MISCELLANEOUS EXERCISE

1. Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of the *x*-axis.

Solution:

Let us consider,

If $\vec{r}$ is a unit vector in the XY-plane, then $\vec{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$.

Here, θ is the angle made by the unit vector with the positive direction of the x-axis.

Hence, for $\theta = 30°$ we have:

$$\vec{r} = \cos 30°\hat{i} + \sin 30°\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

Therefore, the required unit vector is $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

2. Find the scalar components and magnitude of the vector joining the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2).

Solution:

First, let us consider,

The vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ can be found out by:

$$\overrightarrow{PQ} = \text{Position vector of Q} - \text{Position vector of P}$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Therefore, the scalar components and the magnitude of the vector joining the given points are respectively $\{(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)\}$ and $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

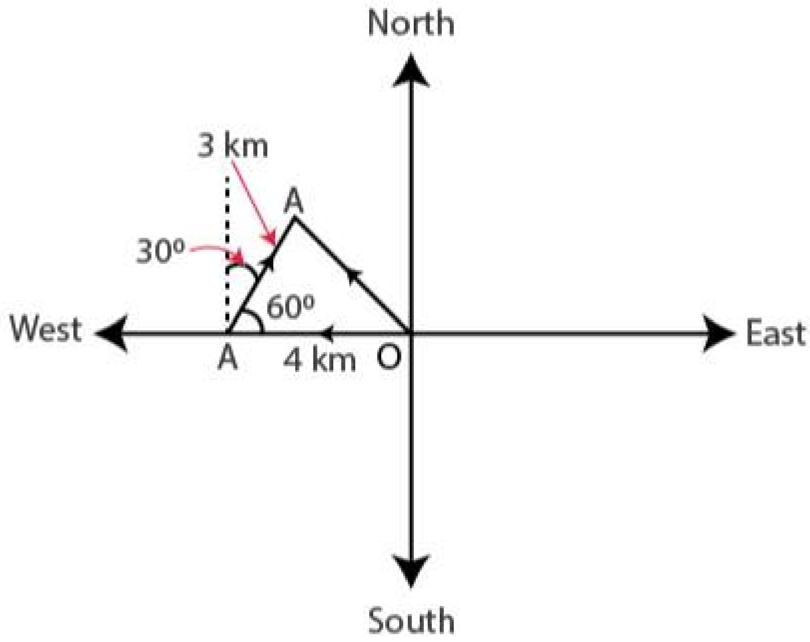
3. A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution:

It is given that

Let O and B be the initial and final positions of the girl, respectively.

Then, the girl's position can be shown as



$\overrightarrow{OA} = -4\hat{i}$

$\overrightarrow{AB} = \hat{i}\left|\overrightarrow{AB}\right|\cos 60° + \hat{j}\left|\overrightarrow{AB}\right|\sin 60°$

$= \hat{i}3\times\frac{1}{2} + \hat{j}3\times\frac{\sqrt{3}}{2}$

$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

By the Triangle law of vector addition, we have

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \left(-4\hat{i}\right) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right)$

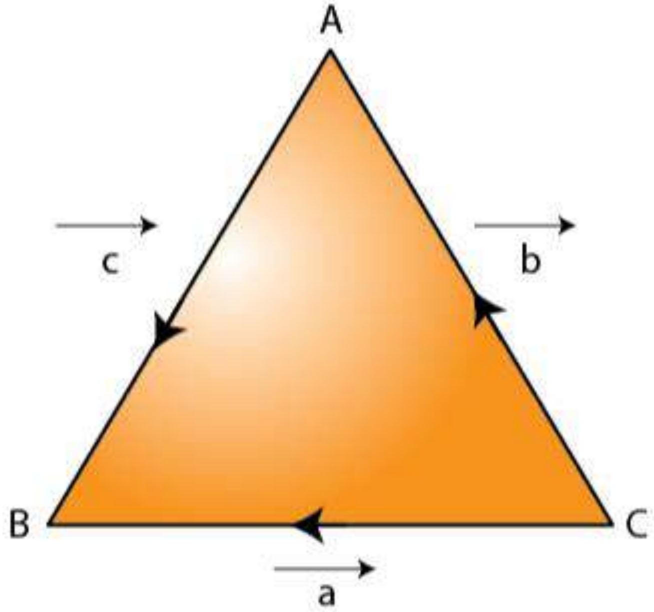
$= \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

$= \left(\frac{-8+3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

Therefore, the girl's displacement from her initial point of departure is

$\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$.



4. If $\vec{a} = \vec{b} + \vec{c}$, then, is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Solution:

It is given that,

In ΔABC, let $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, and $\overrightarrow{AB} = \vec{c}$ (as shown in the following figure).

So, by the Triangle law of vector addition, we have $\vec{a} = \vec{b} + \vec{c}$.

And, we know that $|\vec{a}|$, $|\vec{b}|$, and $|\vec{c}|$ represent the sides of ΔABC.

Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$\therefore |\vec{a}| < |\vec{b}| + |\vec{c}|$

Therefore, it is not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$.

5. Find the value of *x* for which $x\left(\hat{i} + \hat{j} + \hat{k}\right)$ is a unit vector.

Solution:

We know,

Given $x\left(\hat{i}+\hat{j}+\hat{k}\right)$ is a unit vector.

So, $\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right|=1$.

Now,

$$\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right|=1$$
$$\sqrt{x^2+x^2+x^2}=1$$
$$\sqrt{3x^2}=1$$
$$\sqrt{3}\,x=1$$
$$x=\pm\frac{1}{\sqrt{3}}$$

Therefore, the required value of x is $\pm\frac{1}{\sqrt{3}}$

6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors

$\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$.

Solution:

Let us consider that the

Given vectors,

$\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$

Let $\vec{c}$ be the resultant of $\vec{a}$ and $\vec{b}$.

Then,

$$\vec{c}=\vec{a}+\vec{b}=(2+1)\hat{i}+(3-2)\hat{j}+(-1+1)\hat{k}=3\hat{i}+\hat{j}$$
$$|\vec{c}|=\sqrt{3^2+1^2}=\sqrt{9+1}=\sqrt{10}$$
$$\therefore \hat{c}=\frac{\vec{c}}{|\vec{c}|}=\frac{\left(3\hat{i}+\hat{j}\right)}{\sqrt{10}}$$

Therefore, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is

$$\pm5\cdot\hat{c}=\pm5\cdot\frac{1}{\sqrt{10}}\left(3\hat{i}+\hat{j}\right)=\pm\frac{3\sqrt{10}\hat{i}}{2}\pm\frac{\sqrt{10}}{2}\hat{j}.$$

7. If $\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$, find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

Solution:

Let us consider the given vectors,

Given,

$\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$

$$2\vec{a}-\vec{b}+3\vec{c}=2\left(\hat{i}+\hat{j}+\hat{k}\right)-\left(2\hat{i}-\hat{j}+3\hat{k}\right)+3\left(\hat{i}-2\hat{j}+\hat{k}\right)$$

$$=2\hat{i}+2\hat{j}+2\hat{k}-2\hat{i}+\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}$$

$$=3\hat{i}-3\hat{j}+2\hat{k}$$

$$\left|2\vec{a}-\vec{b}+3\vec{c}\right|=\sqrt{3^2+(-3)^2+2^2}=\sqrt{9+9+4}=\sqrt{22}$$

Therefore, the unit vector along $2\vec{a}-\vec{b}+3\vec{c}$ is

$$\frac{2\vec{a}-\vec{b}+3\vec{c}}{\left|2\vec{a}-\vec{b}+3\vec{c}\right|}=\frac{3\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{22}}=\frac{3}{\sqrt{22}}\hat{i}-\frac{3}{\sqrt{22}}\hat{j}+\frac{2}{\sqrt{22}}\hat{k}.$$

8. Show that the points A (1, –2, –8), B (5, 0, –2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

Solution:

First, let us consider,

Given points are: A (1, –2, –8), B (5, 0, –2), and C (11, 3, 7).

$\therefore \overrightarrow{AB} = (5-1)\hat{i} + (0+2)\hat{j} + (-2+8)\hat{k} = 4\hat{i} + 2\hat{j} + 6\hat{k}$

$\overrightarrow{BC} = (11-5)\hat{i} + (3-0)\hat{j} + (7+2)\hat{k} = 6\hat{i} + 3\hat{j} + 9\hat{k}$

$\overrightarrow{AC} = (11-1)\hat{i} + (3+2)\hat{j} + (7+8)\hat{k} = 10\hat{i} + 5\hat{j} + 15\hat{k}$

$\left|\overrightarrow{AB}\right| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16+4+36} = \sqrt{56} = 2\sqrt{14}$

$\left|\overrightarrow{BC}\right| = \sqrt{6^2 + 3^2 + 9^2} = \sqrt{36+9+81} = \sqrt{126} = 3\sqrt{14}$

$\left|\overrightarrow{AC}\right| = \sqrt{10^2 + 5^2 + 15^2} = \sqrt{100+25+225} = \sqrt{350} = 5\sqrt{14}$

$\therefore \left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$

Therefore, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda : 1$. So, we have:

$$\overrightarrow{OB} = \frac{\lambda \overrightarrow{OC} + \overrightarrow{OA}}{(\lambda+1)}$$

$$5\hat{i} - 2\hat{k} = \frac{\lambda\left(11\hat{i} + 3\hat{j} + 7\hat{k}\right) + \left(\hat{i} - 2\hat{j} - 8\hat{k}\right)}{\lambda+1}$$

$$(\lambda+1)\left(5\hat{i} - 2\hat{k}\right) = 11\lambda\hat{i} + 3\lambda\hat{j} + 7\lambda\hat{k} + \hat{i} - 2\hat{j} - 8\hat{k}$$

$$5(\lambda+1)\hat{i} - 2(\lambda+1)\hat{k} = (11\lambda+1)\hat{i} + (3\lambda-2)\hat{j} + (7\lambda-8)\hat{k}$$

On equating the corresponding components, we have:

$$5(\lambda+1) = 11\lambda + 1$$

$$5\lambda + 5 = 11\lambda + 1$$

$$6\lambda = 4$$

$$\lambda = \frac{4}{6} = \frac{2}{3}$$

Therefore, point B divides AC in the ratio $2:3$.

Given points are: A (1, −2, −8), B (5, 0, −2), and C (11, 3, 7).

$\therefore \overrightarrow{AB} = (5-1)\hat{i} + (0+2)\hat{j} + (-2+8)\hat{k} = 4\hat{i} + 2\hat{j} + 6\hat{k}$

$\overrightarrow{BC} = (11-5)\hat{i} + (3-0)\hat{j} + (7+2)\hat{k} = 6\hat{i} + 3\hat{j} + 9\hat{k}$

$\overrightarrow{AC} = (11-1)\hat{i} + (3+2)\hat{j} + (7+8)\hat{k} = 10\hat{i} + 5\hat{j} + 15\hat{k}$

$\left|\overrightarrow{AB}\right| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16+4+36} = \sqrt{56} = 2\sqrt{14}$

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Therefore, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda : 1$. So, we have:

$$\overrightarrow{OB} = \frac{\lambda\overrightarrow{OC} + \overrightarrow{OA}}{(\lambda+1)}$$

$$5\hat{i} - 2\hat{k} = \frac{\lambda\left(11\hat{i} + 3\hat{j} + 7\hat{k}\right) + \left(\hat{i} - 2\hat{j} - 8\hat{k}\right)}{\lambda+1}$$

$$(\lambda+1)\left(5\hat{i} - 2\hat{k}\right) = 11\lambda\hat{i} + 3\lambda\hat{j} + 7\lambda\hat{k} + \hat{i} - 2\hat{j} - 8\hat{k}$$

$$5(\lambda+1)\hat{i} - 2(\lambda+1)\hat{k} = (11\lambda+1)\hat{i} + (3\lambda-2)\hat{j} + (7\lambda-8)\hat{k}$$

On equating the corresponding components, we have:

$5(\lambda+1) = 11\lambda + 1$

$5\lambda + 5 = 11\lambda + 1$

$6\lambda = 4$

$\lambda = \frac{4}{6} = \frac{2}{3}$

Therefore, point B divides AC in the ratio $2:3$.

9. Find the position vector of a point R, which divides the line joining two points P and Q, whose position vectors are $\left(2\vec{a} + \vec{b}\right)$ and $\left(\vec{a} - 3\vec{b}\right)$ externally in the ratio 1: 2. Also, show that P is the midpoint of the line segment RQ.

Solution:

We know,

Given, $\overrightarrow{OP} = 2\vec{a} + \vec{b},\ \overrightarrow{OQ} = \vec{a} - 3\vec{b}$.

Also, given that point R divides a line segment joining two points P and Q externally in the ratio 1: 2. So, on using the section formula, we have

$$\overrightarrow{OR} = \frac{2(2\vec{a}+\vec{b})-(\vec{a}-3\vec{b})}{2-1} = \frac{4\vec{a}+2\vec{b}-\vec{a}+3\vec{b}}{1} = 3\vec{a}+5\vec{b}$$

Hence, the position vector of point R is $3\vec{a}+5\vec{b}$.

Now,

Position vector of the mid-point of RQ = $\dfrac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$$= \frac{(\vec{a}-3\vec{b})+(3\vec{a}+5\vec{b})}{2}$$

$$= 2\vec{a}+\vec{b}$$

$$= \overrightarrow{OP}$$

Therefore, P is the mid-point of the line segment RQ.

10. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$.

Find the unit vector parallel to its diagonal. Also, find its area.

Solution:

First, let us consider,

Adjacent sides of a parallelogram are given as: $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$

We know that, the diagonal of a parallelogram is given by $\vec{a} + \vec{b}$.

$\vec{a} + \vec{b} = (2+1)\hat{i} + (-4-2)\hat{j} + (5-3)\hat{k} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

Hence, the unit vector parallel to the diagonal is

$$\frac{\vec{a}+\vec{b}}{\left|\vec{a}+\vec{b}\right|} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + (-6)^2 + 2^2}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{9+36+4}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7} = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}.$$

So, the area of parallelogram ABCD = $\left|\vec{a} \times \vec{b}\right|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= \hat{i}(12+10) - \hat{j}(-6-5) + \hat{k}(-4+4)$$

$$= 22\hat{i} + 11\hat{j}$$

$$= 11\left(2\hat{i} + \hat{j}\right)$$

$$\therefore \left|\vec{a} \times \vec{b}\right| = 11\sqrt{2^2 + 1^2} = 11\sqrt{5}$$

Therefore, the area of the parallelogram is $11\sqrt{5}$ square units.

Adjacent sides of a parallelogram are given as: $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$

We know that, the diagonal of a parallelogram is given by $\vec{a} + \vec{b}$.

$$\vec{a} + \vec{b} = (2+1)\hat{i} + (-4-2)\hat{j} + (5-3)\hat{k} = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Hence, the unit vector parallel to the diagonal is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + (-6)^2 + 2^2}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{9+36+4}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7} = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}.$$

So, the area of parallelogram ABCD $= |\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= \hat{i}(12+10) - \hat{j}(-6-5) + \hat{k}(-4+4)$$

$$= 22\hat{i} + 11\hat{j}$$

$$= 11(2\hat{i} + \hat{j})$$

$$\therefore |\vec{a} \times \vec{b}| = 11\sqrt{2^2 + 1^2} = 11\sqrt{5}$$

Therefore, the area of the parallelogram is $11\sqrt{5}$ square units.

11. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Solution:

First,

Let's assume a vector to be equally inclined to axes OX, OY, and OZ at angle α.

Then, the direction cosines of the vector are cos α, cos α, and cos α.

Now, we know that

$$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$

$$3\cos^2\alpha = 1$$

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

Therefore, the direction cosines of the vector, which are equally inclined to the axes, are

$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Hence, proved.

12. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}.\vec{d} = 15$.

Solution:

Assume,

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.

As $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we have:

$\vec{d} \cdot \vec{a} = 0$

$d_1 + 4d_2 + 2d_3 = 0 \quad \text{...(i)}$

And,

$\vec{d} \cdot \vec{b} = 0$

$3d_1 - 2d_2 + 7d_3 = 0 \quad \text{...(ii)}$

Also, given that:

$\vec{c} \cdot \vec{d} = 15$

$2d_1 - d_2 + 4d_3 = 15 \quad \text{...(iii)}$

On solving (i), (ii), and (iii), we obtain

$d_1 = \frac{160}{3}, d_2 = -\frac{5}{3}$ and $d_3 = -\frac{70}{3}$

$\therefore \vec{d} = \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k} = \frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$

Therefore, the required vector is $\frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.

As $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we have:

$\vec{d} \cdot \vec{a} = 0$

$d_1 + 4d_2 + 2d_3 = 0 \qquad ...(i)$

And,

$\vec{d} \cdot \vec{b} = 0$

$3d_1 - 2d_2 + 7d_3 = 0 \qquad ...(ii)$

Also, given that:

$\vec{c} \cdot \vec{d} = 15$

$2d_1 - d_2 + 4d_3 = 15 \qquad ...(iii)$

On solving (i), (ii), and (iii), we obtain

$d_1 = \frac{160}{3}, d_2 = -\frac{5}{3}$ and $d_3 = -\frac{70}{3}$

$\therefore \vec{d} = \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k} = \frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$

Therefore, the required vector is $\frac{1}{3}\left(160\hat{i} - 5\hat{j} - 70\hat{k}\right)$

13. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ .

Solution:

Let's consider the

Sum of the given vectors is given by,

$$\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$$

$$=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

Hence, unit vector along $\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$ is given as:

$$\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{4+4\lambda+\lambda^2+36+4}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$$

Scalar product of $\left(\hat{i}+\hat{j}+\hat{k}\right)$ with this unit vector is 1.

$$\left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\sqrt{\lambda^2+4\lambda+44}=\lambda+6$$

$$\lambda^2+4\lambda+44=(\lambda+6)^2$$

$$\lambda^2+4\lambda+44=\lambda^2+12\lambda+36$$

$$8\lambda=8$$

$$\lambda=1$$

Therefore, the value of A is 1.

14. If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$.

Solution:

Let's assume,

As $\vec{a}, \vec{b}$, and $\vec{c}$ are mutually perpendicular vectors, we have
$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0.$
Given that:
$|\vec{a}| = |\vec{b}| = |\vec{c}|$
Let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a}, \vec{b}$, and $\vec{c}$ at angles θ_1, θ_2, and θ_3 respectively.
So, we have

$$\cos\theta_1 = \frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{a}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|} = \frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|}$$

$$= \frac{\left|\vec{a}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|} \qquad \left[\vec{b}\cdot\vec{a} = \vec{c}\cdot\vec{a} = 0\right]$$

$$= \frac{\left|\vec{a}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|}$$

$$\cos\theta_2 = \frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|} = \frac{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\cdot\left|\vec{b}\right|}$$

$$= \frac{\left|\vec{b}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\cdot\left|\vec{b}\right|} \qquad \left[\vec{a}\cdot\vec{b} = \vec{c}\cdot\vec{b} = 0\right]$$

$$= \frac{\left|\vec{b}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|}$$

$$\cos\theta_3 = \frac{\left(\vec{a}+\vec{b}+\vec{c}\right)\cdot\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|} = \frac{\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|}$$

$$= \frac{\left|\vec{c}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|} \qquad \left[\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c} = 0\right]$$

$$= \frac{\left|\vec{c}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|}$$

Now, as $|\vec{a}| = |\vec{b}| = |\vec{c}|$, $\cos\theta_1 = \cos\theta_2 = \cos\theta_3$.
$\therefore \theta_1 = \theta_2 = \theta_3$
Therefore, the vector $\left(\vec{a}+\vec{b}+\vec{c}\right)$ is equally inclined to $\vec{a}, \vec{b}$, and $\vec{c}$.

Hence proved.

15. Prove that $\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right) = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$.

Solution:

It is given that

Required to prove:

$(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$

$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=|\vec{a}|^2+|\vec{b}|^2$ [Distributivity of scalar products over addition]

$|\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2$ $[\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (Scalar product is commutative)$]$

$2\vec{a}\cdot\vec{b}=0$

$\vec{a}\cdot\vec{b}=0$ $[\vec{a}\neq\vec{0},\ \vec{b}\neq\vec{0}$ (Given)$]$

Therefore, $\vec{a}$ and $\vec{b}$ are perpendicular.

Hence, proved.

16. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b}\geq 0$ only when

(A) $0<\theta<\frac{\pi}{2}$ **(B)** $0\leq\theta\leq\frac{\pi}{2}$

(C) $0<\theta<\pi$ **(D)** $0\leq\theta\leq\pi$

Solution:

Explanation:

Let's assume θ to be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors so that $|\vec{a}|$ and $|\vec{b}|$ are positive

We also know, $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

So,

$\vec{a}\cdot\vec{b}\geq 0$

$|\vec{a}||\vec{b}|\cos\theta\geq 0$

$\cos\theta\geq 0$ $[|\vec{a}|$ and $|\vec{b}|$ are positive$]$

$0\leq\theta\leq\frac{\pi}{2}$

Therefore, $\vec{a}.\vec{b}\geq 0$ when $0\leq\theta\leq\frac{\pi}{2}$.

The correct answer is B.

17. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ **(B)** $\theta = \frac{\pi}{3}$ **(C)** $\theta = \frac{\pi}{2}$ **(D)** $\theta = \frac{2\pi}{3}$

Solution:

Explanation:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ be the angle between them.

Then, $|\vec{a}| = |\vec{b}| = 1$.

Now, $\vec{a}+\vec{b}$ is a unit vector if $|\vec{a}+\vec{b}| = 1$.

$$|\vec{a}+\vec{b}| = 1$$
$$(\vec{a}+\vec{b})^2 = 1$$
$$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$$
$$\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 1$$
$$|\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1$$
$$1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1$$
$$1 + 2.1.1\cos\theta + 1 = 1$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = \frac{2\pi}{3}$$

Therefore, $\vec{a}+\vec{b}$ is a unit vector if $\theta = \frac{2\pi}{3}$.

Hence the correct answer is D.

18. The value of $\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}.(\hat{i}\times\hat{j})$ **is**

(A) 0 (B) –1 (C) 1 (D) 3

Solution:

Explanation:

It is given that,

$$\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{i}\times\hat{k}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)$$
$$=\hat{i}\cdot\hat{i}+\hat{j}\cdot\left(-\hat{j}\right)+\hat{k}\cdot\hat{k}$$
$$=1-\hat{j}\cdot\hat{j}+1$$
$$=1-1+1$$
$$=1$$

Hence, the correct answer is C.

19. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $\left|\vec{a}.\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Solution:

Explanation:

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

$$\left|\vec{a}\cdot\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$$
$$|\vec{a}|\left|\vec{b}\right|\cos\theta=|\vec{a}|\left|\vec{b}\right|\sin\theta$$
$$\cos\theta=\sin\theta \qquad \left[|\vec{a}| \text{ and } \left|\vec{b}\right| \text{ are positive}\right]$$
$$\tan\theta=1$$
$$\theta=\frac{\pi}{4}$$

Thus, $\left|\vec{a}.\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$ when θ is equal to $\frac{\pi}{4}$

So, the correct answer is B.