{1}{3}$$

By multiplying

$$\Rightarrow \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)!} = \frac{1}{3}$$

$$\Rightarrow \frac{(r-1)(n-r+1)!}{(n-r+2)(n-r+1)!} = \frac{1}{3}$$

On simplifying we get

$\Rightarrow \dfrac{(r-1)}{(n-r+2)} = \dfrac{1}{3}$

$\Rightarrow 3r - 3 = n - r + 2$

$\Rightarrow n - 4r + 5 = 0$…………1

Also

$\dfrac{\text{the coefficient of } r^{th} \text{ term}}{\text{coefficient of } r+1^{th} \text{ term}} = \dfrac{3}{5}$

$\Rightarrow \dfrac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \dfrac{3}{5}$

On rearranging we get

$\Rightarrow \dfrac{n!}{(r-1)!(n-r+1)!} \times \dfrac{r!(n-r)!}{n!} = \dfrac{3}{5}$

By multiplying

$\Rightarrow \dfrac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)!} = \dfrac{3}{5}$

$\Rightarrow \dfrac{r(n-r)!}{(n-r+1)!} = \dfrac{3}{5}$

$\Rightarrow \dfrac{r(n-r)!}{(n-r+1)(n-r)!} = \dfrac{3}{5}$

On simplifying we get

$\Rightarrow \dfrac{r}{(n-r+1)} = \dfrac{3}{5}$

$\Rightarrow 5r = 3n - 3r + 3$

$\Rightarrow 8r - 3n - 3 = 0$…………2

We have 1 and 2 as

$n - 4r \pm 5 = 0$…………1

$8r - 3n - 3 = 0$……………2

Multiplying equation 1 by number 2

$2n - 8r + 10 = 0$……………….3

Adding equations 2 and 3

$2n - 8r + 10 = 0$

$-3n - 8r - 3 = 0$

$\Rightarrow -n = -7$

$n = 7$ and $r = 3$

11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

The general term for binomial $(1+x)^{2n}$ is

$T_{r+1} = {}^{2n}C_r\, x^r$ …………………..1

To find the coefficient of x^n

$r = n$

$T_{n+1} = {}^{2n}C_n\, x^n$

The coefficient of $x^n = {}^{2n}C_n$

The general term for binomial $(1+x)^{2n-1}$ is

$T_{r+1} = {}^{2n-1}C_r\, x^r$

To find the coefficient of x^n

Putting $n = r$

$T_{r+1} = {}^{2n-1}C_r\, x^n$

The coefficient of $x^n = {}^{2n-1}C_n$

We have to prove

Coefficient of x^n in $(1+x)^{2n} = 2$ coefficient of x^n in $(1+x)^{2n-1}$

Consider LHS $= {}^{2n}C_n$

$$= \frac{2n!}{n!\,(2n-n)!}$$

$$= \frac{2n!}{n!\,(n)!}$$

Again consider RHS = $2 \times {}^{2n-1}C_n$

$$= 2 \times \frac{(2n-1)!}{n!\,(2n-1-n)!}$$

$$= 2 \times \frac{(2n-1)!}{n!\,(n-1)!}$$

Now multiplying and dividing by n we get

$$= 2 \times \frac{(2n-1)!}{n!\,(n-1)!} \times \frac{n}{n}$$

$$= \frac{2n(2n-1)!}{n!\,n(n-1)!}$$

$$= \frac{2n!}{n!\,n!}$$

From above equations LHS = RHS

Hence the proof.

12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here, a = 1, b = x and n = m

Putting the value

$T_{r+1} = {}^mC_r\, 1^{m-r}\, x^r$

$= {}^mC_r\, x^r$

We need the coefficient of x^2

$\therefore$ putting r = 2

$T_{2+1} = {}^mC_2\, x^2$

The coefficient of $x^2 = {}^mC_2$

Given that coefficient of $x^2 = {}^mC_2 = 6$

$$\Rightarrow \frac{m!}{2!(m-2)!} = 6$$

$$\Rightarrow \frac{m(m-1)(m-2)!}{2\times1\times(m-2)!} = 6$$

$\Rightarrow$ m (m – 1) = 12

$\Rightarrow$ m^2– m – 12 =0

$\Rightarrow$ m^2– 4m + 3m – 12 =0

$\Rightarrow$ m (m – 4) + 3 (m – 4) = 0

$\Rightarrow$ (m+3) (m – 4) = 0

$\Rightarrow$ m = – 3, 4

We need the positive value of m, so m = 4