MISCELLANEOUS EXERCISE

1. For each of the differential equations given below, indicate its order and degree (if defined)

(i) $(d^2y/dx^2) + 5x(dy/dx)^2 - 6y = \log x$

(ii) $(dy/dx)^3 - 4(dy/dx)^2 + 7y = \sin x$

(iii) $(d^4y/dx^4) - \sin(d^3y/dx^3) = 0$

Solution:

(i) $(d^2y/dx^2) + 5x(dy/dx)^2 - 6y = \log x$

Rearranging the given equation, we get

$(d^2y/dx^2) + 5x(dy/dx)^2 - 6y - \log x = 0$

Hence, the highest order derivative present in the given differential equation is d^2y/dx^2.

Therefore, the order is 2.

Also, the highest power raised to d^2y/dx^2 is 1.

Hence, the degree is 1.

(ii) $(dy/dx)^3 - 4(dy/dx)^2 + 7y = \sin x$

Rearranging the given equation, we get

$(dy/dx)^3 - 4(dy/dx)^2 + 7y - \sin x = 0$

Hence, the highest order derivative present in the given differential equation is dy/dx.

Therefore, the order is 1.

And the highest power raised to dy/dx is 3.

Hence, the degree is 3.

(iii) $(d^4y/dx^4) - \sin(d^3y/dx^3) = 0$

The highest order derivative present in the given differential equation is d^4y/dx^4. Hence, the order is 4.

Since the given differential equation is not a polynomial equation, the degree of the equation is not defined.

2. For each of the exercises given below, verify that the given function (implicit or explicit) is a solution to the corresponding differential equation.

(i) $y = ae^x + be^{-x} + x^2$: $x(d^2y/dx^2) + 2(dy/dx) - xy + x^2 - 2 = 0$

(ii) $y = e^x(a\cos x + b\sin x)$: $(d^2y/dx^2) - 2(dy/dx) + 2y = 0$

(iii) $y = x \sin 3x$: $(d^2y/dx^2) + 9y - 6 \cos 3x = 0$

(iv) $x^2 = 2y^2 \log y$: $(x^2 + y^2)(dy/dx) - xy = 0$

Solution:

(i) $y = ae^x + be^{-x} + x^2$: $x (d^2y/dx^2) + 2(dy/dx) - xy + x^2 - 2 = 0$

Given: $y = ae^x + be^{-x} + x^2$

Differentiate the function with respect to x, and we get

$dy/dx = ae^x - be^{-x} + 2x$ … (1)

Now, again differentiate with respect to x, and we get

$d^2y/dx^2 = ae^x + be^{-x} + 2$ …(2)

To check whether the given function is the solution of the given differential equation, substitute (1) and (2) in the given differential equation.

L.H.S of the given differential equation

$= x (d^2y/dx^2) + 2(dy/dx) - xy + x^2$

Now, substituting the values, we get

$= x (ae^x + be^{-x} + 2) + 2 (ae^x - be^{-x} + 2x) - x (ae^x + be^{-x} + x^2) + x^2 - 2$

$= (xae^x + x be^{-x} + 2x) + (2ae^x - 2be^{-x} + 4x) - (xae^x + xbe^{-x} + x^3) + x^2 - 2$

On simplifying the above equation, we get

$= 2ae^x - 2be^{-x} - x^3 + x^2 + 6x - 2 \neq 0$

Hence, L.H.S $\neq$ R.H.S.

Therefore, the given function is not a solution to the corresponding differential equation.

(ii) $y = e^x (a \cos x + b \sin x)$: $(d^2y/dx^2) - 2(dy/dx) + 2y = 0$

Given: $y = e^x (a \cos x + b \sin x)$

The given function can be written as follows:

$y = e^x a \cos x + e^x b \sin x$

Differentiating the function on both sides, we get

$dy/dx = (a + b)e^x \cos x + (b - a)e^x \sin x$ …(1)

Again, differentiate the above equation on both sides with respect to x, and we get

$d^2y/dx^2 = [(a + b) (d/dx) (e^x \cos x] + [(b-a) (d/dx) (e^x \sin x]$

$d^2y/dx^2 = [(a+ b) (e^x \cos x – e^x \sin x)] + [(b- a)(e^x \sin x + e^x \cos x)]$

$d^2y/dx^2 = e^x [(a+ b) (\cos x – \sin x) + (b- a)(\sin x + \cos x)]$

On simplifying the above equation, we get

$d^2y/dx^2 = 2e^x(b \cos x – a \sin x)$ …(2)

Now, substitute (1) and (2) in the given differential equation.

L.H.S = $(d^2y/dx^2) – 2(dy/dx) + 2y$

$= [2e^x(b \cos x – a \sin x)] – 2[(a + b)e^x \cos x + (b – a)e^x \sin x] + 2 e^x(a \cos x + b \sin x)$

$= e^x [(2b \cos x – 2a \sin x)- (2a \cos x + 2b \cos x) – (2b \sin x – 2a \sin x) + (2a \cos x + 2b \sin x)]$

$= e^x [2b \cos x – 2a \sin x- 2a \cos x – 2b \cos x – 2b \sin x + 2a \sin x + 2a \cos x + 2b \sin x]$

$= e^x[0]$

= 0 = R.H.S

As L.H.S = R.H.S, the given function is the solution of the corresponding differential equation.

(iii) $y = x \sin 3x$: $(d^2y/dx^2) + 9y – 6 \cos 3x = 0$

Given: $y = x \sin 3x$

Now, differentiating the given function with respect to x, and we get

$dy/dx = \sin 3x + x. \cos 3x. 3$

$dy/dx = \sin 3x + 3x \cos 3x$ …(1)

Again differentiate (1) with respect to x, we get

$d^2y/dx^2 = (d/dx) (\sin 3x) + 3 (d/dx) (x \cos 3x)$

$d^2y/dx^2 = 3 \cos 3x + 3 [\cos 3x + x (- \sin 3x). 3]$

On simplifying the above equation, we get

$d^2y/dx^2 = 6 \cos 3x – 9x \sin 3x$ …(2)

Now, substitute (1) and (2) in the given differential equation, and we get the following:

L.H.S = $(d^2y/dx^2) + 9y – 6 \cos 3x$

$= (6 \cos 3x – 9x \sin 3x) + 9(x \sin 3x) – 6 \cos 3x$

$= 6 \cos 3x – 9x \sin 3x + 9x \sin 3x – 6 \cos 3x$

= 0 = R.H.S

As L.H.S = R.H.S, the given function is the solution of the corresponding differential equation.

(iv) $x^2 = 2y^2 \log y$: $(x^2 + y^2)(dy/dx) – xy = 0$

Given: $x^2 = 2y^2 \log y$

Now, differentiate the function with respect to x, and we get

$2x = 2 (d/dx) (y^2 \log y)$

On simplifying the above equation, we get

$x = (d/dx) (y^2 \log y)$

$x = [2y \log y .(dy/dx) + y^2. (1/y). (dy/dx)]$

$x = (dy/dx)[2y \log y + y]$

Hence, we get

$dy/dx = x / [y(1 + 2 \log y)]$ …(1)

Now, substitute (1) in the given differential equation.

L.H.S = $(x^2 + y^2)(dy/dx) – xy$

$= [2y^2 \log y + y^2] . [x / [y(1 + 2 \log y)]] – xy$

$= [y^2(2 \log y + 1)] . [x / [y(1 + 2 \log y)]] – xy$

$= xy – xy$

= 0 = R.H.S

As L.H.S = R.H.S, the given function is the solution of the corresponding differential equation.

3. Form the differential equation representing the family of curves given by $(x- a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

Solution:

Given equation: $(x- a)^2 + 2y^2 = a^2$

The given equation can be written as:

$\Rightarrow x^2 + a^2 – 2ax + 2y^2 = a^2$

On rearranging the above equation, we get

$\Rightarrow 2y^2 = 2ax – x^2$ …(1)

Now, differentiate equation (1) with respect to x,

⇒ 2 . 2y (dy/dx) = 2a – 2x

⇒ 2y(dy/dx) = (2a – 2x) /2

⇒ dy/dx = (a-x)/2y

⇒ dy/dx = $(2ax – 2x^2)$ / 4xy … (2)

From equation (1), we get

$2ax = 2y^2 + x^2$

Substitute the value in equation (2), and we get

dy/dx = $[2y^2 + x^2 – 2x^2]/4xy$

dy/dx = $(2y^2 – x^2) / 4xy$

Therefore, the differential equation representing the family of curves given by $(x- a)^2 + 2y^2 = a^2$ is $(2y^2 – x^2) / 4xy$.

4. Prove that $x^2 – y^2 = C (x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 – 3xy^2) dx = (y^3 – 3x^2y) dy$, where C is a parameter.

Solution:

Given differential equation: $(x^3 – 3xy^2) dx = (y^3 – 3x^2y) dy$

The equation can be rewritten as:

dy/dx = $(x^3 – 3xy^2) / (y^3 – 3x^2y)$ …(1)

The above equation is a homogeneous equation.

To simplify the equation, let us assume y = vx.

⇒ (d/dx) y = (d/dx) (vx)

⇒ dy/dx = v + x(dv/dx) …(2)

Using equations (1) and (2), we get

⇒v + x(dv/dx) = $(x^3 – 3xy^2) / (y^3 – 3x^2y)$

⇒v + x(dv/dx) = $(x^3 – 3x(vx)^2) / ((vx)^3 – 3x^2(vx))$

⇒v + x(dv/dx) = $[(1-3v^2)/(v^3-3v)]$

⇒x(dv/dx) = $[(1-3v^2)/(v^3-3v)] – v$

On simplifying the above equation, we get

$\Rightarrow x(dv/dx) = (1-v^4)/(v^3 - 3v)$

Rearranging the above equation,

$\Rightarrow [(v^3 - 3v)/(1-v^4)]dv = (dx/x)$.

Integrate both sides, and we get

$$\Rightarrow \int \left(\frac{v^3-3v}{1-v^4}\right)dv = logx + logC' \ldots(3)$$

$$\Rightarrow \int \left(\frac{v^3-3v}{1-v^4}\right)dv = \int \frac{v^3dv}{1-v^4} - 3\int \frac{vdv}{1-v^4}$$

$$\Rightarrow \int \left(\frac{v^3-3v}{1-v^4}\right)dv = I_1 - 3I_2 \ldots(4)$$

Where $I_1 = \int [(v^3dv)/(1-v^4)]$ and $I_2 = \int [(vdv)/(1-v^4)]$

Now, let us assume $1- v^4 = t$

Hence, we get

$\Rightarrow (d/dv)(1-v^4) = (dt/dv)$

$\Rightarrow -4v^3 = dt/dv$

$\Rightarrow v^3\, dv = -dt/4$

Now,

$I_1 = \int -(dt/4t) = (-¼) \log t = -(¼) \log(1-v^4) \ldots(5)$

Similarly,

$I_2 = \int [(vdv)/(1-v^4)] = \int [(vdv)/(1-(v^2)^2)]$

Assume that $v^2 = p$

Hence, we get

$\Rightarrow (d/dv)v^2 = dp/dv$

$\Rightarrow 2v = dp/dv$

$\Rightarrow vdv = dp/2$

Now,

$I_2 = (½) ∫ [dp/(1-p^2)]$

$$I_2 = \frac{1}{2\times 2} log\left|\frac{1+p}{1-p}\right|$$

$$I_2 = \frac{1}{4} log\left|\frac{1+v^2}{1-v^2}\right| \ldots (6)$$

Using equations (4), (5) and (6), we get

$$\int \left(\frac{v^3-3v}{1-v^4}\right)dv = -\frac{1}{4}log(1-v^4) - \frac{3}{4}log\left|\frac{1+v^2}{1-v^2}\right| \ldots (7)$$

Now, using equations (2) and (7)

$$-\frac{1}{4}log(1-v^4) - \frac{3}{4}log\left|\frac{1+v^2}{1-v^2}\right| = logx + logc'$$

$$-\frac{1}{4}log\left[(1-v^4)\left|\frac{1+v^2}{1-v^2}\right|^3\right] = logC'x$$

$$-\frac{1}{4}log\left[(1-v^2)(1+v^2)\left|\frac{1+v^2}{1-v^2}\right|^3\right] = logC'x$$

$$\Rightarrow \frac{(1+v^2)^4}{(1-v^2)^2} = (C'x)^{-4}$$

Now, replace v with y/x in the above equation and simplify it.

Hence, we get

$\Rightarrow (x^2 – y^2)^2 = C'^4 (x^2 + y^2)^4$

Now, take the square root on both sides of the above equation, and we get

$\Rightarrow (x^2 – y^2) = C'^2 (x^2 + y^2)^2$

$\Rightarrow (x^2 – y^2) = C' (x^2 + y^2)^2$, where $C = C'^2$

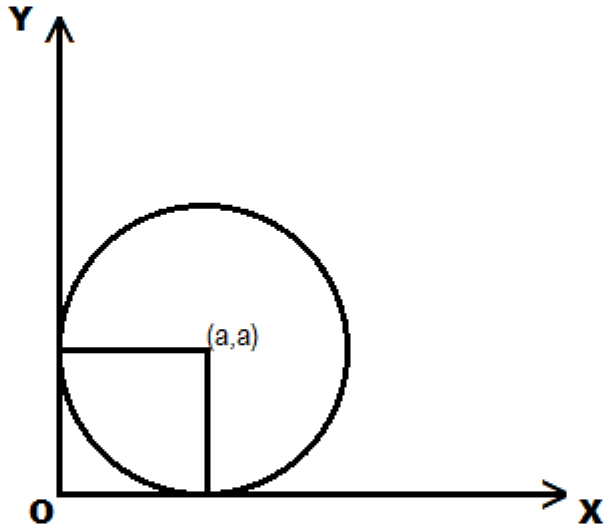
Hence, $x^2 – y^2 = C (x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 – 3xy^2) dx = (y^3 – 3x^2y) dy$, it is proved.

5. Form the differential equation of the family of circles in the first quadrant, which touches the coordinate axes.

Solution:

We know that the equation of a circle with centre (a, a) and radius "a" in the first quadrant, which touches the coordinate axes, is:

$(x-a)^2 + (y-a)^2 = a^2$ …(1)



Differentiating the above equation of both sides with respect to x, we get

$\Rightarrow 2(x-a) + 2(y-a)(dy/dx) = 0$

Now, the equation can be written as

$\Rightarrow (x-a) + (y-a)y' = 0$

$\Rightarrow (x-a) + yy' – ay' = 0$

$\Rightarrow x + yy' -a(1+ y') = 0$

$\Rightarrow x + yy' = a (1 + y')$

Rearranging the above equation, we get

$\Rightarrow a = (x + yy') / (1+y')$

Now, substitute the value of "a" in equation (1), and we get

$$\Rightarrow \left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2 + \left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2 = \left[\frac{x+yy'}{1+y'}\right]^2$$

$$\Rightarrow \left[\frac{(x-y)y'}{(1+y')}\right]^2 + \left[\frac{y-x}{1+y'}\right]^2 = \left[\frac{x+yy'}{1+y'}\right]^2$$

On simplifying the above equation, we get

$\Rightarrow (x-y)^2 . y'^2 + (y-x)^2 = (x+ yy')^2$

Therefore, the differential equation of the family of circles in the first quadrant, which touches the coordinate axes, is $(x-y)^2 . y'^2 + (y-x)^2 = (x+ yy')^2$.

6. Find the general solution of the differential equation $(dy/dx) + \sqrt{[(1-y^2)/(1-x^2)]} = 0$.

Solution:

Given: $(dy/dx) + \sqrt{[(1-y^2)/(1-x^2)]} = 0$.

The given differential equation can be written as $dy/dx = -\sqrt{[(1-y^2)/(1-x^2)]}$

$\Rightarrow dy /\sqrt{(1-y^2)} = -dx/\sqrt{(1-x^2)}$

Now, integrate both sides, and we get

$\Rightarrow \sin^{-1} y = - \sin^{-1} x + C$

Now, rearrange the equation, and we get

$\Rightarrow \sin^{-1} x + \sin^{-1} y = C$.

7. Show that the general solution of the differential equation $(dy/dx) + [(y^2 + y + 1) / (x^2 + x + 1)] = 0$ is given by $(x + y + 1) = A (1 - x - y - 2xy)$, where A is the parameter.

Solution:

Given differential equation: $(dy/dx) + [(y^2 + y + 1) / (x^2 + x + 1)] = 0$

Rearranging the given equation, we get

$dy/dx = - [(y^2 + y + 1) / (x^2 + x + 1)]$

$\Rightarrow dy/(y^2 + y + 1) = - dx/ (x^2 + x + 1)$

Now, integrate both sides, and we get

$\Rightarrow \int [dy/(y^2 + y + 1)] = -\int [dx/ (x^2 + x + 1)]$

$$\Rightarrow \int \frac{dy}{(y+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} = -\int \frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \frac{2}{\sqrt{3}} tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] = -\frac{2}{\sqrt{3}} tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + C$$

$$\Rightarrow tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right] + tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right] = \frac{\sqrt{3}C}{2}$$

On simplifying the above equation, we get

$$\Rightarrow \left(\frac{\frac{2x+2y+2}{\sqrt{3}}}{1-\frac{4xy+2x+2y+1}{3}}\right) = tan\left[\frac{\sqrt{3}C}{2}\right]$$

$$\Rightarrow \left(\frac{\frac{2x+2y+2}{\sqrt{3}}}{\frac{3-(4xy+2x+2y+1)}{3}}\right) = C_1 \text{ Where, } C_1 = tan\left[\frac{\sqrt{3}C}{2}\right]$$

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$$\Rightarrow \frac{\sqrt{3}(2x+2y+2)}{3-(4xy+2x+2y+1)} = C_1$$

⇒ $2\sqrt{3}(x + y + 1) = C_1 (3 – 4xy + 2x + 2y + 1)$

⇒ $2\sqrt{3}(x + y + 1) = C_1 (2 – 4xy – 2x – 2y)$

⇒ $2\sqrt{3}(x + y + 1) = 2C_1 (1 – 2xy – x – y)$

⇒ $\sqrt{3}(x + y + 1) = C_1 (1 – 2xy – x – y)$

⇒ $\sqrt{3}(x + y + 1) = C_1 (1 – x – y – 2xy)$

⇒ $(x + y + 1) = (C_1/\sqrt{3})(1 – x – y – 2xy)$

⇒ $(x + y + 1) = A (1 – x – y – 2xy)$, where $A = (C_1/\sqrt{3})$

Hence, proved.

8. Find the equation of the curve passing through the point (0, π/4) whose differential equation is sin x cos y dx + cos x sin y dy = 0.

Solution:

The given differential equation is sin x cos y dx + cos x sin y dy = 0.

It can also be written as:

⇒ (sin x cos y dx + cos x sin y dy) / cos x cos y = 0.

We know that sin x / cos = tan x,

And simplify the above equation

⇒ tan x dx + tan y dy = 0

⇒ log (sec x) + log (sec y) = log C

⇒ log (sec x. sec y) = log C

On simplification, we get

sec x sec y = C

It is given that the curve passes through the point (0, π/4).

⇒ 1 × √2 = C

⇒ C = √2

Hence,

sec x × sec y = √2

⇒ sec x × (1/cos y) = √2

⇒ cos y = sec x / √2

Hence, the equation of the curve passing through the point (0, π/4) whose differential

equation is sin x cos y dx + cos x sin y dy = 0 is cos y = sec x /√2.

9. Find the particular solution of the differential equation $(1 + e^{2x})$ dy + $(1 + y^2)e^x$ dx = 0, given that y = 1 when x = 0.

Solution:

Given differential equation: $(1 + e^{2x})$ dy + $(1 + y^2)e^x$ dx = 0

Rearranging the equation, we get

⇒ $[dy/(1+y^2)] + [(e^x\ dx)/(1 + e^{2x})] = 0$

Integrating both sides of the equation, we get

$\tan^{-1} y + \int [(e^x\, dx) / (1+e^{2x})] = C$ …(1)

Let $e^x = t$, and hence, $e^{2x} = t^2$

$(d/dx)(e^x) = (dt/dx)$

$\Rightarrow e^x = dt/dx$

$\Rightarrow e^x dx = dt$

Substituting the value in equation (1), we get

$\tan^{-1} y + \int [(dt) / (1+t^2)] = C$

$\Rightarrow \tan^{-1} y + \tan^{-1} t = C$

$\Rightarrow \tan^{-1} y + \tan^{-1} (e^x) = C$

If x = 0 and y = 1, we get

$\Rightarrow \tan^{-1} 1 + \tan^{-1} (e^0) = C$

$\Rightarrow \tan^{-1} 1 + \tan^{-1} 1 = C$

$\Rightarrow (\pi/4) + (\pi/4) = C$

$\Rightarrow C = \pi/2$

Hence, $\tan^{-1} y + \tan^{-1} (e^x) = \pi/2$, which is the particular solution of the given differential equation.

10. Solve the differential equation $y\, e^{x/y}\, dx = (xe^{x/y} + y^2)dy$ ($y \neq 0$).

Solution:

Given: $y\, e^{x/y}\, dx = (xe^{x/y} + y^2)dy$

Rearranging the given equation, we get

$y\, e^{x/y} (dx/dx) = xe^{x/y} + y^2$

$\Rightarrow e^{x/y} [y(dx/dy) – x] = y^2$

$\Rightarrow [e^{x/y} [y(dx/dy) – x]]/y^2 = 1$ ….(1)

Assume that $e^{x/y} = z$

Differentiate with respect to y, and we get

$(d/dy)(e^{x/y}) = dz/dy$

$\Rightarrow e^{x/y} [(y(dx/dy) – x)/y^2] = dz/dy$ …(2)

Comparing equations (1) and (2), we get

⇒dz/dy = 1

⇒ dz = dy

Now, integrating both sides, we get

⇒ z = y + C

⇒ $e^{x/y}$ = y + C, which is the solution of the given differential equation.

11. Find a particular solution of the differential equation (x – y) (dx + dy) = dx – dy,

given that y = –1, when x = 0. (Hint: put x – y = t)

Solution:

Given differential equation: (x – y) (dx + dy) = dx – dy

On simplifying the above equation, we get

⇒ (dy/dx) = (1-x+y)/ (x -y +1)

⇒ (dy/dx) = [1 – (x-y)] / [1 + (x-y)] …(1)

Given: x – y = t …(2)

(d/dx) (x – y) = dt/dx

⇒ 1 – (dy/dx) = dt/dx

⇒ 1 – (dt/dx) = dy/dx …(3)

Using the equations (1), (2) and (3), we get

⇒ 1 – (dt/dx) =(1-t)/(1+t)

⇒ dt/dx = 1 – [(1-t)/(1+t)]

On simplification, we get

⇒dt/dx = 2t / (1+t)

⇒ [(1+t)/t]dt = 2 dx

⇒ [1 + (1/t)]dt = 2dx

Now, integrating both sides, we get

⇒ t + log |t| = 2x + C

$\Rightarrow$ (x-y) + log |x-y| = 2x + C

$\Rightarrow$ log |x-y| = x + y + C

When x = 0 and y = -1, we get

$\Rightarrow$ log 1 = 0 – 1 + C

$\Rightarrow$ C = 1

Hence, log |x-y| = x + y + 1.

Therefore, log |x-y| = x + y +1 is a particular solution of the differential equation (x – y) (dx + dy) = dx – dy.

12. Solve the differential equation [($e^{-2\sqrt{x}}$ /√x) – (y/√x)](dx/dy) = 1 (x ≠ 0).

Solution:

Given: [($e^{-2\sqrt{x}}$ /√x) – (y/√x)](dx/dy) = 1

Rearranging the given equation, we get

$\Rightarrow$ dy/dx = ($e^{-2\sqrt{x}}$ /√x) – (y/√x)

$\Rightarrow$ (dy/dx) + (y/√x) = ($e^{-2\sqrt{x}}$ /√x)

The above equation is a linear equation of the form (dy/dx) + Py = Q

Where, P = 1/√x and Q = $e^{-2\sqrt{x}}$/√x

Now, I.F = $e^{\int P\,dx} = e^{\int 1/\sqrt{x}\,dx} = e^{2\sqrt{x}}$

Hence, the general solution of the given differential equation is:

y. (I.F) = ∫ (Q × I.F) dx + C

Now, substituting the values, we get

$\Rightarrow$ $ye^{2\sqrt{x}}$ = ∫ [($e^{-2\sqrt{x}}$/√x) ×$e^{2\sqrt{x}}$]dx + C

$\Rightarrow$$ye^{2\sqrt{x}}$ = ∫ (1/√x) dx + C

$\Rightarrow$$ye^{2\sqrt{x}}$ = 2√x +C, which is the solution of the given differential equation.

13. Find a particular solution of the differential equation (dy/dx) + y cot x = 4x cosec x, (x≠ 0), given that y = 0 when x = π/2.

Solution:

Given: (dy/dx) + y cot x = 4x cosec x

The given equation is a linear differential equation of the form (dy/dx) + Py = Q

Where

P = cot x and Q = 4x cosec x

Now, I.F = $e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log|\sin x|}$ = sin x

Hence, the general solution of the given differential equation is:

y. (I.F) = ∫ (Q × I.F) dx + C

⇒ y sin x = ∫ (4x cosec x × sin x) dx + C

⇒ y sin x = 4 ∫x dx + C

⇒ y sin x = 4 $(x^2/2)$ + C

⇒ y sin x =$2x^2$ + C

when x = π/2 and y = 0,

Substituting the values in the above equation, we get

⇒ 0 =$2(\pi/2)^2$ + C

⇒ 0 = $2(\pi^2/4)$ + C

⇒ 0 = $\pi^2/2$ + C

⇒ C = – $\pi^2/2$

Hence, y sin x =$2x^2$ – $(\pi^2/2)$

Therefore, the particular solution of the differential equation (dy/dx) + y cot x = 4x cosec x is y sin x =$2x^2$ – $(\pi^2/2)$.

14. Find a particular solution of the differential equation (x + 1) (dy/dx) = $2e^{-y}$ – 1. Given that y = 0 when x = 0.

Solution:

Given differential equation: (x + 1) (dy/dx) = $2e^{-y}$ – 1

Rearranging the equation, we get

⇒ dy/$2e^{-y}$ – 1 = dx/(x + 1)

⇒ (e^y dy)/(2-e^y) = dx/(x+1)

Integrate on both sides, we get

∫ [(e^y dy)/(2-e^y)] = log |x+1| + log C …(1)

Assume that 2-e^y = t

$\Rightarrow$ (d/dy) $(2-e^y)$ = dt/dy

$\Rightarrow -e^y$ = dt/dy

$\Rightarrow -e^y$ dy = dt

Substituting the value in equation (1), we get

$\Rightarrow\int$ [(dt)/(t)] = log |x+1| + log C

$\Rightarrow$ – log |t| = log |C (x+1)|

$\Rightarrow$ – log $|2-e^y|$ = log |C (x+1)|

$\Rightarrow$ 1/ $(2-e^y)$ = C (x+1)

$\Rightarrow 2-e^y$ = 1/[C(x+1)]

When x = 0 and y = 0, we get

$\Rightarrow$ 2 – 1 = 1/C

We get C = 1.

Therefore, $2-e^y$ = 1/[1(x+1)]

$\Rightarrow 2-e^y$ = 1/(x+1)

$\Rightarrow e^y$ = 2 – [1/(x+1)]

On simplification, we get

e^y = (2x +1) / (x+1)

$\Rightarrow$ y = log |(2x +1) / (x+1)|, where x ≠ -1.

Therefore, the particular solution of the differential equation (x + 1) (dy/dx) = $2e^{-y}$-1 is y = log |(2x +1) / (x+1)|, where x ≠ -1.

15. The population of a village increases continuously at a rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

Solution:

Let us assume that the population at any instant (t) be y.

Also, given that the rate of Increase in population is proportional to the number of inhabitants at any instant.

(dy/dx) $\propto$ y

$\Rightarrow$ (dy/dx) = ky

⇒ dy/y = kdt (Where k is a constant)

Now, integrating both sides of the above equation, we get

log y = kt + C … (1)

In 1999, t = 0 and y = 20000, we get log 20000 = C … (2)

In 2004, t = 5 and y = 25000, we get 1og 25000=k.5 + C

⇒ log 25000 = 5k + Iog 20000

⇒ 5k = log(25000/20000) = log(5/4)

⇒ k = (⅕) log(5/4) …(3)

In 2009, t = 10 years.

Substitute the values of k, t and C in (1), and we get

log y = 10 [(⅕) log(5/4)] + log 20000

On simplification, we get

y = (20000)(5/4)(5/4)

y = 31250

Therefore, the population of the village in 2009 was 31250.

16. The general solution of the differential equation [(y dx – x dy)/y] = 0 is:

1. **xy = C (B) x = Cy² (C) y = Cx (D) y = Cx²**

Solution:

The differential equation is [(y dx – x dy)/y] = 0.

The given equation can be written as:

(ydx / y) – (xdy/y) = 0

Thus, we get

dx = xdy /y

dx/x = dy/y

(1/x)dx – (1/y)dy = 0

Now, integrating the above equation on both sides, we get

log |x| – log |y| = log k

⇒ log |x/y| = log k

⇒ x/y = k

⇒ y = (1/k) x

⇒ y = Cx [Where C = 1/k]

Hence, the correct answer is option (C) y = Cx.

17. The general solution of a differential equation of the type (dx/dy) + P_1x = Q_1 is:

A. $ye^{\int P_1 dy} = \int (Q_1 e^{\int P_1 dy}) dy + C$
B. $ye^{\int P_1 dx} = \int (Q_1 e^{\int P_1 dx}) dx + C$
C. $xe^{\int P_1 dy} = \int (Q_1 e^{\int P_1 dy}) dy + C$
D. $xe^{\int P_1 dx} = \int (Q_1 e^{\int P_1 dx}) dx + C$

Solution:

As we know, the integrating factor of the differential equation (dx/dy) + P_1x = Q_1 is $e^{\int P1\, dy}$.

Hence,

⇒ x. (I.F) = ∫ (Q_1 × I.F) dy + C

⇒ x. $e^{\int P1\, dy}$ = ∫ (Q_1 × $e^{\int P1\, dy}$) dy + C

Hence, the correct answer is option (C).

$$xe^{\int P_1 dy} = \int (Q_1 e^{\int P_1 dy}) dy + C$$

18. The general solution of the differential equation e^x dy + (y e^x + 2x) dx = 0 is:

A. **x e^y + x^2 = C**
B. **x e^y + y^2 = C**
C. **y e^x + x^2 = C**
D. **y e^y + x^2 = C**

Solution:

The correct answer is option (C) y e^x + x^2 = C

Explanation:

The given differential equation is $e^x dy + (y e^x + 2x) dx = 0$

$\Rightarrow e^x (dy/dx) + ye^x + 2x = 0$

Hence, we get

$\Rightarrow (dy/dx) + y = -2xe^{-x}$.

The above equation is the linear differential equation of the form $(dy/dx) + Py = Q$, where $P = 1$ and $Q = -2xe^{-x}$.

Now,

$I.F = e^{\int P\,dx} = e^{\int dx} = e^x$.

$\Rightarrow y. (I.F) = \int (Q \times I.F)\, dx + C$

$\Rightarrow y. e^x = \int (-2xe^{-x} \times e^x)\, dx + C$

$\Rightarrow ye^x = \int -2x\, dx + C$

$\Rightarrow ye^x = -x^2 + C$

On rearranging the above equation, we get

$\Rightarrow ye^x + x^2 = C$

Hence, option (C) is the general solution of the given differential equation.