EXERCISE 9.1

Write the first five terms of each of the sequences in Exercises 1 to 6 whose nth terms are:

1. $a_n = n (n + 2)$

Solution:

Given,

n^{th} term of a sequence $a_n = n (n + 2)$

On substituting n = 1, 2, 3, 4, and 5, we get the first five terms

$a_1 = 1(1 + 2) = 3$

$a_2 = 2(2 + 2) = 8$

$a_3 = 3(3 + 2) = 15$

$a_4 = 4(4 + 2) = 24$

$a_5 = 5(5 + 2) = 35$

Hence, the required terms are 3, 8, 15, 24, and 35.

2. $a_n = n/n+1$

Solution:

Given the n^{th} term, $a_n = n/n+1$

On substituting n = 1, 2, 3, 4, 5, we get

$$a_1 = \frac{1}{1+1} = \frac{1}{2},\ a_2 = \frac{2}{2+1} = \frac{2}{3},\ a_3 = \frac{3}{3+1} = \frac{3}{4},\ a_4 = \frac{4}{4+1} = \frac{4}{5},\ a_5 = \frac{5}{5+1} = \frac{5}{6}$$

Hence, the required terms are 1/2, 2/3, 3/4, 4/5 and 5/6.

3. $a_n = 2^n$

Solution:

Given the n^{th} term, $a_n = 2^n$

On substituting n = 1, 2, 3, 4, 5, we get

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 32$

Hence, the required terms are 2, 4, 8, 16, and 32.

4. $a_n = (2n - 3)/6$

Solution:

Given the n^{th} term, $a_n = (2n - 3)/6$

On substituting $n = 1, 2, 3, 4, 5$, we get

$$a_1 = \frac{2\times1-3}{6} = \frac{-1}{6}$$
$$a_2 = \frac{2\times2-3}{6} = \frac{1}{6}$$
$$a_3 = \frac{2\times3-3}{6} = \frac{3}{6} = \frac{1}{2}$$
$$a_4 = \frac{2\times4-3}{6} = \frac{5}{6}$$
$$a_5 = \frac{2\times5-3}{6} = \frac{7}{6}$$

Hence, the required terms are -1/6, 1/6, 1/2, 5/6 and 7/6..

5. $a_n = (-1)^{n-1}\, 5^{n+1}$

Solution:

Given the n^{th} term, $a_n = (-1)^{n-1}\, 5^{n+1}$

On substituting $n = 1, 2, 3, 4, 5$, we get

$$a_1 = (-1)^{1-1}\, 5^{1+1} = 5^2 = 25$$
$$a_2 = (-1)^{2-1}\, 5^{2+1} = -5^3 = -125$$
$$a_3 = (-1)^{3-1}\, 5^{3+1} = 5^4 = 625$$
$$a_4 = (-1)^{4-1}\, 5^{4+1} = -5^5 = -3125$$
$$a^5 = (-1)^{5-1}\, 5^{5+1} = 5^6 = 15625$$

Hence, the required terms are 25, –125, 625, –3125, and 15625.

6.

$$a_n = n\frac{n^2+5}{4}$$

Solution:

On substituting n = 1, 2, 3, 4, 5, we get the first 5 terms.

$$a_1 = 1\cdot\frac{1^2+5}{4} = \frac{6}{4} = \frac{3}{2}$$
$$a_2 = 2\cdot\frac{2^2+5}{4} = 2\cdot\frac{9}{4} = \frac{9}{2}$$
$$a_3 = 3\cdot\frac{3^2+5}{4} = 3\cdot\frac{14}{4} = \frac{21}{2}$$
$$a_4 = 4\cdot\frac{4^2+5}{4} = 21$$
$$a_5 = 5\cdot\frac{5^2+5}{4} = 5\cdot\frac{30}{4} = \frac{75}{2}$$

Hence, the required terms are 3/2, 9/2, 21/2, 21 and 75/2.

Find the indicated terms in each of the sequences in Exercises 7 to 10 whose n^{th} terms are:

7. $a_n = 4n – 3$; a_{17}, a_{24}

Solution:

Given,

The n^{th} term of the sequence is $a_n = 4n – 3$

On substituting $n = 17$, we get

$a_{17} = 4(17) – 3 = 68 – 3 = 65$

Next, on substituting $n = 24$, we get

$a_{24} = 4(24) – 3 = 96 – 3 = 93$

8. $a_n = n^2/2^n$; a^7

Solution:

Given,

The n^{th} term of the sequence is $a_n = n^2/2^n$

Now, on substituting $n = 7$, we get

$a_7 = 7^2/2^7 = 49/ 128$

9. $a_n = (-1)^{n-1} n^3$; a_9

Solution:

Given,

The n^{th} term of the sequence is $a_n = (-1)^{n-1} n^3$

On substituting $n = 9$, we get

$a_9 = (-1)^{9-1} (9)^3 = 1 \times 729 = 729$

$$a_n = \frac{n(n-2)}{n+3}; a_{20}$$

10.

Solution:

On substituting $n = 20$, we get

$$a_{20} = \frac{20(20-2)}{20+3} = \frac{20(18)}{23} = \frac{360}{23}$$

Write the first five terms of each of the sequences in Exercises 11 to 13 and obtain the corresponding series:

11. $a_1 = 3$, $a_n = 3a_{n-1} + 2$ for all $n > 1$

Solution:

Given, $a_n = 3a_{n-1} + 2$ and $a_1 = 3$

Then,

$a_2 = 3a_1 + 2 = 3(3) + 2 = 11$

$a_3 = 3a_2 + 2 = 3(11) + 2 = 35$

$a_4 = 3a_3 + 2 = 3(35) + 2 = 107$

$a_5 = 3a_4 + 2 = 3(107) + 2 = 323$

Thus, the first 5 terms of the sequence are 3, 11, 35, 107 and 323.

Hence, the corresponding series is

3 + 11 + 35 + 107 + 323 …….

12. $a_1 = -1$, $a_n = a_{n-1}/n$, $n \geq 2$

Solution:

Given,

$a_n = a_{n-1}/n$ and $a_1 = -1$

Then,

$a_2 = a_1/2 = -1/2$

$a_3 = a_2/3 = -1/6$

$a_4 = a_3/4 = -1/24$

$a_5 = a_4/5 = -1/120$

Thus, the first 5 terms of the sequence are -1, -1/2, -1/6, -1/24 and -1/120.

Hence, the corresponding series is

-1 + (-1/2) + (-1/6) + (-1/24) + (-1/120) + …….

13. $a_1 = a_2 = 2$, $a_n = a_{n-1} – 1$, $n > 2$

Solution:

Given,

$a_1 = a_2$, $a_n = a_{n-1} – 1$

Then,

$a_3 = a_2 – 1 = 2 – 1 = 1$

$a_4 = a_3 – 1 = 1 – 1 = 0$

$a_5 = a_4 – 1 = 0 – 1 = -1$

Thus, the first 5 terms of the sequence are 2, 2, 1, 0 and -1.

The corresponding series is

2 + 2 + 1 + 0 + (-1) + ……

14. The Fibonacci sequence is defined by

$1 = a_1 = a_2$ and $a_n = a_{n-1} + a_{n-2}$, $n > 2$

Find a_{n+1}/a_n, for n = 1, 2, 3, 4, 5

Solution:

Given,

$1 = a_1 = a_2$

$a_n = a_{n-1} + a_{n-2}$, $n > 2$

So,

$a_3 = a_2 + a_1 = 1 + 1 = 2$

$a_4 = a_3 + a_2 = 2 + 1 = 3$

$a_5 = a_4 + a_3 = 3 + 2 = 5$

$a_6 = a_5 + a_4 = 5 + 3 = 8$

Thus,

$$\text{For } n = 1,\ \frac{a_n + 1}{a_n} = \frac{a_2}{a_1} = \frac{1}{1} = 1$$

$$\text{For } n = 2,\ \frac{a_n + 1}{a_n} = \frac{a_3}{a_2} = \frac{2}{1} = 2$$

$$\text{For } n = 3,\ \frac{a_n + 1}{a_n} = \frac{a_4}{a_3} = \frac{3}{2}$$

$$\text{For } n = 4,\ \frac{a_n + 1}{a_n} = \frac{a_5}{a_4} = \frac{5}{3}$$

$$\text{For } n = 5,\ \frac{a_n + 1}{a_n} = \frac{a_6}{a_5} = \frac{8}{5}$$

EXERCISE 9.2

1. Find the sum of odd integers from 1 to 2001.

Solution:

The odd integers from 1 to 2001 are 1, 3, 5, …1999, 2001.

It clearly forms a sequence in A.P.

Where the first term, $a = 1$

The common difference, $d = 2$

Now,

$a + (n - 1)d = 2001$

$1 + (n-1)(2) = 2001$

$2n – 2 = 2000$

$2n = 2000 + 2 = 2002$

$n = 1001$

We know,

$S_n = n/2 [2a + (n-1)d]$

$$S_n = \frac{1001}{2}\left[2\times 1+(1001-1)\times 2\right]$$
$$= \frac{1001}{2}\left[2+1000\times 2\right]$$
$$= \frac{1001}{2}\times 2002$$
$$= 1001\times 1001$$
$$= 1002001$$

Therefore, the sum of odd numbers from 1 to 2001 is 1002001.

2. Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

Solution:

The natural numbers lying between 100 and 1000, which are multiples of 5, are 105, 110, … 995.

It clearly forms a sequence in A.P.

Where the first term, $a = 105$

The common difference, $d = 5$

Now,

$a + (n - 1)d = 995$

$105 + (n - 1)(5) = 995$

$105 + 5n - 5 = 995$

$5n = 995 - 105 + 5 = 895$

$n = 895/5$

$n = 179$

We know,

$S_n = n/2\ [2a + (n-1)d]$

$$S_n = \frac{179}{2}\left[2(105) + (179-1)(5)\right]$$
$$= \frac{179}{2}\left[2(105) + (178)(5)\right]$$
$$= 179\left[105 + (89)5\right]$$
$$= (179)(105 + 445)$$
$$= (179)(550)$$
$$= 98450$$

Therefore, the sum of all natural numbers lying between 100 and 1000, which are multiples of 5, is 98450.

3. In an A.P, the first term is 2, and the sum of the first five terms is one-fourth of the next five terms. Show that the 20th term is –112.

Solution:

Given,

The first term (a) of an A.P = 2

Let's assume d is the common difference of the A.P.

So, the A.P. will be 2, $2 + d$, $2 + 2d$, $2 + 3d$, …

Then,

Sum of first five terms = $10 + 10d$

Sum of next five terms = 10 + 35*d*

From the question, we have

10 + 10d = ¼ (10 + 35d)

40 + 40d = 10 + 35d

30 = -5d

d = -6

a_{20} = a + (20 – 1)d = 2 + (19) (-6) = 2 – 114 = -112

Therefore, the 20^{th} term of the A.P. is –112.

4. How many terms of the A.P. -6, -11/2, -5, …. are needed to give the sum –25?

Solution:

Let's consider the sum of *n* terms of the given A.P. as –25.

We known that,

S_n = n/2 [2a + (n-1)d]

where *n* = number of terms, *a* = first term, and *d* = common difference

So here, *a* = –6

d = -11/2 + 6 = (-11 + 12)/ 2 = 1/2

Thus, we have

$$-25 = \frac{n}{2}\left[2\times(-6)+(n-1)\left(\frac{1}{2}\right)\right]$$
$$-50 = n\left[-12+\frac{n}{2}-\frac{1}{2}\right]$$
$$-50 = n\left[-\frac{25}{2}+\frac{n}{2}\right]$$
$$-100 = n(-25+n)$$
$$n^2 - 25n + 100 = 0$$
$$n^2 - 5n - 20n + 100 = 0$$
$$n(n-5) - 20(n-5) = 0$$
$$n = 20 \text{ or } 5$$

5. In an A.P., if p^{th} term is 1/q and q^{th} term is 1/p, prove that the sum of first *pq* terms is ½ (pq + 1) where p ≠ q.

Solution:

We know that the general term of an A.P is given by: $a_n = a + (n-1)d$

From the question, we have

$$p^{th} \text{ term} = a_p = a + (p-1)d = \frac{1}{q} \qquad ...(1)$$

$$q^{th} \text{ term} = a_q = a + (q-1)d = \frac{1}{p} \qquad ...(2)$$

Subtracting (2) from (1), we have

$$(p-1)d - (q-1)d = \frac{1}{q} - \frac{1}{p}$$

$$(p-1-q+1)d = \frac{p-q}{pq}$$

$$(p-q)d = \frac{p-q}{pq}$$

$$d = \frac{1}{pq}$$

Using the value of d in (1), we get

$$a + (p-1)\frac{1}{pq} = \frac{1}{q}$$

$$\Rightarrow a = \frac{1}{q} - \frac{1}{q} + \frac{1}{pq} = \frac{1}{pq}$$

$$S_{pq} = \frac{pq}{2}\left[2a + (pq-1)d\right]$$

$$= \frac{pq}{2}\left[\frac{2}{pq} + (pq-1)\frac{1}{pq}\right]$$

$$= 1 + \frac{1}{2}(pq-1)$$

$$= \frac{1}{2}pq + 1 - \frac{1}{2} = \frac{1}{2}pq + \frac{1}{2}$$

$$= \frac{1}{2}(pq+1)$$

Therefore, the sum of first pq terms of the A.P is $\frac{1}{2}(pq+1)$

6. If the sum of a certain number of terms of the A.P. 25, 22, 19, … is 116. Find the last term.

Solution:

Given A.P.,

25, 22, 19, …

Here,

First term, a = 25 and

Common difference, d = 22 – 25 = -3

Also given, the sum of a certain number of terms of the A.P. is 116.

The number of terms is n.

So, we have

S_n = n/2 [2a + (n-1)d] = 116

116 = n/2 [2(25) + (n-1)(-3)]

116 x 2 = n [50 – 3n + 3]

232 = n [53 – 3n]

232 = 53n – $3n^2$

$3n^2$ – 53n + 232 = 0

$3n^2$ – 24n – 29n+ 232 = 0

3n(n – 8) – 29(n – 8) = 0

(3n – 29) (n – 8) = 0

Hence,

n = 29/3 or n = 8

As n can only be an integral value, n = 8

Thus, the 8^{th} term is the last term of the A.P.

a_8 = 25 + (8 – 1)(-3)

= 25 – 21

= 4

7. Find the sum to *n* terms of the A.P., whose k^{th} term is 5*k* + 1.

Solution:

Given, the k^{th} term of the A.P. is 5*k* + 1.

k^{th} term = $a_k = a + (k – 1)d$

And,

$a + (k - 1)d = 5k + 1$

$a + kd - d = 5k + 1$

On comparing the coefficient of k, we get $d = 5$

$a - d = 1$

$a - 5 = 1$

$\Rightarrow a = 6$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$= \frac{n}{2}\left[2(6) + (n-1)(5)\right]$$
$$= \frac{n}{2}\left[12 + 5n - 5\right]$$
$$= \frac{n}{2}(5n + 7)$$

8. If the sum of n terms of an A.P. is ($pn + qn^2$), where p and q are constants, find the common difference.

Solution:

We know that,

$S_n = n/2\ [2a + (n-1)d]$

From the question, we have

$$\frac{n}{2}\left[2a + (n-1)d\right] = pn + qn^2$$
$$\frac{n}{2}\left[2a + nd - d\right] = pn + qn^2$$
$$na + n^2\frac{d}{2} - n\cdot\frac{d}{2} = pn + qn^2$$

On comparing the coefficients of n^2 on both sides, we get

$d/2 = q$

Hence, $d = 2q$

Therefore, the common difference of the A.P. is $2q$.

9. The sums of *n* terms of two arithmetic progressions are in the ratio 5*n* + 4: 9*n* + 6. Find the ratio of their 18th terms.

Solution:

Let a_1, a_2, and d_1, d_2 be the first terms and the common difference of the first and second arithmetic progression, respectively.

Then, from the question, we have

$$\frac{\text{Sum of } n \text{ terms of first A.P.}}{\text{Sum of } n \text{ terms of second A.P.}} = \frac{5n+4}{9n+6}$$

$$\frac{\frac{n}{2}\left[2a_1+(n-1)d_1\right]}{\frac{n}{2}\left[2a_2+(n-1)d_2\right]} = \frac{5n+4}{9n+6}$$

$$\frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{5n+4}{9n+6} \qquad \ldots(1)$$

Substituting $n = 35$ in (1), we get

$$\frac{2a_1+34d_1}{2a_2+34d_2} = \frac{5(35)+4}{9(35)+6}$$

$$\frac{a_1+17d_1}{a_2+17d_2} = \frac{179}{321} \qquad \ldots(2)$$

$$\frac{18^{\text{th}} \text{ term of first A.P.}}{18^{\text{th}} \text{ term of second A.P}} = \frac{a_1+17d_1}{a_2+17d_2} \qquad \ldots(3)$$

From (2) and (3), we have

$$\frac{18^{\text{th}} \text{ term of first A.P.}}{18^{\text{th}} \text{ term of second A.P.}} = \frac{179}{321}$$

Therefore, the ratio of 18^{th} term of both the A.P.s is 179: 321.

10. If the sum of the first *p* terms of an A.P. is equal to the sum of the first *q* terms, then find the sum of the first (*p* + *q*) terms.

Solution:

Let's take *a* and *d* to be the first term and the common difference of the A.P., respectively.

Then, it is given that

$$S_p = \frac{p}{2}\left[2a + (p-1)d\right]$$

$$S_q = \frac{q}{2}\left[2a + (q-1)d\right]$$

From the question, we have

$$\frac{p}{2}\left[2a + (p-1)d\right] = \frac{q}{2}\left[2a + (q-1)d\right]$$

$$p\left[2a + (p-1)d\right] = q\left[2a + (q-1)d\right]$$

$$2ap + pd(p-1) = 2aq + qd(q-1)$$

$$2a(p-q) + d\left[p(p-1) - q(q-1)\right] = 0$$

$$2a(p-q) + d\left[p^2 - p - q^2 + q\right] = 0$$

$$2a(p-q) + d\left[(p-q)(p+q) - (p-q)\right] = 0$$

$$2a(p-q) + d\left[(p-q)(p+q-1)\right] = 0$$

$$2a + d(p+q-1) = 0$$

$$\Rightarrow d = \frac{-2a}{p+q-1} \qquad \text{..........} \ (i)$$

So the sum of (p + q) terms will be,

$$S_{p+q} = \frac{p+q}{2}\left[2a + (p+q-1)\cdot d\right]$$

$$S_{p+q} = \frac{p+q}{2}\left[2a + (p+q-1)\left(\frac{-2a}{p+q-1}\right)\right] \quad \text{[From (i)]}$$

$$= \frac{p+q}{2}\left[2a - 2a\right]$$

$$= 0$$

Therefore, the sum of (p + q) terms of the A.P. is 0.

11. Sum of the first *p, q* and *r* terms of an A.P. are *a, b* and *c,* respectively.

$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0$$ **Prove that**

Solution:

Let a_1 and d be the first term and the common difference of the A.P., respectively.

Then, according to the question, we have

$$S_p = \frac{p}{2}\left[2a_1 + (p-1)d\right] = a$$

$$\Rightarrow 2a_1 + (p-1)d = \frac{2a}{p} \quad ...(1)$$

$$S_q = \frac{q}{2}\left[2a_1 + (q-1)d\right] = b$$

$$\Rightarrow 2a_1 + (q-1)d = \frac{2b}{q} \quad ...(2)$$

$$S_r = \frac{r}{2}\left[2a_1 + (r-1)d\right] = c$$

$$\Rightarrow 2a_1 + (r-1)d = \frac{2c}{r} \quad ...(3)$$

Now, subtracting (2) from (1), we get

$$(p-1)d - (q-1)d = \frac{2a}{p} - \frac{2b}{q}$$

$$d(p-1-q+1) = \frac{2aq-2bp}{pq}$$

$$d(p-q) = \frac{2aq-2bp}{pq}$$

$$d = \frac{2(aq-bp)}{pq(p-q)} \quad ....(4)$$

Then, subracting (3) from (2), we get

$$(q-1)d-(r-1)d=\frac{2b}{q}-\frac{2c}{r}$$

$$d(q-1-r+1)=\frac{2b}{q}-\frac{2c}{r}$$

$$d(q-r)=\frac{2br-2qc}{qr}$$

$$d=\frac{2(br-qc)}{qr(q-r)} \qquad ...(5)$$

On equating both the values of d obtained in (4) and (5), we get

$$\frac{aq-bp}{pq(p-q)}=\frac{br-qc}{qr(q-r)}$$

$$\frac{aq-bp}{p(p-q)}=\frac{br-qc}{r(q-r)}$$

$$r(q-r)(aq-bp)=p(p-q)(br-qc)$$

$$r(aq-bp)(q-r)=p(br-qc)(p-q)$$

$$(aqr-bpr)(q-r)=(bpr-cpq)(p-q)$$

Dividing both sides by pqr, we have

$$\left(\frac{a}{p}-\frac{b}{q}\right)(q-r)=\left(\frac{b}{q}-\frac{c}{r}\right)(p-q)$$

$$\frac{a}{p}(q-r)-\frac{b}{q}(q-r+p-q)+\frac{c}{r}(p-q)=0$$

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

Hence, the given result is proved.

12. The ratio of the sums of m and n terms of an A.P. is m^2: n^2. Show that the ratio of the m^{th} and the n^{th} term is $(2m – 1)$: $(2n – 1)$.

Solution:

Let’s consider that a and b are the first term and the common difference of the A.P., respectively.

Then, from the question, we have

$$\frac{\text{Sum of m terms}}{\text{Sum of n terms}} = \frac{m^2}{n^2}$$

$$\frac{\frac{m}{2}\left[2a+(m-1)d\right]}{\frac{n}{2}\left[2a+(n-1)d\right]} = \frac{m^2}{n^2}$$

$$\frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n} \quad \text{....... (1)}$$

Putting $m = 2m - 1$ and $n = 2n - 1$ in (1), we get

$$\frac{2a+(2m-2)d}{2a+(2n-2)d} = \frac{2m-1}{2n-1}$$

$$\Rightarrow \frac{a+(m-1)d}{a+(n-1)d} = \frac{2m-1}{2n-1} \quad \text{...... (2)}$$

Now,

$$\frac{m^{th} \text{ term of A.P.}}{n^{th} \text{ term of A.P.}} = \frac{a+(m-1)d}{a+(n-1)d} \quad \text{...... (3)}$$

From (2) and (3), we have

$$\frac{m^{th} \text{ term of A.P}}{n^{th} \text{term of A.P}} = \frac{2m-1}{2n-1}$$

Hence, the given result is proved.

13. If the sum of *n* terms of an A.P. is $3n^2 + 5n$ and its m^{th} term is 164, find the value of *m*.

Solution:

Let's consider *a* and *b* to be the first term and the common difference of the A.P., respectively.

$a_m = a + (m - 1)d = 164$ … (1)

The sum of the terms is given by,

$S_n = n/2\ [2a + (n-1)d]$

$\frac{n}{2}[2a + nd - d] = 3n^2 + 5n$

$na + \frac{d}{2}n^2 - \frac{d}{2}n = 3n^2 + 5n$

$\frac{d}{2}n^2 + (a - \frac{d}{2})n = 3n^2 + 5n$

On comparing the coefficient of n^2 on both sides, we get

$\frac{d}{2} = 3$

$\Rightarrow d = 6$

On comparing the coefficient of n on both sides, we get

$a - \frac{d}{2} = 5$

$a - 3 = 5$

$a = 8$

Hence, from (1), we get

$8 + (m - 1) 6 = 164$

$(m - 1) 6 = 164 - 8 = 156$

$m - 1 = 26$

$m = 27$

Therefore, the value of m is 27.

14. Insert five numbers between 8 and 26 such that the resulting sequence is an A.P.

Solution:

Let's assume A_1, A_2, A_3, A_4, and A_5 to be five numbers between 8 and 26 such that 8, A_1, A_2, A_3, A_4, A_5, 26 are in an A.P.

Here, we have,

$a = 8$, $b = 26$, $n = 7$

So,

$26 = 8 + (7 - 1) d$

$6d = 26 - 8 = 18$

$d = 3$

Now,

$A_1 = a + d = 8 + 3 = 11$

$A_2 = a + 2d = 8 + 2 \times 3 = 8 + 6 = 14$

$A_3 = a + 3d = 8 + 3 \times 3 = 8 + 9 = 17$

$A_4 = a + 4d = 8 + 4 \times 3 = 8 + 12 = 20$

$A_5 = a + 5d = 8 + 5 \times 3 = 8 + 15 = 23$

Therefore, the required five numbers between 8 and 26 are 11, 14, 17, 20, and 23.

15. If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between *a* and *b*, then find the value of *n*.

Solution:

The A.M between a and b is given by (a + b)/2

Then, according to the question,

$$\frac{a+b}{2} = \frac{a^n + b^n}{a^{n-1} + b^{n-1}}$$
$$(a+b)(a^{n-1} + b^{n-1}) = 2(a^n + b^n)$$
$$a^n + ab^{n-1} + ba^{n-1} + b^n = 2a^n + 2b^n$$
$$ab^{n-1} + a^{n-1}b = a^n + b^n$$
$$ab^{n-1} - b^n = a^n - a^{n-1}b$$
$$b^{n-1}(a-b) = a^{n-1}(a-b)$$
$$b^{n-1} = a^{n-1}$$
$$\left(\frac{a}{b}\right)^{n-1} = 1 = \left(\frac{a}{b}\right)^0$$
$$n-1 = 0$$
$$n = 1$$

Thus, the value of n is 1.

16. Between 1 and 31, *m* numbers have been inserted in such a way that the resulting sequence is an A.P. and the ratio of 7th and $(m – 1)^{th}$ numbers is 5: 9. Find the value of *m*.

Solution:

Let's consider $a_1, a_2, \ldots a_m$ be *m* numbers such that $1, a_1, a_2, \ldots a_m, 31$ is an A.P.

And here,

$a = 1$, $b = 31$, $n = m + 2$

So, $31 = 1 + (m + 2 – 1)(d)$

$30 = (m + 1)d$

d = 30/ (m + 1) ……. (1)

Now,

$a_1 = a + d$

$a_2 = a + 2d$

$a_3 = a + 3d$ …

Hence, $a_7 = a + 7d$

$a_{m-1} = a + (m - 1) d$

According to the question, we have

$$\frac{a+7d}{a+(m-1)d}=\frac{5}{9}$$

$$\frac{1+7\left(\frac{30}{(m+1)}\right)}{1+(m-1)\left(\frac{30}{m+1}\right)}=\frac{5}{9} \qquad \text{[From (1)]}$$

$$\frac{m+1+7(30)}{m+1+30(m-1)}=\frac{5}{9}$$

$$\frac{m+1+210}{m+1+30m-30}=\frac{5}{9}$$

$$\frac{m+211}{31m-29}=\frac{5}{9}$$

$$9m+1899=155m-145$$

$$155m-9m=1899+145$$

$$146m=2044$$

$$m=14$$

Therefore, the value of m is 14.

17. A man starts repaying a loan as the first instalment of Rs. 100. If he increases the instalment by Rs 5 every month, what amount will he pay in the 30^{th} instalment?

Solution:

Given,

The first instalment of the loan is Rs 100.

The second instalment of the loan is Rs 105, and so on as the instalment increases by Rs 5 every month.

Thus, the amount that the man repays every month forms an A.P.

And then, A.P. is 100, 105, 110, …

Where the first term, $a = 100$

Common difference, $d = 5$

So, the 30th term in this A.P. will be

$A_{30} = a + (30 - 1)d$

$= 100 + (29)(5)$

$= 100 + 145$

$= 245$

Therefore, the amount to be paid in the 30th instalment will be Rs 245.

18. The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of the sides of the polygon.

Solution:

It's understood from the question that the angles of the polygon will form an A.P. with a common difference $d = 5°$ and first term $a = 120°$.

And we know that the sum of all angles of a polygon with n sides is $180° (n - 2)$.

Thus, we can say

$S_n = 180°(n-2)$

$\frac{n}{2}\left[2a+(n-1)d\right] = 180°(n-2)$

$\frac{n}{2}\left[240°+(n-1)5°\right] = 180(n-2)$

$n\left[240+(n-1)5\right] = 360(n-2)$

$240n+5n^2-5n = 360n-720$

$5n^2+235n-360n+720=0$

$5n^2-125n+720=0$

$n^2-25n+144=0$

$n^2-16n-9n+144=0$

$n(n-16)-9(n-16)=0$

$(n-9)(n-16)=0$

$n=9 \text{ or } 16$

Thus, a polygon having 9 and 16 sides will satisfy the condition in the question.

EXERCISE 9.3 PAGE NO: 192

1. Find the 20th and *n*th terms of the G.P. 5/2, 5/4, 5/8, ………

Solution:

Given G.P. is 5/2, 5/4, 5/8, ………

Here, a = First term = 5/2

r = Common ratio = (5/4)/(5/2) = ½

Thus, the 20th term and nth term

$$a_{20} = ar^{20-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{19} = \frac{5}{(2)(2)^{19}} = \frac{5}{(2)^{20}}$$

$$a_n = ar^{n-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{5}{(2)(2)^{n-1}} = \frac{5}{(2)^n}$$

2. Find the 12th term of a G.P. whose 8th term is 192, and the common ratio is 2.

Solution:

Given,

The common ratio of the G.P., $r = 2$

And, let a be the first term of the G.P.

Now,

$a_8 = ar^{8-1} = ar^7$

$ar^7 = 192$

$a(2)^7 = 192$

$a(2)^7 = (2)^6 (3)$

So,

$$a = \frac{(2)^6 \times 3}{(2)^7} = \frac{3}{2}$$

Hence,

$$a_{12} = ar^{12-1} = \left(\frac{3}{2}\right)(2)^{11} = (3)(2)^{10} = 3072$$

3. The 5th, 8th and 11th terms of a G.P. are *p*, *q* and *s*, respectively. Show that $q^2 = ps$.

Solution:

Let's take a to be the first term and r to be the common ratio of the G.P.

Then, according to the question, we have

$a_5 = a\, r^{5-1} = a\, r^4 = p$ … (i)

$a_8 = a\, r^{8-1} = a\, r^7 = q$ … (ii)

$a_{11} = a\, r^{11-1} = a\, r^{10} = s$ … (iii)

Dividing equation (ii) by (i), we get

$$\frac{ar^7}{ar^4} = \frac{q}{p}$$

$$r^3 = \frac{q}{p} \quad \text{..... (iv)}$$

On dividing equation (iii) by (ii), we get

$$\frac{ar^{10}}{ar^7} = \frac{s}{q}$$

$$r^3 = \frac{s}{q} \quad \text{..... (v)}$$

Equating the values of r^3 obtained in (iv) and (v), we get

$$\frac{q}{p} = \frac{s}{q}$$

$$q^2 = ps$$

Hence proved.

4. The 4th term of a G.P. is the square of its second term, and the first term is –3. Determine its 7th term.

Solution:

Let's consider a to be the first term and r to be the common ratio of the G.P.

Given, $a = -3$

And we know that,

$a_n = ar^{n-1}$

So, $a_4 = ar^3 = (-3)\, r^3$

$a_2 = a\, r^1 = (-3)\, r$

Then, from the question, we have

$(-3)\, r^3 = [(-3)\, r]^2$

$\Rightarrow -3r^3 = 9\, r^2$

$\Rightarrow r = -3$

$a_7 = a\, r^{7-1} = a\, r^6 = (-3)\,(-3)^6 = -(3)^7 = -2187$

Therefore, the seventh term of the G.P. is –2187.

5. Which term of the following sequences:

(a) 2, 2√2, 4,… is 128 ? (b) √3, 3, 3√3,… is 729 ?

(c) 1/3, 1/9, 1/27, … is 1/19683 ?

Solution:

(a) The given sequence, 2, 2√2, 4,…

We have,

a = 2 and r = 2√2/2 = √2

Taking the nth term of this sequence as 128, we have

$$a_n = a\,r^{n-1}$$
$$(2)\left(\sqrt{2}\right)^{n-1} = 128$$
$$(2)(2)^{\frac{n-1}{2}} = (2)^7$$
$$(2)^{\frac{n-1}{2}+1} = (2)^7$$
$$\frac{n-1}{2} + 1 = 7$$
$$\frac{n-1}{2} = 6$$
$$n - 1 = 12$$
$$n = 13$$

Therefore, the 13th term of the given sequence is 128.

(ii) Given the sequence, √3, 3, 3√3,…

We have,

a = √3 and r = 3/√3 = √3

Taking the nth term of this sequence to be 729, we have

$$a_n = ar^{n-1}$$
$$\therefore ar^{n-1} = 729$$
$$(\sqrt{3})(\sqrt{3})^{n-1} = 729$$
$$(3)^{\frac{1}{2}}(3)^{\frac{n-1}{2}} = (3)^6$$
$$(3)^{\frac{1}{2}+\frac{n-1}{2}} = (3)^6$$

Equating the exponents, we have

$$\frac{1}{2}+\frac{n-1}{2} = 6$$
$$\frac{1+n-1}{2} = 6$$
$$\therefore n = 12$$

Therefore, the 12th term of the given sequence is 729.

(iii) Given sequence, 1/3, 1/9, 1/27, …

a = 1/3 and r = (1/9)/(1/3) = 1/3

Taking the nth term of this sequence to be 1/19683, we have

$$a_n = ar^{n-1}$$
$$\therefore ar^{n-1} = \frac{1}{19683}$$
$$\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1} = \frac{1}{19683}$$
$$\left(\frac{1}{3}\right)^{n} = \left(\frac{1}{3}\right)^{9}$$
$$n = 9$$

Therefore, the 9th term of the given sequence is 1/19683.

6. For what values of *x*, the numbers -2/7, x, -7/2 are in G.P?

Solution:

The given numbers are -2/7, x, -7/2

Common ratio = x/(-2/7) = -7x/2

Also, common ratio = (-7/2)/x = -7/2x

$$\therefore \frac{-7x}{2} = \frac{-7}{2x}$$
$$x^2 = \frac{-2\times 7}{-2\times 7} = 1$$
$$x = \sqrt{1}$$
$$x = \pm 1$$

Therefore, for $x = \pm 1$, the given numbers will be in G.P.

7. Find the sum to 20 terms in the geometric progression 0.15, 0.015, 0.0015 …

Solution:

Given G.P., 0.15, 0.015, 0.00015, …

Here, $a = 0.15$ and r = 0.015/0.15 = 0.1

We know that, $S_n = \frac{a(1-r^n)}{1-r}$

$$\therefore S_{20} = \frac{0.15\left[1-(0.1)^{20}\right]}{1-0.1}$$
$$= \frac{0.15}{0.9}\left[1-(0.1)^{20}\right]$$
$$= \frac{15}{90}\left[1-(0.1)^{20}\right]$$
$$= \frac{1}{6}\left[1-(0.1)^{20}\right]$$

8. Find the sum to *n* terms in the geometric progression √7, √21, 3√7, ….

Solution:

The given G.P. is √7, √21, 3√7, ….

Here,

a = √7 and

$$r = \frac{\sqrt{21}}{\sqrt{7}} = \sqrt{3}$$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\therefore S_n = \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}}$$

$$= \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} \qquad \text{(By rationalizing)}$$

$$= \frac{\sqrt{7}\left(1+\sqrt{3}\right)\left[1-\left(\sqrt{3}\right)^n\right]}{1-3}$$

$$= \frac{-\sqrt{7}\left(1+\sqrt{3}\right)}{2}\left[1-(3)^{\frac{n}{2}}\right]$$

$$= \frac{\sqrt{7}\left(1+\sqrt{3}\right)}{2}\left[(3)^{\frac{n}{2}}-1\right]$$

9. Find the sum to *n* terms in the geometric progression 1, -a, a^2, $-a^3$ …. (if a ≠ -1)

Solution:

The given G.P. is 1, -a, a^2, $-a^3$ ….

Here, the first term = a_1 = 1

And the common ratio = $r = -a$

We know that,

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

$$\therefore S_n = \frac{1\left[1-(-a)^n\right]}{1-(-a)} = \frac{\left[1-(-a)^n\right]}{1+a}$$

10. Find the sum to *n* terms in the geometric progression x^3, x^5, x^7, … (if x ≠ ±1)

Solution:

Given G.P. is x^3, x^5, x^7, …

Here, we have $a = x^3$ and $r = x^5/x^3 = x^2$

We know that, $S_n = \dfrac{a(1-r^n)}{1-r}$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{x^3\left[1-(x^2)^n\right]}{1-x^2} = \frac{x^3(1-x^{2n})}{1-x^2}$$

11. Evaluate: $\sum_{k=1}^{11}(2+3^k)$

Solution:

$$\sum_{k=1}^{11}(2+3^k) = \sum_{k=1}^{11}(2) + \sum_{k=1}^{11}3^k = 2(11) + \sum_{k=1}^{11}3^k = 22 + \sum_{k=1}^{11}3^k \qquad ...(1)$$

$$\sum_{k=1}^{11}3^k = 3^1 + 3^2 + 3^3 + ... + 3^{11}$$

We can see that, the terms of this sequence 3, 3^2, 3^3, ... forms a G.P

And, we know

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_{11} = \frac{3\left[(3)^{11}-1\right]}{3-1}$$

$$S_{11} = \frac{3}{2}(3^{11}-1)$$

$$\therefore \sum_{k=1}^{11}3^k = \frac{3}{2}(3^{11}-1)$$

On substituting the above value in equation (1), we get

$$\sum_{k=1}^{11}(2+3^k) = 22 + \frac{3}{2}(3^{11}-1)$$

12. The sum of the first three terms of a G.P. is 39/10, and their product is 1. Find the common ratio and the terms.

Solution:

Let a/r, a, ar be the first three terms of the G.P.

a/r + a + ar = 39/10 …… (1)

(a/r) (a) (ar) = 1 …….. (2)

From (2), we have

$a^3 = 1$

Hence, a = 1 [Considering real roots only]

Substituting the value of a in (1), we get

$1/r + 1 + r = 39/10$

$(1 + r + r^2)/r = 39/10$

$10 + 10r + 10r^2 = 39r$

$10r^2 – 29r + 10 = 0$

$10r^2 – 25r – 4r + 10 = 0$

$5r(2r – 5) – 2(2r – 5) = 0$

$(5r – 2) (2r – 5) = 0$

Thus,

r = 2/5 or 5/2

Therefore, the three terms of the G.P. are 5/2, 1 and 2/5.

13. How many terms of G.P. 3, 3^2, 3^3, … are needed to give the sum 120?

Solution:

Given G.P. is 3, 3^2, 3^3, …

Let's consider that *n* terms of this G.P. be required to obtain the sum 120.

We know that,

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

Here, $a = 3$ and $r = 3$

$$S_n = 120 = \frac{3\left(3^n - 1\right)}{3-1}$$

$$120 = \frac{3\left(3^n - 1\right)}{2}$$

$$\frac{120 \times 2}{3} = 3^n - 1$$

$$3^n - 1 = 80$$

$$3^n = 81$$

$$3^n = 3^4$$

Equating the exponents, we get $n = 4$

Therefore, four terms of the given G.P. are required to obtain the sum 120.

14. The sum of the first three terms of a G.P. is 16, and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to *n* terms of the G.P.

Solution:

Let's assume the G.P. to be $a, ar, ar^2, ar^3, \dots$

Then, according to the question, we have

$a + ar + ar^2 = 16$ and $ar^3 + ar^4 + ar^5 = 128$

$a(1 + r + r^2) = 16$ … (1) and,

$ar^3(1 + r + r^2) = 128$ … (2)

Dividing equation (2) by (1), we get

$$\frac{ar^3\left(1+r+r^2\right)}{a\left(1+r+r^2\right)} = \frac{128}{16}$$

$r^3 = 8$

$r = 2$

Now, using $r = 2$ in (1), we get

$a(1 + 2 + 4) = 16$

$a(7) = 16$

$a = 16/7$

Now, the sum of terms is given as

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow S_n = \frac{16}{7}\frac{(2^n - 1)}{2 - 1} = \frac{16}{7}(2^n - 1)$$

15. Given a G.P. with a = 729 and 7th term 64, determine S_7.

Solution:

Given,

$a = 729$ and $a_7 = 64$

Let r be the common ratio of the G.P.

Then, we know that, $a_n = a\, r^{n-1}$

$a_7 = ar^{7-1} = (729)r^6$

$\Rightarrow 64 = 729\, r^6$

$r^6 = 64/729$

$r^6 = (2/3)^6$

r = 2/3

And we know that

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

So,

$$S_7 = \frac{729\left[1 - \left(\frac{2}{3}\right)^7\right]}{1 - \frac{2}{3}}$$

$$= 3 \times 729\left[1 - \left(\frac{2}{3}\right)^7\right]$$

$$= (3)^7\left[\frac{(3)^7 - (2)^7}{(3)^7}\right]$$

$$= (3)^7 - (2)^7$$

$$= 2187 - 128$$

$$= 2059$$

16. Find a G.P. for which the sum of the first two terms is –4 and the fifth term is 4 times the third term.

Solution:

Consider *a* to be the first term and *r* to be the common ratio of the G.P.

Given, $S_2 = -4$

Then, from the question, we have

$$S_2 = -4 = \frac{a(1-r^2)}{1-r} \qquad ...(1)$$

And,

$a_5 = 4 \times a_3$

$ar^4 = 4ar^2$

$r^2 = 4$

$r = \pm 2$

Using the value of r in (1), we have

$$-4 = \frac{a\left[1-(2)^2\right]}{1-2} \text{ for } r = 2$$
$$-4 = \frac{a(1-4)}{-1}$$
$$-4 = a(3)$$
$$a = \frac{-4}{3}$$
$$\text{Also, } -4 = \frac{a\left[1-(-2)^2\right]}{1-(-2)} \text{ for } r = -2$$
$$-4 = \frac{a(1-4)}{1+2}$$
$$-4 = \frac{a(-3)}{3}$$
$$a = 4$$

Therefore, the required G.P is

-4/3, -8/3, -16/3, …. Or 4, -8, 16, -32, ……

17. If the 4th, 10th and 16th terms of a G.P. are *x, y* and *z*, respectively. Prove that *x, y, and z* are in G.P.

Solution:

Let a be the first term and r be the common ratio of the G.P.

According to the given condition,

$a_4 = a r^3 = x$ … (1)

$a_{10} = a r^9 = y$ … (2)

$a_{16} = a r^{15} = z$ … (3)

On dividing (2) by (1), we get

$$\frac{y}{x} = \frac{ar^9}{ar^3} \Rightarrow \frac{y}{x} = r^6$$

And, on dividing (3) by (2), we get

$$\frac{z}{y} = \frac{ar^{15}}{ar^9} \Rightarrow \frac{z}{y} = r^6$$

$$\frac{y}{x} = \frac{z}{y}$$

Therefore, x, y, z are in G. P.

18. Find the sum to *n* terms of the sequence, 8, 88, 888, 8888…

Solution:

Given sequence: 8, 88, 888, 8888…

This sequence is not a G.P.

But, it can be changed to G.P. by writing the terms as

$S_n = 8 + 88 + 888 + 8888 + \ldots\ldots\ldots\ldots\ldots$ to n terms

$$= \frac{8}{9}[9+99+999+9999+\text{..........to } n \text{ terms}]$$

$$= \frac{8}{9}\left[(10-1)+(10^2-1)+(10^3-1)+(10^4-1)+\text{........to } n \text{ terms}\right]$$

$$= \frac{8}{9}\left[(10+10^2+\text{......} n \text{ terms})-(1+1+1+\text{....} n \text{ terms})\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n-1)}{9}-n\right]$$

$$= \frac{80}{81}(10^n-1)-\frac{8}{9}n$$

19. Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, 1/2.

Solution:

The required sum = 2 x 128 + 4 x 32 + 8 x 8 + 16 x 2 + 32 x ½

= 64[4 + 2 + 1 + ½ + $1/2^2$]

Now, it's seen that

4, 2, 1, ½, $1/2^2$ is a G.P.

With the first term, a = 4

Common ratio, r =1/2

We know,

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\therefore S_5 = \frac{4\left[1-\left(\frac{1}{2}\right)^5\right]}{1-\frac{1}{2}} = \frac{4\left[1-\frac{1}{32}\right]}{\frac{1}{2}} = 8\left(\frac{32-1}{32}\right) = \frac{31}{4}$$

Therefore, the required sum = 64(31/4) = (16)(31) = 496

20. Show that the products of the corresponding terms of the sequences a, ar, ar^2, …ar^{n-1} and A, AR, AR^2, … AR^{n-1} form a G.P, and find the common ratio.

Solution:

To be proved: The sequence, aA, $arAR$, ar^2AR^2, …$ar^{n-1}AR^{n-1}$, forms a G.P.

Now, we have

$$\frac{\text{Second term}}{\text{First term}} = \frac{arAR}{aA} = rR$$
$$\frac{\text{Third term}}{\text{Second term}} = \frac{ar^2AR^2}{arAR} = rR$$

Therefore, the above sequence forms a G.P., and the common ratio is rR.

21. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

Solution:

Consider a to be the first term and r to be the common ratio of the G.P.

Then,

$a_1 = a$, $a_2 = ar$, $a_3 = ar^2$, $a_4 = ar^3$

From the question, we have

$a_3 = a_1 + 9$

$ar^2 = a + 9$ … (i)

$a_2 = a_4 + 18$

$ar = ar^3 + 18$ … (ii)

So, from (1) and (2), we get

$a(r^2 – 1) = 9$ … (iii)

$ar(1– r^2) = 18$ … (iv)

Now, dividing (4) by (3), we get

$$\frac{ar\left(1-r^2\right)}{a\left(r^2-1\right)} = \frac{18}{9}$$

-r = 2

r = -2

On substituting the value of r in (i), we get

$4a = a + 9$

$3a = 9$

$\therefore a = 3$

Therefore, the first four numbers of the G.P. are 3, $3(-2)$, $3(-2)^2$, and $3(-2)^3$

i.e., 3¸–6, 12, and –24.

22. If the p^{th}, q^{th} and r^{th} terms of a G.P. are *a, b* and *c*, respectively. Prove that $a^{q-r} b^{r-p} c^{p-q} = 1$

Solution:

Let's take A to be the first term and R to be the common ratio of the G.P.

Then, according to the question, we have

$AR^{p-1} = a$

$AR^{q-1} = b$

$AR^{r-1} = c$

Then,

$a^{q-r} b^{r-p} c^{p-q}$

$= A^{q-r} \times R^{(p-1)(q-r)} \times A^{r-p} \times R^{(q-1)(r-p)} \times A^{p-q} \times R^{(r-1)(p-q)}$

$= Aq^{-r+r-p+p-q} \times R^{(pr-pr-q+r)+(rq-r+p-pq)+(pr-p-qr+q)}$

$= A^0 \times R^0$

$= 1$

Hence proved.

23. If the first and the n^{th} term of a G.P. are *a* and *b*, respectively, and if *P* is the product of *n* terms, prove that $P^2 = (ab)^n$.

Solution:

Given the first term of the G.P is *a*, and the last term is *b*.

Thus,

The G.P. is $a, ar, ar^2, ar^3, \ldots ar^{n-1}$, where r is the common ratio.

Then,

$b = ar^{n-1}$ … (1)

P = Product of n terms

$= (a)(ar)(ar^2) \ldots (ar^{n-1})$

$= (a \times a \times \ldots a)(r \times r^2 \times \ldots r^{n-1})$

$= a^n r^{1+2+\ldots(n-1)}$ … (2)

Here, 1, 2, …$(n – 1)$ is an A.P.

So,

$$1 + 2 + \ldots\ldots\ldots + (n-1) = \frac{n-1}{2}\left[2+(n-1-1)\times 1\right] = \frac{n-1}{2}\left[2+n-2\right] = \frac{n(n-1)}{2}$$

And, the product of n terms P is given by,

$$\begin{aligned} P &= a^n r^{\frac{n(n-1)}{2}} \\ \therefore P^2 &= a^{2n} r^{n(n-1)} \\ &= \left[a^2 r^{(n-1)}\right]^n \\ &= \left[a \times ar^{n-1}\right]^n \\ &= (ab)^n \qquad \left[\text{Using } (1)\right] \end{aligned}$$

24. Show that the ratio of the sum of the first n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

Solution:

Let a be the first term and r be the common ratio of the G.P.

$$\text{Sum of first n terms} = \frac{a(1-r^n)}{(1-r)}$$

Since there are n terms from $(n+1)^{th}$ to $(2n)^{th}$ term,

Sum of terms from$(n + 1)^{th}$ to $(2n)^{th}$ term

$$= \frac{a_{n+1}(1-r^n)}{(1-r)}$$

$a_{n+1} = ar^{n+1-1} = ar^n$

Thus, the required ratio =

$$\frac{a\left(1-r^n\right)}{\left(1-r\right)} \times \frac{\left(1-r\right)}{ar^n\left(1-r^n\right)}$$

$$= \frac{1}{r^n}$$

Thus, the ratio of the sum of the first n terms of a G.P. to the sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

25. If *a, b, c* and *d* are in G.P., show that $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

Solution:

Given a, b, c, d are in G.P.

So, we have

$bc = ad$ … (1)

$b^2 = ac$ … (2)

$c^2 = bd$ … (3)

Taking the R.H.S., we have

R.H.S.

$= (ab + bc + cd)^2$

$= (ab + ad + cd)^2$ [Using (1)]

$= [ab + d(a + c)]^2$

$= a^2b^2 + 2abd(a + c) + d^2(a + c)^2$

$= a^2b^2 + 2a^2bd + 2acbd + d^2(a^2 + 2ac + c^2)$

$= a^2b^2 + 2a^2c^2 + 2b^2c^2 + d^2a^2 + 2d^2b^2 + d^2c^2$ [Using (1) and (2)]

$= a^2b^2 + a^2c^2 + a^2c^2 + b^2c^2 + b^2c^2 + d^2a^2 + d^2b^2 + d^2b^2 + d^2c^2$

$= a^2b^2 + a^2c^2 + a^2d^2 + b^2 \times b^2 + b^2c^2 + b^2d^2 + c^2b^2 + c^2 \times c^2 + c^2d^2$

[Using (2) and (3) and rearranging terms]
$= a^2(b^2 + c^2 + d^2) + b^2(b^2 + c^2 + d^2) + c^2(b^2 + c^2 + d^2)$

$= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

= L.H.S.

Thus, L.H.S. = R.H.S.

Therefore,

$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$

26. Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

Solution:

Let's assume G_1 and G_2 to be two numbers between 3 and 81 such that the series 3, G_1, G_2, 81 forms a G.P.

And let a be the first term and r be the common ratio of the G.P.

Now, we have the 1st term as 3 and the 4th term as 81.

$81 = (3)(r)^3$

$r^3 = 27$

$\therefore r = 3$ (Taking real roots only)

For $r = 3$,

$G_1 = ar = (3)(3) = 9$

$G_2 = ar^2 = (3)(3)^2 = 27$

Therefore, the two numbers which can be inserted between 3 and 81 so that the resulting sequence becomes a G.P are 9 and 27.

27. Find the value of n so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Solution:

We know that,

The G. M. of a and b is given by $\sqrt{ab}$.

Then from the question, we have

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \sqrt{ab}$$

By squaring both sides, we get

$$\frac{\left(a^{n+1}+b^{n+1}\right)^2}{\left(a^n+b^n\right)^2} = ab$$

$$a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2} = (ab)\left(a^{2n} + 2a^nb^n + b^{2n}\right)$$

$$a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2} = a^{2n+1}b + 2a^{n+1}b^{n+1} + ab^{2n+1}$$

$$a^{2n+2} + b^{2n+2} = a^{2n+1}b + ab^{2n+1}$$

$$a^{2n+2} - a^{2n+1}b = ab^{2n+1} - b^{2n+2}$$

$$a^{2n+1}(a-b) = b^{2n+1}(a-b)$$

$$\left(\frac{a}{b}\right)^{2n+1} = 1 = \left(\frac{a}{b}\right)^0$$

$$2n+1 = 0 \qquad \text{(Equting the exponents)}$$

$$n = \frac{-1}{2}$$

28. The sum of two numbers is 6 times their geometric mean; show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

Solution:

Consider the two numbers to be *a* and *b*.

Then, G.M. = √ab.

From the question, we have

$a+b=6\sqrt{ab}$ ...(1)

$\Rightarrow (a+b)^2 = 36(ab)$

Also,

$(a-b)^2=(a+b)^2-4ab=36ab-4ab=32ab$

$\Rightarrow a-b=\sqrt{32}\sqrt{ab}$

$=4\sqrt{2}\sqrt{ab}$ ...(2)

On adding (1) and (2), we get

$2a=\left(6+4\sqrt{2}\right)\sqrt{ab}$

$a=\left(3+2\sqrt{2}\right)\sqrt{ab}$

Substituting the value of a in (1), we get

$b=6\sqrt{ab}-\left(3+2\sqrt{2}\right)\sqrt{ab}$

$b=\left(3-2\sqrt{2}\right)\sqrt{ab}$

$$\frac{a}{b}=\frac{\left(3+2\sqrt{2}\right)\sqrt{ab}}{\left(3-2\sqrt{2}\right)\sqrt{ab}}=\frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

Therefore, the required ratio is $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$

29. If A and G be A.M. and G.M., respectively, between two positive numbers, prove that the

$$A \pm \sqrt{(A+G)(A-G)}$$

numbers are.

Solution:

Given that *A* and *G* are A.M. and G.M. between two positive numbers.

And, let these two positive numbers be *a* and *b*.

So,

$$AM = A = \frac{a+b}{2} \quad ...(1)$$

$$GM = G = \sqrt{ab} \quad ...(2)$$

From (1) and (2), we get

$a + b = 2A$ … (3)

$ab = G^2$ … (4)

Substituting the value of a and b from (3) and (4) in the identity $(a – b)^2 = (a + b)^2 – 4ab$, we have

$(a – b)^2 = 4A^2 – 4G^2 = 4 (A^2–G^2)$

$(a – b)^2 = 4 (A + G) (A – G)$

$$(a-b) = 2\sqrt{(A+G)(A-G)} \quad ...(5)$$

From (3) and (5), we get

$$2a = 2A + 2\sqrt{(A+G)(A-G)}$$

$$\Rightarrow a = A + \sqrt{(A+G)(A-G)}$$

Substituting the value of a in (3), we have

$$b = 2A - A - \sqrt{(A+G)(A-G)} = A - \sqrt{(A+G)(A-G)}$$

Therefore, the two numbers are $A \pm \sqrt{(A+G)(A-G)}$.

30. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of the 2nd hour, 4th hour and *n*th hour?

Solution:

Given the number of bacteria doubles every hour. Hence, the number of bacteria after every hour will form a G.P.

Here we have, $a = 30$ and $r = 2$

So, $a_3 = ar^2 = (30)(2)^2 = 120$

Thus, the number of bacteria at the end of 2nd hour will be 120.

And, $a_5 = ar^4 = (30)(2)^4 = 480$

The number of bacteria at the end of 4th hour will be 480.

$a_{n+1} = ar^n = (30)\, 2^n$

Therefore, the number of bacteria at the end of nth hour will be $30(2)^n$.

31. What will Rs 500 amount to in 10 years after its deposit in a bank which pays an annual interest rate of 10% compounded annually?

Solution:

Given,

The amount deposited in the bank is Rs 500.

At the end of first year, amount = Rs 500(1 + 1/10) = Rs 500 (1.1)

At the end of 2nd year, amount = Rs 500 (1.1) (1.1)

At the end of 3rd year, amount = Rs 500 (1.1) (1.1) (1.1) and so on….

Therefore,

The amount at the end of 10 years = Rs 500 (1.1) (1.1) … (10 times)

= Rs $500(1.1)^{10}$

32. If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Solution:

Let's consider the roots of the quadratic equation to be *a* and *b*.

Then, we have

$$\text{A.M.} = \frac{a+b}{2} = 8 \Rightarrow a+b = 16 \qquad ...(1)$$
$$\text{G.M.} = \sqrt{ab} = 5 \Rightarrow ab = 25 \qquad ...(2)$$

We know that,

A quadratic equation can be formed as,

$x^2 - x$ (Sum of roots) + (Product of roots) = 0

$x^2 - x(a + b) + (ab) = 0$

$x^2 - 16x + 25 = 0$ [Using (1) and (2)]

Therefore, the required quadratic equation is $x^2 - 16x + 25 = 0$

EXERCISE 9.4

Find the sum to n terms of each of the series in Exercises 1 to 7.

1. 1 × 2 + 2 × 3 + 3 × 4 + 4 × 5 + …

Solution:

Given series is 1 × 2 + 2 × 3 + 3 × 4 + 4 × 5 + …

It's seen that,

n^{th} term, $a_n = n\,(n + 1)$

Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k(k+1)$$
$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left(\frac{2n+1}{3}+1\right)$$
$$= \frac{n(n+1)}{2}\left(\frac{2n+4}{3}\right)$$
$$= \frac{n(n+1)(n+2)}{3}$$

2. 1 × 2 × 3 + 2 × 3 × 4 + 3 × 4 × 5 + …

Solution:

Given series is 1 × 2 × 3 + 2 × 3 × 4 + 3 × 4 × 5 + …

It's seen that,

n^{th} term, $a_n = n\,(n + 1)\,(n + 2)$

$= (n^2 + n)\,(n + 2)$

$= n^3 + 3n^2 + 2n$

Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k$$
$$= \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$$
$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$
$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{2} + n(n+1)$$
$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 2n + 1 + 2\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{n^2 + n + 4n + 6}{2}\right]$$
$$= \frac{n(n+1)}{4}(n^2 + 5n + 6)$$
$$= \frac{n(n+1)}{4}(n^2 + 2n + 3n + 6)$$
$$= \frac{n(n+1)[n(n+2) + 3(n+2)]}{4}$$
$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

3. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$

Solution:

Given series is $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$

It's seen that,

n^{th} term, $a_n = (2n + 1)\, n^2 = 2n^3 + n^2$

Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} = \left(2k^3 + k^2\right) = 2\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2$$

$$= 2\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n^2(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+5n+1}{3}\right]$$

$$= \frac{n(n+1)\left(3n^2+5n+1\right)}{6}$$

4. Find the sum to *n* terms of the series

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots$$

Solution:

Given series is, $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots$

It's seen that,

n^{th} term, $a_n = \frac{1}{n(n+1)} = \frac{1}{n}-\frac{1}{n+1}$ (By partial fractions)

$a_1 = \frac{1}{1}-\frac{1}{2}$

$a_2 = \frac{1}{2}-\frac{1}{3}$

$a_3 = \frac{1}{3}-\frac{1}{4}\ \ldots$

$a_n = \frac{1}{n}-\frac{1}{n+1}$

On adding the above terms column wise, we get

$$a_1+a_2+\ldots+a_n = \left[\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}\right]-\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\frac{1}{n+1}\right]$$

$$\therefore S_n = 1-\frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

5. Find the sum to *n* terms of the series $5^2 + 6^2 + 7^2 + \ldots + 20^2$

Solution:

Given series is $5^2 + 6^2 + 7^2 + \ldots + 20^2$

It's seen that,

n^{th} term, $a_n = (n + 4)^2 = n^2 + 8n + 16$

Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(k^2 + 8k + 16\right)$$
$$= \sum_{k=1}^{n} k^2 + 8\sum_{k=1}^{n} k + \sum_{k=1}^{n} 16$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{8n(n+1)}{2} + 16n$$

Now, its found that

16^{th} term is $(16 + 4)^2 = 20^2$

Thus,

$$S_{16} = \frac{16(16+1)(2\times 16+1)}{6} + \frac{8\times 16\times(16+1)}{2} + 16\times 16$$
$$= \frac{(16)(17)(33)}{6} + \frac{(8)\times 16\times(16+1)}{2} + 16\times 16$$
$$= \frac{(16)(17)(33)}{6} + \frac{(8)(16)(17)}{2} + 256$$
$$= 1496 + 1088 + 256$$
$$= 2840$$

Hence, $5^2 + 6^2 + 7^2 + \ldots\ldots + 20^2 = 2840$

6. Find the sum to *n* terms of the series 3 × 8 + 6 × 11 + 9 × 14 +…

Solution:

Given series is 3 × 8 + 6 × 11 + 9 × 14 + …

It’s found out that,

a_n = (n^{th} term of 3, 6, 9 …) × (n^{th} term of 8, 11, 14, …)

= $(3n)(3n + 5)$

= $9n^2 + 15n$

Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(9k^2 + 15k\right)$$
$$= 9\sum_{k=1}^{n} k^2 + 15\sum_{k=1}^{n} k$$
$$= 9 \times \frac{n(n+1)(2n+1)}{6} + 15 \times \frac{n(n+1)}{2}$$
$$= \frac{3n(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2}$$
$$= \frac{3n(n+1)}{2}(2n+1+5)$$
$$= \frac{3n(n+1)}{2}(2n+6)$$
$$= 3n(n+1)(n+3)$$

7. Find the sum to *n* terms of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$

Solution:

Given series is $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$

Finding the n^{th} term, we have

$a_n = (1^2 + 2^2 + 3^2 + \ldots\ldots + n^2)$

$$= \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{n(2n^2+3n+1)}{6}$$
$$= \frac{2n^3+3n^2+n}{6}$$
$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Now, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k$$
$$= \sum_{k=1}^{n}\left(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k\right)$$
$$= \frac{1}{3}\sum_{k=1}^{n}k^3 + \frac{1}{2}\sum_{k=1}^{n}k^2 + \frac{1}{6}\sum_{k=1}^{n}k$$
$$= \frac{1}{3}\frac{n^2(n+1)^2}{(2)^2} + \frac{1}{2}\times\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}\times\frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{6}\left[\frac{n(n+1)}{2} + \frac{(2n+1)}{2} + \frac{1}{2}\right]$$
$$= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+1+1}{2}\right]$$
$$= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+2}{2}\right]$$
$$= \frac{n(n+1)}{6}\left[\frac{n(n+1)+2(n+1)}{2}\right]$$
$$= \frac{n(n+1)}{6}\left[\frac{(n+1)(n+2)}{2}\right]$$
$$= \frac{n(n+1)^2(n+2)}{12}$$

8. Find the sum to n terms of the series whose n^{th} term is given by $n(n + 1)(n + 4)$.

Solution:

Given,

$a_n = n(n + 1)(n + 4) = n(n^2 + 5n + 4) = n^3 + 5n^2 + 4n$

Now, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k$$
$$= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+20n+10+24}{6}\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{3n^2+23n+34}{6}\right]$$
$$= \frac{n(n+1)(3n^2+23n+34)}{12}$$

9. Find the sum to *n* terms of the series whose n^{th} term is given by $n^2 + 2^n$

Solution:

Given,

The n^{th} term of the series as

$a_n = n^2 + 2^n$

Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} k^2 + 2^k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k \qquad (1)$$

Consider $\sum_{k=1}^{n} 2^k = 2^1 + 2^2 + 2^3 + \ldots$

The above series 2, 2^2, 2^3, … is a G.P. with both the first term and common ratio equal to 2.

$$\therefore \sum_{k=1}^{n} 2^k = \frac{(2)\left[(2)^n - 1\right]}{2-1} = 2(2^n - 1) \qquad (2)$$

Therefore, from (1) and (2), we obtain

$$S_n = \sum_{k=1}^{n} k^2 + 2(2^n - 1) = \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

10. Find the sum to *n* terms of the series whose n^{th} term is given by $(2n – 1)^2$

Solution:

Given,

The n^{th} term of the series as:

$a_n = (2n - 1)^2 = 4n^2 - 4n + 1$

Then, the sum of n terms of the series can be expressed as

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(4k^2 - 4k + 1\right) \\
&= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \\
&= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n \\
&= n\left[\frac{2\left(2n^2+3n+1\right)}{3} - 2(n+1) + 1\right] \\
&= n\left[\frac{4n^2+6n+2-6n-6+3}{3}\right] \\
&= n\left[\frac{4n^2-1}{3}\right] \\
&= \frac{n(2n+1)(2n-1)}{3}
\end{aligned}
$$

MISCELLANEOUS EXERCISE

1. Show that the sum of $(m + n)^{th}$ and $(m – n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Solution:

Let's take a and d to be the first term and the common difference of the A.P., respectively.

We know that the k^{th} term of an A. P. is given by

$a_k = a + (k – 1) d$

So, $a_{m+n} = a + (m + n – 1) d$

And, $a_{m-n} = a + (m – n – 1) d$

$a_m = a + (m – 1) d$

Thus,

$a_{m+n} + a_{m-n} = a + (m + n – 1) d + a + (m – n – 1) d$

$= 2a + (m + n – 1 + m – n – 1) d$

$= 2a + (2m – 2) d$

$= 2a + 2 (m – 1) d$

$= 2 [a + (m – 1) d]$

$= 2a_m$

Therefore, the sum of $(m + n)^{th}$ and $(m – n)^{th}$ terms of an A.P. is equal to twice the m^{th} term

2. If the sum of three numbers in A.P. is 24 and their product is 440, find the numbers.

Solution:

Let's consider the three numbers in A.P. as $a – d$, a, and $a + d$.

Then, from the question we have

$(a – d) + (a) + (a + d) = 24$ … (i)

$3a = 24$

$\therefore a = 8$

And,

$(a – d) a (a + d) = 440$ … (ii)

$(8 - d)(8)(8 + d) = 440$

$(8 - d)(8 + d) = 55$

$64 - d^2 = 55$

$d^2 = 64 - 55 = 9$

$\therefore d = \pm 3$

Thus,

When $d = 3$, the numbers are 5, 8, and 11 and

When $d = -3$, the numbers are 11, 8, and 5.

Therefore, the three numbers are 5, 8, and 11.

3. Let the sum of n, $2n$, $3n$ terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3 = 3(S_2 - S_1)$.

Solution:

Let's take a and d to be the first term and the common difference of the A.P., respectively.

So, we have

$$S_1 = \frac{n}{2}[2a + (n-1)d] \quad \ldots(1)$$

$$S_2 = \frac{2n}{2}[2a + (2n-1)d] = n[2a + (2n-1)d] \quad \ldots(2)$$

$$S_3 = \frac{3n}{2}[2a + (3n-1)d] \quad \ldots(3)$$

From (1) and (2), we get

$$S_2 - S_1 = n[2a + (2n-1)d] - \frac{n}{2}[2a + (n-1)d]$$

$$= n\left\{\frac{4a + 4nd - 2d - 2a - nd + d}{2}\right\}$$

$$= n\left[\frac{2a + 3nd - d}{2}\right]$$

$$= \frac{n}{2}[2a + (3n-1)d]$$

Now,

$$3(S_2 - S_1) = \frac{3n}{2}[2a + (3n-1)d] = S_3 \quad \text{[From (3)]}$$

Hence proved.

4. Find the sum of all numbers between 200 and 400, which are divisible by 7.

Solution:

First, let's find the numbers between 200 and 400, which are divisible by 7.

The numbers are:

203, 210, 217, … 399

Here, the first term, $a = 203$

Last term, $l = 399$ and

Common difference, $d = 7$

Let's consider the number of terms of the A.P. to be n.

Hence, $a_n = 399 = a + (n-1)\, d$

$399 = 203 + (n-1)\, 7$

$7\,(n-1) = 196$

$n-1 = 28$

$n = 29$

Then, the sum of 29 terms of the A.P is given by:

$$\begin{aligned}\therefore S_{29} &= \frac{29}{2}(203+399)\\ &= \frac{29}{2}(602)\\ &= (29)(301)\\ &= 8729\end{aligned}$$

Therefore, the required sum is 8729.

5. Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

Solution:

First let's find the integers from 1 to 100, which are divisible by 2.

And they are 2, 4, 6… 100.

Clearly, this forms an A.P. with the first term and common difference both equal to 2.

So, we have

$100 = 2 + (n-1)\, 2$

$n = 50$

Hence, the sum is

$$\begin{aligned} 2+4+6+...+100 &= \frac{50}{2}\left[2(2)+(50-1)(2)\right] \\ &= \frac{50}{2}[4+98] \\ &= (25)(102) \\ &= 2550 \end{aligned}$$

Now, the integers from 1 to 100, which are divisible by 5, are 5, 10… 100.

This also forms an A.P. with the first term and common difference both equal to 5.

So, we have

$100 = 5 + (n - 1)\ 5$

$5n = 100$

$n = 20$

Hence, the sum is

$$\begin{aligned} 5+10+...+100 &= \frac{20}{2}\left[2(5)+(20-1)5\right] \\ &= 10\left[10+(19)5\right] \\ &= 10[10+95] = 10\times 105 \\ &= 1050 \end{aligned}$$

Lastly, the integers, which are divisible by both 2 and 5, are 10, 20, … 100.

And this also forms an A.P. with the first term and common difference both equal to 10.

So, we have

$100 = 10 + (n - 1)\ (10)$

$100 = 10n$

$n = 10$

$$\begin{aligned} 10+20+...+100 &= \frac{10}{2}\left[2(10)+(10-1)(10)\right] \\ &= 5[20+90] = 5(110) = 550 \end{aligned}$$

Thus, the required sum = 2550 + 1050 – 550 = 3050

Therefore, the sum of the integers from 1 to 100, which are divisible by 2 or 5, is 3050.

6. Find the sum of all two-digit numbers, which, when divided by 4, yields 1 as the remainder.

Solution:

We have to first find the two-digit numbers, which, when divided by 4, yield 1 as the remainder.

They are 13, 17, … 97.

As it's seen that this series forms an A.P. with the first term (a) 13 and common difference (d) 4.

Let n be the number of terms of the A.P.

We know that the n^{th} term of an A.P. is given by

$a_n = a + (n - 1)\, d$

So, $97 = 13 + (n - 1)(4)$

$4(n - 1) = 84$

$n - 1 = 21$

$n = 22$

Now, the sum of n terms of an A.P. is given by,

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$\begin{aligned} \therefore S_{22} &= \frac{22}{2}\left[22(13) + (22-1)(4)\right] \\ &= 11\left[26 + 84\right] \\ &= 1210 \end{aligned}$$

Therefore, the required sum is 1210.

7. If *f* is a function satisfying *f(x + y) = f(x) f(y)* for all x, y ∈ N such that $f(1) = 3 \text{ and } \sum_{x=1}^{n} f(x) = 120$, find the value of *n*.

Solution:

Given that,

$f(x + y) = f(x) \times f(y)$ for all $x, y \in N$ … (1)

$f(1) = 3$

Taking $x = y = 1$ in (1), we have

$f(1 + 1) = f(2) = f(1)\,f(1) = 3 \times 3 = 9$

Similarly,

$f(1 + 1 + 1) = f(3) = f(1 + 2) = f(1)\,f(2) = 3 \times 9 = 27$

And, $f(4) = f(1 + 3) = f(1)\,f(3) = 3 \times 27 = 81$

Thus, $f(1), f(2), f(3), \ldots$, that is $3, 9, 27, \ldots$, forms a G.P. with the first term and common ratio both equal to 3.

We know that sum of terms in G.P is given by,

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

And it's given that,

$$\sum_{x=1}^{n} f(x) = 120$$

Hence, the sum of the terms of the function is 120.

$$120 = \frac{3\left(3^n - 1\right)}{3 - 1}$$
$$120 = \frac{3}{2}\left(3^n - 1\right)$$
$$3^n - 1 = 80$$
$$3^n = 81 = 3^4$$
$$\therefore n = 4$$

Therefore, the value of n is 4.

8. The sum of some terms of G.P. is 315, whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

Solution:

Given that the sum of some terms in a G.P is 315.

Let the number of terms be n.

We know that the sum of terms is

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

Given that the first term a is 5 and the common ratio r is 2.

$$315 = \frac{5\left(2^n - 1\right)}{2 - 1}$$
$$2^n - 1 = 63$$
$$2^n = 64 = \left(2\right)^6$$
$$n = 6$$

Hence, the last term of the G.P = 6th term = $ar^{6-1} = (5)(2)^5 = (5)(32) = 160$

Therefore, the last term of the G.P. is 160.

9. The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

Solution:

Let's consider a and r to be the first term and the common ratio of the G.P., respectively.

Given, $a = 1$

$a_3 = ar^2 = r^2$

$a_5 = ar^4 = r^4$

Then, from the question we have

$r^2 + r^4 = 90$

$r^4 + r^2 - 90 = 0$

$$r^2 = \frac{-1 + \sqrt{1 + 360}}{2} = \frac{-1 \pm \sqrt{361}}{2} = \frac{-1 \pm 19}{2} = -10 \text{ or } 9$$
$$\therefore r = \pm 3 \qquad \left(\text{Taking real roots}\right)$$

Therefore, the common ratio of the G.P. is ±3.

10. The sum of three numbers in G.P. is 56. If we subtract 1, 7, and 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

Solution:

Let's consider the three numbers in G.P. to be a, ar, and ar^2.

Then from the question, we have

$a + ar + ar^2 = 56$

$a(1 + r + r^2) = 56$

$\Rightarrow a = \frac{56}{1+r+r^2}$ … (1)

Also, given

$a - 1$, $ar - 7$, $ar^2 - 21$ form an A.P.

So, $(ar - 7) - (a - 1) = (ar^2 - 21) - (ar - 7)$

$ar - a - 6 = ar^2 - ar - 14$

$ar^2 - 2ar + a = 8$

$ar^2 - ar - ar + a = 8$

$a(r^2 + 1 - 2r) = 8$

$a(r - 1)^2 = 8$ … (2)

$\Rightarrow \frac{56}{1+r+r^2}(r-1)^2 = 8$ [Using (1)]

$7(r^2 - 2r + 1) = 1 + r + r^2$

$7r^2 - 14r + 7 - 1 - r - r^2 = 0$

$6r^2 - 15r + 6 = 0$

$6r^2 - 12r - 3r + 6 = 0$

$6r(r - 2) - 3(r - 2) = 0$

$(6r - 3)(r - 2) = 0$

r = 2, 1/2

When $r = 2$, $a = 8$

When r = ½, a = 32

Thus,

When $r = 2$, the three numbers in G.P. are 8, 16, and 32.

When r = 1/2, the three numbers in G.P. are 32, 16, and 8.

Therefore, in either case, the required three numbers are 8, 16, and 32.

11. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Solution:

Let's consider the terms in the G.P.to be $T_1, T_2, T_3, T_4, \ldots T_{2n}$.

The number of terms = $2n$

Then, from the question we have

$T_1 + T_2 + T_3 + \ldots + T_{2n} = 5\,[T_1 + T_3 + \ldots + T_{2n-1}]$

$T_1 + T_2 + T_3 + \ldots + T_{2n} - 5\,[T_1 + T_3 + \ldots + T_{2n-1}] = 0$

$T_2 + T_4 + \ldots + T_{2n} = 4\,[T_1 + T_3 + \ldots + T_{2n-1}]$ …… (1)

Now, let the terms in G.P. be $a, ar, ar^2, ar^3, \ldots$

Then (1) becomes,

$\frac{ar\left(r^n - 1\right)}{r - 1} = \frac{4 \times a\left(r^n - 1\right)}{r - 1}$ [Using sum of terms in G.P.]

ar = 4a

r = 4

Thus, the common ratio of the G.P. is 4.

12. The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

Solution:

Let's consider the terms in A.P. to be $a, a + d, a + 2d, a + 3d, \ldots a + (n - 2)\,d, a + (n - 1)d$.

From the question, we have

Sum of first four terms = $a + (a + d) + (a + 2d) + (a + 3d) = 4a + 6d$

Sum of last four terms = $[a + (n - 4)\,d] + [a + (n - 3)\,d] + [a + (n - 2)\,d] + [a + n - 1)\,d]$

$= 4a + (4n - 10)\,d$

Then, according to the given condition,

$4a + 6d = 56$

$4(11) + 6d = 56$ [Since $a = 11$ (given)]

$6d = 12$

$d = 2$

Hence, $4a + (4n - 10)\,d = 112$

$4(11) + (4n - 10)2 = 112$

$(4n - 10)2 = 68$

$4n - 10 = 34$

$4n = 44$

$n = 11$

Therefore, the number of terms of the A.P. is 11.

13. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}\left(x \neq 0\right)$, then show that *a*, *b*, *c* and *d* are in G.P.

Solution:

Given,

$$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx}$$

On cross-multiplying, we have

$$(a+bx)(b-cx) = (b+cx)(a-bx)$$
$$ab - acx + b^2x - bcx^2 = ab - b^2x + acx - bcx^2$$
$$2b^2x = 2acx$$
$$b^2 = ac$$
$$\frac{b}{a} = \frac{c}{b} \qquad \ldots(1)$$

$\frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ Also, given

On cross-multiplying, we have

$$(b+cx)(c-dx)=(b-cx)(c+dx)$$
$$bc-bdx+c^2x-cdx^2=bc+bdx-c^2x-cdx^2$$
$$2c^2x=2bdx$$
$$c^2=bd$$
$$\frac{c}{d}=\frac{d}{c} \qquad ...(2)$$

From (1) and (2), we get

b/a = c/b = d/c

Therefore, a, b, c and d are in G.P.

14. Let S be the sum, P the product and R the sum of reciprocals of *n* terms in a G.P. Prove that $P^2R^n = S^n$

Solution:

Let the terms in G.P. be $a, ar, ar^2, ar^3, \ldots ar^{n-1} \ldots$

From the question, we have

$$S = \frac{a(r^n - 1)}{r - 1}$$

$$P = a^n \times r^{1+2+...+n-1}$$

$$= a^n \; r^{\frac{n(n-1)}{2}} \qquad \left[\because \text{ Sum of first } n \text{ natural numbers is } n\frac{(n+1)}{2}\right]$$

$$R = \frac{1}{a} + \frac{1}{ar} + ... + \frac{1}{ar^{n-1}}$$

$$= \frac{r^{n-1} + r^{n-2} + ...r + 1}{ar^{n-1}}$$

$$= \frac{1(r^n - 1)}{(r-1)} \times \frac{1}{ar^{n-1}} \qquad [\because 1, r, ... \; r^{n-1} \text{ forms a G.P}]$$

$$= \frac{r^n - 1}{ar^{n-1}(r-1)}$$

$$\therefore P^2R^n = a^{2n}r^{n(n-1)} \frac{(r^n - 1)^n}{a^n r^{n(n-1)} (r-1)^n}$$

$$= \frac{a^n (r^n - 1)^n}{(r-1)^n}$$

$$= \left[\frac{a(r^n - 1)}{(r-1)}\right]^n$$

$$= S^n$$

Hence, $P^2 R^n = S^n$

15. The p^{th}, q^{th} and r^{th} terms of an A.P. are *a, b, c,* respectively.

Show that (q – r) a + (r – p) b + (p -q) c = 0

Solution:

Let's assume *t* and *d* to be the first term and the common difference of the A.P., respectively.

Then the n^{th} term of the A.P. is given by $a_n = t + (n - 1) d$

Thus,

$a_p = t + (p - 1) d = a$ … (1)

$a_q = t + (q - 1) d = b$ … (2)

$a_r = t + (r - 1)\, d = c \quad \ldots (3)$

On subtracting equation (2) from (1), we get

$(p - 1 - q + 1)\, d = a - b$

$(p - q)\, d = a - b$

$$d = \frac{a-b}{p-q} \qquad \ldots(4)$$

On subtracting equation (3) from (2), we get

$(q - 1 - r + 1)\, d = b - c$

$(q - r)\, d = b - c$

$$d = \frac{b-c}{q-r} \qquad \ldots(5)$$

Equating both the values of d obtained in (4) and (5), we get

$$\frac{a-b}{p-q} = \frac{b-c}{q-r}$$
$$(a-b)(q-r) = (b-c)(p-q)$$
$$aq - bq - ar + br = bp - bq - cp + cq$$
$$bp - cp + cq - aq + ar - br = 0$$
$$(-aq + ar) + (bp - br) + (-cp + cq) = 0 \qquad \text{(By rearranging terms)}$$
$$-a(q-r) - b(r-p) - c(p-q) = 0$$
$$a(q-r) + b(r-p) + c(p-q) = 0$$

Therefore, the given result is proved.

16. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

Solution:

Given, $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

$$b\left(\frac{1}{c}+\frac{1}{a}\right)-a\left(\frac{1}{b}+\frac{1}{c}\right)=c\left(\frac{1}{a}+\frac{1}{b}\right)-b\left(\frac{1}{c}+\frac{1}{a}\right)$$

$$\frac{b(a+c)}{ac}-\frac{a(b+c)}{bc}=\frac{c(a+b)}{ab}-\frac{b(a+c)}{ac}$$

$$\frac{b^2a+b^2c-a^2b-a^2c}{abc}=\frac{c^2a+c^2b-b^2a-b^2c}{abc}$$

$$b^2a-a^2b+b^2c-a^2c=c^2a-b^2a+c^2b-b^2c$$

$$ab(b-a)+c(b^2-a^2)=a(c^2-b^2)+bc(c-b)$$

$$ab(b-a)+c(b-a)(b+a)=a(c-b)(c+b)+bc(c-b)$$

$$(b-a)(ab+cb+ca)=(c-b)(ac+ab+bc)$$

$$b-a=c-b$$

Therefore, a, b and c are in A.P.

17. If a, b, c, d are in G.P, prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P.

Solution:

Given a, b, c, and d are in G.P.

So, we have

$\therefore b^2 = ac$ … (i)

$c^2 = bd$ … (ii)

$ad = bc$ … (iii)

Required to prove $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P. i.e.,

$(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$

Taking L.H.S.

$(b^n + c^n)^2 = b^{2n} + 2b^nc^n + c^{2n}$

$= (b^2)^n + 2b^nc^n + (c^2)^n$

$= (ac)^n + 2b^nc^n + (bd)^n$ [Using (i) and (ii)]

$= a^n c^n + b^nc^n + b^n c^n + b^n d^n$

$= a^n c^n + b^nc^n + a^n d^n + b^n d^n$ [Using (iii)]

$= c^n (a^n + b^n) + d^n (a^n + b^n)$

$= (a^n + b^n)(c^n + d^n)$

= R.H.S.

Therefore, $(a^n + b^n)$, $(b^n + c^n)$, and $(c^n + d^n)$ are in G.P

– Hence proved.

18. If a and b are the roots of $x^2 – 3x + p = 0$ and c, dare roots of $x^2 – 12x + q = 0$, where $a, b, c, d,$ form a G.P. Prove that $(q + p)$: $(q – p)$ = 17:15.

Solution:

Given a and b are the roots of $x^2 – 3x + p = 0$

So, we have $a + b = 3$ and $ab = p$ … (i)

Also, c and d are the roots of $x^2 – 12x + q = 0$

So, $c + d = 12$ and $cd = q$ … (ii)

And given a, b, c, d are in G.P.

Let's take $a = x, b = xr, c = xr^2, d = xr^3$

From (i) and (ii), we get

$x + xr = 3$

$x (1 + r) = 3$

And,

$xr^2 + xr^3 =12$

$xr^2 (1 + r) = 12$

On dividing, we get

$$\frac{xr^2(1+r)}{x(1+r)} = \frac{12}{3}$$
$$r^2 = 4$$
$$r = \pm 2$$

When r = 2, x = 3/(1 + 2) = 3/3 = 1

When r = -2, x = 3/(1 – 2) = 3/-1 = -3

Case I:

When $r = 2$ and $x =1$,

$ab = x^2r = 2$

$cd = x^2r^5 = 32$

$$\frac{q+p}{q-p} = \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15}$$
$$(q+p):(q-p) = 17:15$$

Case II:

When $r = -2$, $x = -3$,

$ab = x^2r = -18$

$cd = x^2r^5 = -288$

$$\frac{q+p}{q-p} = \frac{-288-18}{-288+18} = \frac{-306}{-270} = \frac{17}{15}$$
$$(q+p):(q-p) = 17:15$$

Therefore, in both the cases, we get $(q + p)$: $(q - p)$ = 17:15\

19. The ratio of the A.M and G.M. of two positive numbers, *a* and *b*, is *m*: *n*. Show that $a:b = \left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$**.**

Solution:

Let the two numbers be *a* and *b*.

A.M = (a + b)/ 2 and G.M. = √ab

From the question, we have

$$\frac{a+b}{2\sqrt{ab}}=\frac{m}{n}$$

$$\frac{(a+b)^2}{4(ab)}=\frac{m^2}{n^2}$$

$$(a+b)^2=\frac{4ab\,m^2}{n^2}$$

$$(a+b)=\frac{2\sqrt{ab}\,m}{n} \qquad ...(1)$$

By using this in identity $(a - b)^2 = (a + b)^2 - 4ab$, we get

$$(a-b)^2=\frac{4ab\,m^2}{n^2}-4ab=\frac{4ab\left(m^2-n^2\right)}{n^2}$$

$$(a-b)=\frac{2\sqrt{ab}\sqrt{m^2-n^2}}{n} \qquad ...(2)$$

Adding (1) and (2), we get

$$2a=\frac{2\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$a=\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

Substituting the value of a in (1), we get

$$b=\frac{2\sqrt{ab}}{n}m-\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$=\frac{\sqrt{ab}}{n}m-\frac{\sqrt{ab}}{n}\sqrt{m^2-n^2}$$

$$=\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)$$

$$\therefore a:b=\frac{a}{b}=\frac{\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)}{\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)}=\frac{\left(m+\sqrt{m^2-n^2}\right)}{\left(m-\sqrt{m^2-n^2}\right)}$$

Therefore, $a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$

20. If *a, b, c* are in A.P,; *b, c, d* are in G.P and 1/c, 1/d, 1/e are in A.P. prove that *a, c, e* are in G.P.

Solution:

Given *a*, *b*, *c* are in A.P.

Hence, $b - a = c - b$ … (1)

And, given that *b*, *c*, *d* are in G.P.

So, $c^2 = bd$ … (2)

Also, 1/c, 1/d, 1/e are in A.P.

So,

$$\frac{1}{d}-\frac{1}{c}=\frac{1}{e}-\frac{1}{d}$$
$$\frac{2}{d}=\frac{1}{c}+\frac{1}{e} \qquad \ldots(3)$$

Now, required to prove that *a*, *c*, *e* are in G.P. i.e., $c^2 = ae$

From (1), we have

2b = a + c

b = (a + c)/ 2

And from (2), we have

d = c^2/ b

On substituting these values in (3), we get

$$\frac{2b}{c^2}=\frac{1}{c}+\frac{1}{e}$$
$$\frac{2(a+c)}{2c^2}=\frac{1}{c}+\frac{1}{e}$$
$$\frac{a+c}{c^2}=\frac{e+c}{ce}$$
$$\frac{a+c}{c}=\frac{e+c}{e}$$
$$(a+c)e=(e+c)c$$
$$ae+ce=ec+c^2$$
$$c^2=ae$$

Therefore, *a*, *c*, and *e* are in G.P.

21. Find the sum of the following series up to *n* terms:

(i) 5 + 55 + 555 + … (ii) .6 + .66 + . 666 + …

Solution:

(i) Given, 5 + 55 + 555 + …

Let S_n = 5 + 55 + 555 + ….. up to *n* terms

$$= \frac{5}{9}[9+99+999+\ldots\text{to n terms}]$$
$$= \frac{5}{9}\left[(10-1)+(10^2-1)+(10^3-1)+\ldots\text{to n terms}\right]$$
$$= \frac{5}{9}\left[(10+10^2+10^3+\ldots\text{n terms})-(1+1+\ldots\text{ n terms})\right]$$
$$= \frac{5}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$$
$$= \frac{5}{9}\left[\frac{10(10^n-1)}{9}-n\right]$$
$$= \frac{50}{81}(10^n-1)-\frac{5n}{9}$$

(ii) Given, .6 + .66 + . 666 + …

Let S_n = 06. + 0.66 + 0.666 + … up to *n* terms

$$= 6[0.1+0.11+0.111+\ldots\text{to n terms}]$$
$$= \frac{6}{9}[0.9+0.99+0.999+\ldots\text{to n terms}]$$
$$= \frac{6}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\ldots\text{to n terms}\right]$$
$$= \frac{2}{3}\left[(1+1+\ldots\text{n terms})-\frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\ldots\text{n terms}\right)\right]$$
$$= \frac{2}{3}\left[n-\frac{1}{10}\left(\frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}}\right)\right]$$
$$= \frac{2}{3}n-\frac{2}{30}\times\frac{10}{9}(1-10^{-n})$$
$$= \frac{2}{3}n-\frac{2}{27}(1-10^{-n})$$

22. Find the 20th term of the series 2 × 4 + 4 × 6 + 6 × 8 + … + *n* terms.

Solution:

Given series is $2 \times 4 + 4 \times 6 + 6 \times 8 + \ldots$ n terms

$\therefore$ n^{th} term $= a_n = 2n \times (2n + 2) = 4n^2 + 4n$

The 20^{th} term,

$a_{20} = 4\,(20)^2 + 4(20) = 4\,(400) + 80 = 1600 + 80 = 1680$

Therefore, the 20^{th} term of the series is 1680.

23. Find the sum of the first *n* terms of the series: 3 + 7 + 13 + 21 + 31 + …

Solution:

The given series is $3 + 7 + 13 + 21 + 31 + \ldots$

$S = 3 + 7 + 13 + 21 + 31 + \ldots + a_{n-1} + a_n$

$S = 3 + 7 + 13 + 21 + \ldots. + a_{n-2} + a_{n-1} + a_n$

On subtracting both equations, we get

$S - S = [3 + (7 + 13 + 21 + 31 + \ldots + a_{n-1} + a_n)] - [(3 + 7 + 13 + 21 + 31 + \ldots + a_{n-1}) + a_n]$

$S - S = 3 + [(7 - 3) + (13 - 7) + (21 - 13) + \ldots + (a_n - a_{n-1})] - a_n$

$0 = 3 + [4 + 6 + 8 + \ldots (n-1) \text{ terms}] - a_n$

$a_n = 3 + [4 + 6 + 8 + \ldots (n-1) \text{ terms}]$

$$\Rightarrow a_n = 3+\left(\frac{n-1}{2}\right)\left[2\times 4+(n-1-1)2\right]$$
$$= 3+\left(\frac{n-1}{2}\right)\left[8+(n-2)2\right]$$
$$= 3+\frac{(n-1)}{2}(2n+4)$$
$$= 3+(n-1)(n+2)$$
$$= 3+\left(n^2+n-2\right)$$
$$= n^2+n+1$$
$$\therefore \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2+\sum_{k=1}^{n} k+\sum_{k=1}^{n} 1$$
$$= \frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}+n$$
$$= n\left[\frac{(n+1)(2n+1)+3(n+1)+6}{6}\right]$$
$$= n\left[\frac{2n^2+3n+1+3n+3+6}{6}\right]$$
$$= n\left[\frac{2n^2+6n+10}{6}\right]$$
$$= \frac{n}{3}\left(n^2+3n+5\right)$$

24. If S_1, S_2, S_3 are the sum of the first *n* natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3(1 + 8S_1)$.

Solution:

From the question, we have

$$S_1 = \frac{n(n+1)}{2}$$

$$S_3 = \frac{n^2(n+1)^2}{4}$$

$$\text{Here, } S_3(1+8S_1) = \frac{n^2(n+1)^2}{4}\left[1+\frac{8n(n+1)}{2}\right]$$

$$= \frac{n^2(n+1)^2}{4}\left[1+4n^2+4n\right]$$

$$= \frac{n^2(n+1)^2}{4}(2n+1)^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \qquad ...(1)$$

$$\text{Also, } 9S_2^2 = 9\frac{\left[n(n+1)(2n+1)\right]^2}{(6)^2}$$

$$= \frac{9}{36}\left[n(n+1)(2n+1)\right]^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \qquad ...(2)$$

Therefore, from (1) and (2), we have $9S_2^2 = S_3(1 + 8S_1)$.

25. Find the sum of the following series up to *n* terms: $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+...$

Solution:

The n^{th} term of the given series is $\dfrac{1^3+2^3+3^3+...+n^3}{1+3+5+...+(2n-1)} = \dfrac{\left[\dfrac{n(n+1)}{2}\right]^2}{1+3+5+...+(2n-1)}$

Here, $1, 3, 5, \ldots (2n-1)$ is an A.P. with first term a, last term $(2n-1)$ and number of terms as n
So,

$$1+3+5+.....+(2n-1) = \frac{n}{2}\left[2\times1+(n-1)2\right] = n^2$$

And,

$$a_n = \frac{n^2(n+1)^2}{4n^2} = \frac{(n+1)^2}{4} = \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4}$$

Thus,

$$S_n = \sum_{K=1}^{n} a_K = \sum_{K=1}^{n}\left(\frac{1}{4}K^2 + \frac{1}{2}K + \frac{1}{4}\right)$$

$$= \frac{1}{4}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2} + \frac{1}{4}n$$

$$= \frac{n\left[(n+1)(2n+1)+6(n+1)+6\right]}{24}$$

$$= \frac{n\left[2n^2+3n+1+6n+6+6\right]}{24}$$

$$= \frac{n(2n^2+9n+13)}{24}$$

26. Show that

$$\frac{1\times2^2+2\times3^2+...+n\times(n+1)^2}{1^2\times2+2^2\times3+...+n^2\times(n+1)} = \frac{3n+5}{3n+1}$$

Solution:

n^{th} term of the numerator $= n(n+1)^2 = n^3 + 2n^2 + n$

n^{th} term of the denominator $= n^2(n+1) = n^3 + n^2$

$$\frac{1\times 2^2+2\times 3^2+....+n\times(n+1)^2}{1^2\times 2+2^2\times 3+....+n^2\times(n+1)}=\frac{\sum_{K=1}^{n}a_K}{\sum_{K=1}^{n}a_K}=\frac{\sum_{K=1}^{n}\left(K^3+2K^2+K\right)}{\sum_{K=1}^{n}\left(K^3+K^2\right)} \quad ...(1)$$

Here, $\sum_{K=1}^{n}\left(K^3+2K^2+K\right)$

$$=\frac{n^2(n+1)^2}{4}+\frac{2\,n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2}{3}(2n+1)+1\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+8n+4+6}{6}\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+11n+10\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+6n+5n+10\right]$$

$$=\frac{n(n+1)}{12}\left[3n(n+2)+5(n+2)\right]$$

$$=\frac{n(n+1)(n+2)(3n+5)}{12} \quad ...(2)$$

$$\text{Also, } \sum_{K=1}^{n}\left(K^3+K^2\right)=\frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+4n+2}{6}\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+7n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+6n+n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n(n+2)+1(n+2)\right]$$

$$=\frac{n(n+1)(n+2)(3n+1)}{12} \qquad \ldots(3)$$

From (1), (2) and (3), we obtain

$$\frac{1\times2^2+2\times3^2+\ldots+n\times(n+1)^2}{1^2\times2+2^2\times3+\ldots+n^2\times(n+1)}=\frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}$$

$$=\frac{n(n+1)(n+2)(3n+5)}{n(n+1)(n+2)(3n+1)}=\frac{3n+5}{3n+1}$$

Hence proved.

27. A farmer buys a used tractor for Rs 12,000. He pays Rs 6,000 cash and agrees to pay the balance in annual instalments of Rs 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

Solution:

Given, the farmer pays Rs 6000 in cash.

So, the unpaid amount = Rs 12000 – Rs 6000 = Rs 6000

From the question, the interest paid annually will be

12% of 6000, 12% of 5500, 12% of 5000, …, 12% of 500

Hence, the total interest to be paid = 12% of 6000 + 12% of 5500 + 12% of 5000 + … + 12% of 500

= 12% of (6000 + 5500 + 5000 + … + 500)

= 12% of (500 + 1000 + 1500 + … + 6000)

It's seen that the series 500, 1000, 1500 … 6000 is an A.P. with the first term and common difference both equal to 500.

Let's take the number of terms of the A.P. to be n.

So, $6000 = 500 + (n - 1)\,500$

$1 + (n - 1) = 12$

$n = 12$

Now,

The sum of the A.P = 12/2 [2(500) + (12 – 1)(500)] = 6 [1000 + 5500] = 6(6500) = 39000

Hence, the total interest to be paid = 12% of (500 + 1000 + 1500 + … + 6000)

= 12% of 39000 = Rs 4680

Therefore, the tractor will cost the farmer = (Rs 12000 + Rs 4680) = Rs 16680

28. Shamshad Ali buys a scooter for Rs 22,000. He pays Rs 4,000 cash and agrees to pay the balance in annual instalments of Rs 1,000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Solution:

Given, Shamshad Ali buys a scooter for Rs 22000 and pays Rs 4000 in cash.

So, the unpaid amount = Rs 22000 – Rs 4000 = Rs 18000

From the question, it's understood that the interest paid annually is

10% of 18000, 10% of 17000, 10% of 16000 … 10% of 1000

Hence, the total interest to be paid = 10% of 18000 + 10% of 17000 + 10% of 16000 + … + 10% of 1000

= 10% of (18000 + 17000 + 16000 + … + 1000)

= 10% of (1000 + 2000 + 3000 + … + 18000)

It's seen that 1000, 2000, 3000 … 18000 form an A.P. with the first term and common difference both equal to 1000.

Let the number of terms be n.

So, $18000 = 1000 + (n - 1)(1000)$

$n = 18$

Now, the sum of the A.P is given by

$$\therefore 1000+2000+\ldots.+18000=\frac{18}{2}\left[2(1000)+(18-1)(1000)\right]$$
$$=9\left[2000+17000\right]$$
$$=171000$$

Thus,

Total interest paid = 10% of (18000 + 17000 + 16000 + … + 1000)

= 10% of Rs 171000 = Rs 17100

Therefore, the cost of scooter = Rs 22000 + Rs 17100 = Rs 39100

29. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail it to four different persons with the instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when the 8^{th} set of the letter is mailed.

Solution:

It's seen that,

The numbers of letters mailed forms a G.P.: 4, 4^2, … 4^8

Here, first term = 4 and common ratio = 4

And the number of terms = 8

The sum of *n* terms of a G.P. is given by

$$S_n=\frac{a\left(r^n-1\right)}{r-1}$$
$$\therefore S_8=\frac{4\left(4^8-1\right)}{4-1}=\frac{4(65536-1)}{3}=\frac{4(65535)}{3}=4(21845)=87380$$

Also, given that the cost to mail one letter is 50 paisa.

Hence, cost of mailing 87380 letters = Rs 87380 x (50/100) = Rs 43690 = Rs 43690

Therefore, the amount spent when the 8^{th} set of the letter is mailed will be Rs 43,690.

30. A man deposited Rs 10,000 in a bank at the rate of 5% simple interest annually. Find the amount in the 15^{th} year since he deposited the amount and also calculate the total amount after 20 years.

Solution:

Given, the man deposited Rs 10000 in a bank at the rate of 5% simple interest annually.

Hence, the interest in first year = (5/100) x Rs 10000 = Rs 500

So, the amount in the 15th year = Rs $10000+\underbrace{500+500+....+500}_{14\text{times}}$

= Rs 10000 + 14 × Rs 500

= Rs 10000 + Rs 7000

= Rs 17000

And the amount after 20 years = $\text{Rs } 10000+\underbrace{500+500+....+500}_{20\text{times}}$

= Rs 10000 + 20 × Rs 500

= Rs 10000 + Rs 10000

= Rs 20000

Therefore, the amount in the 15th year is Rs 17,000, and the total amount after 20 years will be Rs 20,000.

31. A manufacturer reckons that the value of a machine, which costs him Rs 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

Solution:

Given, the cost of the machine = Rs 15625

Also, given that the machine depreciates by 20% every year.

Hence, its value after every year is 80% of the original cost, i.e., 4/5th of the original cost.

Therefore, the value at the end of 5 years = $15625\times\underbrace{\frac{4}{5}\times\frac{4}{5}\times....\times\frac{4}{5}}_{5\text{times}}$

= 5 × 1024 = 5120

Thus, the value of the machine at the end of 5 years will be Rs 5,120.

32. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on the second day, 4 more workers dropped out on the third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Solution:

Let's assume x to be the number of days in which 150 workers finish the work.

Then, from the question, we have

$150x$ = 150 + 146 + 142 + …. (x + 8) terms

The series 150 + 146 + 142 + …. (x + 8) terms is an A.P.

With first term (a) = 150, common difference (d) = –4 and number of terms (n) = (x + 8)

Now, finding the sum of terms

$$150x = \frac{(x+8)}{2}\left[2(150)+(x+8-1)(-4)\right]$$
$$150x = (x+8)\left[150+(x+7)(-2)\right]$$
$$150x = (x+8)(150-2x-14)$$
$$150x = (x+8)(136-2x)$$
$$75x = (x+8)(68-x)$$
$$75x = 68x - x^2 + 544 - 8x$$
$$x^2 + 75x - 60x - 544 = 0$$
$$x^2 + 15x - 544 = 0$$
$$x^2 + 32x - 17x - 544 = 0$$
$$x(x+32) - 17(x+32) = 0$$
$$(x-17)(x+32) = 0$$
$$x = 17 \text{ or } x = -32$$

As x cannot be negative [Number of days is always a positive quantity]

x = 17

Hence, the number of days in which the work should have been completed is 17.

But, due to the dropping out of workers, the number of days in which the work is completed

= (17 + 8) = 25