1. Find $\left|\vec{a} \times \vec{b}\right|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

Solution:

It is given that,

$$\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k} \text{ and } \vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= \hat{i}(-14+14) - \hat{j}(2-21) + \hat{k}(-2+21) = 19\hat{j} + 19\hat{k}$$

Therefore,

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{(19)^2 + (19)^2} = \sqrt{2 \times (19)^2} = 19\sqrt{2}$$

2. Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Solution:

It is given that,

$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

So, we have

$\vec{a} + \vec{b} = 4\hat{i} + 4\hat{j}$, $\vec{a} - \vec{b} = 2\hat{i} + 4\hat{k}$

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = \hat{i}(16) - \hat{j}(16) + \hat{k}(-8) = 16\hat{i} - 16\hat{j} - 8\hat{k}$$

Thus,

$$\left| (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) \right| = \sqrt{16^2 + (-16)^2 + (-8)^2}$$

$$= \sqrt{2^2 \times 8^2 + 2^2 \times 8^2 + 8^2}$$

$$= 8\sqrt{2^2 + 2^2 + 1} = 8\sqrt{9} = 8 \times 3 = 24$$

Therefore, the unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is given by the relation,

$$= \pm \frac{(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})}{\left| (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) \right|} = \pm \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24}$$

$$= \pm \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3} = \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}$$

3. If a unit vector $\vec{a}$ makes an angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the compounds of $\vec{a}$.

Solution:

First,

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$\Rightarrow \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

As $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

Also given, that $\vec{a}$ makes angles $\dfrac{\pi}{3}$ with $\hat{i}$, $\dfrac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$.

Then, we have

$$\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{2} = a_1 \qquad \left[|\vec{a}| = 1\right]$$

$$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = a_2 \qquad \left[|\vec{a}| = 1\right]$$

Also, $\cos\theta = \dfrac{a_3}{|\vec{a}|}$.

$$\Rightarrow a_3 = \cos\theta$$

Now,

$$|a| = 1$$

$$\sqrt{a_1^2 + a_2^2 + a_3^2} = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$$\frac{3}{4} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$$

Thus, $\theta = \dfrac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$

4. Show that

$$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right)$$

Solution:

First, consider the LHS,

We have

$$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$$

$$=\left(\vec{a}-\vec{b}\right)\times\vec{a}+\left(\vec{a}-\vec{b}\right)\times\vec{b} \qquad \text{[By distributivity of vector product over addition]}$$

$$=\vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b} \qquad \text{[Again, by distributivity of vector product over addition]}$$

$$=\vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0}$$

$$=2\left(\vec{a}\times\vec{b}\right)$$

5. Find λ and μ if $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$.

Solution:

It is given that

Given,

$$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k}$$

$$\hat{i}\left(6\mu-27\lambda\right)-\hat{j}\left(2\mu-27\right)+\hat{k}\left(2\lambda-6\right)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have

$$6\mu-27\lambda=0$$

$$2\mu-27=0$$

$$2\lambda-6=0$$

Now,

$$2\lambda-6=0\Rightarrow\lambda=3$$

$$2\mu-27=0\Rightarrow\mu=\frac{27}{2}$$

Thus, $\lambda=3$ and $\mu=\frac{27}{2}$.

6. Given that $\vec{a}\cdot\vec{b}=0$ **and** $\vec{a}\times\vec{b}=\vec{0}$. **What can you conclude about the vectors** $\vec{a}$ and $\vec{b}$?
Solution:
It is given that,

$\vec{a} \cdot \vec{b} = 0$

Then,

(i) Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$, or $\vec{a} \perp \vec{b}$ $\left(\text{in case } \vec{a} \text{ and } \vec{b} \text{ are non-zero}\right)$

$\vec{a} \times \vec{b} = 0$

(ii) Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$, or $\vec{a} \parallel \vec{b}$ $\left(\text{in case } \vec{a} \text{ and } \vec{b} \text{ are non-zero}\right)$

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Therefore , $|\vec{a}| = 0$ or $|\vec{b}| = 0$.

7. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \ b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \ c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show that $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Solution:

It is given that

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \ \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$(\vec{b} + \vec{c}) = (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$

Now, $\vec{a} \times (\vec{b} + \vec{c}) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$

$= \hat{i}\left[a_2(b_3 + c_3) - a_3(b_2 + c_2)\right] - \hat{j}\left[a_1(b_3 + c_3) - a_3(b_1 + c_1)\right] + \hat{k}\left[a_1(b_2 + c_2) - a_2(b_1 + c_1)\right]$

$= \hat{i}\left[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2\right] + \hat{j}\left[-a_1b_3 - a_1c_3 + a_3b_1 + a_3c_1\right] + \hat{k}\left[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\right]$ ...(1)

And,

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= \hat{i}\left[a_2b_3 - a_3b_2\right] + \hat{j}\left[b_1a_3 - a_1b_3\right] + \hat{k}\left[a_1b_2 - a_2b_1\right]$ \qquad (2)

$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \hat{i}\left[a_2c_3 - a_3c_2\right] + \hat{j}\left[a_3c_1 - a_1c_3\right] + \hat{k}\left[a_1c_2 - a_2c_1\right]$ \qquad (3)

On adding (2) and (3), we get

$(\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) = \hat{i}\left[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2\right] + \hat{j}\left[b_1a_3 + a_3c_1 - a_1b_3 - a_1c_3\right] + \hat{k}\left[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\right]$ \qquad (4)

From (1) and (4), we obtain

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

- Hence proved.

8. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

Solution:

First, let us consider,

Take any parallel non-zero vectors so that $\vec{a} \times \vec{b} = \vec{0}$.
Let $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}, \ \vec{b} = 4\hat{i} + 6\hat{j} + 8\hat{k}$.
Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} = \hat{i}(24-24) - \hat{j}(16-16) + \hat{k}(12-12) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Now, it's seen that

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

$$\therefore \vec{a} \neq \vec{0}$$

$$|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116}$$

$$\therefore \vec{b} \neq \vec{0}$$

Thus, the converse of the given statement need not be true.

9. Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Solution:

We know

Given A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5) are the vertices of triangle ABC.
The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of △ABC are given as:

$$\overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} = -\hat{i} + 2\hat{j}$$

Now,

Area of △ABC $= \dfrac{1}{2} \left| \overrightarrow{AB} \times \overrightarrow{BC} \right|$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore \left| \overrightarrow{AB} \times \overrightarrow{BC} \right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

Therefore, the area of △ABC is $\dfrac{\sqrt{61}}{2}$ square units.

10. Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Solution:

Let us consider,

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a} \times \vec{b}\right|$.
Now, the adjacent sides are given as:

$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{20^2 + 5^2 + 5^2} = \sqrt{400 + 25 + 25} = 15\sqrt{2}$$

Therefore, the area of the given parallelogram is $15\sqrt{2}$ square units.

11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \dfrac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{2}$

Solution:

Explanation:

Given, $|\vec{a}| = 3$ and $|\vec{b}| = \dfrac{\sqrt{2}}{3}$.

We know that $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$

Now, $\vec{a} \times \vec{b}$ is a unit vector if $|\vec{a} \times \vec{b}| = 1$.

$$|\vec{a} \times \vec{b}| = 1$$

$$\left| |\vec{a}||\vec{b}| \sin\theta \, \hat{n} \right| = 1$$

$$|\vec{a}||\vec{b}| |\sin\theta| = 1$$

$$3 \times \dfrac{\sqrt{2}}{3} \times \sin\theta = 1$$

$$\sin\theta = \dfrac{1}{\sqrt{2}}$$

$$\theta = \dfrac{\pi}{4}$$

Thus, $\vec{a} \times \vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{4}$.

So, the correct answer is B.

12. Area of a rectangle having vertices A, B, C, and D with position vectors
$$-\hat{i} + \dfrac{1}{2}\hat{j} + 4\hat{k}, \ \hat{i} + \dfrac{1}{2}\hat{j} + 4\hat{k}, \ \hat{i} - \dfrac{1}{2}\hat{j} + 4\hat{k} \text{ and } -\hat{i} - \dfrac{1}{2}\hat{j} + 4\hat{k} \text{, respectively is}$$

(A) $\dfrac{1}{2}$

(B) 1

(C) 2

(D) 4

Solution:

Explanation:

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$$\overrightarrow{OA} = -\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \quad \overrightarrow{OB} = \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \quad \overrightarrow{OC} = \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}, \quad \overrightarrow{OD} = -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$$

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as.

$$\overrightarrow{AB} = (1+1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = 2\hat{i}$$

$$\overrightarrow{BC} = (1-1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = -\hat{j}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = \hat{k}(-2) = -2\hat{k}$$

$$\Rightarrow \left| \overrightarrow{AB} \times \overrightarrow{BC} \right| = 2$$

We know that, the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left| \vec{a} \times \vec{b} \right|$.

Thus, the area of the given rectangle is $\left| \overrightarrow{AB} \times \overrightarrow{BC} \right| = 2$ square units.
So, the correct answer is C.