## EXERCISE 8.1

Expand each of the expressions in Exercises 1 to 5.

1. $(1-2 x)^{5}$

## Solution:

From binomial theorem expansion, we can write as
$(1-2 x)^{5}$
$={ }^{5} \mathrm{C}_{0}(1)^{5}-{ }^{5} \mathrm{C}_{1}(1)^{4}(2 \mathrm{x})+{ }^{5} \mathrm{C}_{2}(1)^{3}(2 \mathrm{x})^{2}-{ }^{5} \mathrm{C}_{3}(1)^{2}(2 \mathrm{x})^{3}+{ }^{5} \mathrm{C}_{4}(1)^{1}(2 \mathrm{x})^{4}-{ }^{5} \mathrm{C}_{5}(2 \mathrm{x})^{5}$
$=1-5(2 x)+10(4 x)^{2}-10\left(8 x^{3}\right)+5\left(16 x^{4}\right)-\left(32 x^{5}\right)$
$=1-10 x+40 x^{2}-80 x^{3}+80 x^{4}-32 x^{5}$
2. $\left(\frac{2}{x}-\frac{x}{2}\right)^{5}$

## Solution:

From the binomial theorem, the given equation can be expanded as

$$
\begin{aligned}
\left(\frac{2}{x}-\frac{x}{2}\right)^{5}= & { }^{5} C_{0}\left(\frac{2}{x}\right)^{5}-{ }^{5} C_{1}\left(\frac{2}{x}\right)^{4}\left(\frac{x}{2}\right)+{ }^{5} C_{2}\left(\frac{2}{x}\right)^{3}\left(\frac{x}{2}\right)^{2} \\
& -{ }^{5} C_{3}\left(\frac{2}{x}\right)^{2}\left(\frac{x}{2}\right)^{3}+{ }^{5} C_{4}\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^{4}-{ }^{5} C_{5}\left(\frac{x}{2}\right)^{5} \\
= & \frac{32}{x^{5}}-5\left(\frac{16}{x^{4}}\right)\left(\frac{x}{2}\right)+10\left(\frac{8}{x^{3}}\right)\left(\frac{x^{2}}{4}\right)-10\left(\frac{4}{x^{2}}\right)\left(\frac{x^{3}}{8}\right)+5\left(\frac{2}{x}\right)\left(\frac{x^{4}}{16}\right)-\frac{x^{5}}{32} \\
= & \frac{32}{x^{5}}-\frac{40}{x^{3}}+\frac{20}{x}-5 x+\frac{5}{8} x^{3}-\frac{x^{5}}{32}
\end{aligned}
$$

## 3. $(2 x-3)^{6}$

## Solution:

From the binomial theorem, the given equation can be expanded as

$$
\begin{aligned}
& (2 x-3)^{6}={ }^{6} \mathrm{C}_{0}(2 x)^{6}-{ }^{6} \mathrm{C}_{1}(2 x)^{5}(3)+{ }^{6} \mathrm{C}_{1}(2 x)^{4}(3)^{2}-{ }^{4} \mathrm{C}_{3}(2 x)^{3}(3)^{3} \\
& =64 x^{6}-6\left(32 x^{5}\right)(3)+15\left(16 x^{4}\right)(9)-20\left(8 x^{3}\right)(27) \\
& +15\left(4 x^{2}\right)(81)-6(2 x)(243)+729 \\
& =64 x^{6}-576 x^{5}+2160 x^{4}-4320 x^{3}+4860 x^{2}-2916 x+729
\end{aligned}
$$

4. $\left(\frac{x}{3}+\frac{1}{x}\right)^{5}$

## Solution:

From the binomial theorem, the given equation can be expanded as
$\left(\frac{x}{3}+\frac{1}{x}\right)^{5}={ }^{5} C_{0}\left(\frac{x}{3}\right)^{5}+{ }^{3} C_{1}\left(\frac{x}{3}\right)^{4}\left(\frac{1}{x}\right)+{ }^{3} C_{2}\left(\frac{x}{3}\right)^{3}\left(\frac{1}{x}\right)^{2}$
$=\frac{x^{3}}{243}+5\left(\frac{x^{4}}{81}\right)\left(\frac{1}{x}\right)+10\left(\frac{x^{3}}{27}\right)\left(\frac{1}{x^{2}}\right)+10\left(\frac{x^{2}}{9}\right)\left(\frac{1}{x^{3}}\right)+5\left(\frac{x}{3}\right)\left(\frac{1}{x^{4}}\right)+\frac{1}{x^{5}}$
$=\frac{x^{5}}{243}+\frac{5 x^{3}}{81}+\frac{10 x}{27}+\frac{10}{9 x}+\frac{5}{3 x^{3}}+\frac{1}{x^{3}}$
5. $\left(x+\frac{1}{x}\right)^{6}$

## Solution:

From the binomial theorem, the given equation can be expanded as

$$
\begin{aligned}
& \left(\mathrm{x}+\frac{1}{\mathrm{x}}\right)^{6}={ }^{6} \mathrm{C}_{0}(\mathrm{x})^{6}+{ }^{6} \mathrm{C}_{1}(\mathrm{x})^{\prime}\left(\frac{1}{\mathrm{x}}\right)+{ }^{6} \mathrm{C}_{2}(\mathrm{x})^{4}\left(\frac{1}{\mathrm{x}}\right)^{2} \\
& +{ }^{6} \mathrm{C}_{3}(\mathrm{x})^{3}\left(\frac{1}{\mathrm{x}}\right)^{3}+{ }^{6} \mathrm{C}_{4}(\mathrm{x})^{2}\left(\frac{1}{\mathrm{x}}\right)^{4}+{ }^{6} \mathrm{C}_{3}(\mathrm{x})\left(\frac{1}{\mathrm{x}}\right)^{5}+{ }^{6} \mathrm{C}_{6}\left(\frac{1}{\mathrm{x}}\right)^{6} \\
& =\mathrm{x}^{4}+6(\mathrm{x})^{3}\left(\frac{1}{\mathrm{x}}\right)+15(\mathrm{x})^{4}\left(\frac{1}{x^{2}}\right)+20(\mathrm{x})^{3}\left(\frac{1}{x^{3}}\right)+15(\mathrm{x})^{2}\left(\frac{1}{\mathrm{x}^{4}}\right)+6(\mathrm{x})\left(\frac{1}{x^{5}}\right)+\frac{1}{\mathrm{x}^{6}} \\
& =x^{6}+6 x^{4}+15 x^{2}+20+\frac{15}{x^{2}}+\frac{6}{x^{4}}+\frac{1}{x^{6}}
\end{aligned}
$$

## 6. Using the binomial theorem, find (96) ${ }^{3}$.

Solution:
Given (96) ${ }^{3}$
96 can be expressed as the sum or difference of two numbers, and then the binomial theorem can be applied.

The given question can be written as $96=100-4$
$(96)^{3}=(100-4)^{3}$
$\left.={ }^{3} \mathrm{C}_{0}(100)\right)^{3}-{ }^{3} \mathrm{C}_{1}(100)^{2}(4)-{ }^{3} \mathrm{C}_{2}(100)(4)^{2}-{ }^{3} \mathrm{C}_{3}(4)^{3}$
$=(100)^{3}-3(100)^{2}(4)+3(100)(4)^{2}-(4)^{3}$
$=1000000-120000+4800-64$
$=884736$
7. Using the binomial theorem, find (102) ${ }^{5}$.

## Solution:

Given (102) ${ }^{5}$
102 can be expressed as the sum or difference of two numbers, and then the binomial theorem can be applied.
The given question can be written as $102=100+2$
$(102)^{5}=(100+2)^{5}$
$={ }^{5} \mathrm{C}_{0}(100)^{5}+{ }^{5} \mathrm{C}_{1}(100)^{4}(2)+{ }^{5} \mathrm{C}_{2}(100)^{3}(2)^{2}+{ }^{5} \mathrm{C}_{3}(100)^{2}(2)^{3}+{ }^{5} \mathrm{C}_{4}(100)(2)^{4}+{ }^{5} \mathrm{C}_{5}(2)^{5}$
$=(100)^{5}+5(100)^{4}(2)+10(100)^{3}(2)^{2}+5(100)(2)^{3}+5(100)(2)^{4}+(2)^{5}$
$=1000000000+1000000000+40000000+80000+8000+32$
$=11040808032$
8. Using the binomial theorem, find (101) ${ }^{4}$.

## Solution:

Given (101) ${ }^{4}$
101 can be expressed as the sum or difference of two numbers, and then the binomial theorem can be applied.
The given question can be written as $101=100+1$
$(101)^{4}=(100+1)^{4}$
$={ }^{4} \mathrm{C}_{0}(100)^{4}+{ }^{4} \mathrm{C}_{1}(100)^{3}(1)+{ }^{4} \mathrm{C}_{2}(100)^{2}(1)^{2}+{ }^{4} \mathrm{C}_{3}(100)(1)^{3}+{ }^{4} \mathrm{C}_{4}(1)^{4}$
$=(100)^{4}+4(100)^{3}+6(100)^{2}+4(100)+(1)^{4}$
$=100000000+4000000+60000+400+1$
$=104060401$
9. Using the binomial theorem, find (99) ${ }^{5} \mathrm{~m}$.

## Solution:

Given (99) ${ }^{5}$
99 can be written as the sum or difference of two numbers then the binomial theorem can be applied.
The given question can be written as $99=100-1$
$(99)^{5}=(100-1)^{5}$
$={ }^{5} \mathrm{C}_{0}(100)^{5}-{ }^{5} \mathrm{C}_{1}(100)^{4}(1)+{ }^{5} \mathrm{C}_{2}(100)^{3}(1)^{2}-{ }^{5} \mathrm{C}_{3}(100)^{2}(1)^{3}+{ }^{5} \mathrm{C}_{4}(100)(1)^{4}-{ }^{5} \mathrm{C}_{5}(1)^{5}$
$=(100)^{5}-5(100)^{4}+10(100)^{3}-10(100)^{2}+5(100)-1$
$=1000000000-5000000000+10000000-100000+500-1$
$=9509900499$
10. Using Binomial Theorem, indicate which number is larger (1.1) ${ }^{10000}$ or 1000.

## Solution:

By splitting the given 1.1 and then applying the binomial theorem, the first few terms of $(1.1)^{10000}$ can be obtained as
$(1.1)^{10000}=(1+0.1)^{10000}$
$=(1+0.1)^{10000} \mathrm{C}_{1}(1.1)+$ other positive terms
$=1+10000 \times 1.1+$ other positive terms
$=1+11000+$ other positive terms
$>1000$
$(1.1)^{10000}>1000$
11. Find $(\mathbf{a}+\mathbf{b})^{4}-(\mathbf{a}-\mathbf{b})^{4}$. Hence, evaluate
$(\sqrt{3}+\sqrt{2})^{4}-(\sqrt{3}-\sqrt{2})^{4}$.

## Solution:

Using the binomial theorem, the expression $(\mathrm{a}+\mathrm{b})^{4}$ and $(\mathrm{a}-\mathrm{b})^{4}$ can be expanded

$$
\begin{aligned}
& (\mathrm{a}+\mathrm{b})^{4}={ }^{4} \mathrm{C}_{0} \mathrm{a}^{4}+{ }^{4} \mathrm{C}_{1} \mathrm{a}^{3} \mathrm{~b}+{ }^{4} \mathrm{C}_{2} \mathrm{a}^{2} \mathrm{~b}^{2}+{ }^{4} \mathrm{C}_{3} \mathrm{ab}^{3}+{ }^{4} \mathrm{C}_{4} \mathrm{~b}^{4} \\
& (\mathrm{a}-\mathrm{b})^{4}={ }^{4} \mathrm{C}_{0} \mathrm{a}^{4}-{ }^{4} \mathrm{C}_{1} \mathrm{a}^{3} \mathrm{~b}+{ }^{4} \mathrm{C}_{2} \mathrm{a}^{2} \mathrm{~b}^{2}-{ }^{4} \mathrm{C}_{3} \mathrm{ab}^{3}+{ }^{4} \mathrm{C}_{4} \mathrm{~b}^{4}
\end{aligned}
$$

$$
\text { Now }(\mathrm{a}+\mathrm{b})^{4}-(\mathrm{a}-\mathrm{b})^{4}={ }^{4} \mathrm{C}_{0} \mathrm{a}^{4}+{ }^{4} \mathrm{C}_{1} \mathrm{a}^{3} \mathrm{~b}+{ }^{4} \mathrm{C}_{2} \mathrm{a}^{2} \mathrm{~b}^{2}+{ }^{4} \mathrm{C}_{3} \mathrm{ab}^{3}+{ }^{4} \mathrm{C}_{4} \mathrm{~b}^{4}-\left[{ }^{4} \mathrm{C}_{0} \mathrm{a}^{4}-{ }^{4} \mathrm{C}_{1} \mathrm{a}^{3} \mathrm{~b}+{ }^{4} \mathrm{C}_{2} \mathrm{a}^{2} \mathrm{~b}^{2}-{ }^{4} \mathrm{C}_{3} \mathrm{ab}^{3}+{ }^{4} \mathrm{C}_{4} \mathrm{~b}^{4}\right]
$$

$$
=2\left({ }^{4} \mathrm{C}_{1} \mathrm{a}^{3} \mathrm{~b}+{ }^{4} \mathrm{C}_{3} \mathrm{ab}^{3}\right)
$$

$=2\left(4 a^{3} b+4 a b^{3}\right)$
$=8 a b\left(a^{2}+b^{2}\right)$
Now by substituting $a=\sqrt{ } 3$ and $b=\sqrt{ } 2$, we get
$(\sqrt{ } 3+\sqrt{ } 2)^{4}-(\sqrt{ } 3-\sqrt{ } 2)^{4}=8(\sqrt{ } 3)(\sqrt{ } 2)\left\{(\sqrt{ } 3)^{2}+(\sqrt{ } 2)^{2}\right\}$
$=8(\sqrt{ } 6)(3+2)$
$=40 \sqrt{ } 6$
12. Find $(x+1)^{6}+(x-1)^{6}$. Hence or otherwise evaluate
$(\sqrt{2}+1)^{6}+(\sqrt{2}-1)^{6}$

## Solution:

Using binomial theorem, the expressions $(x+1)^{6}$ and $(x-1)^{6}$ can be expressed as
$(\mathrm{X}+1)^{6}={ }^{6} \mathrm{C}_{0} \mathrm{X}^{6}+{ }^{6} \mathrm{C}_{1} \mathrm{X}^{5}+{ }^{6} \mathrm{C}_{2} \mathrm{X}^{4}+{ }^{6} \mathrm{C}_{3} \mathrm{X}^{3}+{ }^{6} \mathrm{C}_{4} \mathrm{X}^{2}+{ }^{6} \mathrm{C}_{5} \mathrm{X}+{ }^{6} \mathrm{C}_{6}$
$(\mathrm{x}-1))^{6}={ }^{6} \mathrm{C}_{0} \mathrm{X}^{6}-{ }^{6} \mathrm{C}_{1} \mathrm{X}^{5}+{ }^{6} \mathrm{C}_{2} \mathrm{X}^{4}-{ }^{6} \mathrm{C}_{3} \mathrm{X}^{3}+{ }^{6} \mathrm{C}_{4} \mathrm{X}^{2}-{ }^{6} \mathrm{C}_{5} \mathrm{X}+{ }^{6} \mathrm{C}_{6}$
Now, $(\mathrm{x}+1)^{6}-(\mathrm{x}-1)^{6}={ }^{6} \mathrm{C}_{0} \mathrm{X}^{6}+{ }^{6} \mathrm{C}_{1} \mathrm{X}^{5}+{ }^{6} \mathrm{C}_{2} \mathrm{X}^{4}+{ }^{6} \mathrm{C}_{3} \mathrm{X}^{3}+{ }^{6} \mathrm{C}_{4} \mathrm{X}^{2}+{ }^{6} \mathrm{C}_{5} \mathrm{X}+{ }^{6} \mathrm{C}_{6}-\left[{ }^{6} \mathrm{C}_{0} \mathrm{X}^{6}-{ }^{6} \mathrm{C}_{1} \mathrm{X}^{5}+{ }^{6} \mathrm{C}_{2} \mathrm{X}^{4}-{ }^{6} \mathrm{C}_{3} \mathrm{X}^{3}+{ }^{6} \mathrm{C}_{4} \mathrm{X}^{2}-\right.$ $\left.{ }^{6} \mathrm{C}_{5} \mathrm{X}+{ }^{6} \mathrm{C}_{6}\right]$
$=2\left[{ }^{6} \mathrm{C}_{0} \mathrm{X}^{6}+{ }^{6} \mathrm{C}_{2} \mathrm{X}^{4}+{ }^{6} \mathrm{C}_{4} \mathrm{X}^{2}+{ }^{6} \mathrm{C}_{6}\right]$
$=2\left[x^{6}+15 x^{4}+15 x^{2}+1\right]$
Now by substituting $x=\sqrt{ }$, we get
$(\sqrt{ } 2+1)^{6}-(\sqrt{ } 2-1)^{6}=2\left[(\sqrt{ } 2)^{6}+15(\sqrt{ } 2)^{4}+15(\sqrt{ } 2)^{2}+1\right]$
$=2(8+15 \times 4+15 \times 2+1)$
$=2(8+60+30+1)$
$=2$ (99)
$=198$
13. Show that $9^{n+1}-8 n-9$ is divisible by 64 whenever $n$ is a positive integer.

## Solution:

In order to show that $9^{n+1}-8 n-9$ is divisible by 64 , it has to be shown that $9^{n+1}-8 n-9=64 k$, where $k$ is some natural number.

Using the binomial theorem,
$(1+\mathrm{a})^{\mathrm{m}}={ }^{\mathrm{m}} \mathrm{C}_{0}+{ }^{\mathrm{m}} \mathrm{C}_{1} \mathrm{a}+{ }^{\mathrm{m}} \mathrm{C}_{2} \mathrm{a}^{2}+\ldots .+{ }^{\mathrm{m}} \mathrm{C}_{\mathrm{m}} \mathrm{a}^{\mathrm{m}}$
For $\mathrm{a}=8$ and $\mathrm{m}=\mathrm{n}+1$ we get
$(1+8)^{n+1}={ }^{n+1} C_{0}+{ }^{n+1} C_{1}(8)+{ }^{n+1} C_{2}(8)^{2}+\ldots .+{ }^{n+1} C_{n+1}(8)^{n+1}$
$9^{n+1}=1+(n+1) 8+8^{2}\left[{ }^{[n+1} C_{2}+{ }^{n+1} C_{3}(8)+\ldots .+{ }^{n+1} C_{n+1}(8)^{n-1}\right]$
$9^{n+1}=9+8 n+64\left[{ }^{[n+1} C_{2}+{ }^{n+1} C_{3}(8)+\ldots .+{ }^{n+1} C^{n+1}(8)^{n-1}\right]$
$9^{n+1}-8 n-9=64 k$
Where $\left.\mathrm{k}={ }^{[n+1} \mathrm{C}_{2}+{ }^{n+1} \mathrm{C}_{3}(8)+\ldots .+{ }^{n+1} \mathrm{C}_{\mathrm{n+1}}(8)^{\mathrm{n}-1}\right]$ is a natural number
Thus, $9^{n+1}-8 n-9$ is divisible by 64 whenever $n$ is a positive integer.
Hence proved.
14. Prove that
$\sum_{\mathrm{r}=0}^{\mathrm{n}} 3^{\mathrm{r}} \mathrm{C}_{\mathrm{r}}=4^{\mathrm{n}}$
Solution:

## By Binomial Theorem

$\sum_{r=0}^{n}\binom{n}{r} a^{n-r} b^{r}=(a+b)^{n}$
On right side we need $4^{n}$ so we will put the values as, Putting $b=3 \& a=1$ in the above equation, we get
$\sum_{\mathrm{r}=0}^{\mathrm{n}}\binom{\mathrm{n}}{\mathrm{r}}(1)^{\mathrm{n}-\mathrm{r}}(3)^{\mathrm{r}}=(1+3)^{\mathrm{n}}$
$\sum_{r=0}^{n}\binom{n}{r}(1)(3)^{r}=(4)^{n}$
$\sum_{r=0}^{n}\binom{n}{r}(3)^{r}=(4)^{n}$

## Hence Proved.

## EXERCISE 8.2

Find the coefficient of

1. $x^{5}$ in $(x+3)^{8}$

## Solution:

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1}={ }^{n} C_{r} a^{n+t} b^{r}$
Here $\mathrm{x}^{5}$ is the $\mathrm{T}_{\mathrm{r}+1}$ term so $\mathrm{a}=\mathrm{x}, \mathrm{b}=3$ and $\mathrm{n}=8$
$\mathrm{T}_{\mathrm{r}+1}={ }^{8} \mathrm{C}_{\mathrm{r}} \mathrm{X}^{8 \mathrm{r}} 3^{\mathrm{r}}$ (i)

To find out $\mathrm{X}^{5}$
We have to equate $\mathrm{x}^{5}=\mathrm{x}^{8 .}$
$\Rightarrow \mathrm{r}=3$
Putting the value of r in (I), we get

$$
T_{3+1}={ }^{8} C_{3} x^{8-3} 3^{3}
$$

$$
\mathrm{T}_{4}=\frac{8!}{3!5!} \times \mathrm{x}^{5} \times 27
$$

$=1512 \mathrm{x}^{5}$
Hence the coefficient of $\mathrm{X}^{5}=1512$.
2. $a^{5} b^{7}$ in $(a-2 b)^{12}$

## Solution:

The general term $\mathrm{T}_{\mathrm{r}+1}$ in the binomial expansion is given by $\mathrm{T}_{\mathrm{r}+1}={ }^{n} \mathrm{C}_{\mathrm{r}} \mathrm{a}^{\text {ner }} \mathrm{b}^{r}$
Here $\mathrm{a}=\mathrm{a}, \mathrm{b}=-2 \mathrm{~b} \& \mathrm{n}=12$
Substituting the values, we get
$\mathrm{T}_{\mathrm{r}+1}={ }^{12} \mathrm{C}_{\mathrm{r}} \mathrm{a}^{12 . r}(-2 \mathrm{~b})^{\mathrm{r}}$. $\qquad$
To find $\mathrm{a}^{5}$
We equate $\mathrm{a}^{12 \cdot x}=\mathrm{a}^{5}$
$r=7$
Putting $\mathrm{r}=7$ in (i)
$\mathrm{T}_{8}={ }^{12} \mathrm{C}_{7} \mathrm{a}^{5}(-2 \mathrm{~b})^{7}$
$T_{8}=\frac{12!}{7!5!} \times a^{5} \times(-2)^{7} b^{7}$
$=-101376 a^{5} b^{7}$
Hence, the coefficient of $\mathrm{a}^{5} \mathrm{~b}^{7}=-101376$.
Write the general term in the expansion of
3. $\left(x^{2}-y\right)^{6}$

Solution:
The general term $\mathrm{T}_{\mathrm{r}+1}$ in the binomial expansion is given by
$\mathrm{T}_{\mathrm{r}+1}={ }^{\mathrm{n}} \mathrm{C}_{\mathrm{r}} \mathrm{a}^{\mathrm{n}-\mathrm{r}} \mathrm{b}^{\mathrm{r}}$ $\qquad$
Here, $a=x^{2}, n=6$ and $b=-y$
Putting values in (i)
$\mathrm{T}_{\mathrm{r}+1}={ }^{6} \mathrm{C}_{\mathrm{r}} \mathrm{X}^{2(6-\mathrm{r})}(-1)^{\mathrm{r}} \mathrm{y}^{\mathrm{r}}$
$=\frac{6!}{r!(6-r)!} \times x^{12-2 r} \times(-1)^{r} \times y^{r}$
$=-1^{r} \frac{6!}{r!(6-r)!} \times x^{12-2 r} \times y^{r}$
$=-1^{r}{ }^{6} c_{r} \cdot X^{12-2 r} \cdot y^{r}$
4. $\left(\mathrm{x}^{2}-\mathrm{y} \mathrm{x}\right)^{12}, \mathrm{x} \neq 0$

## Solution:

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1}={ }^{n} C_{r} a^{n-r} b^{r}$
Here $\mathrm{n}=12, \mathrm{a}=\mathrm{x}^{2}$ and $\mathrm{b}=-\mathrm{y} \mathrm{x}$
Substituting the values, we get
$T_{n+1}={ }^{12} C_{r} \times X^{2(12-r)}(-1)^{r} y^{r} X^{r}$
$=\frac{12!}{r!(12-r)!} \times x^{24-2 r}-1^{r} y^{r} x^{r}$
$=-1^{r} \frac{12!}{r!(12-r)!} x^{24-r} y^{r}$
$=-1^{1212} \mathrm{c}_{\mathrm{r}} \cdot \mathrm{X}^{24-2 x} \cdot \mathrm{y}^{\mathrm{r}}$
5. Find the 4 th term in the expansion of $(x-2 y)^{12}$.

## Solution:

The general term $\mathrm{T}_{\mathrm{r}+1}$ in the binomial expansion is given by $\mathrm{T}_{\mathrm{r}+1}={ }^{n} \mathrm{C}_{\mathrm{r}} \mathrm{a}^{\text {ntr }} \mathrm{b}^{r}$
Here, $a=x, n=12, r=3$ and $b=-2 y$
By substituting the values, we get
$\mathrm{T}_{4}={ }^{12} \mathrm{C}_{3} \mathrm{x}^{9}(-2 \mathrm{y})^{3}$
$=\frac{12!}{3!9!} \times \mathrm{x}^{9} \times-8 \times \mathrm{y}^{3}$
$=-\frac{12 \times 11 \times 10 \times 8}{3 \times 2 \times 1} \times \mathrm{x}^{9} \mathrm{y}^{3}$
$=-1760 x^{9} y^{3}$
6. Find the $13^{\text {th }}$ term in the expansion of

$$
\left(9 x-\frac{1}{3 \sqrt{x}}\right)^{18}, x \neq 0
$$

Solution:

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1}={ }^{n} C_{r} a^{n-r} b^{r}$
Here $a=9 x, b=-\frac{1}{3 \sqrt{x}} n=18$ and $r=12$
Putting values
$T_{13}=\frac{18!}{12!6!} 9 \mathrm{x}^{18-12}\left(-\frac{1}{3 \sqrt{\mathrm{x}}}\right)^{12}$
$=\frac{(18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12!)}{12!\times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 3^{12} \times \mathrm{x}^{6} \times \frac{1}{\mathrm{x}^{6}} \times \frac{1}{3^{12}}$
$=18564$
Find the middle terms in the expansions of
7. $\left(3-\frac{x^{3}}{6}\right)^{7}$

## Solution:

Here $\mathrm{n}=7$ so there would be two middle terms given by

$$
\left(\frac{\mathrm{n}+1^{\text {th }}}{2}\right) \text { term }=4^{\text {th }} \text { and }\left(\frac{\mathrm{n}+1}{2}+1\right)^{\text {th }} \text { term }=5^{\text {th }}
$$

We have

$$
\mathrm{a}=3, \mathrm{n}=7 \text { and } \mathrm{b}=-\frac{\mathrm{x}^{3}}{6}
$$

For $T_{4}, r=3$
The term will be
$T_{r+1}={ }^{n} C_{r} a^{n-r} b^{r}$
$T_{4}=\frac{7!}{3!} 3^{4}\left(-\frac{x^{3}}{6}\right)^{3}$
$=-\frac{7 \times 6 \times 5 \times 4}{3 \times 2 \times 1} \times 3^{4} \times \frac{\mathrm{x}^{9}}{2^{3} 3^{3}}$
$=-\frac{105}{8} \mathrm{x}^{9}$
For $\mathrm{T}_{5}$ term, $\mathrm{r}=4$
The term $\mathrm{T}_{\mathrm{r}+1}$ in the binomial expansion is given by
$T_{r+1}={ }^{n} C_{r} a^{n-r} b^{r}$
$T_{5}=\frac{7!}{4!3!} 3^{3}\left(-\frac{x^{3}}{6}\right)^{4}$
$=\frac{7 \times 6 \times 5 \times 4!}{4!3!} \times \frac{3^{3}}{2^{4} 3^{4}} \times \mathrm{x}^{3}=\frac{35 \mathrm{x}^{12}}{48}$
8. $\left(\frac{x}{3}+9 y\right)^{10}$

Solution:

Here n is even so the middle term will be given by $\left(\frac{\mathrm{n}+1}{2}\right)^{\text {th }}$ term $=6^{\text {th }}$ term
The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1}={ }^{n} C_{r} a^{n-r} b^{r}$
Now $\mathrm{a}=\frac{\mathrm{x}}{3}, \mathrm{~b}=9 \mathrm{y}, \mathrm{n}=10$ and $\mathrm{r}=5$
Substituting the values
$T_{6}=\frac{10!}{5!5!} \times\left(\frac{x}{3}\right)^{5} \times(9 y)^{5}$
$T_{6}=\frac{10!}{5!5!} \times\left(\frac{x}{3}\right)^{5} \times(9 y)^{5}$
$=\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!\times 5 \times 4 \times 3 \times 2 \times 1} \times \frac{x^{5}}{3^{5}} \times 3^{10} \times y^{5}$
$=61236 x^{5} y^{5}$
9. In the expansion of $(1+a)^{m+n}$, prove that coefficients of $a^{m}$ and $a^{\mathrm{n}}$ are equal.

## Solution:

We know that the general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1}={ }^{n} C_{r} a^{n-r} b^{r}$
Here $\mathrm{n}=\mathrm{m}+\mathrm{n}, \mathrm{a}=1$ and $\mathrm{b}=\mathrm{a}$
Substituting the values in the general form
$\mathrm{T}_{\mathrm{r}+1}={ }^{\mathrm{m}+\mathrm{n}} \mathrm{C}_{\mathrm{r}} 1^{\mathrm{m}+\mathrm{n}-\mathrm{r}} \mathrm{a}^{\mathrm{r}}$
$={ }^{m+n} C_{r} a^{r}$.
Now, we have that the general term for the expression is,
$\mathrm{T}_{\mathrm{r}+1}={ }^{\mathrm{m}+\mathrm{n}} \mathrm{C}_{\mathrm{r}} \mathrm{ar}^{\mathrm{r}}$
Now, for coefficient of $\mathrm{a}^{\mathrm{m}}$
$\mathrm{T}_{\mathrm{m}+1}={ }^{\mathrm{m}+\mathrm{n}} \mathrm{C}_{\mathrm{m}} \mathrm{a}^{\mathrm{m}}$

Hence, for the coefficient of $a^{m}$, the value of $r=m$

So, the coefficient is ${ }^{m+n} C_{m}$

Similarly, the coefficient of $\mathrm{a}^{\mathrm{n}}$ is ${ }^{\mathrm{m+n}} \mathrm{C}_{\mathrm{n}}$
${ }^{m+n} C_{m}=\frac{(m+n)!}{m!n!}$
And also, ${ }^{m+n} C_{n}=\frac{(m+n)!}{m!n!}$
The coefficient of $a^{m}$ and $a^{n}$ are same that is $\frac{(m+n)!}{m!n!}$
10. The coefficients of the $(r-1)^{\text {th }}, r^{\text {th }}$ and $(r+1)^{\text {th }}$ terms in the expansion $o f(x+1)^{n}$ are in the ratio 1:3:5. Find $n$ and $r$.

## Solution:

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1}={ }^{n} C_{r} a^{n-r} b^{r}$
Here, the binomial is $(1+\mathrm{x})^{\mathrm{n}}$ with $\mathrm{a}=1, \mathrm{~b}=\mathrm{x}$ and $\mathrm{n}=\mathrm{n}$
The $(r+1)^{\text {th }}$ term is given by
$T_{(r+1)}={ }^{n} C_{r} 1^{n-r} X^{r}$
$T_{(r+1)}={ }^{n} C_{r} X^{r}$
The coefficient of $(r+1)^{\text {th }}$ term is ${ }^{n} C_{r}$
The $r^{\text {th }}$ term is given by $(r-1)^{\text {th }}$ term
$\mathrm{T}_{(\mathrm{r}+1-1)}={ }^{\mathrm{n}} \mathrm{C}_{\mathrm{r}-1} \mathrm{X}^{\mathrm{r}-1}$
$\mathrm{T}_{\mathrm{r}}={ }^{\mathrm{n}} \mathrm{C}_{\mathrm{r}-1} \mathrm{X}^{\mathrm{r}-1}$
$\therefore$ the coefficient of $\mathrm{r}^{\text {th }}$ term is ${ }^{\mathrm{n}} \mathrm{C}_{\mathrm{r}-1}$
For $(\mathrm{r}-1)^{\mathrm{th}}$ term, we will take $(\mathrm{r}-2)^{\mathrm{th}}$ term
$\mathrm{T}_{\mathrm{r}-2+1}={ }^{\mathrm{n}} \mathrm{C}_{\mathrm{r}-2} \mathrm{X}^{\mathrm{r}-2}$
$\mathrm{T}_{\mathrm{r}-1}={ }^{\mathrm{n}} \mathrm{C}_{\mathrm{r}-2} \mathrm{X}^{\mathrm{r}-2}$
$\therefore$ the coefficient of $(\mathrm{r}-1)^{\text {th }}$ term is ${ }^{\mathrm{n}} \mathrm{C}_{\mathrm{r}-2}$

Given that the coefficient of $(\mathrm{r}-1)^{\text {th }}, \mathrm{r}^{\mathrm{th}}$ and $\mathrm{r}+1^{\text {th }}$ term are in ratio 1:3:5
Therefore,
$\frac{\text { the coefficient of } r-1^{\text {th }} \text { term }}{\text { coefficient of } \mathrm{r}^{\text {th }} \text { term }}=\frac{1}{3}$
$\mathrm{n}_{\substack{\mathrm{r}-2 \\ \mathrm{n}_{\mathrm{r}-1}^{\mathrm{c}}}}=\frac{1}{3}$
$\Rightarrow \frac{\frac{n!}{(r-2)!(n-r+2)!}}{\frac{n!}{(r-1)!(n-r+1)!}}=\frac{1}{3}$
On rearranging we get
$\frac{\mathrm{n}!}{(\mathrm{r}-2)!(\mathrm{n}-\mathrm{r}+2)!} \times \frac{(\mathrm{r}-1)!(\mathrm{n}-\mathrm{r}+1)!}{\mathrm{n}!}=\frac{1}{3}$
By multiplying
$\Rightarrow \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)!}=\frac{1}{3}$
$\Rightarrow \frac{(r-1)(n-r+1)!}{(n-r+2)(n-r+1)!}=\frac{1}{3}$
On simplifying we get
$\Rightarrow \frac{(r-1)}{(n-r+2)}=\frac{1}{3}$
$\Rightarrow 3 r-3=n-r+2$
$\Rightarrow \mathrm{n}-4 \mathrm{r}+5=0$. .1

Also
$\frac{\text { the coefficient of } r^{\text {th }} \text { term }}{\text { coefficient of } r+1^{\text {th }} \text { term }}=\frac{3}{5}$
$\Rightarrow \frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}}=\frac{3}{5}$
On rearranging we get
$\Rightarrow \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!}=\frac{3}{5}$
By multiplying
$\Rightarrow \frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)!}=\frac{3}{5}$
$\Rightarrow \frac{r(n-r)!}{(n-r+1)!}=\frac{3}{5}$
$\Rightarrow \frac{r(n-r)!}{(n-r+1)(n-r)!}=\frac{3}{5}$
On simplifying we get
$\Rightarrow \frac{\mathrm{r}}{(\mathrm{n}-\mathrm{r}+1)}=\frac{3}{5}$
$\Rightarrow 5 \mathrm{r}=3 \mathrm{n}-3 \mathrm{r}+3$
$\Rightarrow 8 \mathrm{r}-3 \mathrm{n}-3=0$. .2

We have 1 and 2 as
$n-4 r \pm 5=0$. $\qquad$1
$8 \mathrm{r}-3 \mathrm{n}-3=0 \ldots \ldots \ldots \ldots \ldots . . .$.
Multiplying equation 1 by number 2
$2 n-8 r+10=0$ .3

Adding equations 2 and 3
$2 \mathrm{n}-8 \mathrm{r}+10=0$
$-3 n-8 r-3=0$
$\Rightarrow-\mathrm{n}=-7$
$\mathrm{n}=7$ and $\mathrm{r}=3$
11. Prove that the coefficient of $x^{n}$ in the expansion of $(1+x)^{2 n}$ is twice the coefficient of $x^{n}$ in the expansion of $(1+$ $\mathbf{x})^{2 n-1}$.

## Solution:

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1}={ }^{n} C_{r} a^{n-r} b^{r}$
The general term for binomial $(1+x)^{2 n}$ is
$\mathrm{T}_{\mathrm{r}+1}={ }^{2 n} \mathrm{C}_{\mathrm{r}} \mathrm{X}^{\mathrm{r}}$ $\qquad$ .. 1

To find the coefficient of $\mathrm{x}^{\mathrm{n}}$
$\mathrm{r}=\mathrm{n}$
$\mathrm{T}_{\mathrm{n}+1}={ }^{2 n} \mathrm{C}_{\mathrm{n}} \mathrm{X}^{\mathrm{n}}$
The coefficient of $\mathrm{X}^{\mathrm{n}}={ }^{2 n} \mathrm{C}_{\mathrm{n}}$
The general term for binomial $(1+x)^{2 n-1}$ is
$\mathrm{T}_{\mathrm{r}+1}={ }^{2 \mathrm{n}-1} \mathrm{C}_{\mathrm{r}} \mathrm{X}{ }^{\mathrm{r}}$
To find the coefficient of $\mathrm{X}^{\mathrm{n}}$
Putting $\mathrm{n}=\mathrm{r}$
$\mathrm{T}_{\mathrm{r}+1}={ }^{2 \mathrm{n}-1} \mathrm{C}_{\mathrm{r}} \mathrm{X}^{\mathrm{n}}$
The coefficient of $X^{n}={ }^{2 n-1} C_{n}$
We have to prove
Coefficient of $x^{n}$ in $(1+x)^{2 n}=2$ coefficient of $x^{n}$ in $(1+x)^{2 n-1}$
Consider LHS $={ }^{2 n} \mathrm{C}_{\mathrm{n}}$

$$
=\frac{2 n!}{n!(2 n-n)!}
$$

$$
=\frac{2 n!}{n!(n)!}
$$

Again consider RHS $=2 \times{ }^{2 n-1} C_{n}$
$=2 \times \frac{(2 n-1)!}{n!(2 n-1-n)!}$
$=2 \times \frac{(2 n-1)!}{n!(n-1)!}$
Now multiplying and dividing by n we get
$=2 \times \frac{(2 n-1)!}{n!(n-1)!} \times \frac{n}{n}$
$=\frac{2 n(2 n-1)!}{n!n(n-1)!}$
$=\frac{2 n!}{n!n!}$
From above equations LHS $=$ RHS
Hence the proof.
12. Find a positive value of $m$ for which the coefficient of $x^{2}$ in the expansion $(1+x)^{m}$ is 6 .

Solution:
The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1}={ }^{n} C_{r} a^{n+r} b^{r}$
Here, $\mathrm{a}=1, \mathrm{~b}=\mathrm{x}$ and $\mathrm{n}=\mathrm{m}$
Putting the value
$\mathrm{T}_{\mathrm{r}+1}={ }^{m} \mathrm{C}_{\mathrm{r}} \mathrm{l}^{\mathrm{mrr}} \mathrm{X}^{r}$
$={ }^{m} \mathrm{C}_{\mathrm{r}} \mathrm{X}^{\mathrm{r}}$

We need the coefficient of $\mathrm{x}^{2}$
$\therefore$ putting $\mathrm{r}=2$
$\mathrm{T}_{2+1}={ }^{\mathrm{m}} \mathrm{C}_{2} \mathrm{X}^{2}$
The coefficient of $\mathrm{x}^{2}={ }^{\mathrm{m}} \mathrm{C}_{2}$
Given that coefficient of $\mathrm{x}^{2}={ }^{\mathrm{m}} \mathrm{C}_{2}=6$
$\Rightarrow \frac{\mathrm{m}!}{2!(\mathrm{m}-2)!}=6$
$\Rightarrow \frac{\mathrm{m}(\mathrm{m}-1)(\mathrm{m}-2)!}{2 \times 1 \times(\mathrm{m}-2)!}=6$
$\Rightarrow \mathrm{m}(\mathrm{m}-1)=12$
$\Rightarrow \mathrm{m}^{2}-\mathrm{m}-12=0$
$\Rightarrow \mathrm{m}^{2}-4 \mathrm{~m}+3 \mathrm{~m}-12=0$
$\Rightarrow \mathrm{m}(\mathrm{m}-4)+3(\mathrm{~m}-4)=0$
$\Rightarrow(\mathrm{m}+3)(\mathrm{m}-4)=0$
$\Rightarrow \mathrm{m}=-3,4$
We need the positive value of m , so $\mathrm{m}=4$

## MISCELLANEOUS EXERCISE

1. Find $a, b$ and $n$ in the expansion of $(a+b)^{n}$ if the first three terms of the expansion are 729,7290 and 30375 , respectively.

## Solution:

We know that $(r+1)^{\text {in }}$ term, $\left(\mathrm{T}_{\mathrm{r}+1}\right)$, in the binomial expansion of $(\mathrm{a}+\mathrm{b})^{\mathrm{n}}$ is given by
$\mathrm{T}_{\mathrm{r}+1}={ }^{\mathrm{n}} \mathrm{C}_{\mathrm{r}} \mathrm{a}^{\mathrm{nt-}} \mathrm{~b}^{\mathrm{r}}$
The first three terms of the expansion are given as 729,7290 and 30375 , respectively. Then we have,
$\mathrm{T}_{1}={ }^{\mathrm{n}} \mathrm{C}_{0} \mathrm{a}^{\mathrm{n}-0} \mathrm{~b}^{0}=\mathrm{a}^{\mathrm{n}}=729 \ldots . .1$
$\mathrm{T}_{2}={ }^{\mathrm{n}} \mathrm{C}_{1} \mathrm{a}^{\mathrm{n}-1} \mathrm{~b}^{1}=\mathrm{na}^{\mathrm{n}-1} \mathrm{~b}=7290 \ldots .2$
$\mathrm{T}_{3}={ }^{\mathrm{n}} \mathrm{C}_{2} \mathrm{a}^{\mathrm{n}-2} \mathrm{~b}^{2}=\{\mathrm{n}(\mathrm{n}-1) / 2\} \mathrm{a}^{\mathrm{n}-2} \mathrm{~b}^{2}=30375 \ldots \ldots 3$
Dividing 2 by 1 , we get

$$
\begin{aligned}
& \frac{n a^{n-1} b}{a^{n}}=\frac{7290}{729} \\
& \frac{n b}{a}=10
\end{aligned}
$$

Dividing 3 by 2 , we get

$$
\begin{aligned}
& \frac{n(n-1) a^{n-2} b^{2}}{2 n a^{n-1} b}=\frac{30375}{7290} \\
& \frac{(n-1) b}{2 a}=\frac{30375}{7290} \\
& \frac{(n-1) b}{a}=\frac{30375}{7290} \times 2=\frac{25}{3} \\
& \frac{(n b)}{a}-\frac{b}{a}=\frac{25}{3} \\
& 10-\frac{b}{a}=\frac{25}{3} \\
& \frac{b}{a}=10-\frac{25}{3}=\frac{5}{3}
\end{aligned}
$$

From 4 and 5, we have
n. $5 / 3=10$
$n=6$

Substituting $\mathrm{n}=6$ in 1 , we get
$\mathrm{a}^{6}=729$
$a=3$
From 5, we have, $b / 3=5 / 3$
$\mathrm{b}=5$
Thus $\mathrm{a}=3, \mathrm{~b}=5$ and $\mathrm{n}=76$
2. Find a if the coefficients of $x^{2}$ and $x^{3}$ in the expansion of $(3+a x)^{9}$ are equal.

Solution:

We know that general term of expansion $(a+b)^{n}$ is
$\mathrm{T}_{\mathrm{r}+1}=\left(\frac{\mathrm{n}}{\mathrm{r}}\right) \mathrm{a}^{\mathrm{n}-\mathrm{r}} \mathrm{b}^{\mathrm{r}}$
For $(3+a x)^{9}$
Putting $a=3, b=a \times \& n=9$
General term of $(3+a x)^{9}$ is

$$
\begin{aligned}
& T_{r+1}=\left(\frac{9}{r}\right) 3^{n-r}(a x)^{r} \\
& T_{r+1}=\left(\frac{9}{r}\right) 3^{n-r} a^{r} x^{r}
\end{aligned}
$$

Since we need to find the coefficients of $x^{2}$ and $x^{3}$, therefore
For $r=2$
$\mathrm{T}_{2+1}=\left(\frac{9}{2}\right) 3^{\mathrm{n}-2} \mathrm{a}^{2} \mathrm{x}^{2}$
Thus, the coefficient of $x^{2}=\left(\frac{9}{2}\right) 3^{n-2} \mathrm{a}^{2}$
For $r=3$
$\mathrm{T}_{3+1}=\left(\frac{9}{3}\right) 3^{\mathrm{n}-3} \mathrm{a}^{3} \mathrm{x}^{3}$
Thus, the coefficient of $x^{3}=\left(\frac{9}{3}\right) 3^{n-3} a^{3}$
Given that coefficient of $x^{2}=$ Coefficient of $x^{3}$
$\Rightarrow\left(\frac{9}{2}\right) 3^{n-2} a^{2}=\left(\frac{9}{3}\right) 3^{n-3} a^{3}$
$\Rightarrow \frac{9!}{2!(9-2)!} \times 3^{n-2} a^{2}=\frac{9!}{3!(9-3)!} \times 3^{n-3} a^{3}$
$\Rightarrow \frac{3^{\mathrm{n}-2} \mathrm{a}^{2}}{3^{\mathrm{n}-3} \mathrm{a}^{3}}=\frac{2!(9-2)!}{3!(9-3)!}$
$\Rightarrow \frac{3^{(\mathrm{n}-2)-(\mathrm{n}-3)}}{a}=\frac{2!7!}{3!6!}$
$\Rightarrow \frac{3}{a}=\frac{7}{3}$
$\therefore \mathrm{a}=9 / 7$
Hence, $a=9 / 7$
3. Find the coefficient of $x^{5}$ in the product $(1+2 x)^{6}(1-x)^{7}$ using binomial theorem.

## Solution:

$(1+2 \mathrm{x})^{6}={ }^{6} \mathrm{C}_{0}+{ }^{6} \mathrm{C}_{1}(2 \mathrm{x})+{ }^{6} \mathrm{C}_{2}(2 \mathrm{x})^{2}+{ }^{6} \mathrm{C}_{3}(2 \mathrm{x})^{3}+{ }^{6} \mathrm{C}_{4}(2 \mathrm{x})^{4}+{ }^{6} \mathrm{C}_{5}(2 \mathrm{x})^{5}+{ }^{6} \mathrm{C}_{6}(2 \mathrm{x})^{6}$
$=1+6(2 \mathrm{x})+15(2 \mathrm{x})^{2}+20(2 \mathrm{x})^{3}+15(2 \mathrm{x})^{4}+6(2 \mathrm{x})^{5}+(2 \mathrm{x})^{6}$
$=1+12 x+60 x^{2}+160 x^{3}+240 x^{4}+192 x^{5}+64 x^{6}$
$(1-\mathrm{x})^{7}={ }^{7} \mathrm{C}_{0}-{ }^{7} \mathrm{C}_{1}(\mathrm{x})+{ }^{7} \mathrm{C}_{2}(\mathrm{x})^{2}-{ }^{7} \mathrm{C}_{3}(\mathrm{x})^{3}+{ }^{7} \mathrm{C}_{4}(\mathrm{x})^{4}-{ }^{7} \mathrm{C}_{5}(\mathrm{x})^{5}+{ }^{7} \mathrm{C}_{6}(\mathrm{x})^{6}-{ }^{7} \mathrm{C}_{7}(\mathrm{x})^{7}$
$=1-7 \mathrm{x}+21 \mathrm{x}^{2}-35 \mathrm{x}^{3}+35 \mathrm{x}^{4}-21 \mathrm{x}^{5}+7 \mathrm{x}^{6}-\mathrm{x}^{7}$
$(1+2 x)^{6}(1-x)^{7}=\left(1+12 x+60 x^{2}+160 x^{3}+240 x^{4}+192 x^{5}+64 x^{6}\right)\left(1-7 x+21 x^{2}-35 x^{3}+35 x^{4}-21 x^{5}+7 x^{6}-x^{7}\right)$
$192-21=171$
Thus, the coefficient of $x^{5}$ in the expression $(1+2 x)^{6}(1-x) 7$ is 171 .
4. If $a$ and $b$ are distinct integers, prove that $a-b$ is a factor of $a^{n}-b^{n}$, whenever $n$ is a positive integer. [Hint write $\mathbf{a}^{\mathrm{n}}=(\mathbf{a}-\mathbf{b}+\mathbf{b})^{\mathrm{n}}$ and expand]

## Solution:

In order to prove that $(\mathrm{a}-\mathrm{b})$ is a factor of $\left(\mathrm{a}^{\mathrm{n}}-\mathrm{b}^{\mathrm{n}}\right)$, it has to be proved that
$a^{n}-b^{n}=k(a-b)$ where $k$ is some natural number.
a can be written as $\mathrm{a}=\mathrm{a}-\mathrm{b}+\mathrm{b}$
$\mathrm{a}^{\mathrm{n}}=(\mathrm{a}-\mathrm{b}+\mathrm{b})^{\mathrm{n}}=[(\mathrm{a}-\mathrm{b})+\mathrm{b}]^{\mathrm{n}}$
$={ }^{n} C_{0}(a-b)^{n}+{ }^{n} C_{1}(a-b)^{n-1} b+$ $\qquad$ $+{ }^{n} \mathrm{C}_{\mathrm{n}} \mathrm{b}^{\mathrm{n}}$
$a^{n}-b^{n}=(a-b)\left[(a-b)^{n-1}+{ }^{n} C_{1}(a-b)^{n-1} b+\right.$ $\qquad$ $\left.+{ }^{n} C_{n} b^{n}\right]$
$\mathrm{a}^{\mathrm{n}}-\mathrm{b}^{\mathrm{n}}=(\mathrm{a}-\mathrm{b}) \mathrm{k}$
Where $\mathrm{k}=\left[(\mathrm{a}-\mathrm{b})^{\mathrm{n}-1}+{ }^{\mathrm{n}} \mathrm{C}_{1}(\mathrm{a}-\mathrm{b})^{\mathrm{n}-1} \mathrm{~b}+\right.$ $\qquad$ $\left.+{ }^{n} C_{n} b^{n}\right]$ is a natural number

This shows that $(a-b)$ is a factor of $\left(a^{n}-b^{n}\right)$, where $n$ is a positive integer.

## 5. Evaluate

$(\sqrt{3}+\sqrt{2})^{6}-(\sqrt{3}-\sqrt{2})^{6}$

## Solution:

Using the binomial theorem, the expression $(a+b)^{6}$ and $(a-b)^{6}$ can be expanded
$(\mathrm{a}+\mathrm{b})^{6}={ }^{6} \mathrm{C}_{0} \mathrm{a}^{6}+{ }^{6} \mathrm{C}_{1} \mathrm{a}^{5} \mathrm{~b}+{ }^{6} \mathrm{C}_{2} \mathrm{a}^{4} \mathrm{~b}^{2}+{ }^{6} \mathrm{C}_{3} \mathrm{a}^{3} \mathrm{~b}^{3}+{ }^{6} \mathrm{C}_{4} \mathrm{a}^{2} \mathrm{~b}^{4}+{ }^{6} \mathrm{C}_{5} \mathrm{ab}^{5}+{ }^{6} \mathrm{C}_{6} \mathrm{~b}^{6}$
$(\mathrm{a}-\mathrm{b})^{6}={ }^{6} \mathrm{C}_{0} \mathrm{a}^{6}-{ }^{6} \mathrm{C}_{1} \mathrm{a}^{5} \mathrm{~b}+{ }^{6} \mathrm{C}_{2} \mathrm{a}^{4} \mathrm{~b}^{2}-{ }^{6} \mathrm{C}_{3} \mathrm{a}^{3} \mathrm{~b}^{3}+{ }^{6} \mathrm{C}_{4} \mathrm{a}^{2} \mathrm{~b}^{4}-{ }^{6} \mathrm{C}_{5} \mathrm{ab}^{5}+{ }^{6} \mathrm{C}_{6} \mathrm{~b}^{6}$
Now $(\mathrm{a}+\mathrm{b})^{6}-(\mathrm{a}-\mathrm{b})^{6}={ }^{6} \mathrm{C}_{0} \mathrm{a}^{6}+{ }^{6} \mathrm{C}_{1} \mathrm{a}^{5} \mathrm{~b}+{ }^{6} \mathrm{C}_{2} \mathrm{a}^{4} \mathrm{~b}^{2}+{ }^{6} \mathrm{C}_{3} \mathrm{a}^{3} \mathrm{~b}^{3}+{ }^{6} \mathrm{C}_{4} \mathrm{a}^{2} \mathrm{~b}^{4}+{ }^{6} \mathrm{C}_{5} \mathrm{ab}^{5}+{ }^{6} \mathrm{C}_{6} \mathrm{~b}^{6}-\left[{ }^{6} \mathrm{C}_{0} \mathrm{a}^{6}-{ }^{6} \mathrm{C}_{1} \mathrm{a}^{5} \mathrm{~b}+{ }^{6} \mathrm{C}_{2} \mathrm{a}^{4} \mathrm{~b}^{2}-\right.$ $\left.{ }^{6} \mathrm{C}_{3} \mathrm{a}^{3} \mathrm{~b}^{3}+{ }^{6} \mathrm{C}_{4} \mathrm{a}^{2} \mathrm{~b}^{4}-{ }^{6} \mathrm{C}_{5} \mathrm{ab}^{5}+{ }^{6} \mathrm{C}_{6} \mathrm{~b}^{6}\right]$

Now by substituting $a=\sqrt{ } 3$ and $b=\sqrt{ } 2$, we get
$(\sqrt{ } 3+\sqrt{ } 2)^{6}-(\sqrt{ } 3-\sqrt{ } 2)^{6}=2\left[6(\sqrt{ } 3)^{5}(\sqrt{ } 2)+20(\sqrt{ } 3)^{3}(\sqrt{ } 2)^{3}+6(\sqrt{ } 3)(\sqrt{ } 2)^{5}\right]$
$=2[54(\sqrt{ } 6)+120(\sqrt{ } 6)+24 \sqrt{ } 6]$
$=2(\sqrt{ } 6)(198)$
$=396 \sqrt{ } 6$
6. Find the value of

$$
\left(a^{2}+\sqrt{a^{2}-1}\right)^{4}+\left(a^{2}-\sqrt{a^{2}-1}\right)^{4}
$$

## Solution:

Firstly the expression $(x+y)^{4}+(x-y)^{4}$ is simplified by using binomial theorem

$$
\begin{aligned}
& (x+y)^{4}={ }^{4} C_{0} x^{4}+{ }^{4} C_{1} x^{3} y+{ }^{4} C_{2} x^{2} y^{2}+{ }^{+} C_{3} x y^{3}+{ }^{4} C_{4} y^{4} \\
& =x^{4}+4 x^{3} y+6 x^{2} y^{2}+4 x y^{3}+y^{4} \\
& (x-y)^{4}={ }^{4} C_{0} x^{4}-{ }^{4} C_{1} x^{3} y+{ }^{4} C_{2} x^{2} y^{2}-{ }^{4} C_{3} x^{3}+{ }^{4} C_{4} y^{4} \\
& =x^{4}-4 x^{3} y+6 x^{2} y^{2}-4 x^{3}+y^{4} \\
& \therefore(x+y)^{4}+(x-y)^{4}=2\left(x^{4}+6 x^{2} y^{2}+y^{4}\right)
\end{aligned}
$$

Putting $\mathrm{x}=\mathrm{a}^{2}$ and $\mathrm{y}=\sqrt{\mathrm{a}^{2}-1}$, we obtain

$$
\begin{aligned}
& \left(a^{2}+\sqrt{a^{2}-1}\right)^{4}+\left(a^{2}-\sqrt{a^{2}-1}\right)^{4} \\
& =2\left[\left(a^{2}\right)^{4}+6\left(a^{2}\right)^{2}\left(\sqrt{a^{2}-1}\right)^{2}+\left(\sqrt{a^{2}-1}\right)^{4}\right] \\
& =2\left[a^{8}+6 a^{4}\left(a^{2}-1\right)+\left(a^{2}-1\right)^{2}\right] \\
& =2\left[a^{8}+6 a^{6}-6 a^{4}+a^{4}-2 a^{2}+1\right] \\
& =2\left[a^{8}+6 a^{6}-5 a^{4}-2 a^{2}+1\right] \\
& =2 a^{8}+12 a^{6}-10 a^{4}-4 a^{2}+2
\end{aligned}
$$

7. Find an approximation of $(0.99)^{5}$ using the first three terms of its expansion.

## Solution:

0.99 can be written as
$0.99=1-0.01$
Now by applying the binomial theorem, we get
$(0.99)^{5}=(1-0.01)^{5}$
$={ }^{5} C_{0}(1)^{5}-{ }^{5} C_{1}(1)^{4}(0.01)+{ }^{5} \mathrm{C}_{2}(1)^{3}(0.01)^{2}$
$=1-5(0.01)+10(0.01)^{2}$
$=1-0.05+0.001$
$=0.951$
8. Find $n$, if the ratio of the fifth term from the beginning to the fifth term from the end, in the expansion
of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^{\mathrm{n}}$, is $\sqrt{6}: 1$
Solution:
In the expansion $(a+b)^{n}$, if $n$ is even then the middle term is $(n / 2+1)^{\text {th }}$ term

$$
\begin{aligned}
& { }^{n} C_{4}(\sqrt[4]{2})^{n-1}\left(\frac{1}{\sqrt[4]{3}}\right)^{4}={ }^{n} C_{4} \frac{(\sqrt[4]{2})^{n}}{(\sqrt[4]{2})^{4}} \cdot \frac{1}{3}={ }^{n} C_{4} \frac{(\sqrt[4]{2})^{n}}{2} \cdot \frac{1}{3}=\frac{n!}{6.4!(n-4)!}(\sqrt[4]{2})^{n} \\
& { }^{n} C_{n-4}(\sqrt[4]{2})^{4}\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}={ }^{n} C_{n-1} \cdot 2 \cdot \frac{(\sqrt[4]{3})^{4}}{(\sqrt[4]{3})^{n}}={ }^{n} C_{n-1} \cdot 2 \cdot \frac{3}{(\sqrt[4]{3})^{n}}=\frac{6 n!}{(n-4)^{4!}!} \cdot \frac{1}{(\sqrt[4]{3})^{n}} \\
& \frac{n!}{6.4!(n-4)!}(\sqrt[4]{2})^{n}: \frac{6 n!}{(n-4)!!4!} \cdot \frac{1}{(\sqrt[4]{3})^{n}}=\sqrt{6}: 1 \\
& \Rightarrow \frac{(\sqrt[4]{2})^{n}}{6}: \frac{6}{(\sqrt[4]{3})^{n}}=\sqrt{6}: 1 \\
& \Rightarrow \frac{(\sqrt[4]{2})^{n}}{6} \times \frac{(\sqrt[4]{3})^{n}}{6}=\sqrt{6} \\
& \Rightarrow(\sqrt[4]{6})^{n}=36 \sqrt{6} \\
& \Rightarrow 6^{\frac{n}{4}}=66^{\frac{5}{2}} \\
& \Rightarrow \frac{n}{4}=\frac{5}{2} \\
& \Rightarrow n=4 \times \frac{5}{2}=10
\end{aligned}
$$

Thus the value of $n=10$
9. Expand using the Binomial Theorem
$\left(1+\frac{\mathrm{x}}{2}-\frac{2}{\mathrm{x}}\right)^{4}, \mathrm{x} \neq 0$

## Solution:

Using the binomial theorem, the given expression can be expanded as

$$
\begin{align*}
& {\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^{4}} \\
& ={ }^{4} C_{0}\left(1+\frac{x}{2}\right)^{4}-{ }^{4} C_{1}\left(1+\frac{x}{2}\right)^{3}\left(\frac{2}{x}\right)+{ }^{4} C_{2}\left(1+\frac{x}{2}\right)^{2}\left(\frac{2}{x}\right)^{2}-{ }^{+} C_{3}\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^{3}+{ }^{4} C_{4}\left(\frac{2}{x}\right)^{4} \\
& =\left(1+\frac{x}{2}\right)^{4}-4\left(1+\frac{x}{2}\right)^{3}\left(\frac{2}{x}\right)+6\left(1+x+\frac{x^{2}}{4}\right)\left(\frac{4}{x^{2}}\right)-4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^{3}}\right)+\frac{16}{x^{4}} \\
& =\left(1+\frac{x}{2}\right)^{4}-\frac{8}{x}\left(1+\frac{x}{2}\right)^{3}+\frac{24}{x^{2}}+\frac{24}{x}+6-\frac{32}{x^{3}}-\frac{16}{x^{2}}+\frac{16}{x^{4}} \\
& =\left(1+\frac{x}{2}\right)^{4}-\frac{8}{x}\left(1+\frac{x}{2}\right)^{3}+\frac{8}{x^{2}}+\frac{24}{x}+6-\frac{32}{x^{3}}+\frac{16}{x^{4}} \tag{1}
\end{align*}
$$

Again by using the binomial theorem to expand the above terms, we get

$$
\begin{align*}
\left(1+\frac{x}{2}\right)^{4} & ={ }^{4} C_{0}(1)^{4}+{ }^{4} C_{1}(1)^{3}\left(\frac{x}{2}\right)+{ }^{4} C_{2}(1)^{2}\left(\frac{x}{2}\right)^{2}+{ }^{4} C_{3}(1)^{1}\left(\frac{x}{2}\right)^{3}+{ }^{4} C_{4}\left(\frac{x}{2}\right)^{4} \\
& =1+4 \times \frac{x}{2}+6 \times \frac{x^{2}}{4}+4 \times \frac{x^{3}}{8}+\frac{x^{4}}{16} \\
& =1+2 x+\frac{3 x^{2}}{2}+\frac{x^{3}}{2}+\frac{x^{4}}{16}  \tag{2}\\
\left(1+\frac{x}{2}\right)^{3} & ={ }^{3} C_{0}(1)^{3}+{ }^{3} C_{1}(1)^{2}\left(\frac{x}{2}\right)+{ }^{3} C_{2}(1)\left(\frac{x}{2}\right)^{2}+{ }^{3} C_{3}\left(\frac{x}{2}\right)^{3} \\
& =1+\frac{3 x}{2}+\frac{3 x^{2}}{4}+\frac{x^{3}}{8} \tag{3}
\end{align*}
$$

From equations 1, 2 and 3, we get

$$
\begin{aligned}
& {\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^{4}} \\
& =1+2 x+\frac{3 x^{2}}{2}+\frac{x^{3}}{2}+\frac{x^{4}}{16}-\frac{8}{x}\left(1+\frac{3 x}{2}+\frac{3 x^{2}}{4}+\frac{x^{3}}{8}\right)+\frac{8}{x^{2}}+\frac{24}{x}+6-\frac{32}{x^{3}}+\frac{16}{x^{4}} \\
& =1+2 x+\frac{3}{2} x^{2}+\frac{x^{3}}{2}+\frac{x^{4}}{16}-\frac{8}{x}-12-6 x-x^{2}+\frac{8}{x^{2}}+\frac{24}{x}+6-\frac{32}{x^{3}}+\frac{16}{x^{4}} \\
& =\frac{16}{x}+\frac{8}{x^{2}}-\frac{32}{x^{3}}+\frac{16}{x^{4}}-4 x+\frac{x^{2}}{2}+\frac{x^{3}}{2}+\frac{x^{4}}{16}-5
\end{aligned}
$$

10. Find the expansion of $\left(3 x^{2}-2 a x+3 a^{2}\right)^{3}$ using binomial theorem.

## Solution:

We know that $(a+b)^{3}=a^{3}+3 a^{2} b+3 a b^{2}+b^{3}$
Putting $\mathrm{a}=3 \mathrm{x}^{2} \& \mathrm{~b}=-\mathrm{a}(2 \mathrm{x}-3 \mathrm{a})$, we get

$$
\begin{aligned}
& {\left[3 x^{2}+(-a(2 x-3 a))\right]^{3}} \\
& =\left(3 x^{2}\right)^{3}+3\left(3 x^{2}\right)^{2}(-a(2 x-3 a))+3\left(3 x^{2}\right)(-a(2 x-3 a))^{2}+(-a(2 x-3 a))^{3} \\
& =27 x^{6}-27 a x^{4}(2 x-3 a)+9 a^{2} x^{2}(2 x-3 a)^{2}-a^{3}(2 x-3 a)^{3} \\
& =27 x^{6}-54 a x^{5}+81 a^{2} x^{4}+9 a^{2} x^{2}\left(4 x^{2}-12 a x+9 a^{2}\right)-a^{3}\left[(2 x)^{3}-(3 a)^{3}-3(2 x)^{2}(3 a)+3(2 x)(3 a)^{2}\right] \\
& =27 x^{6}-54 a x^{5}+81 a^{2} x^{4}+36 a^{2} x^{4}-108 a^{3} x^{3}+81 a^{4} x^{2}-8 a^{3} x^{3}+27 a^{6}+36 a^{4} x^{2}-54 a^{5} x \\
& =27 x^{6}-54 a x^{5}+117 a^{2} x^{4}-116 a^{3} x^{3}+117 a^{4} x^{2}-54 a^{5} x+27 a^{6}
\end{aligned}
$$

Thus, $\left(3 x^{2}-2 a x+3 a^{2}\right)^{3}$
$=27 x^{6}-54 a x^{5}+117 a^{2} x^{4}-116 a^{3} x^{3}+117 a^{4} x^{2}-54 a^{5} x+27 a^{6}$

