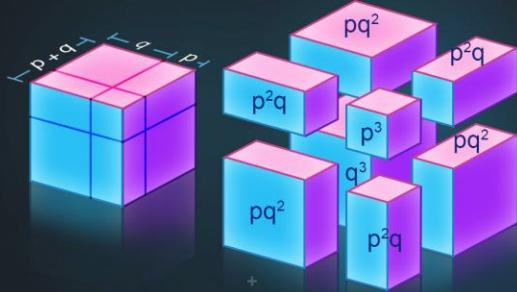


Welcome to



Binomial Theorem



$$(1+x)^n = ?$$

$${}^n C_r = \frac{n!}{r!(n-r)!}$$

Σ



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Session 01

**Mathematical Induction
and its Applications**

$$\sum n = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$\sum n^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

+

+

+

Mathematical Induction:

- Mathematical Induction is a technique which is used to prove a mathematical statement, a formula or a theorem valid for every natural number starting from some initial value.
- Suppose there is a given statement $P(n)$ involving natural number n .
- To prove the statement, we follow these two steps:

Key Takeaways

(i) BASE STEP (Basis):

Prove that statement is true for initial value of n .

(ii) INDUCTIVE STEP:

Prove that if the statement is true for $n = k$

(where k is positive integer greater than the initial value)

then the statement is also true for $n = k + 1$

i.e. truth of $P(k)$ implies truth of $P(k + 1)$.

Then, $P(n)$ is true for all natural numbers given in the statement.

Prove using principle of mathematical induction for all $n \in \mathbb{N}$.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Let the given statement is $P(n)$, i.e.

$$P(n) = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

For $n = 1$,

$$P(1) = 1 = \frac{1(2)}{2} = 1 \text{ which is true}$$

Assume that $P(k)$ is true for some positive integer k , i.e.

$$P(k) = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \cdots (i)$$



Prove using principle of mathematical induction for all $n \in \mathbb{N}$.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$P(n) = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$P(k) = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \cdots (i)$$

Now, prove that $P(k + 1)$ is also true

$$(1 + 2 + 3 + \cdots + k) + (k + 1)$$

$$= \frac{k(k+1)}{2} + (k + 1) = (k + 1) \left(\frac{k}{2} + 1 \right) = \frac{(k+1)(k+2)}{2}$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, statement $P(n)$ is true for all natural numbers n .



Prove using principle of mathematical induction for all $n \in \mathbb{N}$.

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Let the given statement is $P(n)$, i.e.

$$P(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n = 1$,

$$P(1) = 1 = \frac{1(2)(3)}{6} = 1 \text{ which is true}$$

Assume that $P(k)$ is true for some positive integer k , i.e.

$$P(k) = 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \dots (i)$$



Prove using principle of mathematical induction for all $n \in \mathbb{N}$.

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$P(k) = 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \dots (i)$$

Now, prove that $P(k + 1)$ is also true

$$\begin{aligned} & (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k + 1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2 \\ &= (k + 1) \left(\frac{k(2k+1)}{6} + (k + 1) \right) = \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, statement $P(n)$ is true for all natural numbers n .





Prove using principle of mathematical induction for all $n \in \mathbb{N}$.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Let the given statement is $P(n)$, i.e.

$$P(n) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

For $n = 1$, $P(1) = \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$ which is true

Assume that $P(k)$ is true for some positive integer k , i.e.

$$P(k) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k} \cdots (i)$$

Now, prove that $P(k + 1)$ is also true

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k}\right) + \frac{1}{2^{k+1}}$$

$$= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, statement $P(n)$ is true for all natural numbers n .



Prove using principle of mathematical induction for all $n \in \mathbb{N}$.

$$\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$$

Let the given statement is $P(n)$, i.e.

$$P(n) = \frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$$

For $n = 1$,

$$P(1) = \frac{1}{2 \cdot 5} = \frac{1}{10} = \frac{1}{6+4} \text{ which is true}$$

Assume that $P(k)$ is true for some positive integer k , i.e.

$$P(k) = \frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \cdots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4} \cdots (i)$$



Prove using principle of mathematical induction for all $n \in \mathbb{N}$.

$$\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$$



Now, prove that $P(k + 1)$ is also true

$$\left(\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \cdots + \frac{1}{(3k-1)(3k+2)} \right) + \left(\frac{1}{(3k+2)(3k+5)} \right)$$

$$= \frac{k}{6k+4} + \left(\frac{1}{(3k+2)(3k+5)} \right)$$

$$= \frac{1}{3k+2} \left(\frac{k}{2} + \frac{1}{3k+5} \right) = \frac{(k+1)(3k+2)}{2(3k+2)(3k+5)} = \frac{(k+1)}{(6k+10)}$$

+

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

+

Hence, statement $P(n)$ is true for all natural numbers n .

+

For every positive integer n , prove that $3^{2n+2} - 8n - 9$ is divisible by 8.

Let the given statement is $P(n)$, i.e.

$P(n): 3^{2n+2} - 8n - 9$ is divisible by 8

For $n = 1$,

$P(1) = 81 - 17 = 64$ (64 is divisible by 8) which is true

Assume that $P(k)$ is true for some positive integer k , i.e.

$P(k): 3^{2k+2} - 8k - 9$ is divisible by 8

i.e. $3^{2k+2} - 8k - 9 = 8\lambda, \lambda \in \mathbb{N} \cdots (i)$

For every positive integer n , prove that $3^{2n+2} - 8n - 9$ is divisible by 8.

Now, prove that $P(k + 1)$ is also true

$$\begin{aligned} & 3^{2(k+1)+2} - 8(k + 1) - 9 \\ &= 9 \cdot 3^{2k+2} - 8k - 9 - 8 \\ &= 3^{2k+2} + 8 \cdot 3^{2k+2} - 8k - 9 - 8 \\ &= 3^{2k+2} - 8k - 9 + 8 \cdot 3^{2k+2} - 8 \\ &= 8\lambda + 8(3^{2k+2} - 1) \text{ is divisible by 8} \end{aligned}$$

$$P(k): 3^{2k+2} - 8k - 9 = 8\lambda, \lambda \in \mathbb{N}$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, statement $P(n)$ is true for all natural numbers n .



For every positive integer n , prove that $x^{2n} - y^{2n}$ is divisible by $x + y$.

Let the given statement is $P(n)$, i.e.

$P(n)$: $x^{2n} - y^{2n}$ is divisible by $x + y$

For $n = 1$,

$P(1) = x^2 - y^2 = (x - y)(x + y)$ is divisible by $(x + y)$ which is true

Assume that $P(k)$ is true for some positive integer k , i.e.

$P(k)$: $x^{2k} - y^{2k}$ is divisible by $x + y$

i.e. $x^{2k} - y^{2k} = \lambda(x + y), \lambda \in \mathbb{N} \dots (i)$



For every positive integer n , prove that $x^{2n} - y^{2n}$ is divisible by $x + y$.

Now, prove that $P(k + 1)$ is also true

$$\begin{aligned}x^{2(k+1)} - y^{2(k+1)} &= x^2 \cdot x^{2k} - x^2 \cdot y^{2k} + x^2 \cdot y^{2k} - y^2 \cdot y^{2k} \\&= x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2) \\&= \lambda(x + y)x^2 + y^{2k}(x^2 - y^2) \\&= (x + y)(\lambda x^2 + y^{2k}(x - y))\end{aligned}$$

+

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

+

Hence, statement $P(n)$ is true for all natural numbers n .

+

For every positive integer n , prove that $2^n > n$.

Let the given statement is $P(n)$, i.e.

$$P(n): 2^n > n$$

For $n = 1$, $P(1) = 2 > 1$, which is true

Assume that $P(k)$ is true for some positive integer k , i.e.

$$P(k): 2^k > k$$

Now, prove that $P(k + 1)$ is also true

$2^k > k$, multiplying both sides by 2

$$\Rightarrow 2^{k+1} > 2k$$

$$\Rightarrow 2^{k+1} > k + k$$

$$\Rightarrow 2^{k+1} > k + 1$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, statement $P(n)$ is true for all natural numbers n .



Statement: $p(n): n^2 \geq 3n$ is true, then which of the following is correct ?

A

$$\forall n \in \mathbb{N}$$

B

$$\forall n \in \mathbb{N}, n > 1$$

C

$$\forall n \in \mathbb{N}, n \geq 3$$

D

For all odd natural numbers n



Statement: $p(n): n^2 \geq 3n$ is true, then which of the following is correct ?

$p(1): 1 \geq 3$ is not true.

$p(2): 4 \geq 6$ is not true.

$p(3): 9 \geq 9$ is true.

$p(4): 16 \geq 12$ is true.

$p(5): 25 \geq 15$ is true.

By the principle of mathematical induction as shown below that $p(n)$ is true $\forall n \geq 3$

Suppose $p(k): k^2 \geq 3k, k \geq 3$ is true

We have to prove that $p(k+1): (k+1)^2 \geq 3(k+1)$

Here $(k+1)^2 = k^2 + 2k + 1$

$(k+1)^2 \geq 3k + 2k + 1$ (As $k \geq 3$, So $2k + 1 \geq 3$)





Statement: $p(n): n^2 \geq 3n$ is true, then which of the following is correct ?



Here $(k + 1)^2 = k^2 + 2k + 1$

$(k + 1)^2 \geq 3k + 2k + 1$ (As $k \geq 3$, So $2k + 1 \geq 3$)

Since, $(k + 1)^2 \geq 3k + 3 = 3(k + 1)$

$(k + 1)^2 \geq 3(k + 1)$

$p(k + 1)$ is true.

Hence, By the principle of mathematical induction $p(n): n^2 \geq 3n; \forall n \in \mathbb{N}, n \geq 3$ is true

Session 02

Introduction to Binomial Theorem

Key Takeaways

Binomial:

- An algebraic expression containing two dissimilar terms is called a Binomial.

Example:

- $a + b$
- $a^2 - b^2$
- $2x + 3y$

Trinomial:

- An algebraic expression containing three dissimilar terms is called a Trinomial.

Example:

- $3x^2 + 7y^2 + 2$
- $2a^2 - b^2 + 7$
- $2a^2 + 7ab + 9b$

Key Takeaways

Polynomial:

An algebraic expression containing one, two or more dissimilar terms with

- non-zero coefficient (with variables having non-negative integers as exponents) is called a Polynomial.

Example:

$$2a + 5b$$

- $3xy + 5x + z$

$$3x + 5yz - 7z + 11$$



Key Takeaways



Algebraic Identities:

- $(a + b)^0 = 1$
- $(a + b)^1 = a + b$
- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$



Observation:

- $(a + b)^0 = 1$
 - $(a + b)^1 = a + b$
 - $(a + b)^2 = a^2 + 2ab + b^2$
 - $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Note:

- Number of terms = $n + 1$ (If degree is n)
 - Exponent of a decreases from n to 0, and exponent of b increases from 0 to n .
 - In any term, the sum of exponents of a and b is n .



Key Takeaways

Pascal's Triangle:

									$(a + b)^0 = 1$
									$(a + b)^1 = a + b$
									$(a + b)^2 = a^2 + 2ab + b^2$
									$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
									$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
									$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
									$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
1	6	15	20	15	6	1			
1	5	10	10	5	1				



Key Takeaways



Pascal's Triangle:

- Pascal's triangle is used to determine the coefficients in the binomial expansion.
- But this process is quite lengthy when the index (n) is large.
- Hence a general rule to expand $(a + b)^n$ without writing all the terms of Pascal's Triangle is required.



Key Takeaways



Factorial Notation:

- The product of first n natural numbers is denoted by $n!$

$$n! = 1 \times 2 \times 3 \times 4 \times \cdots \times (n - 1) \times n$$

Example:

- $2! = 2 \times 1 = 2$
- $3! = 3 \times 2! = 6$



Key Takeaways



Factorial Notation:

Note:

- We define $0! = 1$
- $n! = n \cdot (n - 1)! = n \times (n - 1) \times (n - 2)!$ and so on..

Example:

- $1! = 1$
- $2! = 2 \times 1 = 2$
- $3! = 3 \times 2! = 6$
- $4! = 4 \times 3! = 4 \times 6 = 24$
- $5! = 5 \times 4! = 5 \times 24 = 120$
- $6! = 6 \times 5! = 6 \times 120 = 720$



If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, then find x .

$$\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}.$$

$$\Rightarrow x = \frac{10!}{8!} + \frac{10!}{9!}.$$

$$\Rightarrow x = \frac{10 \cdot 9 \cdot 8!}{8!} + \frac{10 \cdot 9!}{9!}.$$

$$\Rightarrow x = 10 \cdot 9 + 10 = 100$$

$$n! = n \cdot (n - 1)! = n \cdot (n - 1) \cdot (n - 2)!$$



Key Takeaways



Formulas for Combination:

- n_{C_r} or $C(n, r)$ or $\binom{n}{r}$ denotes the number of combinations (selections) of r things chosen from n distinct things.

$$n_{C_r} = \frac{n!}{r!(n-r)!} \text{ where } 0 \leq r \leq n; \quad n \in \mathbb{N}, r \in \mathbb{W}$$

- $n_{C_0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1;$
- $n_{C_n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = 1;$
- $n_{C_r} = n_{C_{n-r}} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-(n-r))!(n-r)!}$
- $n_{C_r} = n_{C_s} \Rightarrow r = s \text{ or } n = r + s$



Key Takeaways

1

Index $n = 1 \longrightarrow {}^1C_0 \quad {}^1C_1$

Index $n = 2 \longrightarrow {}^2C_0 \quad {}^2C_1 \quad {}^2C_2$

Index $n = 3 \longrightarrow {}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3$

Index $n = 4 \longrightarrow {}^4C_0 \quad {}^4C_1 \quad {}^4C_2 \quad {}^4C_3 \quad {}^4C_4$

Index $n = 5 \longrightarrow {}^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5$

${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ are the binomial coefficients.





Key Takeaways



Important Results:

- ${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$
- $\frac{{}^n C_r}{{}^n C_{r-1}} = \frac{n-r+1}{r}$
- ${}^n C_r = \frac{n}{r} {}^{n-1} C_{r-1} = \frac{n(n-1)}{r(r-1)} {}^{n-2} C_{r-2}$



Find the value of ${}^{10}C_4$

Given: ${}^{10}C_4$, here $n = 10$ and $r = 4$

$$\therefore {}^{10}C_4 = \frac{10!}{4!6!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6!}{4! 6!}$$

$$= \frac{10 \times 9 \times 8 \times 7}{4!}$$

$$= 210$$

$${}^nC_r = \frac{n!}{r!(n-r)!} \text{ where } 0 \leq r \leq n, n \in \mathbb{N}, r \in \mathbb{W}$$



Solve for n , ${}^nC_{n^2-25} = {}^nC_5$

$${}^nC_r = {}^nC_s \Rightarrow r = s \text{ or } n = r + s$$

Given, ${}^nC_{n^2-25} = {}^nC_5$

$$\Rightarrow n^2 - 25 = 5 \quad \{r = s\}$$

$$\Rightarrow n^2 = 30$$

As $n \notin \mathbb{N}$ It will be rejected.

Given, ${}^nC_{n^2-25} = {}^nC_5$

$$\Rightarrow n^2 - 25 + 5 = n \quad \{r + s = n\}$$

$$\Rightarrow n^2 - n - 20 = 0$$

$$\Rightarrow (n - 5)(n - 4) = 0$$

$$\Rightarrow n = 5, n = -4$$



If ${}^nC_9 = {}^nC_8$, then find ${}^nC_{16}$:

Given, ${}^nC_9 = {}^nC_8$

$$\Rightarrow n = 9 + 8 = 17$$

$${}^nC_r = {}^nC_s \Rightarrow r = s \text{ or } n = r + s$$

$$\Rightarrow {}^nC_{16} = {}^{17}C_{16} = 17$$



The value of $\sum_{r=5}^{12} {}^{16}c_r$,

$$\sum_{r=5}^{12} {}^{16}c_r = {}^{16}c_5 + {}^{16}c_6 + {}^{16}c_7 + {}^{16}c_8 + {}^{16}c_9 + {}^{16}c_{10} + {}^{16}c_{11} + {}^{16}c_{12}$$

$$= ({}^{16}c_5 + {}^{16}c_6) + ({}^{16}c_7 + {}^{16}c_8) + ({}^{16}c_9 + {}^{16}c_{10}) + ({}^{16}c_{11} + {}^{16}c_{12})$$

$${}^n c_r + {}^n c_{r-1} = {}^{n+1} c_r$$

$$= {}^{17}c_6 + {}^{17}c_8 + {}^{17}c_{10} + {}^{17}c_{12} = {}^{17}c_{11} + {}^{17}c_9 + {}^{17}c_{10} + {}^{17}c_{12}$$

$$= ({}^{17}c_{11} + {}^{17}c_{12}) + ({}^{17}c_9 + {}^{17}c_{10})$$

$$= {}^{18}c_{12} + {}^{18}c_{10} = {}^{18}c_6 + {}^{18}c_8$$



B

The value of $\sum_{r=5}^{12} {}^{16}c_r$,

A

$${}^{18}c_5 + {}^{18}c_{12}$$

B

$${}^{18}c_6 + {}^{18}c_8$$

C

$${}^{17}c_5 + {}^{17}c_6$$

D

$${}^{17}c_6 + {}^{17}c_8$$

Session 03

**Binomial theorem and its
applications**

Binomial Theorem:

- If $n \in \mathbb{N}$, then

$$(a + b)^n = {}^n c_0 a^n b^0 + {}^n c_1 a^{n-1} b^1 + {}^n c_2 a^{n-2} b^2 + \dots + {}^n c_n a^0 b^n$$

- $(a + b)^n = \sum_{r=0}^n {}^n c_r a^{n-r} b^r$
- Number of terms = $n + 1$.
- ${}^n c_0, {}^n c_1, {}^n c_2, \dots, {}^n c_n$ are called binomial coefficients.
- Index of a decrease from n to 0, and Index of b increase from 0 to n .
- In any term sum of exponents of a and b is n .



Key Takeaways



Binomial Theorem:

- The coefficient equidistant from the beginning and the end are equal.
- k^{th} term from the end is equal to $(n - k + 2)^{th}$ term from the beginning.



For the expansion of $(a + b)^{10}$

1. Number of terms in expansion.
2. Draw the Pascal's Triangle row for above expansion.
3. Using second find the coefficient of a^5b^6 .
4. Using second find the coefficient of a^4b^6 .

Number of terms = $n + 1$.

$\Rightarrow n = 10 \Leftrightarrow$ Number of terms = 11

$$(a + b)^n = \sum_{r=0}^n {}^n c_r a^{n-r} b^r$$

Here, $n - r = 5$ and $r = 6$

Here, $n = 11$ which is not possible.

1 10 45 120 210 252 210 120 45 10 1

For the expansion of $(a + b)^{10}$

1. Number of terms in expansion.
2. Draw the Pascal's Triangle row for above expansion.
3. Using second find the coefficient of a^5b^6 .
4. Using second find the coefficient of a^4b^6 .

$$(a + b)^n = \sum_{r=0}^n {}^n c_r a^{n-r} b^r$$

Here, $n - r = 4$ and $r = 6$

$$\Rightarrow n = 10$$

Here, Coefficient will become ${}^{10}c_6$



If ${}^{18}C_{2r} = {}^{18}C_{r+3}$ then the value(s) of r is

$${}^{18}C_{2r} = {}^{18}C_{r+3}$$

$${}^nC_r = {}^nC_{n-r}$$

So,

$$2r = r + 3 \text{ or } 2r + r + 3 = 18$$

$$\Rightarrow r = 3$$

$$\Rightarrow r = 5$$

If ${}^{18}C_{2r} = {}^{18}C_{r+3}$ then the value(s) of r is

A

2

B

3

C

4

D

6



Key Takeaways



BINOMIAL THEOREM

If $n \in \mathbb{N}$, then

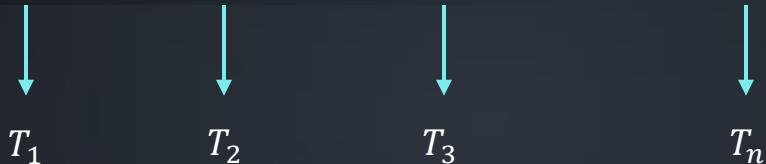
$$\begin{aligned}(a + b)^n &= {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n \\ &= \sum_{r=0}^n {}^nC_r a^{n-r} b^r\end{aligned}$$



General Term

We have,

- $(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n$



- $(r + 1)^{th}$ term in the expansion of $(a + b)^n$ is called General Term.

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$



Expand $\left(x^2 + \frac{3}{x}\right)^4$:



$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_n a^0 b^n$$

$$a = x^2; b = \frac{3}{x}$$

$$\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0 (x^2)^4 + {}^4C_1 (x^2)^3 \frac{3}{x} + {}^4C_2 (x^2)^2 \left(\frac{3}{x}\right)^2 + {}^4C_3 (x^2) \left(\frac{3}{x}\right)^3 + {}^4C_4 \left(\frac{3}{x}\right)^4$$

$$\Rightarrow \left(x^2 + \frac{3}{x}\right)^4 = x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}$$



Find the number of terms in the expansion of $(1 + 3x + 3x^2 + x^3)^{40}$:

$$\begin{aligned}(1 + 3x + 3x^2 + x^3)^{40} &= [(1 + x)^3]^{40} \\ &= (1 + x)^{120}\end{aligned}$$

\therefore Number of terms = $n + 1$

\therefore Number of terms = 121

Find the number of terms in the expansion of $(1 + 3x + 3x^2 + x^3)^{40}$:

A

120

B

119

C

121

D

123

If the third term in the binomial expansion of $(1 + x^{\log_2 x})^5$ is 2560, then a possible value of x is :

Given: $(1 + x^{\log_2 x})^5$ and $T_3 = 2560$

$$\Rightarrow T_{2+1} = 2560$$

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$\Rightarrow {}^5C_2 1^3 \cdot (x^{\log_2 x})^2 = 2560$$

$$\Rightarrow 10 \cdot x^{2 \log_2 x} = 2560$$

$$\Rightarrow x^{2 \log_2 x} = 256$$

$$\Rightarrow 2 \log_2 x = \log_x 256 = 8 \log_x 2 \quad (\text{By taking } \log_x \text{ on both the sides})$$

$$\Rightarrow (\log_2 x)^2 = 4$$

$$\Rightarrow \log_2 x = \pm 2$$

$$\Rightarrow x = 4 \text{ or } \frac{1}{4}$$

If the third term in the binomial expansion of $(1 + x^{\log_2 x})^5$ is 2560,
then a possible value of x is :

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A

$$4\sqrt{2}$$

B

$$\frac{1}{4}$$

C

$$\frac{1}{8}$$

D

$$2\sqrt{2}$$



The ratio of the 5th term from the beginning to the 5th term from the end in

the binomial expansion of $\left(2^{\frac{1}{3}} + \frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$ is:

Given: $\left(2^{\frac{1}{3}} + \frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$

k^{th} term from the end = $(n - k + 2)^{\text{th}}$ term from the beginning.

5th term from the end = $(10 - 5 + 2)^{\text{th}}$ term from the beginning

= 7th term

$$\therefore \frac{T_5}{T_7} = \frac{T_{4+1}}{T_{6+1}} = \frac{{}^{10}C_4 (2^{\frac{1}{3}})^6 \left(\frac{1}{2(3)^{\frac{1}{3}}}\right)^4}{{}^{10}C_6 (2^{\frac{1}{3}})^4 \left(\frac{1}{2(3)^{\frac{1}{3}}}\right)^6}$$

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$



The ratio of the 5th term from the beginning to the 5th term from the end in

the binomial expansion of $\left(2^{\frac{1}{3}} + \frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$ is:

$$= \frac{(2^{\frac{1}{3}})^2}{\left(\frac{1}{2(3)^{\frac{1}{3}}}\right)^2}$$

$$= 2^{\frac{2}{3}} \cdot 2^2 \cdot 3^{\frac{2}{3}}$$

$$= 4 (36)^{\frac{1}{3}}$$

The ratio of the 5th term from the beginning to the 5th term from the end in

the binomial expansion of $\left(2^{\frac{1}{3}} + \frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$ is:

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A

$$1:2(6)^{\frac{1}{3}}$$

B

$$1:4(16)^{\frac{1}{3}}$$

C

$$4(36)^{\frac{1}{3}}:1$$

D

$$2(36)^{\frac{1}{3}}:1$$



The coefficient of x^3 in the expansion of $(x + 1)^5$ is

Using Pascal triangle, we get

$$(x + 1)^5 = {}^5C_0 x^5 + {}^5C_1 x^4 + {}^5C_2 x^3 + \dots + {}^5C_5$$

$$\text{If } n \in \mathbb{N}, \text{ then } (a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n$$

Therefore, the coefficient of x^3 in $(x + 1)^5$ is

$${}^5C_2 = \frac{5 \times 4}{2} = 10$$

The coefficient of x^3 in the expansion of $(x + 1)^5$ is

A

10

B

5

C

1

D

15

The total number of irrational terms in the binomial expansion of $(7^{\frac{1}{5}} - 3^{\frac{1}{10}})^{60}$ is:

Given: $(7^{\frac{1}{5}} - 3^{\frac{1}{10}})^{60}$

$$T_{r+1} = {}^nC_r \ a^{n-r} \ b^r$$

$$\Rightarrow T_{r+1} = {}^{60}C_r \left(7^{\frac{1}{5}}\right)^{60-r} \left(-3^{\frac{1}{10}}\right)^r$$

$$\Rightarrow T_{r+1} = (-1)^r {}^{60}C_r (7)^{12-\frac{r}{5}} \cdot 3^{\frac{r}{10}}$$

The above term will be rational if $\frac{r}{5}$ and $\frac{r}{10}$ are integers

The possible values of r ($0 \leq r \leq 60$) are:

0, 10, 20, 30, 40, 50, 60 (7 values)

\Rightarrow No. of rational terms = 7

\Rightarrow No. of irrational terms = $61 - 7 = 54$



Key Takeaways



MIDDLE TERM(s) IN THE EXPANSION OF $(a + b)^n$

- $(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n$

$n = 4$

- $\Rightarrow (a + b)^4 = {}^4C_0 a^4 b^0 + {}^4C_1 a^3 b^1 + {}^4C_2 a^2 b^2 + {}^4C_3 a b^3 + {}^4C_4 b^4$



Number of terms = 5 = Odd

$n = 5$

- $\Rightarrow (a + b)^5 = {}^5C_0 a^5 b^0 + {}^5C_1 a^4 b^1 + {}^5C_2 a^3 b^2 + {}^5C_3 a^2 b^3 + {}^5C_4 a b^4 + {}^5C_5 b^5$



Number of terms = 6 = Even



Key Takeaways



MIDDLE TERM(s) IN THE EXPANSION OF $(a + b)^n$

Case 1

- If n is even, the number of terms in the expansion is odd.
- There will be only one middle term, which is $\left(\frac{n}{2} + 1\right)^{\text{th}}$ term.



Find the middle term in the expansion of $(a + b)^{10}$.

Number of terms in the expansion = $10 + 1 = 11$

∴ There will be only one middle term, which is $\left(\frac{10}{2} + 1\right)^{\text{th}}$ term.

⇒ 6th term

$$T_6 = T_{5+1}$$

$$\Rightarrow T_6 = {}^{10}C_5 \cdot a^5 \cdot b^5$$



Key Takeaways



MIDDLE TERM(s) IN THE EXPANSION OF $(a + b)^n$

Case 2

- If n is odd, the number of terms in the expansion is even.
- There will be two middle terms, which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms.

Find the middle term in the expansion of $(a + b)^{11}$.

Number of terms in the expansion = $11 + 1 = 12$

∴ There will be two middle terms, that are $\left(\frac{11+1}{2}\right)^{th}$ and $\left(\frac{11+3}{2}\right)^{th}$ terms.

⇒ 6th and 7th terms

$$\Rightarrow T_6 = T_{5+1}$$

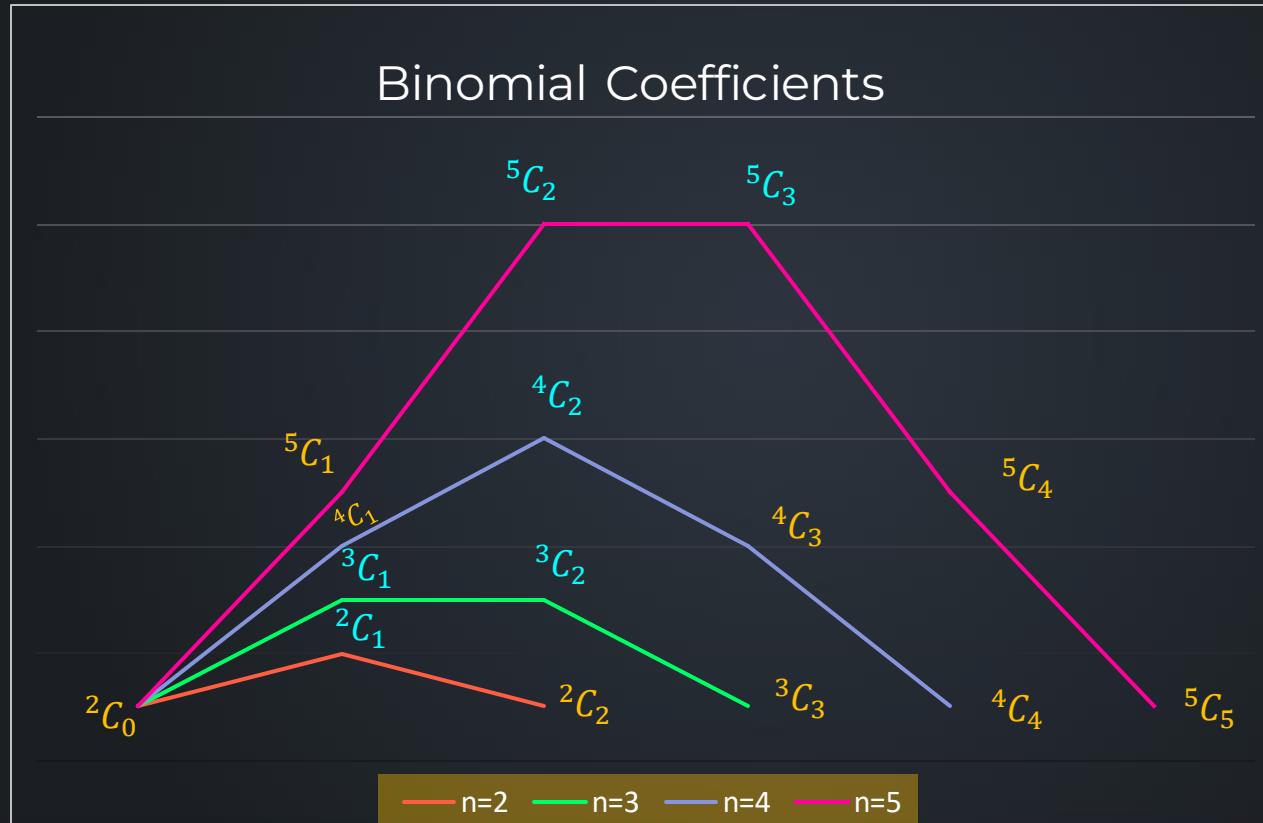
$$\Rightarrow T_6 = {}^{11}C_5 \cdot a^6 \cdot b^5$$

$$\Rightarrow T_7 = T_{6+1}$$

$$\Rightarrow T_7 = {}^{11}C_6 \cdot a^5 \cdot b^6$$

Key Takeaways

MAXIMUM VALUE OF nC_r IN THE EXPANSION OF $(a + b)^n$



The value of nC_r is maximum when

- $r = \frac{n}{2}$, if n is even.
- $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$,
if n is odd.



Find the maximum value of

i. ${}^{10}C_r$

ii. 9C_r

i. ${}^{10}C_r$ is maximum when $r = \frac{10}{2} = 5$

$$\therefore \text{Max. value} = {}^{10}C_5 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{4!} = 252$$

ii. 9C_r is maximum when $r = \frac{9-1}{2}$ or $\frac{9+1}{2} = 4$ (or) 5

$$\therefore \text{Max. value} = {}^9C_5 = {}^9C_4 = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4!} = 126$$



B

The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3} + \frac{3}{x}\right)^8$ equals 5670 is:

Given: $\left(\frac{x^3}{3} + \frac{3}{x}\right)^8$

$n = 8$ (even) \Rightarrow one middle term (is) $T_{\frac{n}{2}+1} = T_5$

$$T_5 = 5670$$

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$\Rightarrow {}^8C_4 \left(\frac{x^3}{3}\right)^4 \left(\frac{3}{x}\right)^4 = 5670$$

$$\Rightarrow 70 \cdot \frac{x^{12}}{3^4} \cdot \frac{3^4}{x^4} = 5670$$

$$\Rightarrow x^8 = 81$$

$$\Rightarrow x = \pm\sqrt{3}$$

So, sum of real values of $x = \sqrt{3} + (-\sqrt{3}) = 0$.

If the middle term in the expansion of $(1 + x)^{2n}$ is $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \cdot k^n \cdot x^n$ ($n \in \mathbb{N}$), then find the value of 'k'.

Given: $(1 + x)^{2n}$

$2n$ is even \Rightarrow one middle term (i.e.) $T_{\frac{2n}{2}+1} = T_{n+1}$

$$T_{n+1} = {}^{2n}C_n \cdot x^n$$

$$= \frac{(2n)!}{n!n!} \cdot x^n$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (2n-1)(2n)}{n! \cdot n!} \cdot x^n$$

$$= \frac{[1 \ 3 \ 5 \ \cdots \ (2n-1)] [2 \ 4 \ 6 \ \cdots \ 2n]}{n! \cdot n!} \cdot x^n$$

$$= \frac{[1 \cdot 3 \cdot 5 \cdots (2n-1)] \cdot 2^n \cdot [1 \cdot 2 \cdot 3 \cdots n]}{n! \cdot n!} \cdot x^n$$

$$T_{n+1} = \frac{[1 \cdot 3 \cdot 5 \cdots (2n-1)]}{n!} \cdot 2^n \cdot x^n$$

$$\therefore k = 2$$



Find the coefficient of x^6y^3 in the expansion of $(x + 2y)^9$.

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$\Rightarrow T_{r+1} = {}^9C_r \cdot (x)^{9-r} \cdot (2y)^r$$

$$\Rightarrow T_{r+1} = {}^9C_r \cdot 2^r \cdot (x)^{9-r} \cdot y^r$$

x^6y^3 occurs in the above term, if $r = 3$

$$\therefore T_4 = {}^9C_3 \cdot 2^3 \cdot x^6y^3$$

$$\Rightarrow \text{Coefficient of } x^6y^3 = {}^9C_3 \cdot 2^3$$

$$\therefore \text{Coefficient of } x^6y^3 = 672.$$

Session 04

**Numerically Greatest
Term**



INDEPENDENT TERM (CONSTANT TERM)

- The term free from variable(s) is called Independent term.

Find the term independent of x in the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^6$:

Given: $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^6$

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$T_{r+1} = {}^6C_r \cdot \left(\frac{3x^2}{2}\right)^{6-r} \cdot \left(\frac{1}{3x}\right)^r$$

$$T_{r+1} = (-1)^n {}^6C_r \cdot \left(\frac{3}{2}\right)^{6-r} \cdot \left(\frac{1}{3}\right)^r \cdot x^{12-3r}$$

Above term will be independent of ' x ', if $12 - 3r = 0 \Rightarrow r = 4$

$$T_5 = (-1)^4 {}^6C_4 \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{1}{3}\right)^4 = \frac{5}{12}$$

Here, the term independent of x is $\frac{5}{12}$.



The term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{x-1}{x - x^{\frac{1}{2}}}\right)^{10}$

where $x \neq 0, 1$ is:

- a.) 120
- b.) 210
- c.) 310
- d.) 4

$$\text{Let } x+1 = \left(x^{\frac{1}{3}}\right)^3 + (1)^3 = \left(x^{\frac{1}{3}} + 1\right)\left(x^{\frac{2}{3}} + 1 - x^{\frac{1}{3}}\right)$$

$$\text{and } x-1 = (\sqrt{x})^2 - (1)^2 = (\sqrt{x}-1)(\sqrt{x}+1)$$

$$\Rightarrow \left(\frac{\left(x^{\frac{1}{3}}+1\right)\left(x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1\right)}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}(\sqrt{x}-1)} \right)$$

$$\Rightarrow \left(\left(x^{\frac{1}{3}} + 1\right) - \left(1 + \frac{1}{\sqrt{x}}\right) \right)^{10}$$

$$\Rightarrow \left(x^{\frac{1}{3}} - x^{-\frac{1}{2}}\right)^{10}$$



The term independent of x in the expansion of $\left(\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{x-1}{x - x^{\frac{1}{2}}}\right)^{10}$

where $x \neq 0, 1$ is:

- a.) 120
- b.) 210
- c.) 310
- d.) 4

$$\text{Let } T_{r+1} = (-1)^{r-10} C_r \left(x^{\frac{1}{3}}\right)^{10-r} \left(x^{-\frac{1}{2}}\right)^r$$

$$\text{Power of } x : \frac{10-r}{3} - \frac{r}{2} = 0 \text{ (for term independent of } x)$$

$$\Rightarrow 20 - 2r - 3r = 0$$

$$\Rightarrow r = 4$$

$$\therefore \text{Term independent of } x = T_{4+1} = T_5$$

$$T_5 = (-1)^4 \cdot {}^{10}C_4 \cdot x^0$$

$$T_5 = 210$$



The positive value of λ for which the coefficient of x^2 in the expansion of $x^2 \left(\sqrt{x} + \frac{\lambda}{x^2} \right)^{10}$ is 720, is:

$$\text{Given : } x^2 \left(\sqrt{x} + \frac{\lambda}{x^2} \right)^{10}$$

$$T_{r+1} = x^2 \left[{}^{10}C_r \cdot (\sqrt{x})^{10-r} \cdot \left(\frac{\lambda}{x^2} \right)^r \right] = {}^{10}C_r \cdot \lambda^r \cdot x^2 \cdot x^{\frac{10-r}{2}} \cdot x^{-2r}$$

$$\Rightarrow T_{r+1} = {}^{10}C_r \cdot \lambda^r \cdot x^{2+\frac{10-r}{2}-2r}$$

$$x^2 \text{ occurs in the above term if, } 2 + \frac{10-r}{2} - 2r = 2 \Rightarrow r = 2$$

$$\therefore T_3 = {}^{10}C_r \cdot \lambda^2 \cdot x^2 \Rightarrow \text{Coefficient of } x^2 = {}^{10}C_2 \lambda^2$$

$${}^{10}C_2 \lambda^2 = 720 \text{ (Given)}$$

$$45\lambda^2 = 720 \Rightarrow \lambda = \pm 4$$

The positive value of λ for which the coefficient of x^2 in the expansion of $x^2 \left(\sqrt{x} + \frac{\lambda}{x^2} \right)^{10}$ is 720, is:

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A

4

B

 $\sqrt{5}$

C

 $2\sqrt{2}$

D

3



Find the coefficient of a^4 in the product $(1 + 2a)^4(2 - a)^5$.

$$(1 + 2a)^4 = {}^4C_0 + {}^4C_1(2a) + {}^4C_2(2a)^2 + {}^4C_3(2a)^3 + {}^4C_4(2a)^4$$

$$(1 + 2a)^4 = 1 + 8a + 24a^2 + 32a^3 + 16a^4$$

$$(2 - a)^5 = {}^5C_0 \cdot 2^5 - {}^5C_1 \cdot 2^4 \cdot a + {}^5C_2 \cdot 2^3 \cdot a^2 - {}^5C_3 \cdot 2^2 \cdot a^3 + {}^5C_4 \cdot 2 \cdot a^4 - {}^5C_5 a^5$$

$$(2 - a)^5 = 32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5$$

$$\Rightarrow (1 + 2a)^4(2 - a)^5 = (1 + 8a + 24a^2 + 32a^3 + 16a^4)(32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5)$$

∴ Coefficient of a^4 in the product is

$$= 1 \times 10 + 8 \times (-40) + 24 \times 80 + 32 \times (-80) + 16 \times 32$$

$$= 10 - 320 + 1920 - 2560 + 512 = -438$$



The term independent of x in the expansion of

$$\left(\frac{1}{60} - \frac{x^8}{81}\right) \left(2x^2 - \frac{3}{x^2}\right)^6 \text{ is:}$$

- a.) -72 b.) 36 c.) -36 d.) -108

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Given: $\left(\frac{1}{60} - \frac{x^8}{81}\right) \left(2x^2 - \frac{3}{x^2}\right)^6$

$$T_{r+1} = \left(\frac{1}{60} - \frac{x^8}{81}\right) \cdot {}^6C_r \cdot (2x^2)^{6-r} \cdot \left(-\frac{3}{x^2}\right)^r$$

$$T_{r+1} = \frac{1}{60} \cdot {}^6C_r \cdot 2^{6-r} \cdot (-3)^r \cdot x^{12-4r} - \frac{1}{81} \cdot {}^6C_r \cdot 2^{6-r} \cdot (-3)^r \cdot x^{20-4r}$$

The term independent of x in $\left(\frac{1}{60} - \frac{x^8}{81}\right) \left(2x^2 - \frac{3}{x^2}\right)^6$

⇒ The power raised to x is 0



The term independent of x in the expansion of

$$\left(\frac{1}{60} - \frac{x^8}{81}\right) \left(2x^2 - \frac{3}{x^2}\right)^6 \text{ is:}$$

- a.) -72 b.) 36 c.) -36 d.) -108

$$T_{r+1} = \underbrace{\frac{1}{60} \cdot {}^6C_r \cdot 2^{6-r} \cdot (-3)^r \cdot x^{12-4r}}_{T'_{r+1}} - \underbrace{\frac{1}{81} \cdot {}^6C_r \cdot 2^{6-r} \cdot (-3)^r \cdot x^{20-4r}}_{T''_{r+1}}$$

Case I. $12 - 4r = 0 \Rightarrow r = 3$

$$\Rightarrow T'_4 = \frac{1}{60} \cdot {}^6C_3 \cdot 2^{6-3} \cdot (-3)^3$$

$$\Rightarrow T'_4 = \frac{1}{60} \cdot 20 \cdot 8 \cdot (-27)$$

$$\Rightarrow T'_4 = -72$$

Case II. $20 - 4r = 0 \Rightarrow r = 5$

$$\Rightarrow T''_6 = -\frac{1}{81} \cdot {}^6C_5 \cdot 2^{6-5} \cdot (-3)^5$$

$$\Rightarrow T''_6 = -\frac{1}{81} \cdot 6 \cdot 2 \cdot (-243)$$

$$\Rightarrow T''_6 = 36$$

\therefore Sum of terms independent of $x \Rightarrow T'_4 + T''_6 = -72 + 36 = -36$



If the sum of the coefficients of the first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^n$ is 559, then find the value n . ($n \in \mathbb{N}$)

$$\left(x - \frac{3}{x^2}\right)^n = {}^nC_0 x^n - {}^nC_1 \cdot x^{n-1} \cdot \frac{3}{x^2} + {}^nC_2 \cdot x^{n-2} \cdot \left(\frac{3}{x^2}\right)^2 \dots$$

Given sum of the coefficients of the first three terms = 559

$$\Rightarrow {}^nC_0 - {}^nC_1 \cdot 3 + {}^nC_2 \cdot 3^2 = 559$$

$$\Rightarrow 1 - 3n + \frac{9n(n-1)}{2} = 559$$

$$\Rightarrow 2 - 6n + 9n^2 - 9n = 1118$$

$$\Rightarrow 9n^2 - 15n - 1116 = 0$$

$$\Rightarrow 3n^2 - 5n - 372 = 0$$

$$\Rightarrow (n - 12)(3n + 31) = 0$$

$$\Rightarrow n = 12 \text{ or } n = -\frac{31}{3} \text{ (not acceptable)}$$



Find the coefficients of x^4 in the expansion of $1 + (1 + x) + (1 + x)^2 + \dots + (1 + x)^9$.

Given, $S = 1 + (1 + x) + (1 + x)^2 + \dots + (1 + x)^9$.

Terms are in G.P. with $a = 1$, $r = (1 + x)$, $n = 10$

$$\therefore S = \frac{(1+x)^{10} - 1}{(1+x) - 1}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\Rightarrow S = \frac{(1+x)^{10} - 1}{x}$$

$$\Rightarrow S = \frac{(1+x)^{10} - 1}{x}$$

Coefficient of x^4 in S = Coefficient of x^5 in $(1 + x)^{10}$

$$= {}^{10}C_5$$

$$\text{Coefficient of } x^r \text{ in } (1 + x)^n = {}^nC_r$$

$$= 252$$



Find the coefficients of x^4 in the expansion of $1 + (1 + x) + (1 + x)^2 + \cdots + (1 + x)^9$.

Alternate Method:

Coefficient of x^4 in $1 + (1 + x) + (1 + x)^2 + (1 + x)^3 + (1 + x)^4 + \cdots + (1 + x)^9$.

Coefficient of x^4 in $(1 + x)^4 + (1 + x)^5 + (1 + x)^6 + \cdots + (1 + x)^9$.

$$= {}^4C_4 + {}^5C_4 + {}^6C_4 + \cdots + {}^9C_4$$

Coefficient of x^r in $(1 + x)^n = {}^nC_r$

$$= {}^5C_5 + {}^5C_4 + {}^6C_4 + {}^7C_4 + \cdots + {}^9C_4$$

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

$$= {}^6C_5 + {}^6C_4 + {}^7C_4 \cdots + {}^9C_4$$

$$= {}^7C_5 + {}^7C_4 + \cdots + {}^9C_4$$

$$= {}^{10}C_5$$

$$= 252$$



If the coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5:10:14, then find the value of n .

Let the three consecutive terms be T_r, T_{r+1}, T_{r+2} in $(1 + x)^{n+5}$

Given: ${}^{n+5}C_{r-1}, {}^{n+5}C_r, {}^{n+5}C_{r+1}$ are in the ratio 5:10:14

$$\therefore \frac{{}^{n+5}C_r}{{}^{n+5}C_{r-1}} = \frac{10}{5} \text{ and } \frac{{}^{n+5}C_{r+1}}{{}^{n+5}C_r} = \frac{14}{10}$$

$$\therefore \frac{n+6-r}{r} = 2 \text{ and } \frac{n+5-r}{r+1} = \frac{7}{5}$$

$$n - 3r + 6 = 0 \quad \cdots (1)$$

$$5n - 12r + 18 = 0 \quad \cdots (2)$$

Solving (1) and (2), we get

$$n = 6$$



Key Takeaways



Numerically greatest term (N.G.T.)

Lets analyze terms in the following expansions:

$$(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n$$

Example 1:

$$\begin{aligned}(2 + 3)^5 &= {}^5C_0 2^5 3^0 + {}^5C_1 2^4 3^1 + {}^5C_2 2^3 3^2 + {}^5C_3 2^2 3^3 + {}^5C_4 2^1 3^4 + {}^5C_5 2^0 3^5 \\&= 32 + 240 + 720 + 1080 + 810 + 243\end{aligned}$$

Greatest Binomial Coefficient = 5C_2 or ${}^5C_3 = 10$

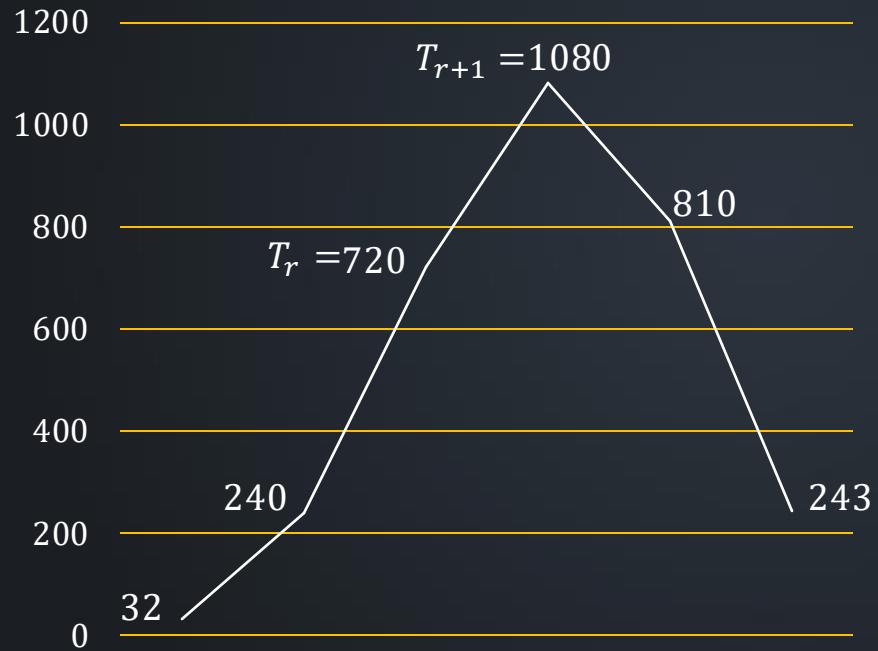
Numerically Greatest Term = 1080



Key Takeaways



Numerically greatest term (N.G.T.)





Key Takeaways



Numerically greatest term (N.G.T.)

Example 2:

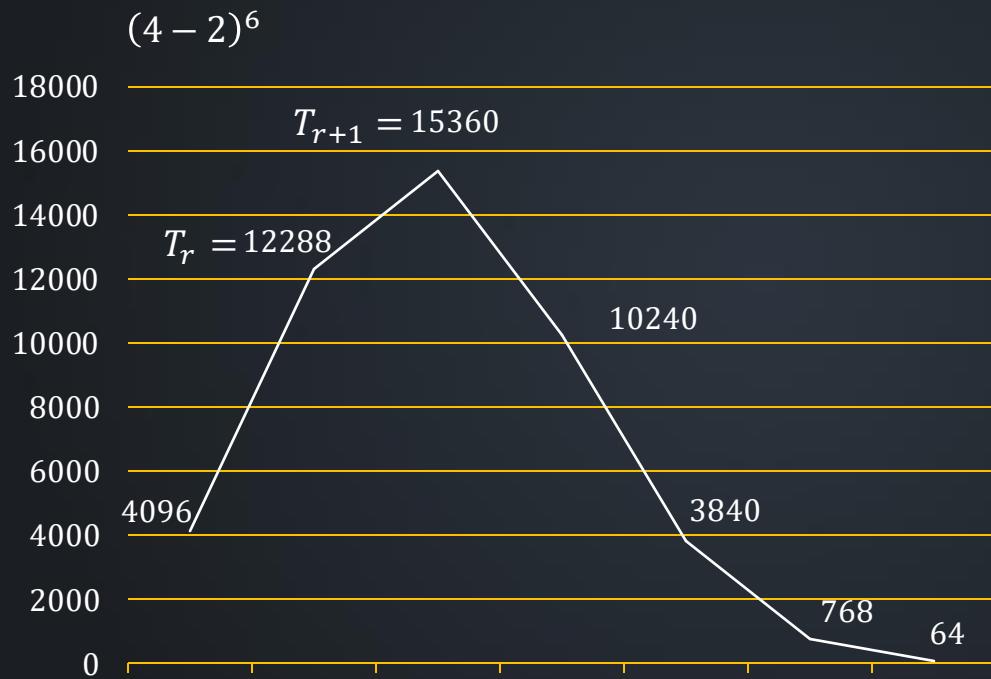
$$\begin{aligned}(4 - 2)^6 &= {}^6C_0 4^6 - {}^6C_1 4^5 2^1 + {}^6C_2 4^4 2^2 - {}^6C_3 4^3 2^3 + \dots + {}^6C_6 2^6 \\ &= 4096 - 12288 + 15360 - 10240 + 3840 - 768 + 64\end{aligned}$$

Greatest Binomial Coefficient = ${}^6C_3 = 20$

Numerically Greatest Term = 15360

Key Takeaways

Numerically greatest term (N.G.T.)





Numerically greatest term (N.G.T.)

- In the expansion of $(a + b)^n$, if we give certain values for 'a' and 'b', then the term having the numerically greatest value is said to be N.G.T.

Let us consider T_r, T_{r+1} in the expansion of $(a + b)^n$

Now, T_{r+1} will be N.G.T., if

$$\left| \frac{T_{r+1}}{T_r} \right| \geq 1 \Rightarrow \left| \frac{nC_r a^{n-r} b^r}{nC_{r-1} a^{n-r+1} b^{r-1}} \right| \geq 1$$

$$\Rightarrow \left(\frac{n-r+1}{r} \right) \left| \frac{b}{a} \right| \geq 1 \Rightarrow \frac{n+1}{r} - 1 \geq \left| \frac{a}{b} \right| \Rightarrow r \leq \frac{n+1}{1 + \left| \frac{a}{b} \right|}$$

$$\Rightarrow r \leq \frac{(n+1)|b|}{|a| + |b|}$$



Case 1:

- If $\frac{(n+1)|b|}{|a|+|b|} = m$ (integer), then $|T_m|, |T_{m+1}|$ will be numerically greatest terms. (both are equal)

Which term(s) in the expansion of $(3 + 5x)^{11}$ when

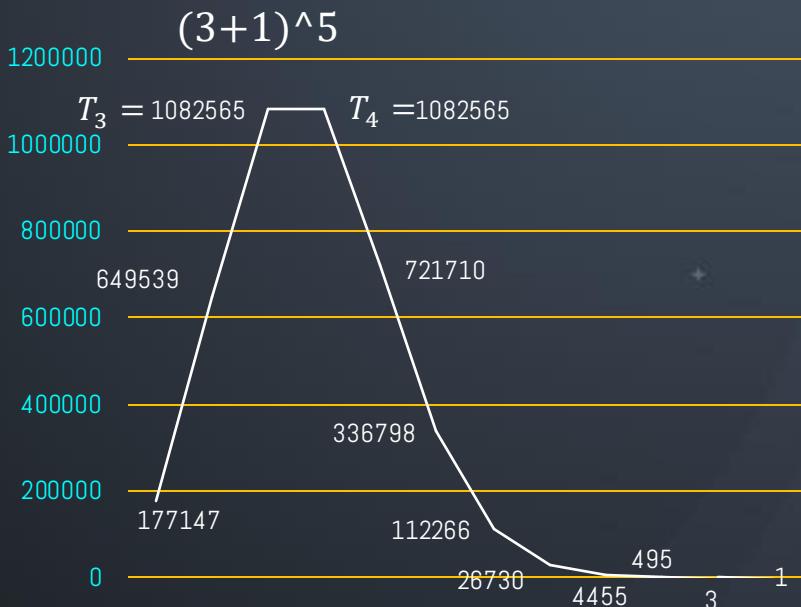
$x = \frac{1}{5}$ is/are numerically greatest ?

Given: $(3 + 5x)^{11}, x = \frac{1}{5}$

$$\therefore n = 11, |a| = 3, |b| = \left|5 \times \frac{1}{3}\right| = 1$$

$$\frac{(n+1)|b|}{|a|+|b|} = \frac{12 \times 1}{4} = 3 \text{ (} m = 3, \text{ integer)}$$

$\therefore |T_3|$ and $|T_4|$ are the numerically greatest terms.





Case 2:

- If $\frac{(n+1)|b|}{|a|+|b|} = m + f$, where m is integer and $0 < f < 1$, then $|T_{m+1}|$ will be numerically greatest term.



Find the numerically greatest term(s) in the expansion of $(7 - 3x)^{25}$, when $x = \frac{1}{3}$.

Given $(7 - 3x)^{25} = (7 + (-3x))^{25}, x = \frac{1}{3}$

$$\therefore n = 25, |a| = 7, |b| = \left| -3 \times \frac{1}{3} \right| = 1$$

$$\frac{(n+1)|b|}{|a|+|b|} = \frac{26 \times 1}{8} = 3.25 = 3 + 0.25 \quad (m = 3)$$

$\therefore |T_{m+1}| = |T_4|$ is the N.G.T.

$$|T_4| = \left| {}^{25}C_3 \cdot (7)^{22} \cdot \left(-3 \times \frac{1}{3} \right)^3 \right|$$

$$= {}^{25}C_3 \cdot (7)^{22}$$



In the expansion of $(2x + 5)^{10}$, if the middle term is the N.G.T, then find the range of 'x'.

Given: $(2x + 5)^{10}$

$\therefore n = 10$, even $\Rightarrow T_{\frac{n}{2}+1} = T_6$ is the middle term

Also, T_6 is the N.G.T

$$\Rightarrow |T_5| < |T_6| \text{ and } |T_6| > |T_7|$$

$$\Rightarrow \left| \frac{T_6}{T_5} \right| > 1 \text{ and } \left| \frac{T_7}{T_6} \right| < 1$$

$$\Rightarrow \left| \frac{{}^{10}C_5 \cdot (2x)^5 \cdot 5^5}{{}^{10}C_4 \cdot (2x)^6 \cdot 5^4} \right| > 1 \text{ and } \left| \frac{{}^{10}C_6 \cdot (2x)^4 \cdot 5^6}{{}^{10}C_5 \cdot (2x)^5 \cdot 5^5} \right| < 1$$

$$\Rightarrow \left| \frac{6}{5} \times \frac{5}{2x} \right| > 1 \text{ and } \left| \frac{5}{6} \times \frac{5}{2x} \right| < 1 \Rightarrow |x| < 3 \text{ and } |x| > \frac{25}{12}$$

$$\Rightarrow \frac{25}{12} < |x| < 3 \Rightarrow x \in \left(-3, -\frac{25}{12} \right) \cup \left(\frac{25}{12}, 3 \right)$$

Session 05

Properties of Binomial Coefficients



Find the remainder when 5^{99} is divisible by 13.

$$5^{99} = 5 \times 5^{98} = 5 \times 25^{49} = 5 \times (26 - 1)^{49}$$

$$(a - b)^n = {}^nC_0 a^n - {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 - \dots + (-1)^n {}^nC_n b^n$$

$$\therefore 5^{99} = 5[{}^{49}C_0 26^{49} - {}^{49}C_1 26^{48} + \dots + {}^{49}C_{48} 26 - {}^{49}C_{49}]$$

$$= 5 \times 26[k] - 5 \quad (K \in I)$$

$$= 13(10k) - 5$$

$$= 13k' + 8$$

$$= 13k' + 13 - 5$$

Hence the remainder is 8.



For all $\forall n \in \mathbb{N}$, $7^n + 5$ is divisible by ____?

$$7^n + 5 = 5 + (1 + 6)^n$$

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$$

$$\therefore 7^n + 5 = 5 + {}^nC_0 + {}^nC_1 \cdot 6 + {}^nC_2 \cdot 6^2 + \cdots + {}^nC_n \cdot 6^n$$

$$= 6 + {}^nC_1 \cdot 6 + {}^nC_1 \cdot 6^2 + \cdots + {}^nC_n \cdot 6^n$$

$$= 6 [1 + {}^nC_1 + {}^nC_1 \cdot 6 + \cdots + {}^nC_n \cdot 6^{n-1}]$$

$$= 6 \times \text{some integer}$$

Hence $7^n + 5$ is divisible by 6, $\forall n \in \mathbb{N}$.



B

For all $\forall n \in \mathbb{N}, 7^n + 5$ is divisible by ____?

A

6

B

4

C

12

D

18

If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then the value of k is equal to ____.

Given: Fractional part of $\frac{2^{403}}{15}$ is $\frac{k}{15}$.

$$\text{Consider } 2^{403} = 2^3 \cdot 2^{400} = 8 \times (16)^{100} = 8 \times (15 + 1)^{100}$$

$$\therefore 2^{403} = 8 \times (1 + 15)^{100}$$

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$$

$$\begin{aligned} &= 8[{}^{100}C_0 + {}^{100}C_1 15 + {}^{100}C_2 15^2 + \cdots + {}^{100}C_{99} 15^{99} + {}^{100}C_{100} 15^{100}] \\ &= 8[15k] + 8 \quad (k \in I) \end{aligned}$$

$$\frac{2^{403}}{15} = 8k + \frac{8}{15} \Rightarrow \text{Fractional Part of } \frac{2^{403}}{15} \text{ is } \frac{8}{15}$$

$$\therefore k = 8$$



Which of the following is/are true for 17^{10} ?

$$17^{10} = (289)^5 = (290 - 1)^5$$

$$(a - b)^n = {}^n C_0 a^n - {}^n C_1 a^{n-1}b + {}^n C_2 a^{n-2}b^2 - \dots + (-1)^n {}^n C_n b^n$$

$$\Rightarrow 17^{10} = {}^5 C_0 (290)^5 - {}^5 C_1 (290)^4 + {}^5 C_2 (290)^3 - {}^5 C_3 (290)^2 + {}^5 C_4 (290) - {}^5 C_5$$

$$= [\underbrace{{}^5 C_0 (290)^5 - {}^5 C_1 (290)^4 + {}^5 C_2 (290)^3 - {}^5 C_3 (290)^2}_\text{multiple of } 10^3] + {}^5 C_4 (290) - {}^5 C_5$$

$$= 5(290) - 1$$

$$= 1450 - 1 = 1449$$

⇒ Last three digits are 449.



If $(7 + 4\sqrt{3})^n = I + f$, where $I, n \in \mathbb{N}$ and $0 < f < 1$, then which of the following is/are true?

$$I + f = (7 + 4\sqrt{3})^n \cdots (i) \quad (0 < f < 1)$$

$$\text{Let } f' = (7 - 4\sqrt{3})^n \cdots (ii) \quad (0 < f' < 1)$$

$$(i) + (ii) \Rightarrow I + f + f'$$

$$\Rightarrow (i) + (ii) = (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$$

$$(a + b)^n + (a - b)^n = 2[nC_0 a^n + nC_2 a^{n-2}b^2 + nC_4 a^{n-4}b^4 + \dots]$$

$$\Rightarrow I + f + f' = 2[nC_0 7^n + nC_2 7^{n-2} (4\sqrt{3})^2 + \dots]$$

$$\Rightarrow I + f + f' = 2K \cdots (iii) \quad (K \in I)$$

$$\Rightarrow f + f' = 2K - I, \text{ which is an integer}$$

$$\Rightarrow f + f' = 1 \cdots (iv) \quad (\because 0 < f + f' < 2)$$



If $(7 + 4\sqrt{3})^n = I + f$, where $I, n \in \mathbb{N}$ and $0 < f < 1$, then which of the following is/are true?

From (iii) $I = 2K - 1$, which is odd

From (iv) $f' = 1 - f$

$$\Rightarrow (I + f)(1 - f) = (7 + 4\sqrt{3})^n (7 - 4\sqrt{3})^n$$

$$= (49 - 48)^n$$

$$\Rightarrow (I + f)(1 - f) = 1$$



Key Takeaways



Properties of Binomial Coefficients:

- In a binomial expansion ${}^nC_0, {}^nC_1, {}^nC_2 \dots, {}^nC_n$ are called binomial coefficients.

Usually they are denoted by $C_0, C_1, C_2, \dots, C_n$

$$(i). C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

Consider $(1 + x)^n = C_0 + C_1x + C_2 x^2 + \dots + C_n x^n$

Put $x = 1, 2^n = C_0 + C_1 + C_2 + \dots + C_n \dots (i)$



Key Takeaways



Properties of Binomial Coefficients:

- In a binomial expansion ${}^nC_0, {}^nC_1, {}^nC_2 \dots, {}^nC_n$ are called binomial coefficients.

Usually they are denoted by $C_0, C_1, C_2, \dots, C_n$

$$(ii). C_0 - C_1 + C_2 - \dots + (-1)^n C_n = 0$$

Consider $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$

Put $x = -1, 0 = C_0 - C_1 + C_2 - \dots + (-1)^n C_n \dots (ii)$



Key Takeaways



Properties of Binomial Coefficients:

- In a binomial expansion ${}^nC_0, {}^nC_1, {}^nC_2 \dots, {}^nC_n$ are called binomial coefficients.

Usually they are denoted by $C_0, C_1, C_2, \dots, C_n$

$$(iii). C_0 + C_2 + C_4 + \dots = 2^{n-1}$$

$$2^n = C_0 + C_1 + C_2 + \dots + C_n \dots (i)$$

$$0 = C_0 - C_1 + C_2 - \dots + (-1)^n C_n \dots (ii)$$

$$(i) + (ii) \Rightarrow 2^n = 2[C_0 + C_2 + C_4 + \dots]$$

$$\Rightarrow C_0 + C_2 + C_4 + \dots = 2^{n-1}$$



Key Takeaways



Properties of Binomial Coefficients:

- In a binomial expansion ${}^nC_0, {}^nC_1, {}^nC_2 \dots, {}^nC_n$ are called binomial coefficients.
Usually they are denoted by $C_0, C_1, C_2, \dots, C_n$

$$(iv). C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

$$2^n = C_0 + C_1 + C_2 + \dots + C_n \dots (i)$$

$$0 = C_0 - C_1 + C_2 - \dots + (-1)^n C_n \dots (ii)$$

$$(i) - (ii) \Rightarrow 2^n = 2[C_1 + C_3 + C_5 + \dots]$$

$$\Rightarrow C_1 + C_3 + C_5 + \dots = 2^{n-1}$$



Which of the following is/are true?

$$(A) {}^{12}C_1 + {}^{12}C_2 + \dots + {}^{12}C_{11} = 4094 \quad (B) {}^{10}C_1 + {}^{10}C_3 + \dots + {}^{10}C_9 = 512 \quad (C) {}^{11}C_0 + {}^{11}C_1 + \dots + {}^{11}C_5 = 2^{10}$$

(A)

$${}^{12}C_1 + {}^{12}C_2 + \dots + {}^{12}C_{11} = 4094$$

$$C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

$$\Rightarrow {}^{12}C_0 + ({}^{12}C_1 + {}^{12}C_2 + \dots + {}^{12}C_{11}) + {}^{12}C_{12} = 2^{12}$$

$$\Rightarrow ({}^{12}C_1 + {}^{12}C_2 + \dots + {}^{12}C_{11}) = 2^{12} - {}^{12}C_0 - {}^{12}C_{12} = 4094$$

(B)

$${}^{10}C_1 + {}^{10}C_3 + \dots + {}^{10}C_9 = 512$$

$$C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

$$\Rightarrow {}^{10}C_1 + {}^{10}C_3 + \dots + {}^{10}C_9 = 2^{10-1} = 2^9 = 512$$



Which of the following is/are true?

$$(A) {}^{12}C_1 + {}^{12}C_2 + \dots + {}^{12}C_{11} = 4094 \quad (B) {}^{10}C_1 + {}^{10}C_3 + \dots + {}^{10}C_9 = 512 \quad (C) {}^{11}C_0 + {}^{11}C_1 + \dots + {}^{11}C_5 = 2^{10}$$

$$(C) {}^{11}C_0 + {}^{11}C_1 + \dots + {}^{11}C_5 = 2^{10}$$

$$\Rightarrow {}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2 + \dots + {}^{11}C_{11} = 2^{11}$$

$$\begin{aligned}\Rightarrow & ({}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2 + {}^{11}C_3 + {}^{11}C_4 + {}^{11}C_5) \\ & + ({}^{11}C_6 + {}^{11}C_7 + {}^{11}C_8 + {}^{11}C_9 + {}^{11}C_{10} + {}^{11}C_{11})\end{aligned}\left.\right\} = 2^{11}$$

$$\Rightarrow 2({}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2 + {}^{11}C_3 + {}^{11}C_4 + {}^{11}C_5) = 2^{11}$$

$$\Rightarrow ({}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2 + {}^{11}C_3 + {}^{11}C_4 + {}^{11}C_5) = 2^{10}$$

Prove that : $({}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \dots + {}^{20}C_{10}) = 2^{19} + \frac{1}{2} {}^{20}C_{10}$

$$C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

$$\Rightarrow {}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \dots + {}^{20}C_{20} = 2^{20}$$

$$\begin{aligned} \Rightarrow & ({}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \dots + {}^{20}C_{10}) \\ & + ({}^{20}C_{10} + {}^{20}C_{11} + {}^{20}C_{12} + \dots + {}^{20}C_{10}) \end{aligned} \quad \left. \right\} = 2^{20} + {}^{20}C_{10}$$

$${}^nC_r = {}^nC_{n-r}$$

$$\Rightarrow 2({}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \dots + {}^{20}C_{10}) = 2^{20} + {}^{20}C_{10}$$

$$\Rightarrow ({}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \dots + {}^{20}C_{10}) = 2^{19} + \frac{1}{2} {}^{20}C_{10}$$



Key Takeaways



Factorial Notation:

Note:

- ${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \dots + {}^{2n+1}C_n = 2^{2n}$
- ${}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \dots + {}^{2n}C_n = \frac{1}{2}(2^{2n} + {}^{2n}C_n)$



The value of ${}^nC_0 + 3 {}^nC_1 + 3^2 {}^nC_2 + \dots + 3^n {}^nC_n$ is _____

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$$

Put $x = 3 \Rightarrow 4^n = {}^nC_0 + {}^nC_1 3 + {}^nC_2 3^2 + \dots + {}^nC_n 3^n$

$$\Rightarrow {}^nC_0 + 3 \cdot {}^nC_1 + 3^2 \cdot {}^nC_2 + \dots + 3^n \cdot {}^nC_n = 4^n = 2^{2n}$$

The value of ${}^nC_0 + 3 {}^nC_1 + 3^2 {}^nC_2 + \dots + 3^n {}^nC_n$ is _____

A

 4^{2n}

B

 2^{2n}

C

 2^n

D

 3^n

The value of $(^{21}C_1 - ^{10}C_1) + (^{21}C_2 - ^{10}C_2) + \dots + (^{21}C_{10} - ^{10}C_{10})$ is ____.

A

$$2^{21} - 2^{11}$$

B

$$2^{20} - 2^9$$

C

$$2^{21} - 2^{10}$$

D

$$2^{20} - 2^{10}$$

The value of $(^{21}C_1 - ^{10}C_1) + (^{21}C_2 - ^{10}C_2) + \dots + (^{21}C_{10} - ^{10}C_{10})$ is ____.

Let the general equation,

$$G = (^{21}C_1 - ^{10}C_1) + (^{21}C_2 - ^{10}C_2) + \dots + (^{21}C_{10} - ^{10}C_{10})$$

$$= (^{21}C_1 + ^{21}C_2 + \dots + ^{21}C_{10}) - (^{10}C_1 + ^{10}C_2 + \dots + ^{10}C_{10})$$

$$= (^{21}C_0 + ^{21}C_1 + ^{21}C_2 + \dots + ^{21}C_{10} - ^{21}C_0) - (^{10}C_0 + ^{10}C_1 + ^{10}C_2 + \dots + ^{10}C_{10} - ^{10}C_0)$$

$$^{2n+1}C_0 + ^{2n+1}C_1 + ^{2n+1}C_2 + \dots + ^{2n+1}C_n = 2^{2n}$$

$$^nC_0 + ^nC_1 + ^nC_2 + \dots + ^nC_n = 2^n$$

$$\therefore G = (2^{20} - 1) - (2^{10} - 1) = 2^{20} - 2^{10}$$

If $a_n = \sum_{r=0}^n \frac{1}{nC_r}$ then $\sum_{r=0}^n \frac{r}{nC_r}$ equals ____.

A

$$(n - 1)a_n$$

B

$$\frac{n}{2} \cdot a_n$$

C

$$n \cdot a_n$$

D

$$2n \cdot a_n$$

Session 06

**Series involving binomial
coefficients**



If n is even, then the last two digits of $99^n + 1, n > 1$, is ?

$$S = 99^n + 1, \quad n > 1, n \in \mathbb{N},$$

$$S = 99^n + 1 = 1 + (100 - 1)^n$$

$$\Rightarrow S = 1 + \{ {}^n C_0 (100)^n - {}^n C_1 (100)^{n-1} + \dots + (-1)^n \}$$

When n is even, we get

$$S = 2 + 100k_1, \quad k_1 \in \mathbb{Z}$$

The last two digits are 02

If n is even, then the last two digits of $99^n + 1, n > 1$, is ?

A

00

B

02

C

11

D

99



Find the value of $C_1 + 2.C_2 + 3.C_3 + \dots + n.C_n$:

$$C_1 + 2.C_2 + 3.C_3 + \dots + n.C_n$$

$$= \sum_{r=1}^n r.C_r = \sum_{r=1}^n r.^n C_r = \sum_{r=1}^n r \cdot \frac{n}{r} \cdot {}^{n-1} C_{r-1}$$

$${}^n C_r = \frac{n}{r} {}^{n-1} C_{r-1}$$

$$= n \sum_{r=1}^n {}^{n-1} C_{r-1}$$

$${}^n C_0 + {}^n C_1 + \dots + {}^n C_n = 2^n$$

$$= n [{}^{n-1} C_0 + {}^{n-1} C_1 + \dots + {}^{n-1} C_{n-1}]$$

$$= n \cdot 2^{n-1}$$



Find the value of $C_1 - 2.C_2 + 3.C_3 - \dots + (-1)^{n+1}.n.C_n$:

$$C_1 - 2.C_2 + 3.C_3 - \dots + (-1)^{n+1}.n.C_n$$

$$= \sum_{r=1}^n (-1)^{r+1}.r.C_r$$

$$= \sum_{r=1}^n (-1)^{r+1}.r.^nC_r$$

$$= \sum_{r=1}^n (-1)^{r+1}.r.\frac{n}{r}.^{n-1}C_{r-1}$$

$$= n \sum_{r=1}^n (-1)^{r+1}.^{n-1}C_{r-1}$$

$$= n [^{n-1}C_0 - ^{n-1}C_1 + ^{n-1}C_2 - \dots]$$

$$= 0$$

$$^nC_r = \frac{n}{r}^{n-1}C_{r-1}$$

$$^nC_0 - ^nC_1 + ^nC_2 - \dots = 0$$



If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then prove that

$$(1.2)C_2 + (2.3)C_3 + \dots + ((n-1).n)C_n = n(n-1)2^{n-2}.$$

$$\text{LHS} = (1.2)C_2 + (2.3)C_3 + \dots + ((n-1).n)C_n$$

$$= \sum_{r=2}^n r(r-1)^n C_r$$

$$= \sum_{r=2}^n r(r-1) \frac{n(n-1)}{r(r-1)} {}^{n-2}C_{r-2}$$

$$= \sum_{r=2}^n n(n-1) {}^{n-2}C_{r-2}$$

$$= n(n-1) \sum_{r=2}^n {}^{n-2}C_{r-2}$$

$$= n(n-1)[{}^{n-2}C_0 + {}^{n-2}C_1 + \dots + {}^{n-2}C_{n-2}]$$

$$= n(n-1)2^{n-2} = \text{RHS}$$

$${}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)} {}^{n-2}C_{r-2}$$



Key Takeaways



Differentiation (Quick revision):

$$\frac{d}{dx} (x^n) = n(x^{n-1})$$

Example: i) $\frac{d}{dx} (x^5) = 5(x^{5-1}) = 5x^4$

ii) $\frac{d}{dx} (x^{-3}) = -3(x^{-3-1}) = -3x^{-4}$

$$\frac{d}{dx} (ax + b)^n = n(ax + b)^{n-1}(a)$$

Example: i) $\frac{d}{dx} (2x + 3)^5 = 5(2x + 3)^{5-1} (2) = 10(2x + 3)^4$

ii) $\frac{d}{dx} (x + 1)^7 = 7(x + 1)^{7-1} (1) = 7(x + 1)^6$

iii) $\frac{d}{dx} (-x + 1)^8 = 8(-x + 1)^{8-1} (-1) = -8(-x + 1)^7$



Alternative Method:

To prove : $(1.2)C_2 + (2.3)C_3 + \dots + ((n-1).n)C_n = n(n-1)2^{n-2}$

The last term of LHS is $n(n-1) C_n$ i.e., $(n-1)n$

The given series is $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$

Differentiating both sides w.r.t. x , we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \dots + nC_nx^{n-1}$$

Again differentiating both sides w.r.t. x , we get

$$n(n-1)(1+x)^{n-2} = 0 + 0 + (1.2)C_2 + (2.3)C_3x + \dots + (n-1) n C_n x^{n-2}$$

Substituting $x = 1$, we get

$$n(n-1)(1+1)^{n-2} = 0 + 0 + (1.2)C_2 + (2.3)C_3 + \dots + (n-1) n C_n$$

$$\Rightarrow (1.2)C_2 + (2.3)C_3 + \dots + (n-1) n C_n = n(n-1)2^{n-2}$$



Find the sum of $C_0 + \frac{C_1}{2} + \frac{C_3}{3} + \dots + \frac{C_n}{n+1}$:

$$C_0 + \frac{C_1}{2} + \frac{C_3}{3} + \dots + \frac{C_n}{n+1}$$

$$= \sum_{r=0}^n \frac{{}^n C_r}{r+1} = \sum_{r=0}^n \frac{1}{(n+1)} \cdot \frac{(n+1)}{(r+1)} \cdot {}^n C_r$$

$$= \frac{1}{(n+1)} \sum_{r=0}^n \frac{(n+1)}{(r+1)} \cdot {}^n C_r$$

$$= \frac{1}{(n+1)} \sum_{r=0}^n {}^{n+1} C_{r+1}$$

$$= \frac{1}{(n+1)} [{}^{(n+1)} C_1 + {}^{(n+1)} C_2 + \dots + {}^{(n+1)} C_{(n+1)}]$$

$${}^n C_r = \frac{n}{r} {}^{n-1} C_{r-1}$$

$${}^n C_0 + {}^n C_1 + \dots + {}^n C_n = 2^n$$

$$= \frac{1}{(n+1)} [2^{n+1} - {}^{(n+1)} C_0]$$

$$= \frac{2^{n+1} - 1}{n+1}$$



Key Takeaways



Integration (Quick revision):

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

Example:

$$\int x^5 dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$$

$$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1} \right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$$

Example:

$$\int_a^b x^3 dx = \left[\frac{x^{3+1}}{3+1} \right]_a^b = \left[\frac{x^4}{4} \right]_a^b = \frac{b^4 - a^4}{4}$$

$$\int_a^b (px + q)^n dx = \left[\frac{(px+q)^{n+1}}{n+1} \cdot \frac{1}{p} \right]_a^b = \frac{1}{p} \left[\frac{(pb+q)^{n+1}}{n+1} - \frac{(pa+q)^{n+1}}{n+1} \right]$$



Key Takeaways



Alternative Method:

To find the sum of: $C_0 + \frac{C_1}{2} + \frac{C_3}{3} + \dots + \frac{C_n}{n+1}$

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$$

Integrating both sides within limits 0 to 1, we get

$$\int_0^1 (1+x)^n dx = \int_0^1 (C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n) dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1 = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1} \right]_0^1$$

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1 = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1} \right]_0^1$$

$$\Rightarrow \frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$



Find the sum of $2C_0 + \frac{2^2}{2} C_1 + \frac{2^3}{3} C_2 + \dots + \frac{2^{11}}{11} C_{10}$: ($C_r = {}^{10}C_r$)

$$2C_0 + \frac{2^2}{2} C_1 + \frac{2^3}{3} C_2 + \dots + \frac{2^{11}}{11} C_{10}$$

$$= \sum_{r=0}^{10} \frac{2^{r+1}}{r+1} \cdot {}^{10}C_r$$

$$= \frac{1}{11} \sum_{r=0}^{10} \frac{11}{r+1} \cdot {}^{10}C_r \cdot 2^{r+1}$$

$$= \frac{1}{11} \sum_{r=0}^{10} \frac{11}{r+1} \cdot {}^{10}C_r \cdot 2^{r+1} = \frac{1}{11} \sum_{r=0}^{10} {}^{11}C_{r+1} \cdot 2^{r+1}$$

$${}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1}$$

$$= \frac{1}{11} [{}^{11}C_1 \cdot 2 + {}^{11}C_2 \cdot 2^2 + \dots + {}^{11}C_{11} \cdot 2^{11}]$$

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 \dots + C_nx^n$$

$$n = 11, x = 2$$

$$= \frac{1}{11} [(1+2)^{11} - {}^{11}C_0] = \frac{3^{11}-1}{11}$$

Find the sum of $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \dots + C_{n-r} C_n$.
Hence find the sum of $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2$:

We know,

$$(1+x)^n = {}^n C_0 + {}^n C_1 x + {}^n C_2 x^2 + \dots + {}^n C_{n-r} x^{n-r} + \dots + {}^n C_n x^n \dots (i)$$

$$(x+1)^n = {}^n C_0 x^n + {}^n C_1 x^{n-1} + \dots + {}^n C_r x^{n-r} + {}^n C_{r+1} x^{n-r-1} + {}^n C_{r+2} x^{n-r-2} + \dots + {}^n C_n \dots (ii)$$

Coefficient of x^{n-r} in $(1+x)^n \cdot (x+1)^n$

= Coefficient of x^{n-r} in $(1+x)^{2n}$

$$\Rightarrow {}^n C_0 \cdot {}^n C_r + {}^n C_1 \cdot {}^n C_{r+1} + {}^n C_2 \cdot {}^n C_{r+2} + \dots + {}^n C_{n-r} \cdot {}^n C_n = {}^{2n} C_{n-r}$$

$$\Rightarrow C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \dots + C_{n-r} C_n = {}^{2n} C_{n-r}$$

$$\Rightarrow C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \dots + C_{n-r} C_n = \frac{(2n)!}{(n-r)!(n+r)!}$$

Find the sum of $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \dots + C_{n-r}C_n$.
Hence find the sum of $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2$:

$$\Rightarrow C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \dots + C_{n-r}C_n = \frac{(2n)!}{(n-r)!(n+r)!}$$

Substituter $r = 0$, we get

$$\Rightarrow C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{(2n)!}{n!n!} = {}^{2n}C_n$$



Key Takeaways



Product of Binomial Coefficient:

- $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \cdots + C_{n-r} C_n = {}^{2n}C_{n-r}$
- $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = \frac{(2n)!}{n!n!} = {}^{2n}C_n$



Key Takeaways



Product of Binomial Coefficient:

$$\bullet \quad C_0^2 - C_1^2 + C_2^2 - \cdots + (-1)^n \ C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}} \cdot {}^nC_{\frac{n}{2}}, & \text{if } n \text{ is even} \end{cases}$$



Key Takeaways



Proof:

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_{n-r} x^{n-r} + \cdots + {}^nC_n x^n \dots (i)$$

$$(x-1)^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} - \cdots + (-1)^n {}^nC_n \dots (ii)$$

Coefficient of x^n in $(1+x)^n \cdot (x-1)^n$

= Coefficient of x^n in $(x^2 - 1)^n$

$$\Rightarrow ({}^nC_0)^2 - ({}^nC_1)^2 + ({}^nC_2)^2 - \cdots + (-1)^n ({}^nC_n)^2$$

In $(x^2 - 1)^n$,

$$T_{r+1} = {}^nC_r \cdot (x^2)^{n-r} \cdot (-1)^r$$



Key Takeaways



$$T_{r+1} = {}^nC_r \cdot (x^2)^{n-r} \cdot (-1)^r$$

For coefficient of x^n , $r = \frac{n}{2}$ (n must be even) and x^n does not occur if n is odd

- $C_0^2 - C_1^2 + C_2^2 - \dots + (-1)^n C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}} \cdot {}^nC_{\frac{n}{2}}, & \text{if } n \text{ is even} \end{cases}$



Key Takeaways



- ${}^m C_r \cdot {}^n C_0 + {}^m C_{r-1} \cdot {}^n C_1 + {}^m C_{r-2} \cdot {}^n C_2 + \dots + {}^m C_0 \cdot {}^n C_r = {}^{(m+n)} C_r$

Find the sum of ${}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \dots + {}^nC_r$:
(where $m, n, r \in \mathbb{N}$ and $r < m, r < n$)

$$(1+x)^m = {}^mC_0 + {}^mC_1 x + \dots + {}^mC_{r-2} x^{r-2} + {}^mC_{r-1} x^{r-1} + {}^mC_r x^r + \dots + {}^mC_m x^m$$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_{n-r} x^{n-r} + \dots + {}^nC_n x^n$$

Coefficient of x^r in $(1+x)^m \cdot (1+x)^n$

= Coefficient of x^r in $(1+x)^{m+n}$

$$\Rightarrow {}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \dots + {}^mC_0 \cdot {}^nC_r = {}^{(m+n)}C_r$$



The value of 'r' for which ${}^{20}C_r \cdot {}^{20}C_0 + {}^{20}C_{r-1} \cdot {}^{20}C_1 + {}^{20}C_{r-2} \cdot {}^{20}C_2 + \dots + {}^{20}C_0 \cdot {}^{20}C_r$ is maximum, is _____

$$(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1 x + \dots + {}^{20}C_{r-2} x^{r-2} + {}^{20}C_{r-1} x^{r-1} + {}^{20}C_r x^r + \dots + {}^{20}C_{20} x^{20}$$

$$(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1 x + {}^{20}C_2 x^2 + \dots + {}^{20}C_r x^r + \dots + {}^{20}C_{20} x^{20}$$

Coefficient of x^r in $(1+x)^{20} \cdot (x+1)^{20}$

= Coefficient of x^r in $(1+x)^{40}$

$$\Rightarrow {}^{20}C_r \cdot {}^{20}C_0 + {}^{20}C_{r-1} \cdot {}^{20}C_1 + \dots + {}^{20}C_0 \cdot {}^{20}C_r = {}^{40}C_r$$

nC_r is maximum when $r = \frac{n}{2}$, if n is even

${}^{40}C_r$ is maximum, when $r = \frac{40}{2} = 20$

The value of ' r ' for which ${}^{20}C_r \cdot {}^{20}C_0 + {}^{20}C_{r-1} \cdot {}^{20}C_1 + {}^{20}C_{r-2} \cdot {}^{20}C_2 + \dots + {}^{20}C_0 \cdot {}^{20}C_r$ is maximum, is _____

A

15

B

10

C

11

D

20

Session 07

**Binomial Theorem for
any index**

If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \dots + a_{40}x^{40}$, then

Find : (a) $a_0 + a_1 + a_2 + \dots + a_{40}$

(b) $a_0 + a_2 + a_4 + \dots + a_{40}$

(c) $a_1 + a_3 + a_5 + \dots + a_{39}$

Let $f(x) = (1 + x + x^2)^{20}$

$$= a_0 + a_1x + a_2x^2 + \dots + a_{40}x^{40}$$

Now ,

$$f(1) = 4^{20} = a_0 + a_1 + a_2 + \dots + a_{40} \dots (i) \rightarrow (a)$$

$$f(-1) = 2^{20} = a_0 - a_1 + a_2 - \dots + a_{40} \dots (ii)$$

$$(i) + (ii) \Rightarrow a_0 + a_1 + a_2 + \dots + a_{40} = 4^{20}$$

+

$$a_0 - a_1 + a_2 - \dots + a_{40} = 2^{20}$$

$$a_0 + a_2 + a_4 + \dots + a_{40} = \frac{4^{20} + 2^{20}}{2} \rightarrow (b)$$

If $(1 + x + 2x^2)^{20} = a_0 + a_1x + a_2x^2 + \dots + a_{40}x^{40}$, then

Find : (a) $a_0 + a_1 + a_2 + \dots + a_{40}$

(b) $a_0 + a_2 + a_4 + \dots + a_{40}$

(c) $a_1 + a_3 + a_5 + \dots + a_{39}$

$$f(1) = 4^{20} = a_0 + a_1 + a_2 + \dots + a_{40} \dots (i) \rightarrow (a)$$

$$f(-1) = 2^{20} = a_0 - a_1 + a_2 - \dots + a_{40} \dots (ii)$$

$$(i) - (ii) \Rightarrow a_0 + a_1 + a_2 + \dots + a_{40} = 4^{20}$$

$$- a_0 - a_1 + a_2 - \dots + a_{40} = 2^{20}$$

$$a_1 + a_3 + a_5 + \dots + a_{39} = \frac{4^{20} - 2^{20}}{2} \rightarrow (c)$$



Key Takeaways



Substitution Method:

If $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, then

- $a_0 + a_1 + a_2 + a_3 + \cdots + a_n = f(1)$

- $a_0 + a_2 + a_4 + \cdots = \frac{f(1) + f(-1)}{2}$

- $a_1 + a_3 + a_5 + \cdots = \frac{f(1) - f(-1)}{2}$



If the sum of the coefficients of all even powers of x in the product $(1 + x + x^2 + \dots + x^{2n})(1 - x + x^2 - x^3 + \dots + x^{2n})$ is 61, then n is equal to ____.

$$f(x) = (1 + x + x^2 + \dots + x^{2n})(1 - x + x^2 - x^3 + \dots + x^{2n})$$

$$\text{sum of the coefficients of all even powers of } x = \left(\frac{f(1) + f(-1)}{2} \right)$$

$$f(1) = (\underbrace{1 + 1 + 1 + \dots + 1}_{2n+1 \text{ terms}})(1) = 2n + 1$$

$$f(-1) = (1 - 1 + 1 - 1 + 1 \dots - 1 + 1)(\underbrace{1 + 1 + 1 + \dots + 1}_{2n+1 \text{ terms}})(1) = 2n + 1$$

$$\text{sum of the coefficients of all even powers of } x = \frac{(2n+1) + (2n+1)}{2} = 2n + 1$$

$$\Rightarrow 2n + 1 = 61$$

$$\therefore n = 30.$$



If n is a positive integer and $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$

Then the value of $a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2$ is equal to

$$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n} \quad \dots (i)$$

Replacing x by $-\frac{1}{x}$, we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \dots + \frac{a_{2n}}{x^{2n}} \quad \dots (ii)$$

Constant (Independent) term in

$$(1 + x + x^2)^n \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2 \quad \dots (iii)$$

$$(1 + x + x^2)^n \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = (1 + x + x^2)^n \left(\frac{x^2 - x + 1}{x^2}\right)^n$$

$$= \frac{[(x^2 + 1)^2 - x^2]^n}{x^{2n}} = \frac{[1 + x^2 + x^4]^n}{x^{2n}}$$



If n is a positive integer and $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$
 Then the value of $a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2$ is equal to

$$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n} \quad \dots (i)$$

\therefore Independent term = Coefficient of x^{2n} in $(1 + x^2 + x^4)^n$

= Coefficient of x^n in $(1 + x + x^2)^n$

$$= a_n$$

From (iii) $a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2 = a_n$

If n is a positive integer and $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$

Then the value of $a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2$ is equal to

A

 a_n

B

 $2a_n$

C

 a_n^2

D

 $2a_n^2$



Key Takeaways



Binomial Theorem for any index

If $n \in \mathbb{Q} - \mathbb{W}$ and $|x| < 1$, then

$$(1 + x)^n = 1 + n \cdot x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

- The number of terms in the above expression is infinite.
- General Term $T_{r+1} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)}{r!} \cdot x^r$
- If ' n ' is a negative integer or a fraction, then

$$(a + b)^n = \begin{cases} a^n \left(1 + \frac{b}{a}\right)^n, & \text{if } \left|\frac{b}{a}\right| < 1 \\ b^n \left(1 + \frac{a}{b}\right)^n, & \text{if } \left|\frac{a}{b}\right| < 1 \end{cases}$$



Find the range of values of x , for which $\frac{1}{\sqrt{5+4x}}$ can be expanded in ascending powers of x .

$$\begin{aligned}\text{Given: } \frac{1}{\sqrt{5+4x}} &= (5+4x)^{-\frac{1}{2}} = 5^{-\frac{1}{2}} \left(1 + \frac{4x}{5}\right)^{-\frac{1}{2}} \\ &= 5^{-\frac{1}{2}} \left(1 + \frac{4x}{5}\right)^{-\frac{1}{2}}\end{aligned}$$

$$(a+b)^n = \begin{cases} a^n \left(1 + \frac{b}{a}\right)^n & , \text{ if } \left|\frac{b}{a}\right| < 1 \\ b^n \left(1 + \frac{a}{b}\right)^n & , \text{ if } \left|\frac{a}{b}\right| < 1 \end{cases}$$

\therefore Expansion is valid only when $\left|\frac{4x}{5}\right| < 1$

$$\Rightarrow |x| < \frac{5}{4} \quad \Rightarrow x \in \left(-\frac{5}{4}, \frac{5}{4}\right)$$



Key Takeaways



Special Case

When n is a negative integer

Let $n = -m$, where $m \in \mathbb{N}$

We have ,

$$(1+x)^n = 1 + n \cdot x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

Now , put $n = -m$

$$\Rightarrow (1+x)^{-m} = 1 - m \cdot x + \frac{m(m+1)}{2!} x^2 - \frac{m(m+1)(m+2)}{3!} x^3 + \dots$$

$$\Rightarrow (1+x)^{-m} = 1 - {}^m C_1 x + {}^{(m+1)} C_2 x^2 - {}^{(m+2)} C_3 x^3 + \dots$$



Key Takeaways



Special Case

$$(1 + x)^{-m} = 1 - {}^m C_1 x + {}^{(m+1)} C_2 x^2 - {}^{(m+2)} C_3 x^3 + \dots$$

$$(1 - x)^m = 1 - m \cdot x + \frac{m(m-1)}{2!} x^2 - \frac{m(m-1)(m-2)}{3!} x^3 + \dots$$

General term , $T_{r+1} = {}^{(m+r-1)} C_r \cdot x^r$

- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$



If $y = \frac{2}{5} + \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{2}{5}\right)^3 + \dots$, then the value of $(y^2 + 2y - 4)$, is _____.

We have $y = \frac{2}{5} + \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{2}{5}\right)^3 + \dots$

$$\Rightarrow 1 + y = 1 + \left(\frac{2}{5}\right) + \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{2}{5}\right)^3 + \dots$$

Let $1 + y = (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$

On equating the given series, we get

$$nx = \frac{2}{5} \quad \dots (i)$$

$$\text{And } \frac{n(n-1)}{2} x^2 = \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2$$

On solving Equations (i) and (ii), we get

$$x = -\frac{4}{5} \text{ and } n = -\frac{1}{2}$$



If $y = \frac{2}{5} + \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{2}{5}\right)^3 + \dots$, then the value of $(y^2 + 2y - 4)$, is _____.

$$\therefore 1 + y = \left(1 - \frac{4}{5}\right)^{\frac{1}{2}}$$

$$\Rightarrow (y + 1) = (5)^{\frac{1}{2}}$$

On squaring both sides, we get

$$y^2 + 2y + 1 = 5$$

$$\Rightarrow y^2 + 2y - 4 = 0$$

\therefore The value of $y^2 + 2y - 4 = 0$

Hence, option (c) is correct.



If $y = \frac{2}{5} + \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{2}{5}\right)^3 + \dots$, then the value of $(y^2 + 2y - 4)$, is _____.

A

1

B

-1

C

0

D

None of the above



The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is

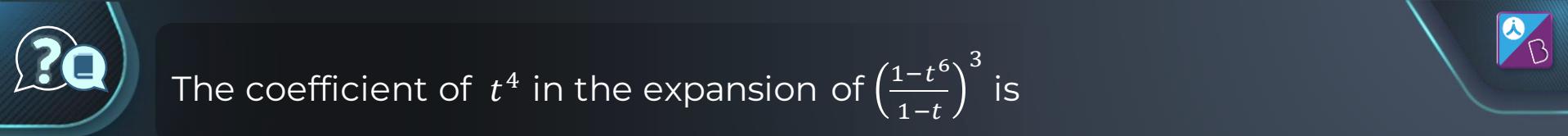
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$$\begin{aligned}\left(\frac{1-t^6}{1-t}\right)^3 &= (1-t^6)^3(1-t)^{-3} \\ &= (1 - 3t^6 + 3t^{12} - t^{18})(1-t)^{-3}\end{aligned}$$

∴ Coefficient of t^4 in $\left(\frac{1-t^6}{1-t}\right)^3$ = Coefficient of t^4 in $(1-t)^{-3}$

Coefficient of x^r in $(1-x)^{-m}$ is ${}^{(m+r-1)}C_r$

$$= {}^{(3+4-1)}C_4 = {}^6C_4 = {}^6C_2 = 15$$



The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is

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A

12

B

15

C

10

D

14

For $|x| < 1$ and $m, n \in \mathbb{W}$, the coefficients of x^n in the expansion of

$\frac{1}{1+x+x^2}$ is

$$\text{Given: } \frac{1}{1+x+x^2}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 \dots$$

$$= \frac{1}{1+x+x^2} \times \frac{(1-x)}{(1-x)}$$

$$= \frac{1-x}{1-x^3}$$

$$= (1-x)(1-x^3)^{-1}$$

$$= (1-x)(1+x^3+x^6+x^9+\dots+x^{3r}+\dots)$$

$$= (1+x^3+x^6+x^9+\dots+x^{3r}+\dots) - (x+x^4+x^7+x^{10}+\dots)$$

A

1, if $n = 3m$

B

1, if $n = 3m+1$

C

0, if $n = 3m-1$

D

-1, if $n = 3m+1$

For $|x| < 1$ and $m, n \in \mathbb{W}$, the coefficients of x^n in the expansion of

$\frac{1}{1+x+x^2}$ is

$$\frac{1}{1+x+x^2} = (1 + x^3 + x^6 + x^9 + \dots + x^{3r} + \dots) - (x + x^4 + x^7 + x^{10} + \dots)$$

In the above expansion, if $m \in \mathbb{W}$

i) Coefficient of $x^{3m} = 1$

ii) Coefficient of $x^{3m+1} = -1$

iii) Coefficient of $x^{3m-1} = 0$ (no term contain x^{3m-1})

Hence options *a*, *b* and *d* are correct.

A

1, if $n = 3m$

B

1, if $n = 3m + 1$

C

0, if $n = 3m - 1$

D

-1, if $n = 3m + 1$

If x is so small such that x^k is neglected for all $k \geq 2$,

then the value of $\frac{(1-2x)^{\frac{1}{3}} + (1+5x)^{-\frac{3}{2}}}{(9+x)^{\frac{1}{2}}}$ is

A

$$1 - \frac{149}{36}x$$

B

$$\frac{4}{3} + \frac{145}{36}x$$

C

$$\frac{2}{3} - \frac{149}{54}x$$

D

$$1 + \frac{149}{36}x$$



If x is so small such that x^k is neglected for all $k \geq 2$,

then the value of $\frac{(1-2x)^{\frac{1}{3}} + (1+5x)^{-\frac{3}{2}}}{(9+x)^{\frac{1}{2}}}$ is

Given: x^k is neglected $\forall k \geq 2$

$$(1+x)^n = 1 + n \cdot x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

$$\frac{(1-2x)^{\frac{1}{3}} + (1+5x)^{-\frac{3}{2}}}{(9+x)^{\frac{1}{2}}}$$

$$= \frac{1 + \frac{1}{3}(-2x) + 1 + \left(-\frac{3}{2}\right)(5x)}{3\left(1 + \frac{x}{9}\right)^{\frac{1}{2}}}$$

$$= \frac{1}{3} \left(2 - \frac{2x}{3} - \frac{15x}{2} \right) \left(1 + \frac{x}{9} \right)^{-\frac{1}{2}}$$

$$= \frac{1}{3} \left(2 - \frac{49x}{6} \right) \left(1 - \frac{x}{18} \right) = \frac{1}{3} \left(2 - \frac{x}{9} - \frac{49x}{6} + \frac{49}{108} x^2 \right)$$



If x is so small such that x^k is neglected for all $k \geq 2$,

then the value of $\frac{(1-2x)^{\frac{1}{3}} + (1+5x)^{-\frac{3}{2}}}{(9+x)^{\frac{1}{2}}}$ is

Given: x^k is neglected $\forall k \geq 2$

$$\frac{(1-2x)^{\frac{1}{3}} + (1+5x)^{-\frac{3}{2}}}{(9+x)^{\frac{1}{2}}}$$

$$= \frac{1}{3} \left(2 - \frac{x}{9} - \frac{49x}{6} + \underbrace{\frac{49}{108}x^2}_{\text{neglected}} \right)$$

neglected

$$= \frac{1}{3} \left(2 - \frac{149x}{18} \right) = \boxed{\frac{2}{3} - \frac{149x}{54}}$$

If $|x| << 1, n \in \mathbb{Q} - \mathbb{W}, x^2$ and its higher power are neglected , then

$$(1+x)^n = 1 + nx$$

+

+

If x is so small such that x^k is neglected for all $k \geq 2$,

then the value of $\frac{(1-2x)^{\frac{1}{3}} + (1+5x)^{-\frac{3}{2}}}{(9+x)^{\frac{1}{2}}}$ is

A

$$1 - \frac{149}{36}x$$

B

$$\frac{4}{3} + \frac{145}{36}x$$

C

$$\frac{2}{3} - \frac{149}{54}x$$

D

$$1 + \frac{149}{36}x$$

Session 08

**Multinomial theorem and
its applications**



Find the sum of $\sum_{s=1}^n \sum_{r=0}^s {}^nC_s \cdot {}^sC_r$ ($r \leq s$)



$$\sum_{r=0}^s \sum_{s=1}^n {}^nC_s \cdot {}^sC_r$$

$$= \sum_{s=1}^n {}^nC_s \cdot ({}^sC_0 + {}^sC_1 + {}^sC_2 + \dots + {}^sC_s)$$

$$= \sum_{s=1}^n {}^nC_s \cdot 2^s$$

$$= {}^nC_1 \cdot 2 + {}^nC_2 \cdot 2^2 + \dots + {}^nC_n \cdot 2^n$$

$$x = 2$$

$$= (1+2)^n - {}^nC_0$$

$$= 3^n - 1$$

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$$



Let $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r$. Then $\frac{a_7}{a_{13}}$ is equal to ____.

We have, $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r \quad \dots (i)$

Replacing x by $\frac{2}{x}$, we get

$$\Rightarrow \left(\frac{8}{x^2} + \frac{6}{x} + 4 \right)^{10} = \sum_{r=0}^{20} a_r \left(\frac{2}{x} \right)^r$$

$$\Rightarrow \frac{2^{10}(2x^2 + 3x + 4)^{10}}{x^{20}} = \sum_{r=0}^{20} a_r 2^r x^{-r}$$

$$\Rightarrow \sum_{r=0}^{20} a_r x^r = \sum_{r=0}^{20} a_r 2^{r-10} x^{20-r} \quad \text{Using (i)}$$

On comparing the coefficient of x^7 from both sides, we get

$$a_7 = a_{13} 2^3 \Rightarrow \frac{a_7}{a_{13}} = 8$$



MULTINOMIAL THEOREM

We know,

$$\bullet (a + b)^n = \sum_{r=0}^n {}^n C_r a^{n-r} b^r, \quad \forall n \in \mathbb{N}$$

Consider $(x_1 + x_2)^n = \sum_{r=0}^n {}^n C_r x_1^{n-r} x_2^r$

$$= \sum_{r=0}^n \frac{n!}{(n-r)! r!} \cdot x_1^{n-r} \cdot x_2^r$$

$$(x_1 + x_2)^n = \sum_{k_1+k_2=n} \frac{n!}{k_1! k_2!} \cdot x_1^{k_1} \cdot x_2^{k_2}$$

where $k_1 = n - r, k_2 = r$ and $k_1 + k_2 = n$



MULTINOMIAL THEOREM

$$\bullet (x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_n=n} \frac{n!}{k_1!k_2!\cdots k_r!} \cdot x_1^{k_1} \cdot x_2^{k_2} \cdots x_r^{k_r}$$

Also, $0 \leq k_1, k_2, \dots, k_r \leq n$, $n \in \mathbb{N}$

And $k_1 + k_2 + \cdots + k_r = n$

Note:

The total number of distinct terms in the expansion of $(x_1 + x_2 + \cdots + x_r)^n$

$$= {}^{(n+r-1)}C_{(r-1)}$$



In the number of terms in the expansion of $(x + y + z)^n$ is 36, then the value of n is ____.

Given: Number of terms in the expansion of $(x + y + z)^n$ is 36.

$$\text{Number of terms} = {}^{(n+r-1)}C_{(r-1)}$$

$$\text{Number of terms in } (x + y + z)^n = {}^{n+3-1}C_{3-1} = {}^{n+2}C_2$$

$$\Rightarrow \frac{(n+2)(n+1)}{2!} = 36$$

$$\Rightarrow (n+2)(n+1) = 72$$

$$\Rightarrow n^2 + 3n - 70 = 0$$

$$\Rightarrow (n-7)(n+10) = 0$$

$$\Rightarrow n = 7 \text{ and } n = -10 \text{ (Rejected)}$$

$$\therefore n = 7$$

In the number of terms in the expansion of $(x + y + z)^n$ is 36,
then the value of n is ____.

A

6

B

8

C

9

D

7

Find the independent term in the expansion of $\left(1 + \frac{x}{2} + \frac{2}{x}\right)^4$.

$$\text{Given: } \left(1 + \frac{x}{2} + \frac{2}{x}\right)^4 = \sum \frac{4!}{k_1! k_2! k_3!} (1)^{k_1} \cdot \left(\frac{x}{2}\right)^{k_2} \cdot \left(\frac{2}{x}\right)^{k_3}$$

(Where $k_1 + k_2 + k_3 = 4$ and $0 \leq k_1, k_2, k_3 \leq 4$)

$$\left(1 + \frac{x}{2} + \frac{2}{x}\right)^4 = \sum \frac{4!}{k_1! k_2! k_3!} (1)^{k_1} \left(\frac{1}{2}\right)^{k_2} (2)^{k_3} x^{k_2 - k_3}$$

For independent terms $k_2 - k_3 = 0 \Rightarrow k_2 = k_3$

Also, $k_1 + k_2 + k_3 = 4$

Possible Cases:

(i) $k_1 = 4, k_2 = 0, k_3 = 0$

(ii) $k_1 = 2, k_2 = 1, k_3 = 1$

(iii) $k_1 = 0, k_2 = 2, k_3 = 2$

+

Find the independent term in the expansion of $\left(1 + \frac{x}{2} + \frac{2}{x}\right)^4$.

Independent term

$$\begin{aligned}&= \sum \frac{4!}{k_1! k_2! k_3!} (1)^{k_1} \left(\frac{1}{2}\right)^{k_2} (2)^{k_3} x^{k_2 - k_3} \\&= \frac{4!}{4!0!0!} (1)^4 \left(\frac{1}{2}\right)^0 (2)^0 + \frac{4!}{2!1!1!} (1)^2 \left(\frac{1}{2}\right)^1 (2)^1 + \frac{4!}{0!2!2!} (1)^0 \left(\frac{1}{2}\right)^2 (2)^2 \\&= 1 + 12 + 6\end{aligned}$$

Independent term = 19.



Find the coefficient of $a^2b^3c^4d$ in the expansion of $(a - b - c + d)^{10}$.

$$(x_1 + x_2 + \dots + x_r)^n = \sum_{k_1+k_2+\dots+k_n=n} \frac{n!}{k_1! k_2! \dots k_r!} \cdot x_1^{k_1} \cdot x_2^{k_2} \cdot \dots \cdot x_r^{k_r}$$

$$(a - b - c + d)^{10} = \sum \frac{10!}{k_1! k_2! k_3! k_4!} \cdot (a)^{k_1} (-b)^{k_2} (-c)^{k_3} (d)^{k_4} \quad (\text{where } k_1 + k_2 + k_3 + k_4 = 10)$$

To find the coefficient of $a^2 b^3 c^4 d$,

Put $k_1 = 2, k_2 = 3, k_3 = 4, k_4 = 1$

$$\therefore \text{Coefficient of } (a)^2 (-b)^3 (-c)^4 (d)^1 = \frac{10!}{2! 3! 4! 1!}$$

$$= 12600$$

⇒ Coefficient of $a^2 b^3 c^4 d = -12600$.



In the expression of $(\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{5})^{10}$ which of the following is/are true ?

$$(\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{5})^{10} = \sum \frac{10!}{k_1! k_2! k_3!} (\sqrt{2})^{k_1} \cdot (\sqrt[3]{3})^{k_2} \cdot (\sqrt[6]{5})^{k_3}$$

Where, $k_1 + k_2 + k_3 = 10$.

For rational terms

k_1 should be multiple of 2

k_2 should be multiple of 3

k_3 should be multiple of 6

Possible cases:

(i) $k_1 = 10, k_2 = 0, k_3 = 0$

(ii) $k_1 = 4, k_2 = 6, k_3 = 0$

(iii) $k_1 = 4, k_2 = 0, k_3 = 6$

A

The number of rational terms is 4.

B

The number of rational terms is 3.

C

Sum of the rational terms is 12632.

D

Sum of the rational terms is 11792.

In the expression of $(\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{5})^{10}$ which of the following is/are true ?

Hence the number of rational terms = 3 $\rightarrow (b)$

Now, Sum of rational terms

$$\begin{aligned}
 &= \sum \frac{10!}{k_1! k_2! k_3!} (\sqrt{2})^{k_1} \cdot (\sqrt[3]{3})^{k_2} \cdot (\sqrt[6]{5})^{k_3} \\
 &= \frac{10!}{10!0!0!} (\sqrt{2})^{10} (\sqrt[3]{3})^0 (\sqrt[6]{5})^0 + \frac{10!}{4!6!0!} (\sqrt{2})^4 (\sqrt[3]{3})^6 (\sqrt[6]{5})^0 \\
 &\quad + \frac{10!}{4!0!6!} (\sqrt{2})^4 (\sqrt[3]{3})^0 (\sqrt[6]{5})^6 \\
 &= 32 + 7560 + 4200
 \end{aligned}$$

So, Sum of rational terms = 11792 $\rightarrow (d)$

A

The number of rational terms is 4.

B

The number of rational terms is 3.

C

Sum of the rational terms is 12632.

D

Sum of the rational terms is 11792.



THANK
YOU