EXERCISE 11.2

1. Show that the three lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ **Are mutually perpendicular.**

Solution:

Let us consider the direction cosines of L_1, L_2 and L_3 be l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3.

We know that

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines,

And θ is the acute angle between the two lines,

Then $\cos \theta = |l_1l_2 + m_1m_2 + n_1n_2|$

If two lines are perpendicular, then the angle between the two is $\theta = 90°$

For perpendicular lines, $| l_1l_2 + m_1m_2 + n_1n_2 | = \cos 90° = 0$, i.e. $| l_1l_2 + m_1m_2 + n_1n_2 | = 0$

So, in order to check if the three lines are mutually perpendicular, we compute $| l_1l_2 + m_1m_2 + n_1n_2 |$ for all the pairs of the three lines.

Firstly let us compute, $| l_1l_2 + m_1m_2 + n_1n_2 |$

$$|l_1l_2 + m_1m_2 + n_1n_2| = \left|\left(\frac{12}{13}\times\frac{4}{13}\right) + \left(\frac{-3}{13}\times\frac{12}{13}\right) + \left(\frac{-4}{13}\times\frac{3}{13}\right)\right| = \frac{48}{13} + \left(\frac{-36}{13}\right) + \left(\frac{-12}{13}\right)$$

$$= \frac{48 + (-48)}{13} = 0$$

So, $L_1 \perp L_2$ …… (1)

Similarly,

Let us compute, $| l_2l_3 + m_2m_3 + n_2n_3 |$

$$|l_2l_3 + m_2m_3 + n_2n_3| = \left|\left(\frac{4}{13}\times\frac{3}{13}\right) + \left(\frac{12}{13}\times\frac{-4}{13}\right) + \left(\frac{3}{13}\times\frac{12}{13}\right)\right| = \frac{12}{13} + \left(\frac{-48}{13}\right) + \frac{36}{13}$$

$$= \frac{(-48)+48}{13} = 0$$

So, $L_2 \perp L_3$ ….. (2)

Similarly,

Let us compute, $|l_3l_1 + m_3m_1 + n_3n_1|$

$$|l_3l_1 + m_3m_1 + n_3n_1| = \left|\left(\frac{3}{13}\times\frac{12}{13}\right) + \left(\frac{-4}{13}\times\frac{-3}{13}\right) + \left(\frac{12}{13}\times\frac{-4}{13}\right)\right| = \frac{36}{13} + \frac{12}{13} + \left(\frac{-48}{13}\right)$$

$$= \frac{48+(-48)}{13} = 0$$

So, $L_1 \perp L_3$ ….. (3)

∴ By (1), (2) and (3), the lines are perpendicular.

L_1, L_2 and L_3 are mutually perpendicular.

2. Show that the line through the points (1, –1, 2), (3, 4, –2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Solution:

Given:

The points (1, –1, 2), (3, 4, –2) and (0, 3, 2), (3, 5, 6).

Let us consider AB be the line joining the points, (1, -1, 2) and (3, 4, -2), and CD be the line through the points (0, 3, 2) and (3, 5, 6).

Now,

The direction ratios, a_1, b_1, c_1 of AB are

(3 – 1), (4 – (-1)), (-2 – 2) = 2, 5, -4.

Similarly,

The direction ratios, a_2, b_2, c_2 of CD are

(3 – 0), (5 – 3), (6 – 2) = 3, 2, 4.

Then, AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$a_1a_2 + b_1b_2 + c_1c_2 = 2(3) + 5(2) + 4(-4)$

= 6 + 10 – 16

= 0

∴ AB and CD are perpendicular to each other.

3. Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (–1, –2, 1), (1, 2, 5).

Solution:

Given:

The points (4, 7, 8), (2, 3, 4) and (–1, –2, 1), (1, 2, 5).

Let us consider AB to be the line joining the points, (4, 7, 8), (2, 3, 4) and CD to be the line through the points (–1, –2, 1), (1, 2, 5).

Now,

The direction ratios, a_1, b_1, c_1 of AB are

(2 – 4), (3 – 7), (4 – 8) = -2, -4, -4.

The direction ratios, a_2, b_2, c_2 of CD are

(1 – (-1)), (2 – (-2)), (5 – 1) = 2, 4, 4.

Then, AB will be parallel to CD, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

So, a_1/a_2 = -2/2 = -1

b_1/b_2 = -4/4 = -1

c_1/c_2 = -4/4 = -1

∴ We can say that,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

-1 = -1 = -1

Hence, AB is parallel to CD where the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (–1, –2, 1), (1, 2, 5)

4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Solution:

Given:

Line passes through the point (1, 2, 3) and is parallel to the vector.

We know that

Vector equation of a line that passes through a given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is

$\vec{r} = \vec{a} + \lambda \vec{b}$.

So, here the position vector of the point (1, 2, 3) is given by

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and the parallel vector is $3\hat{i} + 2\hat{j} - 2\hat{k}$

∴ The vector equation of the required line is:

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} + 2\hat{j} - 2\hat{k}\right),$$

Where λ is constant.

5. Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and $\hat{i} + 2\hat{j} - \hat{k}.$ is in the direction

Solution:

It is given that

Vector equation of a line that passes through a given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$

Here let, $\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

So, the vector equation of the required line is:

$$\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$$

Now the Cartesian equation of a line through a point (x_1, y_1, z_1) and having direction cosines l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

We know that if the direction ratios of the line are a, b, c, then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

The Cartesian equation of a line through a point (x_1, y_1, z_1) and having direction ratios a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Here, $x_1 = 2$, $y_1 = -1$, $z_1 = 4$ and a = 1, b = 2, c = -1

∴ The Cartesian equation of the required line is:

$$\frac{x - 2}{1} = \frac{y - (-1)}{2} = \frac{z - 4}{-1} \Rightarrow \frac{x - 2}{1} = \frac{y + 1}{2} = \frac{z - 4}{-1}$$

6. Find the Cartesian equation of the line which passes through the point (–2, 4, –5) and parallel to the line given by

$$\frac{x + 3}{3} = \frac{y - 4}{5} = \frac{z + 8}{6}.$$

Solution:

Given:

The points (-2, 4, -5)

We know that the Cartesian equation of a line through a point (x_1, y_1, z_1) and having direction ratios a, b, c is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Here, the point (x_1, y_1, z_1) is (-2, 4, -5) and the direction ratio is given by:

a = 3, b = 5, c = 6

∴ The Cartesian equation of the required line is:

$$\frac{x-(-2)}{3}=\frac{y-4}{5}=\frac{z-(-5)}{6} \Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$$

7. The Cartesian equation of a line is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$$ **. Write its vector form.**

Solution:

Given:

The Cartesian equation is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2} \quad \ldots. (1)$$

We know that

The Cartesian equation of a line passing through a point (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

So when comparing this standard form with the given equation, we get

$x_1 = 5$, $y_1 = -4$, $z_1 = 6$ and

l = 3, m = 7, n = 2

The point through which the line passes has the position vector
$\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$ and
The vector parallel to the line is given by $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

Since, vector equation of a line that passes through a given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$

∴ The required line in vector form is given as:

$$\vec{r} = \left(5\hat{i} - 4\hat{j} + 6\hat{k}\right) + \lambda\left(3\hat{i} + 7\hat{j} + 2\hat{k}\right)$$

8. Find the vector and the Cartesian equations of the lines that passes through the origin and (5, –2, 3).

Solution:

Given:
The origin (0, 0, 0) and the point (5, -2, 3)
We know that
The vector equation of as line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$

Here, the position vectors of the two points (0, 0, 0) and (5, -2, 3) are $\vec{a} = 0\hat{i} + 0\hat{j} + 0\hat{k}$ and $\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$, respectively.

∴ The vector equation of the required line is given as:

$$\vec{r} = 0\hat{i} + 0\hat{j} + 0\hat{k} + \lambda\left[\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right) - \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right)\right]$$

$$\vec{r} = \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

Now, by using the formula,
Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

So, the Cartesian equation of the line that passes through the origin (0, 0, 0) and (5, – 2, 3) is

$$\frac{x-0}{5-0}=\frac{y-0}{-2-0}=\frac{z-0}{3-0} \Rightarrow \frac{x}{5}=\frac{y}{-2}=\frac{z}{3}$$

∴ The vector equation is

$$\vec{r}=\lambda\left(5\hat{i}-2\hat{j}+3\hat{k}\right)$$

The Cartesian equation is

$$\frac{x}{5}=\frac{y}{-2}=\frac{z}{3}$$

9. Find the vector and the Cartesian equations of the line that passes through the points (3, –2, –5), (3, –2, 6).

Solution:

Given:

The points (3, -2, -5) and (3, -2, 6)

Firstly let us calculate the vector form:

The vector equation of as line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\left(\vec{b}-\vec{a}\right)$

Here, the position vectors of the two points (3, –2, –5) and (3, –2, 6) are $\vec{a}=3\hat{i}-2\hat{j}-5\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+6\hat{k}$ respectively.

∴ The vector equation of the required line is:

$$\vec{r}=3\hat{i}-2\hat{j}-5\hat{k}+\lambda\left[\left(3\hat{i}-2\hat{j}+6\hat{k}\right)-\left(3\hat{i}-2\hat{j}-5\hat{k}\right)\right]$$

$$\vec{r}=3\hat{i}-2\hat{j}-5\hat{k}+\lambda\left(3\hat{i}-2\hat{j}+6\hat{k}-3\hat{i}+2\hat{j}+5\hat{k}\right)$$

$$\vec{r}=3\hat{i}-2\hat{j}-5\hat{k}+\lambda\left(11\hat{k}\right)$$

Now,
By using the formula,
Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

So, the Cartesian equation of the line that passes through the origin (3, -2, -5) and (3, -2, 6) is

$$\frac{x-3}{3-3}=\frac{y-(-2)}{(-2)-(-2)}=\frac{z-(-5)}{6-(-5)}$$

$$\frac{x-3}{0}=\frac{y+2}{0}=\frac{z+5}{11}$$

∴ The vector equation is

$$\vec{r}=3\hat{i}-2\hat{j}-5\hat{k}+\lambda\left(11\hat{k}\right)$$

The Cartesian equation is

$$\frac{x-3}{0}=\frac{y+2}{0}=\frac{z+5}{11}$$

10. Find the angle between the following pairs of lines:

(i) $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ and

$\vec{r}=7\hat{i}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$

(ii) $\vec{r}=3\hat{i}+\hat{j}-2\hat{k}+\lambda(\hat{i}-\hat{j}-2\hat{k})$ and

$\vec{r}=2\hat{i}-\vec{j}-56\hat{k}+\mu(3\hat{i}-5\hat{j}-4\hat{k})$

Solution:

Let us consider θ be the angle between the given lines.

If θ is the acute angle between $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\overrightarrow{b_2}$ then

$$\cos\theta = \left|\frac{\vec{b_1}\vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}\right| \quad \ldots\ldots (1)$$

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

Here $\overrightarrow{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\overrightarrow{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$

So, from equation (1), we have

$$\cos\theta = \left|\frac{\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right).\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)}{\left|3\hat{i} + 2\hat{j} + 6\hat{k}\right|.\left|\hat{i} + 2\hat{j} + 2\hat{k}\right|}\right| \quad \ldots\ldots (2)$$

We know that,

$$\left|a\hat{i}+b\hat{j}+c\hat{k}\right|=\sqrt{a^2+b^2+c^2}$$

So,

$$\left|3\hat{i}+2\hat{j}+6\hat{k}\right|=\sqrt{3^2+2^2+6^2}=\sqrt{9+4+36}=\sqrt{49}=7$$

And

$$\left|\hat{i}+2\hat{j}+2\hat{k}\right|=\sqrt{1^2+2^2+2^2}=\sqrt{1+4+4}=\sqrt{9}=3$$

Now, we know that

$$\left(a_1\hat{i}+b_1\hat{j}+c_1\hat{k}\right).\left(a_2\hat{i}+b_2\hat{j}+c_2\hat{k}\right)=a_1a_2+b_1b_2+c_1c_2$$

So,

$$\left(3\hat{i}+2\hat{j}+6\hat{k}\right).\left(\hat{i}+2\hat{j}+2\hat{k}\right)=3\times1+2\times2+6\times2=3+4+12=19$$

By (2), we have

$$\cos\theta=\frac{19}{7\times3}=\frac{19}{21}$$

$$\theta=\cos^{-1}\left(\frac{19}{21}\right)$$

So,

$$\left(3\hat{i}+2\hat{j}+6\hat{k}\right).\left(\hat{i}+2\hat{j}+2\hat{k}\right)=3\times1+2\times2+6\times2=3+4+12=19$$

By (2), we have

$$\cos\theta=\frac{19}{7\times3}=\frac{19}{21}$$

$$\theta=\cos^{-1}\left(\frac{19}{21}\right)$$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and

$\vec{r} = 2\hat{i} - \vec{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

Here, $\overrightarrow{b_1} = \hat{i} - \hat{j} - 2\hat{k}$ and $\overrightarrow{b_2} = 3\hat{i} - 5\hat{j} - 4\hat{k}$

So, from (1), we have

$$\cos\theta = \left|\frac{(\hat{i} - \hat{j} - 2\hat{k}).(3\hat{i} - 5\hat{j} - 4\hat{k})}{|\hat{i} - \hat{j} - 2\hat{k}||3\hat{i} - 5\hat{j} - 4\hat{k}|}\right| \quad \text{.... (3)}$$

We know that,

$$|a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$$

So,

$$|\hat{i} - \hat{j} - 2\hat{k}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6} = \sqrt{3} \times \sqrt{2}$$

And

$$|3\hat{i} - 5\hat{j} - 4\hat{k}| = \sqrt{3^2 + (-5)^2 + (-4)^2} = \sqrt{9+25+16} = \sqrt{50} = 5\sqrt{2}$$

Now, we know that

$$(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}).(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = a_1a_2 + b_1b_2 + c_1c_2$$

$$\therefore\ (\hat{i} - \hat{j} - 2\hat{k}).(3\hat{i} - 5\hat{j} - 4\hat{k}) = 1 \times 3 + (-1) \times (-5) + (-2) \times (-4) = 3 + 5 + 8 = 16$$

By (3), we have

$$\cos\theta = \frac{16}{\sqrt{3} \times \sqrt{2} \times 5\sqrt{2}} = \frac{16}{5 \times 2\sqrt{3}} = \frac{8}{5\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

$$\theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

11. Find the angle between the following pair of lines:

(i) $\frac{x-2}{2}=\frac{y-1}{5}-\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}-\frac{z-5}{4}$

(ii) $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}$

Solution:

We know that

If

$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are the equations of two lines, then the acute angle between the two lines is given by

$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$ …… (1)

(i) $\frac{x-2}{2}=\frac{y-1}{5}-\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}-\frac{z-5}{4}$

Here, $a_1 = 2$, $b_1 = 5$, $c_1 = -3$ and

$a_2 = -1$, $b_2 = 8$, $c_2 = 4$

Now,

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}},\ m=\frac{b}{\sqrt{a^2+b^2+c^2}},\ n=\frac{c}{\sqrt{a^2+b^2+c^2}} \quad \text{...... (2)}$$

Here, we know that

$$\sqrt{a_1^2+b_1^2+c_1^2}=\sqrt{2^2+5^2+(-3)^2}=\sqrt{4+25+9}=\sqrt{38}$$

And

$$\sqrt{a_2^2+b_2^2+c_2^2}=\sqrt{(-1)^2+8^2+4^2}=\sqrt{1+64+16}=\sqrt{81}=9$$

So, from equation (2), we have

$$l_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{2}{\sqrt{38}},\ m_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{5}{\sqrt{38}},\ n_1 = \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{-3}{\sqrt{38}}$$

And

$$l_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{-1}{9},\ m_2 = \frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{8}{9},\ n_2 = \frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{4}{9}$$

∴ From equation (1), we have

$$\cos\theta = \left|\left(\frac{2}{\sqrt{38}}\right)\times\left(\frac{-1}{9}\right) + \left(\frac{5}{\sqrt{38}}\right)\times\left(\frac{8}{9}\right) + \left(\frac{-3}{\sqrt{38}}\right)\times\left(\frac{4}{9}\right)\right|$$

$$= \left|\frac{-2+40-12}{9\sqrt{38}}\right| = \left|\frac{40-12}{9\sqrt{38}}\right| = \frac{26}{9\sqrt{38}}$$

$$\theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}$

Here, $a_1 = 2$, $b_1 = 2$, $c_1 = 1$ and

$a_2 = 4$, $b_2 = 1$, $c_2 = 8$

Here, we know that

$$\sqrt{a_1^2+b_1^2+c_1^2}=\sqrt{2^2+2^2+1^2}=\sqrt{4+4+1}=\sqrt{9}=3$$

And

$$\sqrt{a_2^2+b_2^2+c_2^2}=\sqrt{4^2+1^2+8^2}=\sqrt{16+1+64}=\sqrt{81}=9$$

So, from equation (2), we have

$$l_1=\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\frac{2}{3},\ m_1=\frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\frac{2}{3},\ n_1=\frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\frac{1}{3}$$

And

$$l_2=\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}=\frac{4}{9},\ m_2=\frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}}=\frac{1}{9},\ n_2=\frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}=\frac{8}{9}$$

∴ From equation (1), we have

$$\cos\theta=\left|\left(\frac{2}{3}\times\frac{4}{9}\right)+\left(\frac{2}{3}\times\frac{1}{9}\right)+\left(\frac{1}{3}\times\frac{8}{9}\right)\right|=\left|\frac{8+2+8}{27}\right|=\frac{18}{27}=\frac{2}{3}$$

$$\theta=\cos^{-1}\left(\frac{2}{3}\right)$$

∴ From equation (1), we have

$$\cos\theta=\left|\left(\frac{2}{3}\times\frac{4}{9}\right)+\left(\frac{2}{3}\times\frac{1}{9}\right)+\left(\frac{1}{3}\times\frac{8}{9}\right)\right|=\left|\frac{8+2+8}{27}\right|=\frac{18}{27}=\frac{2}{3}$$

$$\theta=\cos^{-1}\left(\frac{2}{3}\right)$$

12. Find the values of p so that the lines

$$\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}\text{ and }\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$$ **are at right angles.**

Solution:

The standard form of a pair of Cartesian lines is:

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2} \quad \ldots.(1)$$

So the given equations can be written according to the standard form, i.e.

$$\frac{-(x-1)}{3}=\frac{7(y-2)}{2p}=\frac{z-3}{2} \text{ and } \frac{-7(x-1)}{3p}=\frac{y-5}{1}=\frac{-(z-6)}{5}$$

$$\frac{x-1}{-3}=\frac{y-2}{2p/7}=\frac{z-3}{2} \text{ and } \frac{x-1}{-3p/7}=\frac{y-5}{1}=\frac{z-6}{-5} \quad \ldots.(2)$$

Now, comparing equation (1) and (2), we get

$$a_1=-3,\ b_1=\frac{2p}{7},\ c_1=2 \text{ and } a_2=\frac{-3p}{7},\ b_2=1,\ c_2=-5$$

So, the direction ratios of the lines are

-3, 2p/7, 2 and -3p/7, 1, -5

Now, as both the lines are at right angles,

So, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(-3) (-3p/7) + (2p/7) (1) + 2 (-5) = 0

9p/7 + 2p/7 – 10 = 0

(9p+2p)/7 = 10

11p/7 = 10

11p = 70

p = 70/11

∴ The value of p is 70/11

13. Show that the lines

$$\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1} \text{ and } \frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$ **are perpendicular to each other.**

Solution:

The equations of the given lines are

$$\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1} \text{ and } \frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

Two lines with direction ratios is given as

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

So the direction ratios of the given lines are 7, -5, 1 and 1, 2, 3

i.e., $a_1 = 7$, $b_1 = -5$, $c_1 = 1$ and

$a_2 = 1$, $b_2 = 2$, $c_2 = 3$

Now, considering

$a_1a_2 + b_1b_2 + c_1c_2 = 7 \times 1 + (-5) \times 2 + 1 \times 3$

= 7 -10 + 3

= – 3 + 3

= 0

∴ The two lines are perpendicular to each other.

14. Find the shortest distance between the lines

$$\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k}) \text{ and}$$

$$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu(2\hat{i}+\hat{j}+2\hat{k})$$

Solution:

We know that the shortest distance between two lines $\vec{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu \overrightarrow{b_2}$ is given as:

$$d = \left| \frac{\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|} \right| \quad \ldots. (1)$$

Here by comparing the equations we get,

$\overrightarrow{a_1} = \hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{b_1} = \hat{i} - \hat{j} + \hat{k}$ and

$\overrightarrow{a_2} = 2\hat{i} - \hat{j} - \hat{k}$, $\overrightarrow{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$

Now,

$$\left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) - \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$$

$$\overrightarrow{a_2} - \overrightarrow{a_1} = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right) = \hat{i} - 3\hat{j} - 2\hat{k} \quad \ldots. (2)$$

$$\left(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right)-\left(x_2\hat{i}+y_2\hat{j}+z_2\hat{k}\right)=(x_1-x_2)\hat{i}+(y_1-y_2)\hat{j}+(z_1-z_2)\hat{k}$$

$$\overrightarrow{a_2}-\overrightarrow{a_1}=\left(2\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+\hat{k}\right)=\hat{i}-3\hat{j}-2\hat{k} \quad \text{.... (2)}$$

Now,

$$\overrightarrow{b_1}\times\overrightarrow{b_2}=\left(\hat{i}-\hat{j}+\hat{k}\right)\times\left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}$$

$$=-3\hat{i}+3\hat{k}$$

$$\Rightarrow \overrightarrow{b_1}\times\overrightarrow{b_2}=-3\hat{i}+3\hat{k} \quad \text{.......... (3)}$$

$$\Rightarrow \left|\overrightarrow{b_1}\times\overrightarrow{b_2}\right|=\sqrt{(-3)^2+3^2}=\sqrt{9+9}=\sqrt{18}=3\sqrt{2} \quad \text{.......... (4)}$$

Now,

$$\left(a_1\hat{i}+b_1\hat{j}+c_1\hat{k}\right).\left(a_2\hat{i}+b_2\hat{j}+c_2\hat{k}\right)=a_1a_2+b_1b_2+c_1c_2$$

$$\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right).\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)=\left(-3\hat{i}+3\hat{k}\right).\left(\hat{i}-3\hat{j}-2\hat{k}\right)=-3-6=-9 \quad \text{.......... (5)}$$

Now, by substituting all the values in equation (1), we get

The shortest distance between the two lines,

$$d = \left|\frac{-9}{3\sqrt{2}}\right|$$

$$= \frac{9}{3\sqrt{2}}$$ [From equation (4) and (5)]

$$= \frac{3}{\sqrt{2}}$$

Let us rationalizing the fraction by multiplying the numerator and denominator by √2, we get

$$d = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{2}$$

∴ The shortest distance is 3√2/2

Let us rationalizing the fraction by multiplying the numerator and denominator by √2, we get

$$d = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

∴ The shortest distance is 3√2/2

15. Find the shortest distance between the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

Solution:

We know that the shortest distance between two lines

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$ is given as:

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}} \quad \dots (1)$$

The standard form of a pair of Cartesian lines is:

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

And the given equations are:

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

Now let us compare the given equations with the standard form we get,

$x_1 = -1, y_1 = -1, z_1 = -1;$

$x_2 = 3, y_2 = 5, z_2 = 7$

$a_1 = 7, b_1 = -6, c_1 = 1;$

$a_2 = 1, b_2 = -2, c_2 = 1$

Now, consider

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 3-(-1) & 5-(-1) & 7-(-1) \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}=\begin{vmatrix} 3+1 & 5+1 & 7+1 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

= 4 (-6 + 2) – 6 (7 - 1) + 8 (-14 + 6)
= 4 (4) – 6 (6) + 8 (-8)
= -16 – 36 – 64
= - 116

Now we shall consider

$$\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}$$

$$= \sqrt{((-6\times1) - (-2\times1))^2 + ((1\times1) - (1\times7))^2 + ((7\times-2) - (1\times-6))^2}$$

$$= \sqrt{(-6+2)^2 + (1-7)^2 + (-14+6)^2} = \sqrt{(-4)^2 + (-6)^2 + (-8)^2}$$

$$= \sqrt{16+36+64} = \sqrt{116}$$

By substituting all the values in equation (1), we get

The shortest distance between the two lines,

$$d = \left|\frac{-116}{\sqrt{116}}\right| = \frac{116}{\sqrt{116}} = \sqrt{116} = 2\sqrt{29}$$

∴ The shortest distance is 2√29

16. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and}$$

$$\vec{r} = 4\hat{i} + 5\hat{j} - 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

Solution:

We know that shortest distance between two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given as:

$$d = \left|\frac{(\vec{b_1}\times\vec{b_2}).(\vec{a_2}-\vec{a_1})}{|\vec{b_1}\times\vec{b_2}|}\right| \quad \text{……….(1)}$$

Here by comparing the equations we get,

$\overrightarrow{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\overrightarrow{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$ and

$\overrightarrow{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}$, $\overrightarrow{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$

Now let us subtract the above equations we get,

$$\left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) - \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$$

$$\overrightarrow{a_2} - \overrightarrow{a_1} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 3\hat{i} + 3\hat{j} + 3\hat{k} \quad \text{.......... (2)}$$

And,

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \left(\hat{i} - 3\hat{j} + 2\hat{k}\right) \times \left(2\hat{i} + 3\hat{j} + \hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \overrightarrow{b_1} \times \overrightarrow{b_2} = -9\hat{i} + 3\hat{j} + 9\hat{k} \quad \text{.......... (3)}$$

$$\Rightarrow \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{(-9)^2 + 3^2 + 9^2} = \sqrt{81 + 9 + 81} = \sqrt{171} = 3\sqrt{19} \quad \text{..... (4)}$$

Now by multiplying equation (2) and (3) we get,

$$\left(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right).\left(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}\right) = a_1a_2 + b_1b_2 + c_1c_2$$

$$\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) = \left(-9\hat{i} + 3\hat{j} + 9\hat{k}\right).\left(3\hat{i} + 3\hat{j} + 3\hat{k}\right) = -27 + 9 + 27 = 9 \quad \text{.... (5)}$$

By substituting all the values in equation (1), we obtain

The shortest distance between the two lines,

$$d = \left|\frac{9}{3\sqrt{19}}\right| = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}}$$

∴ The shortest distance is 3√19

17. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

Solution:

Firstly let us consider the given equations

=> $\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$

$\vec{r} = \hat{i} - t\hat{i} + t\hat{j} - 2\hat{j} + 3\hat{k} - 2t\hat{k}$

$\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + t\left(-\hat{i} + \hat{j} - 2\hat{k}\right)$

=> $\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

$\vec{r} = s\hat{i} + \hat{i} + 2s\hat{j} - \hat{j} - 2s\hat{k} - \hat{k}$

$\vec{r} = \hat{i} - \hat{j} - \hat{k} + s\left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$

So now we need to find the shortest distance between

$\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + t\left(-\hat{i} + \hat{j} - 2\hat{k}\right)$ and $\vec{r} = \hat{i} - \hat{j} - \hat{k} + s\left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$

We know that shortest distance between two lines

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given as:

$$d = \left|\frac{\left(\vec{b_1} \times \vec{b_2}\right).\left(\vec{a_2} - \vec{a_1}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|}\right| \quad \ldots (1)$$

Here by comparing the equations we get,

$\vec{a_1} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b_1} = -\hat{i} + \hat{j} - 2\hat{k}$ and

$\vec{a_2} = \hat{i} - \hat{j} - \hat{k}, \vec{b_2} = \hat{i} + 2\hat{j} - 2\hat{k}$

Since,

$$\left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) - \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$$

So,

$$\vec{a_2} - \vec{a_1} = \left(\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) = \hat{j} - 4\hat{k} \quad \ldots. (2)$$

And,

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \left(-\hat{i} + \hat{j} - 2\hat{k}\right) \times \left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\Rightarrow \overrightarrow{b_1} \times \overrightarrow{b_2} = 2\hat{i} - 4\hat{j} - 3\hat{k} \quad \ldots\ldots\ldots\ (3)$$

$$\Rightarrow \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{4 + 16 + 9} = \sqrt{29} \quad \ldots\ldots\ldots\ (4)$$

Now by multiplying equation (2) and (3) we get,

$$\left(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right).\left(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}\right) = a_1a_2 + b_1b_2 + c_1c_2$$

$$\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) = \left(2\hat{i} - 4\hat{j} - 3\hat{k}\right).\left(\hat{j} - 4\hat{k}\right) = -4 + 12 = 8 \quad \ldots\ldots\ldots\ (5)$$

By substituting all the values in equation (1), we obtain

The shortest distance between the two lines,

$$d = \left|\frac{8}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

∴ The shortest distance is 8√29