

Welcome to



Complex Numbers

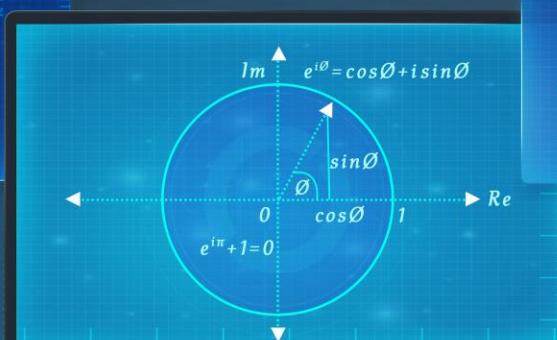


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Session 1

Introduction to Complex Numbers

- $x^2 + 1 = 0$

$$\Rightarrow x^2 = -1$$

$$\Rightarrow x = \pm\sqrt{-1} = \pm i$$

Where i is called **imaginary number**.

+

- An **imaginary number** is a number whose **square** is a real number that is **not positive**.

It is represented by i (iota) such that

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

+

+



B

Key Takeaways

- $\sqrt{ab} = \sqrt{a} \sqrt{b}$ \longrightarrow At least one of a and b is non-negative

$$\sqrt{ab} = -\sqrt{a} \sqrt{b} \longrightarrow a < 0 \text{ & } b < 0$$

- $x^2 + 16 = 0$

$$\Rightarrow x^2 = -16$$

$$\Rightarrow x = \pm\sqrt{-16} = \pm\sqrt{16 \times (-1)}$$

$$\Rightarrow x = \pm\sqrt{16}\sqrt{-1}$$

$$\Rightarrow x = \pm 4i$$



Evaluate $\sqrt{-25} + \sqrt{-49} - \sqrt{-81}$



Solution: $\sqrt{-25} = \sqrt{25 \times (-1)} = 5i$

Similarly $\sqrt{-49} = 7i$ and $\sqrt{-81} = 9i$

$$\therefore \sqrt{-25} + \sqrt{-49} - \sqrt{-81} = 5i + 7i - 9i$$

$$= 3i$$



A

Key Takeaways

Integral power of iota(i)

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = i^2 \cdot i = -i$
- $i^4 = i^2 \cdot i^2 = 1$

For $n \in \mathbb{Z}$:

- $i^{4n} = 1$ for $n \in \mathbb{Z}$
- $i^{4n+1} = i$ for $n \in \mathbb{Z}$
- $i^{4n+2} = -1$ for $n \in \mathbb{Z}$
- $i^{4n+3} = -i$ for $n \in \mathbb{Z}$

- Sum of any four consecutive power of i is zero.

i.e. $i^{4n} + i^{4n+1} + i^{4n+2} + i^{4n+3} = 0 , n \in \mathbb{Z}$



$$\text{Evaluate } \sum_{n=1}^{13} (i^n + i^{n+1})$$

Solution:

$$\sum_{n=1}^{13} (i^n + i^{n+1})$$

$$= (i + i^2 + i^3 + \dots + i^{13}) + (i^2 + i^3 + i^4 + \dots + i^{14})$$

+

$$= i^{13} + i^{14} \quad (\because \text{Sum of first 12 terms is 0})$$

$$= i + i^2$$

$$= i - 1$$

+

+



Find $\prod_{k=1}^{100} i^k$

Solution:

$$\prod_{k=1}^{100} i^k = i \times i^2 \times i^3 \cdots i^{100} = i^{1+2+3+\cdots+100} = i^{\frac{100 \times 101}{2}} = i^{5050} = i^2$$



Find $\frac{i^{587} + i^{629} + i^{777} + i^{847} + i^{995}}{i^{577} + i^{619} + i^{767} + i^{837} + i^{985}} = ?$

Solution:

$$\frac{i^{587} + i^{629} + i^{777} + i^{847} + i^{995}}{i^{577} + i^{619} + i^{767} + i^{837} + i^{985}} = \frac{i^{10} \times (i^{577} + i^{619} + i^{767} + i^{837} + i^{985})}{i^{577} + i^{619} + i^{767} + i^{837} + i^{985}} = i^2 = -1$$



Key Takeaways



Complex Number:

- A number of the form $a + ib$ is called a **complex number** where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.
- It is denoted by $z = a + ib$ here a is called real part represented by $\text{Re}(z)$ and b is called imaginary part represented by $\text{Im}(z)$.
- A set of complex numbers \mathbb{C} can be defined
Where, $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}, i = \sqrt{-1}\}$



Key Takeaways



Note:

- $z = a + ib$ is purely real, if $\text{Im}(z) = 0$ i.e. $b = 0$.
- $z = a + ib$ is purely imaginary, if $\text{Re}(z) = 0$ i.e. $a = 0$.
- $0 = 0 + i \cdot 0$, so 0 is purely real as well as purely imaginary.
- The set of real numbers \mathbb{R} is a proper subset of \mathbb{C} , hence
$$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Equality of Complex Numbers:

- Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$
 $\Rightarrow \operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$



Find (x, y) which satisfy $4x + i(3x - y) = 3 - 6i$, where $x, y \in \mathbb{R}$

Solution:

$$4x + i(3x - y) = 3 - 6i$$

$$\Rightarrow 4x = 3 \text{ and } 3x - y = -6$$

$$\Rightarrow x = \frac{3}{4} \text{ and } \frac{9}{4} - y = -6 \Rightarrow y = \frac{33}{4}$$

$$\therefore (x, y) = \left(\frac{3}{4}, \frac{33}{4}\right)$$

Key Takeaways

Let $z_1 = a + ib, z_2 = c + id$ then:

Addition:

- $$z_1 + z_2 = (a + ib) + (c + id) = \underbrace{(a + c)}_{\text{Re}(z_1 + z_2)} + i\underbrace{(b + d)}_{\text{Im}(z_1 + z_2)}$$

Subtraction:

- $$z_1 - z_2 = (a + ib) - (c + id) = \underbrace{(a - c)}_{\text{Re}(z_1 - z_2)} + i\underbrace{(b - d)}_{\text{Im}(z_1 - z_2)}$$

Multiplication:

- $$z_1 \cdot z_2 = (a + ib) \cdot (c + id) = \underbrace{(ac - bd)}_{\text{Re}(z_1 \cdot z_2)} + i\underbrace{(ad + bc)}_{\text{Im}(z_1 \cdot z_2)}$$

If $P = 2 + i$ and $Q = 1 + i$ then $P \cdot Q$ is

Solution:

$$P \cdot Q = (2 + i)(1 + i)$$

$$= 2 + 2i + i + i^2$$

$$= 1 + 3i$$

A

$$1 - 3i$$

B

$$-1 + 3i$$

C

$$1 + 3i$$

D

$$-1 - 3i$$

Key Takeaways

Division:

$$\bullet \frac{z_1}{z_2} = \frac{(a+ib)}{(c+id)} = \underbrace{\frac{(ac+bd)}{c^2+d^2}}_{\text{Re}\left(\frac{z_1}{z_2}\right)} + i \underbrace{\frac{(bc-ad)}{c^2+d^2}}_{\text{Im}\left(\frac{z_1}{z_2}\right)} \quad (z_2 \neq 0)$$



Express $\frac{1+i}{1-i}$ in form of $a + ib$.

Solution:

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{(1+i)^2}{1-i^2} = \frac{1+i^2+2i}{2}$$

$$= \frac{2i}{2}$$

$$= i$$

Note:

- $(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2$
- $(z_1 - z_2)^2 = z_1^2 + z_2^2 - 2z_1z_2$
- $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$
- $(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$
- $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$

+

+

+



Let $A = \left\{ \theta \in \left(0, \frac{\pi}{2}\right) : \frac{3+2i \sin \theta}{1-2i \sin \theta} \text{ is purely imaginary} \right\}$. Then, the sum of the elements in A is ____.

Solution:

Given:

$$A = \left\{ \theta \in \left(0, \frac{\pi}{2}\right) : \frac{3+2i \sin \theta}{1-2i \sin \theta} \text{ is purely imaginary} \right\}$$

$$\text{Let } z = \frac{3+2i \sin \theta}{1-2i \sin \theta} \times \frac{1+2i \sin \theta}{1+2i \sin \theta} = \frac{(3-4 \sin^2 \theta)+(8 \sin^2 \theta)i}{1+4 \sin^2 \theta}$$

z is purely imaginary if $\operatorname{Re}(z) = 0$

$$\Rightarrow \frac{3-4 \sin^2 \theta}{1+4 \sin^2 \theta} = 0$$

$$\Rightarrow 3 - 4 \sin^2 \theta = 0$$

$$\Rightarrow \sin^2 \theta = \frac{3}{4}$$

$$\Rightarrow \sin \theta = \pm \frac{\sqrt{3}}{2}$$



Let $A = \left\{ \theta \in \left(0, \frac{\pi}{2}\right) : \frac{3+2i \sin \theta}{1-2i \sin \theta} \text{ is purely imaginary} \right\}$. Then, the sum of the elements in A is ____.



Solution:

$$\Rightarrow \sin \theta = \pm \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}, \left(\because \theta \in \left(0, \frac{\pi}{2}\right) \right)$$

+

$$\text{Hence, } A = \left\{ \frac{\pi}{3} \right\}$$

+

$$\therefore \text{Sum of elements in } A = \frac{\pi}{3}$$

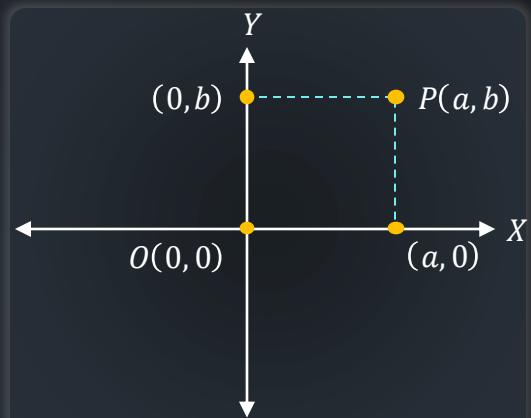
+

+

Key Takeaways

GEOMETRIC REPRESENTATION AND ARGAND PLANE:

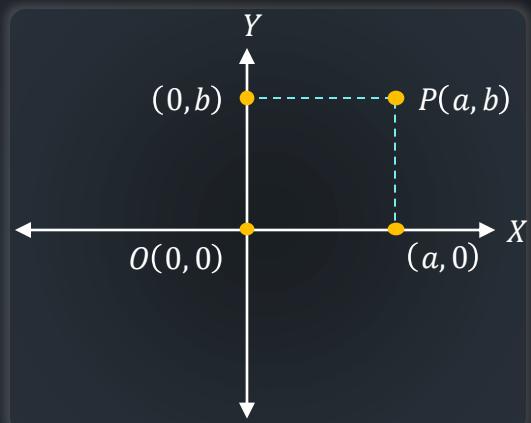
- A complex number $z = a + ib$ can be represented by a unique point $P(a, b)$ in the cartesian plane referred to a pair of rectangular axes.
- $0 + i0$ represents the origin point $O(0, 0)$.
- A purely real number a , i.e., $a + i0$ represented by the point $(a, 0)$ on x –axis (called **real axis**).



Key Takeaways

GEOMETRIC REPRESENTATION AND ARGAND PLANE:

- A purely imaginary number b , i.e., $0 + bi$ represented by the point $(0, b)$ on y –axis (called **imaginary axis**).
- The plane representing complex numbers as points is called **Argand Plane/ Complex plane/ Gaussian Plane**.

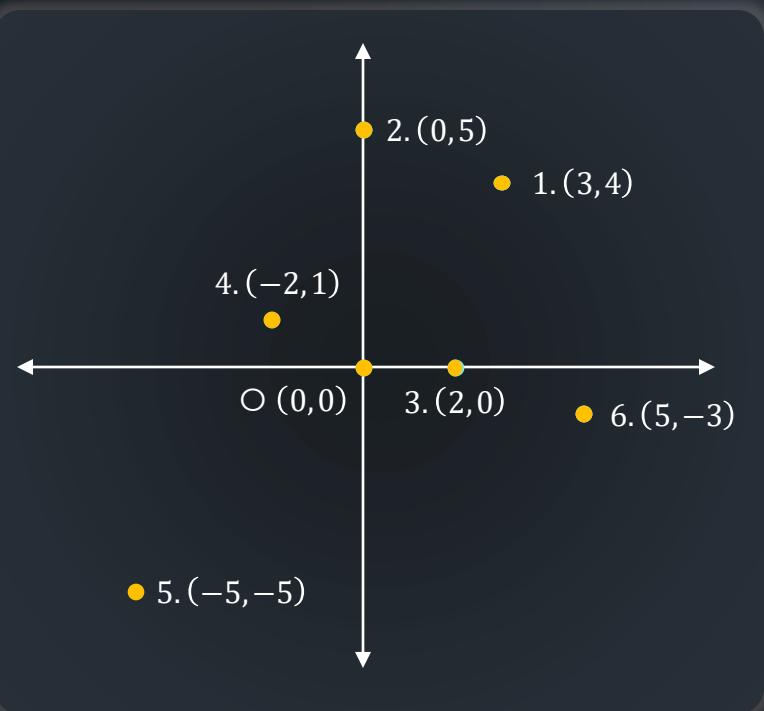




Mark these complex numbers as points on the Argand plane.

- (i). $3 + 4i$
- (ii). $5i$
- (iii). 2
- (iv). $-2 + i$
- (v). $-5 - 5i$
- (vi). $5 - 3i$

Solution:



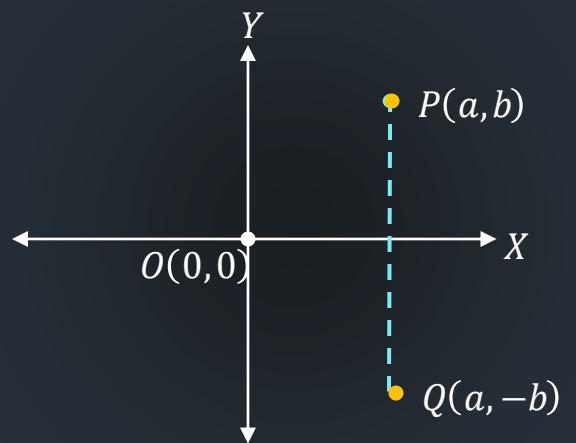
Session 2

Modulus and Conjugate of Complex Number

Key Takeaways

Conjugate of a complex Number:

- Conjugate of a complex number $z = a + ib$ is denoted by \bar{z} and is defined as $\bar{z} = a - ib$.
- \bar{z} is obtained by changing the sign of the imaginary part of z .
- If P represents z and Q represents \bar{z} in the Argand plane, then $P \equiv (a, b)$, $Q \equiv (a, -b)$.
- $Q(\bar{z})$ is the reflection of $P(z)$ in the real axis.



Examples:

i) If $z = 3 + 4i$, then $\bar{z} = 3 - 4i$.

ii) If $z = i - 5$, then $\bar{z} = -5 - i$.

iii) If $z = 5$, then $\bar{z} = 5$.

iv) If $z = -2i$, then $\bar{z} = 2i$.

Properties of conjugate:

Let $z = a + ib$ and $\bar{z} = a - ib$.

I. $\overline{(\bar{z})} = z$

II. $z + \bar{z} = 2\operatorname{Re}(z) = 2a$

III. $z - \bar{z} = 2i\operatorname{Im}(z) = 2bi$

IV. $z\bar{z} = [\operatorname{Re}(z)]^2 + [\operatorname{Im}(z)]^2 = a^2 + b^2$

V. $z = \bar{z} \Leftrightarrow z$ is purely real

VI. $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

VII. $(\overline{z_1 + z_2}) = \bar{z}_1 + \bar{z}_2$

Properties of conjugate:

Viii. $(\overline{z_1 - z_2}) = \bar{z}_1 - \bar{z}_2$

IX. $(\overline{z_1 \cdot z_2}) = \bar{z}_1 \cdot \bar{z}_2$

X. $\left(\overline{\frac{z_1}{z_2}}\right) = \frac{\bar{z}_1}{\bar{z}_2} \quad (z_2 \neq 0)$

XI. $(\overline{z_1 + z_2 + \dots + z_n}) = \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n$
 $\Rightarrow (\overline{z + z + z + \dots + z}) = \bar{z} + \bar{z} + \dots + \bar{z} \Rightarrow \overline{n z} = n \bar{z}$

XII. $(\overline{z_1 \cdot z_2 \dots z_n}) = \bar{z}_1 \cdot \bar{z}_2 \dots \bar{z}_n$
 $\Rightarrow (\overline{z \cdot z \cdot z \dots z}) = \bar{z} \cdot \bar{z} \dots \bar{z} \Rightarrow (\overline{z^n}) = (\bar{z})^n$

XIII. If $W = f(x + iy)$, then $\overline{W} = f(x - iy)$, where $x, y \in \mathbb{R}$

i.e for conjugate replace i with $-i$.



If $(x + iy)^5 = 4 + 5i$, ($x, y \in \mathbb{R}$) then $(y + ix)^5 = \underline{\hspace{2cm}}$.

Given, $(x + iy)^5 = 4 + 5i$

Taking conjugate on both sides

$$\Rightarrow \overline{(x + iy)^5} = \overline{4 + 5i}$$

$$\Rightarrow (x - iy)^5 = 4 - 5i$$

$$\Rightarrow (-i)^5 \left(y + \frac{x}{-i}\right)^5 = 4 - 5i$$

$$\Rightarrow -i (y + ix)^5 = 4 - 5i$$

$$\Rightarrow (y + ix)^5 = \frac{4}{-i} + 5$$

$$\Rightarrow (y + ix)^5 = 5 + 4i$$

Key Takeaways

Modulus of complex number:

- Let $z = x + iy \equiv P(x, y)$ in the Argand plane. Then modulus of complex number represented as $|z|$, where $|z| = \sqrt{x^2 + y^2} = OP$.
 $|z|$ represents the distance of $P(z)$ from Origin.
- $z = 0 \Leftrightarrow |z| = 0$.

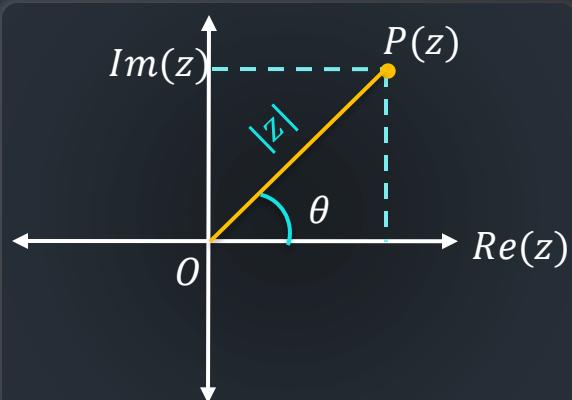
Examples:

(i). If $z = 3 + 4i$, then $|z| = \sqrt{3^2 + 4^2} = 5$

(ii). If $z = 5$, then $|z| = \sqrt{5^2 + 0^2} = 5$

(iii). If $z = 3i = 0 + 3i$, then $|z| = \sqrt{0^2 + 3^2} = 3$

(iv). If $z = 0 = 0 + 0i$, then $|z| = \sqrt{0^2 + 0^2} = 0$





If the equation, $x^2 + bx + 45 = 0$ ($b \in \mathbb{R}$) has conjugate complex roots and they satisfy $|z + 1| = 2\sqrt{10}$, then

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Given $x^2 + bx + 45 = 0$ ($b \in \mathbb{R}$)

Let roots of the equation be $p + iq$

Then, sum of roots = $2p = -b$

Product of roots = $p^2 + q^2 = 45$

As $p + iq$ lie on $|z + 1| = 2\sqrt{10}$, we get

$$(p + 1)^2 + q^2 = 40$$

$$\Rightarrow p^2 + q^2 + 2p + 1 = 40$$

$$\Rightarrow 45 - b + 1 = 40$$

$$\Rightarrow b = 6$$

$$\Rightarrow b^2 - b = 30$$

A

$$b^2 + b = 12$$

B

$$b^2 - b = 42$$

C

$$b^2 - b = 30$$

D

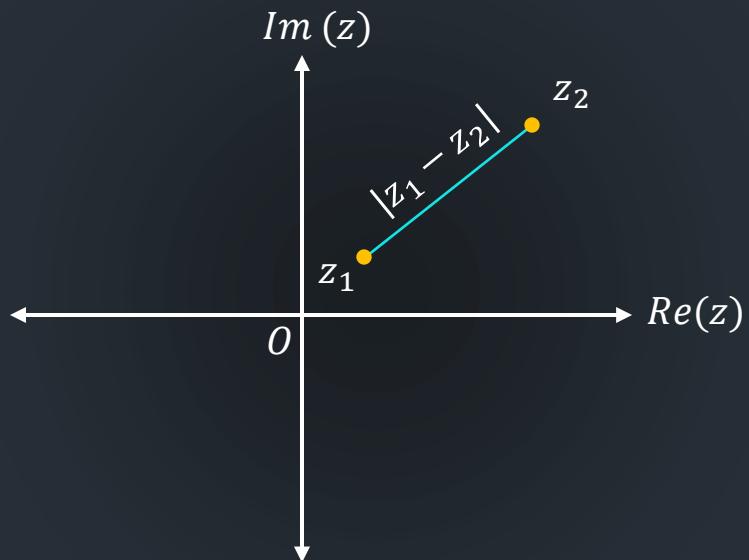
$$b^2 + b = 72$$



Key Takeaways



- If z_1 and z_2 are two complex numbers represented by P and Q in the Argand plane, then $PQ = |z_1 - z_2|$





Let z be a complex number such that $|z| + z = 3 + i$,
where $i = \sqrt{-1}$. Then $|z|$ is equal to :

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$$|z| + z = 3 + i \quad \dots(i)$$

$$z = x + iy$$

$$\Rightarrow |z| = \sqrt{x^2 + y^2}$$

From equation first

$$\sqrt{x^2 + y^2} + x + iy = 3 + i$$

$$\left(\sqrt{x^2 + y^2} + x\right) + iy = 3 + i$$

Comparing imaginary part

$$\Rightarrow y = 1$$

Comparing real part

A

$$\frac{5}{3}$$

B

$$\frac{5}{4}$$

C

$$\frac{\sqrt{34}}{3}$$

D

$$\frac{\sqrt{41}}{4}$$



Let z be a complex number such that $|z| + z = 3 + i$,
where $i = \sqrt{-1}$. Then $|z|$ is equal to :

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$$\sqrt{x^2 + 1} + x = 3$$

$$\Rightarrow \sqrt{x^2 + 1} = 3 - x$$

$$\Rightarrow x^2 + 1 = (3 - x)^2$$

$$\Rightarrow x = \frac{4}{3}$$

$$\Rightarrow \text{Thus, } |z| = \sqrt{x^2 + y^2}$$

$$\Rightarrow |z| = \sqrt{\left(\frac{4}{3}\right)^2 + 1^2}$$

$$\Rightarrow |z| = \frac{5}{3}$$

A

$$\frac{5}{3}$$

B

$$\frac{5}{4}$$

C

$$\frac{\sqrt{34}}{3}$$

D

$$\frac{\sqrt{41}}{4}$$

Key Takeaways

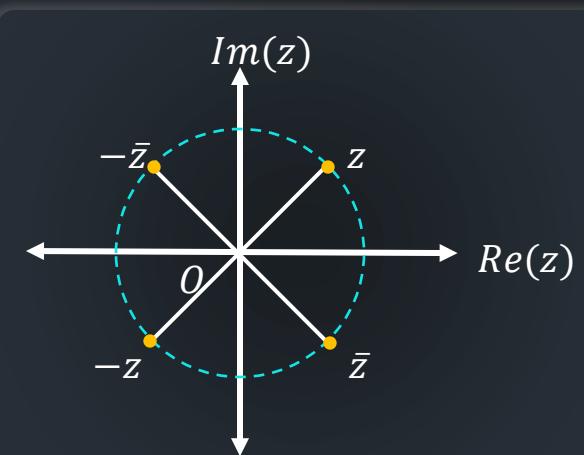
Properties of Modulus:

(i). For any complex number z , $|z| = |-z| = |\bar{z}| = |-\bar{z}|$

(ii). If $z = 0 \Rightarrow |z| = 0$

(iii). Let $z = x + iy$, $-|z| \leq Re(z) \leq |z|$

and $-|z| \leq Im(z) \leq |z|$



Properties of Modulus:

(iv). Let $z = x + iy$, $z\bar{z} = |z|^2$

$$\Rightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$
 If z is unimodular i.e $|z| = 1$, then $\bar{z} = \frac{1}{z}$.

(v). Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2, \dots, z_n = x_n + iy_n$

Then, $|z_1 z_2| = |z_1| |z_2|$

In general, $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$.

(vi). $|z^n| = |z|^n$

(vii). $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$



If $(2 + i)(2 + 2i)(2 + 3i) \dots \dots \dots (2 + ni) = x + iy$ then $5 \cdot 8 \cdot 13 \dots \dots (4 + n^2)$ is equal to _____. A B

Solution:

$$\text{Given, } (2 + i)(2 + 2i)(2 + 3i) \dots \dots \dots (2 + ni) = x + iy$$

$$\Rightarrow |(2 + i)(2 + 2i)(2 + 3i) \dots \dots \dots (2 + ni)| = |x + iy|$$

$$\Rightarrow |2 + i||2 + 2i||2 + 3i| \dots \dots \dots |2 + ni| = x + iy$$

$$\Rightarrow \sqrt{2^2 + 1^2} \sqrt{2^2 + 2^2} \sqrt{2^2 + 3^2} \dots \dots \dots \sqrt{2^2 + n^2} = \sqrt{x^2 + y^2}$$

Squaring both sides

$$\Rightarrow 5 \cdot 8 \cdot 7 \dots (4^2 + n^2) = x^2 + y^2$$



If z_1, z_2, z_3 are complex numbers such that

$$|z_1| = |z_2| = |z_3| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right|, \text{ then find } |z_1 + z_2 + z_3|.$$

Solution:

$$\text{Given, } |z_1| = |z_2| = |z_3| = 1$$

$$\Rightarrow |z_1|^2 = |z_2|^2 = |z_3|^2 = 1$$

$$\Rightarrow z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = 1$$

$$\Rightarrow \bar{z}_1 = \frac{1}{z_1}, \bar{z}_2 = \frac{1}{z_2}, \bar{z}_3 = \frac{1}{z_3}$$

$$\text{Also given, } \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1$$

$$\Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$$

$$\Rightarrow \overline{|z_1 + z_2 + z_3|} = 1$$

$$\Rightarrow |z_1 + z_2 + z_3| = 1$$



Prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2)$

Solution:

$$\begin{aligned}|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\&= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\&= (z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2) \\&= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} \\&\Rightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2)\end{aligned}$$



Key Takeaways

I. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \cdot \bar{z}_2)$

II. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1 \cdot \bar{z}_2)$

III. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

IV. $|az_1 - bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

V. $|z_1 + z_2| \neq |z_1| + |z_2|$

VI. $|z_1 - z_2| \neq |z_1| - |z_2|$



If $\frac{z-a}{z+a}$ ($a \in \mathbb{R}$) is a purely imaginary number and $|z| = 2$,
then a value of a is

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$$\text{Let } u = \frac{z-a}{z+a}$$

As, u is purely imaginary

$$\text{So, } u + \bar{u} = 0$$

$$\Rightarrow \frac{z-a}{z+a} + \frac{\bar{z}-a}{\bar{z}+a} = 0$$

$$\Rightarrow z\bar{z} + za - \bar{z}a - a^2 + z\bar{z} - za + \bar{z}a - a^2 = 0$$

$$\Rightarrow 2z\bar{z} - 2a^2 = 0$$

$$\Rightarrow |z|^2 = a^2$$

$$\Rightarrow a = \pm |z|$$

$$\Rightarrow a = \pm 2$$

A

$\frac{1}{2}$

B

2

C

$\sqrt{2}$

D

1

Session 3

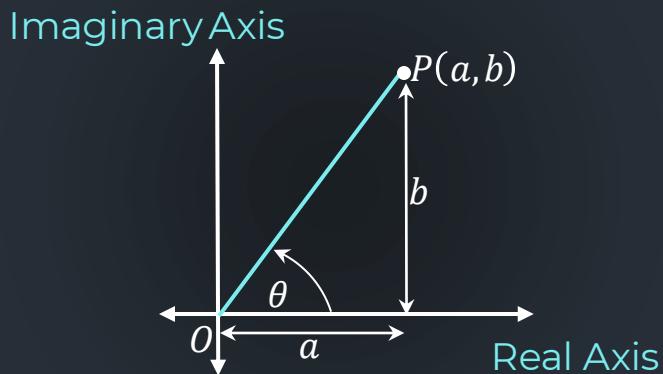
Argument and Different Forms of Complex Number

Key Takeaways

Argument (Amplitude) Of A Complex Number:

- Let $P \equiv (a, b)$ be a point representing a non-zero complex number $z = a + ib$ in the argand plane.
- If OP makes an angle θ with the positive real axis, then θ is called the **argument** or **amplitude** of z and written as $\arg(z) = \theta$.

$$\tan \theta = \frac{b}{a}$$





Key Takeaways



Principal Argument:

- The unique value of θ such that $-\pi < \theta \leq \pi$ is called **principal argument**.
- Unless otherwise stated, $\arg(z)$ refers to the principal value of z .

General Argument:

- General values of argument of z are given by $2n\pi + \theta, n \in \mathbb{Z}$.
- Any two consecutive arguments of the same complex number differ by 2π .
- If $z = 0 + 0i$, then $\arg(z)$ is not defined.



Key Takeaways



Working Rule For Finding Principal Argument:

Let $z = a + i b$ ($a, b \in \mathbb{R}$ and $a \neq 0, b \neq 0$)

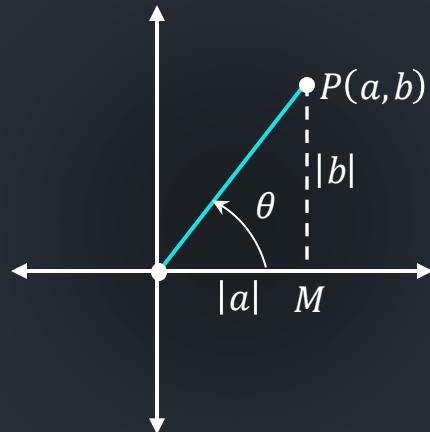
First compute α such that $\tan \alpha = \frac{|b|}{|a|}$.

Let θ represent the principal argument of z .

CASE I : $a > 0, b > 0$

z lies in first quadrant.

$$\arg(z) = \theta = \alpha$$





Key Takeaways



Working Rule For Finding Principal Argument:

Let $z = a + i b$ ($a, b \in \mathbb{R}$ and $a \neq 0, b \neq 0$)

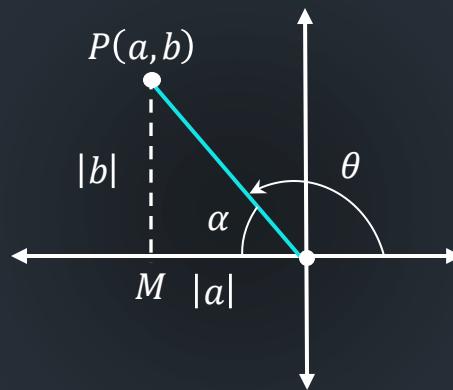
First compute α such that $\tan \alpha = \frac{|b|}{|a|}$.

Let θ represent the principal argument of z .

CASE II : $a < 0, b > 0$

z lies in second quadrant.

$$\arg(z) = \theta = \pi - \alpha$$





Working Rule For Finding Principal Argument:

Let $z = a + i b$ ($a, b \in \mathbb{R}$ and $a \neq 0, b \neq 0$)

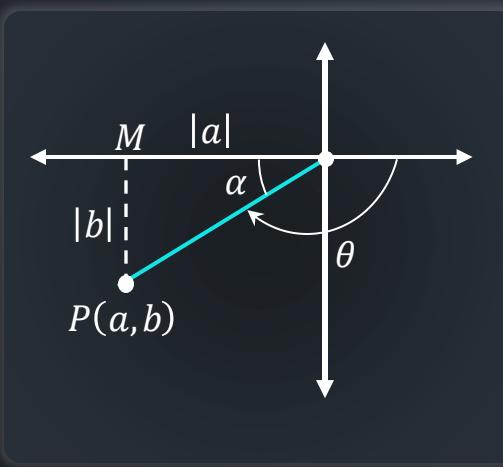
First compute α such that $\tan \alpha = \frac{|b|}{|a|}$.

Let θ represent the principal argument of z .

CASE III: $a < 0, b < 0$

z lies in third quadrant.

$$\arg(z) = \theta = \alpha - \pi$$





Working Rule For Finding Principal Argument:

Let $z = a + i b$ ($a, b \in \mathbb{R}$ and $a \neq 0, b \neq 0$)

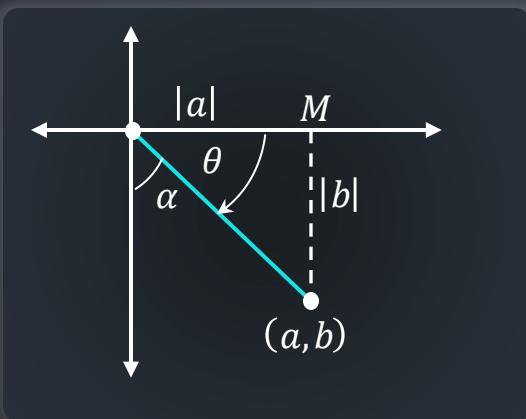
First compute α such that $\tan \alpha = \frac{|b|}{|a|}$.

Let θ represent the principal argument of z .

CASE IV: $a > 0, b < 0$

z lies in fourth quadrant.

$$\arg(z) = \theta = -\alpha$$



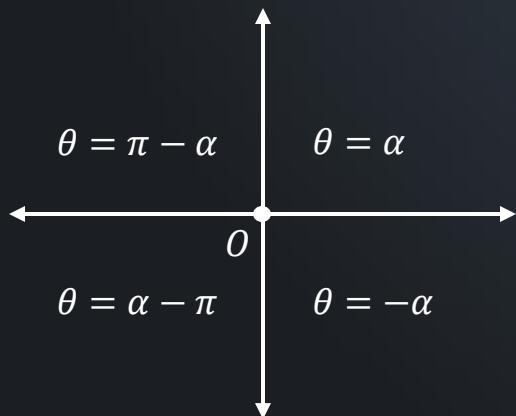


Working Rule For Finding Principal Argument:

Let $z = a + i b$ ($a, b \in \mathbb{R}$ and $a \neq 0, b \neq 0$)

First compute α such that $\tan \alpha = \frac{|b|}{|a|}$.

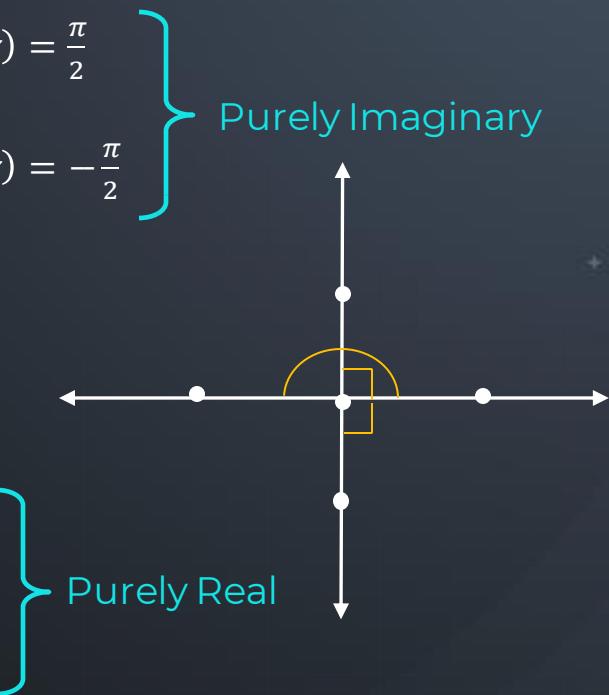
Let θ represent the principal argument of z .



For $z = a + i b$ ($a, b \in \mathbb{R}$):

- $a = 0, b > 0 \Rightarrow z$ lies on $+ve$ imaginary axis $\Rightarrow \arg(z) = \frac{\pi}{2}$

- $a = 0, b < 0 \Rightarrow z$ lies on $-ve$ imaginary axis $\Rightarrow \arg(z) = -\frac{\pi}{2}$



- $a > 0, b = 0 \Rightarrow z$ lies on $+ve$ real axis $\Rightarrow \arg(z) = 0$

- $a < 0, b = 0 \Rightarrow z$ lies on $-ve$ real axis $\Rightarrow \arg(z) = \pi$

Key Takeaways

Different Forms Of A Complex Number:

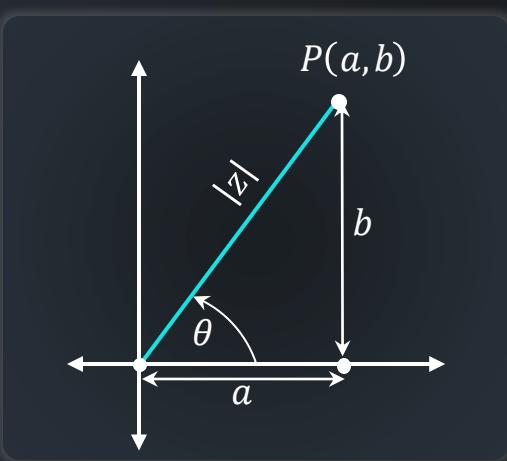
i) Cartesian Form/Algebraic Form/ Geometrical Form

$z = a + ib \equiv (a, b)$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$

$Re(z) = a$ and $Im(z) = b$

$$\Rightarrow |z| = \sqrt{a^2 + b^2}$$

$$\tan \theta = \frac{b}{a}$$





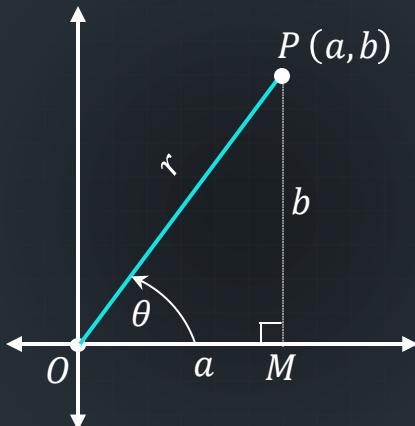
Key Takeaways



ii) Polar Form/Trigonometric Form

Let $z = a + ib$, $|z| = r$ and $\arg(z) = \theta$

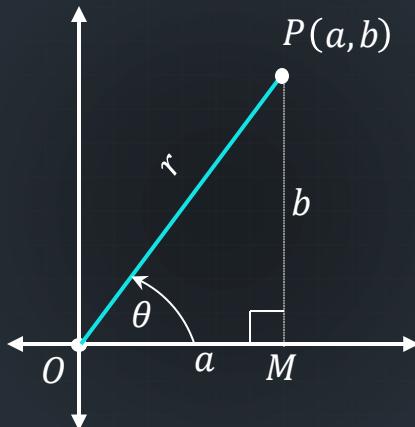
$$z = r (\cos \theta + i \sin \theta)$$



- $z = r(\cos \theta + i \sin \theta)$
- $\bar{z} = r(\cos \theta - i \sin \theta)$
- $|\cos \theta + i \sin \theta| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$
- $|\cos \theta - i \sin \theta| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$

iii) Euler Form:

- For complex number, $z = a + ib = r(\cos \theta + i \sin \theta)$,
- Euler's form is $z = |z|e^{i\theta}$
- $e^{i\theta} = \cos \theta + i \sin \theta$ where $\theta = \arg(z)$



If z is a complex number of unit modulus and argument θ , then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals:

$$|z| = 1 \text{ and } \arg(z) = \theta$$

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$$\text{So, } z = e^{i\theta}$$

$$\arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg\left(\frac{1+z}{1+\frac{1}{z}}\right)$$

$$= \arg(z)$$

$$= \theta$$

A

$$-\theta$$

B

$$\frac{\pi}{2} - \theta$$

C

$$\theta$$

D

$$\pi - \theta$$



For any integer k , if $a_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, then value of the

expression $\frac{\sum_{k=1}^{12} |a_{k+1} - a_k|}{\sum_{k=1}^3 |a_{4k-1} - a_{4k-2}|}$ is _____.

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$$a_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right) = e^{i\left(\frac{k\pi}{7}\right)}$$

$$\Rightarrow a_{k+1} - a_k = e^{i\left(\frac{(k+1)\pi}{7}\right)} - e^{i\left(\frac{k\pi}{7}\right)}$$

$$= e^{i\left(\frac{k\pi}{7}\right)} \left(e^{\frac{i\pi}{7}} - 1 \right)$$

$$= e^{i\left(\frac{k\pi}{7}\right)} e^{\frac{i\pi}{14}} \left(e^{\frac{i\pi}{14}} - e^{-\frac{i\pi}{14}} \right) = e^{i\left(\frac{(2k+1)\pi}{14}\right)} \times \left[2i \sin\left(\frac{\pi}{14}\right) \right]$$

$$\Rightarrow |a_{k+1} - a_k| = 2 \sin\left(\frac{\pi}{14}\right)$$

Replacing k with $4k - 2$ in the above expression,

$$|a_{4k-1} - a_{4k-2}| = 2 \sin\left(\frac{\pi}{14}\right)$$



For any integer k , if $a_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, then value of the

expression $\frac{\sum_{k=1}^{12} |a_{k+1} - a_k|}{\sum_{k=1}^3 |a_{4k-1} - a_{4k-2}|}$ is _____.

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Hence the value of the expression is,

$$\frac{\sum_{k=1}^{12} |a_{k+1} - a_k|}{\sum_{k=1}^3 |a_{4k-1} - a_{4k-2}|} = \frac{24}{6} = 4$$

Key Takeaways

Properties of Argument:

Let $z_1 = |z_1|e^{i\theta_1}$, $z_2 = |z_2|e^{i\theta_2}$, $z_3 = |z_3|e^{i\theta_3}, \dots$, $z_n = |z_n|e^{i\theta_n}$

$$(i) \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi, k \in \mathbb{Z}$$

Proof:

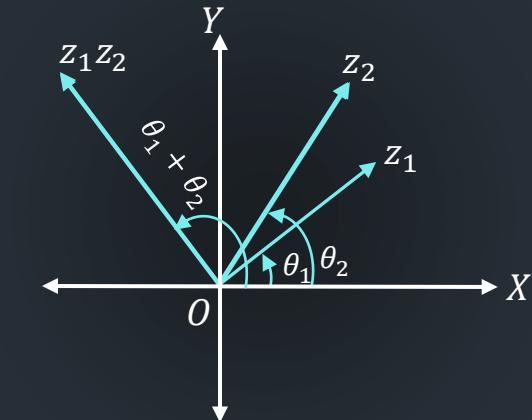
$$\Rightarrow z_1 z_2 = |z_1|e^{i\theta_1} \cdot |z_2|e^{i\theta_2}$$

$$\Rightarrow z_1 z_2 = |z_1||z_2|e^{i(\theta_1 + \theta_2)}$$

$$\Rightarrow z_1 z_2 = |z|e^{i\theta} \text{ where } |z| = |z_1||z_2|, \theta = \theta_1 + \theta_2$$

$$\Rightarrow \arg(z_1 z_2) = \theta_1 + \theta_2 + 2k\pi, k \in \mathbb{Z}$$

$$\Rightarrow \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi, k \in \mathbb{Z}$$





Key Takeaways



Note:

- $\arg(z_1 z_2 z_3 \cdots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \cdots + \arg(z_n) + 2k\pi, k \in \mathbb{Z}$
- If $z_1 = z_2 = z_3 = \cdots z_n = z$, then $\arg(z^n) = n \arg(z) + 2k\pi, k \in \mathbb{Z}$



If $\arg(z_1) = 160^\circ$ and $\arg(z_2) = 80^\circ$, then $\arg(z_1 z_2) = \underline{\hspace{2cm}}$.

Given: $\arg(z_1) = 160^\circ$ and $\arg(z_2) = 80^\circ$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi, k \in \mathbb{Z}$$

$$\therefore \arg(z_1 z_2) = 160^\circ + 80^\circ + 2k\pi, k \in \mathbb{Z}$$

Here, $\theta_1 + \theta_2 \notin (-\pi, \pi]$

$$\Rightarrow \arg(z_1 z_2) = 240^\circ - 2\pi \quad (k = -1 \text{ for principal argument})$$

$$\Rightarrow \arg(z_1 z_2) = -120^\circ$$

$$\Rightarrow \arg(z_1 z_2) = -\frac{2\pi}{3}$$

$$\therefore \arg(z_1 z_2) = -\frac{2\pi}{3}$$



If $z = 1 + i$, then $\arg(z^{50}) = \underline{\hspace{2cm}}$.



Given: $z = 1 + i$

$$= \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

$$= \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$\therefore \arg(z) = \frac{\pi}{4}$$

$$\arg(z^n) = n \arg(z) + 2k\pi, k \in \mathbb{Z}$$

$$\therefore \arg(z^{50}) = 50 \cdot \frac{\pi}{4} + 2k\pi, k \in \mathbb{Z}$$

$\Rightarrow k = -6$ for principal argument

$$\Rightarrow \arg(z^{50}) = \frac{25\pi}{2} - 12\pi$$

$$\therefore \arg(z^{50}) = \frac{\pi}{2}$$

Key Takeaways

Properties of Argument:

$$(ii) \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi, k \in \mathbb{Z}$$

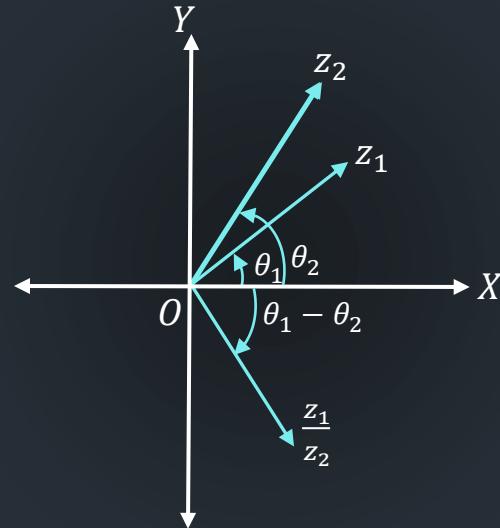
Proof:

$$\frac{z_1}{z_2} = \frac{|z_1|e^{i\theta_1}}{|z_2|e^{i\theta_2}} = \left|\frac{z_1}{z_2}\right| e^{i(\theta_1 - \theta_2)}$$

$$\Rightarrow \frac{z_1}{z_2} = |z|ie^\theta, \text{ where } |z| = \left|\frac{z_1}{z_2}\right|, \quad \theta = \theta_1 - \theta_2$$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 + 2k\pi, k \in \mathbb{Z}$$

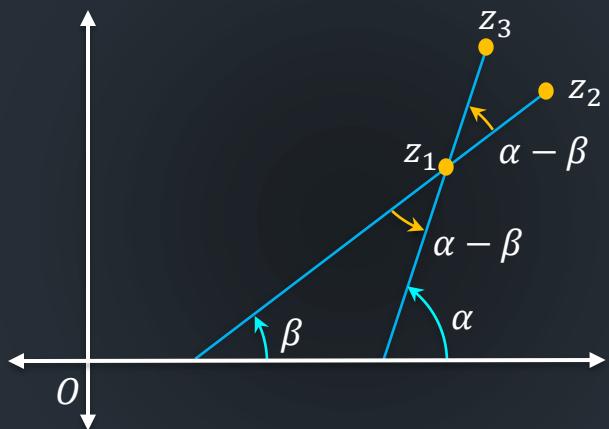
$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi, k \in \mathbb{Z}$$



Angle between two lines

Note:

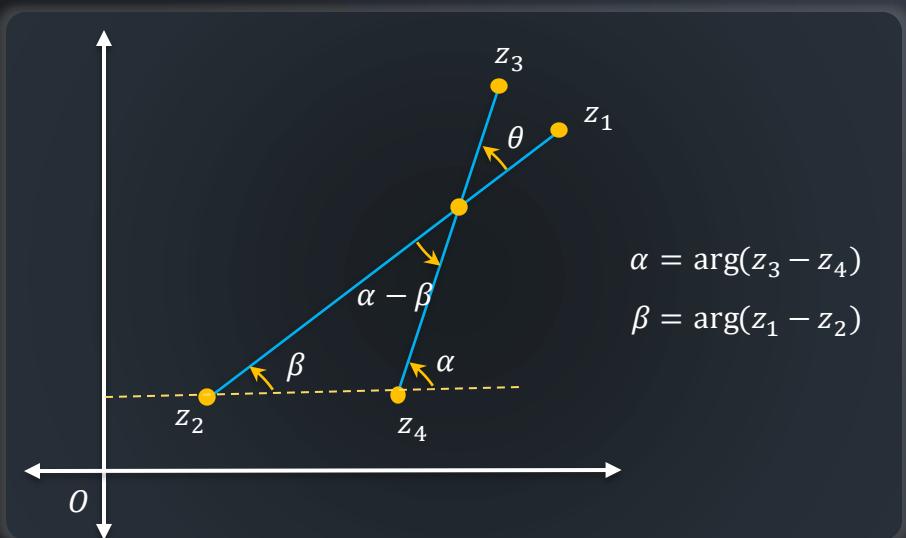
- Angle between two lines = $\alpha - \beta = \text{Arg}(z_3 - z_1) - \text{Arg}(z_2 - z_1) = \text{Arg}\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$



Angle between two lines

Note:

- Angle between two lines joining z_1, z_2 and $z_3, z_4 = \text{Arg}\left(\frac{z_3 - z_4}{z_1 - z_2}\right)$





If z and ω are two complex numbers such that $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \frac{3\pi}{2}$,
then $\arg\left(\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega}\right)$ is:
(here $\arg z$ denotes the principal argument of complex number z)

$$z = r e^{i\theta} \quad \therefore \omega = \frac{1}{r} e^{i\left(\theta - \frac{3\pi}{2}\right)}$$

$$\Rightarrow \bar{z} = r e^{-i\theta}$$

$$\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega} = \frac{1-2e^{-i\theta} \cdot e^{i\left(-\frac{3\pi}{2} + \theta\right)}}{1+3e^{-i\theta} \cdot e^{i\left(-\frac{3\pi}{2} + \theta\right)}}$$

$$= \frac{1-2i}{1+3i}$$

$$\therefore \arg\left(\frac{1-2i}{1+3i}\right) = \arg\left(-\frac{5}{10} - \frac{5}{10}i\right) = -\frac{3\pi}{4}$$

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A

$\frac{3\pi}{4}$

B

$-\frac{3\pi}{4}$

C

$\frac{\pi}{4}$

D

$-\frac{\pi}{4}$



If z_1, z_2 are complex numbers such that $\operatorname{Re}(z_1) = |z_1 - 1|$,
 $\operatorname{Re}(z_2) = |z_2 - 1|$ and $\arg(z_1 - z_2) = \frac{\pi}{6}$, then $\operatorname{Im}(z_1 + z_2)$ is equal to:

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Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\operatorname{Re}(z_1) = |z_1 - 1|$$

$$x_1^2 = (x_1 - 1)^2 + y_1^2$$

$$\Rightarrow y_1^2 - 2x_1 + 1 = 0 \quad \dots\dots (i)$$

$$\operatorname{Re}(z_2) = |z_2 - 1|$$

$$x_2^2 = (x_2 - 1)^2 + y_2^2$$

$$\Rightarrow y_2^2 - 2x_2 + 1 = 0 \quad \dots\dots (ii)$$

Subtracting equation (ii) from (i), we get

A

$2\sqrt{3}$

B

$\frac{2}{\sqrt{3}}$

C

$\frac{1}{\sqrt{3}}$

D

$\frac{\sqrt{3}}{2}$



If z_1, z_2 are complex numbers such that $\operatorname{Re}(z_1) = |z_1 - 1|$,
 $\operatorname{Re}(z_2) = |z_2 - 1|$ and $\arg(z_1 - z_2) = \frac{\pi}{6}$, then $\operatorname{Im}(z_1 + z_2)$ is equal to:

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Subtracting equation (ii) from (i), we get

$$\Rightarrow (y_1^2 - y_2^2) - 2(x_1 - x_2) = 0$$

$$\Rightarrow (y_1 - y_2)(y_1 + y_2) = 2(x_1 - x_2)$$

$$\Rightarrow (y_1 + y_2) = \frac{2(x_1 - x_2)}{(y_1 - y_2)}$$

$$\arg(z_1 - z_2) = \frac{\pi}{6}$$

$$\Rightarrow \tan^{-1} \left(\frac{(y_1 - y_2)}{(x_1 - x_2)} \right) = \frac{\pi}{6}$$

$$\Rightarrow \frac{(y_1 - y_2)}{(x_1 - x_2)} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow (y_1 + y_2) = 2\sqrt{3}$$

A

$2\sqrt{3}$

B

$\frac{2}{\sqrt{3}}$

C

$\frac{1}{\sqrt{3}}$

D

$\frac{\sqrt{3}}{2}$



Consider two points of non-zero conjugate complex numbers (z_1, z_2) and (z_3, z_4) . Then the principal value of $\arg\left(\frac{z_1}{z_3}\right) + \arg\left(\frac{z_2}{z_4}\right)$ is _____.

Given : $z_2 = \overline{z_1}$, $z_4 = \overline{z_3}$

$$\arg\left(\frac{z_1}{z_3}\right) + \arg\left(\frac{z_2}{z_4}\right) = \arg\left(\frac{z_1 \cdot z_2}{z_3 \cdot z_4}\right) = \arg\left(\frac{z_1 \cdot \overline{z_1}}{z_3 \cdot \overline{z_3}}\right)$$

$$= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right)$$

Here, $\frac{|z_1|^2}{|z_3|^2}$ is a positive real number and we know that Argument of a positive real number is zero.

$$\therefore \arg\left(\frac{z_1}{z_3}\right) + \arg\left(\frac{z_2}{z_4}\right) = 0$$

Session 4

De Moivre's Theorem

Key Takeaways

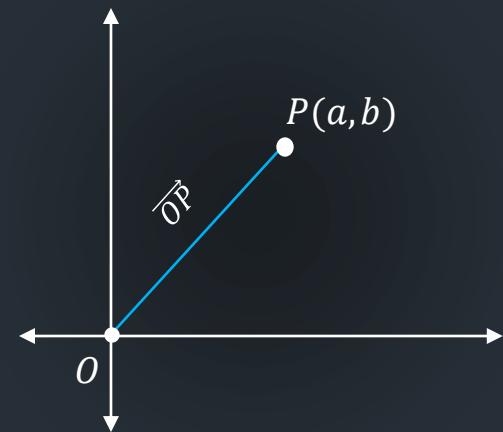
Vector form (Vectorial Representation)

Every complex number can be considered as the **position vector of a point**.

If the point $P(a, b)$ represents the complex number z .

$$\Rightarrow z = a + ib$$

$$\text{Then } \overrightarrow{OP} = z \text{ and } |\overrightarrow{OP}| = |z|$$





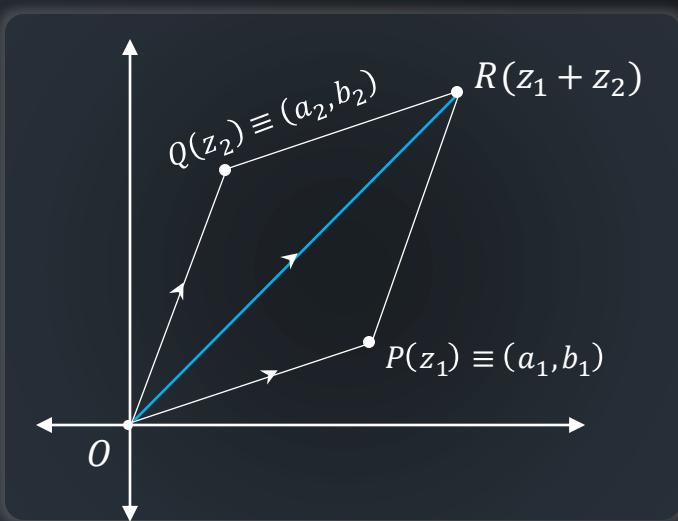
Geometrical Representation of Fundamental Operations:

(1) Addition Of Complex Numbers

Let $\overrightarrow{OP} = z_1 = a_1 + ib_1$, $\overrightarrow{OQ} = z_2 = a_2 + ib_2$

Now $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$ (Parallelogram law)

$$\therefore \overrightarrow{OR} = z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$



Key Takeaways

Geometrical Representation of Fundamental Operations:

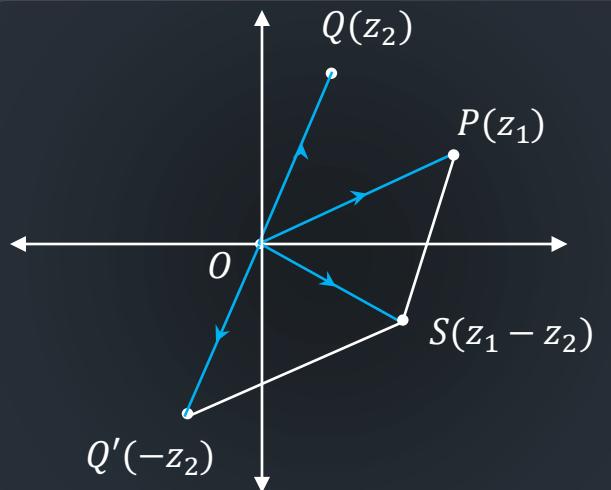
(2) Subtraction Of Complex Numbers

Let $\overrightarrow{OP} = z_1$, $\overrightarrow{OQ} = z_2$

$$\Rightarrow \overrightarrow{OQ'} = -z_2$$

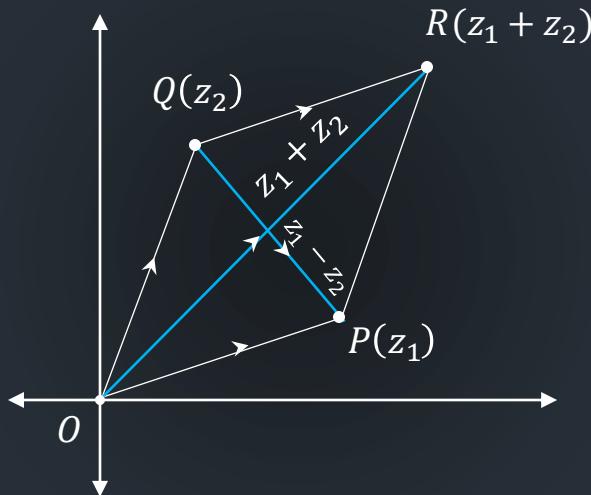
Now $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OQ'}$ (Parallelogram law)

$$\therefore \overrightarrow{OS} = z_1 - z_2$$



Key Takeaways

Geometrical Representation of Fundamental Operations:



Key Takeaways

Geometrical Representation of Fundamental Operations:

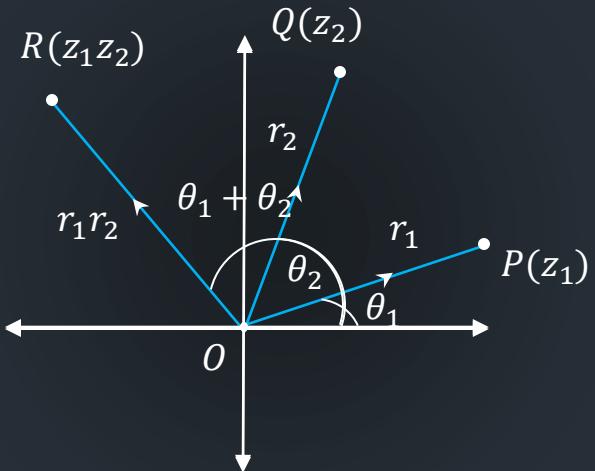
(3) Multiplication Of Complex Numbers

Let $z_1 = \overrightarrow{OP} = r_1 e^{i\theta_1}$, $z_2 = \overrightarrow{OQ} = r_2 e^{i\theta_2}$ be complex numbers represented by P and Q .

$$z_1 z_2 = \overrightarrow{OR} = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2})$$

$$= (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

$$\begin{aligned} \arg(z_1 z_2) &= \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2) + 2n\pi; n \in \mathbb{Z} \\ &\in (-\pi, \pi] \end{aligned}$$



Key Takeaways

Geometrical Representation of Fundamental Operations:

(4) Division Of Complex Numbers

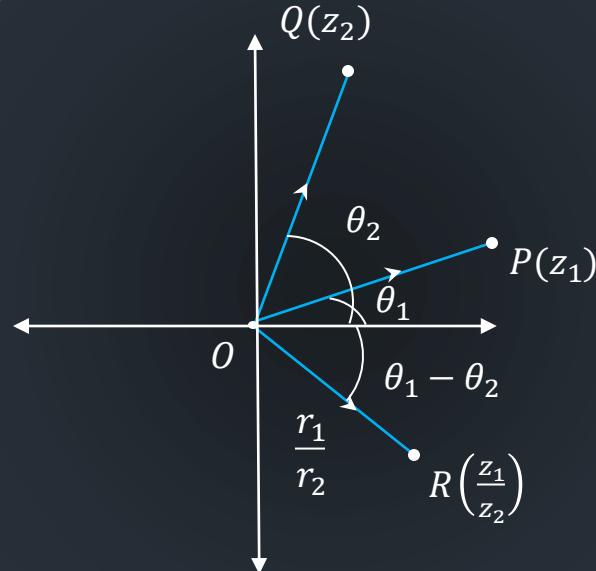
Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ be complex numbers represented by P and Q .

$$\frac{z_1}{z_2} = \frac{(r_1 e^{i\theta_1})}{(r_2 e^{i\theta_2})}$$

$$= \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 + 2n\pi; n \in \mathbb{Z}$$

$$\theta_1 - \theta_2 + 2n\pi \in (-\pi, \pi]$$





Key Takeaways



Triangle Inequalities:

Let z_1, z_2 be two complex numbers represented by the points P and Q in argand plane.

- $\| |z_1| - |z_2| \| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
- $\| |z_1| - |z_2| \| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

Key Takeaways

Triangle Inequalities:

Let z_1, z_2 be two complex numbers represented by the points P and Q in argand plane.

$$\text{Let } \overrightarrow{OP} = z_1, \quad \overrightarrow{OQ} = z_2$$

$$\Rightarrow \overrightarrow{OR} = z_1 + z_2, \quad \overrightarrow{QP} = z_1 - z_2$$

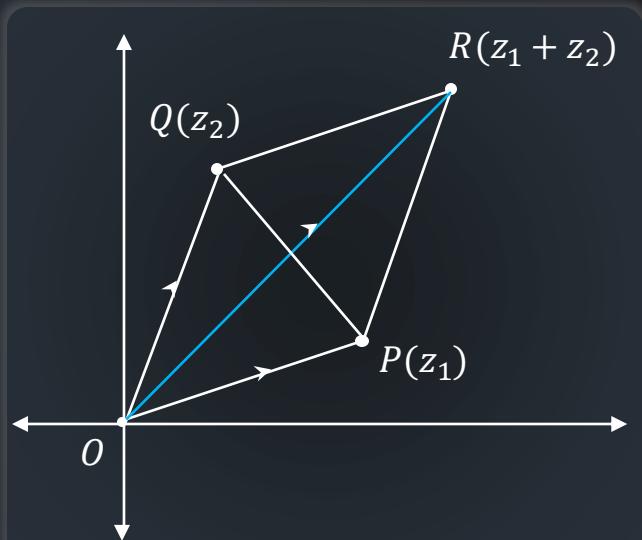
In $\triangle OPR$

Sum of two sides is always greater than third side.

$$|OP| + |PR| \geq |OR| \text{ i.e, } |z_1| + |z_2| \geq |z_1 + z_2|$$

Difference of two sides is always less than third side.

$$|OP| - |PR| \leq |OR| \text{ i.e, } |z_1| - |z_2| \leq |z_1 + z_2|$$



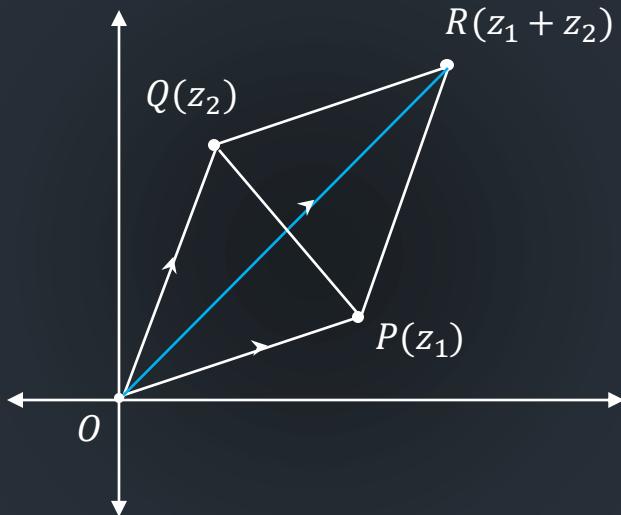
Key Takeaways

Triangle Inequalities:

$$\therefore | |z_1| - |z_2| | \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

Similarly, from $\triangle OPQ$, we have

$$\therefore | |z_1| - |z_2| | \leq |z_1 - z_2| \leq |z_1| + |z_2|$$



Note:

$$|z_1 + z_2| = |z_1| + |z_2|$$

$$||z_1| - |z_2|| = |z_1 - z_2|$$

Holds, if origin, z_1, z_2 are Collinear.

z_1 and z_2 lies on the same side of origin.

Also, argument between z_1 and z_2 will be zero.

Origin lies between z_1 and z_2 .

Also, argument between z_1 and z_2 will be π .



If $|z - (5 + 7i)| = 9$, then find the greatest and least value of $|z - 2 - 3i|$.

Method-I

$$|z - (5 + 7i)| = 9$$

$$\text{Now, } |z - 2 - 3i| = \underbrace{|z - (5 + 7i)|}_{z_1} + \underbrace{|(3 + 4i)|}_{z_2}$$

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\therefore ||z - (5 + 7i)| - |3 + 4i|| \leq |z - 2 - 3i| \leq |z - (5 + 7i)| + |3 + 4i|$$

$$\Rightarrow |9 - 5| \leq |z - 2 - 3i| \leq 9 + 5$$



If $|z - (5 + 7i)| = 9$, then find the greatest and least value of $|z - 2 - 3i|$.

$$\Rightarrow 4 \leq |z - 2 - 3i| \leq 14$$
$$|z - 2 - 3i|_{min} = 4$$
$$|z - 2 - 3i|_{max} = 14$$

Method-II

$$|z - (5 + 7i)| = 9$$

$$\text{Let } z = x + iy$$

$$\Rightarrow (x - 5)^2 + (y - 7)^2 = 9^2$$

$\therefore z$ lies on the circle with centre $C(5, 7)$ and radius, $r = 9$.



If $|z - (5 + 7i)| = 9$, then find the greatest and least value of $|z - 2 - 3i|$.



$$\Rightarrow (x - 5)^2 + (y - 7)^2 = 9^2$$

$$|z - 2 - 3i| = |z - (2 + 3i)|$$

→ Distance between z and $P(2,3)$

Distance between z_1 and z_2 = $|z_1 - z_2|$

∴ Minimum distance : $AP = |CP - r| = |5 - 9| = 4$

Maximum distance : $BP = |CP + r| = 5 + 9 = 14$



If $|z_1 - 1| \leq 1$, $|z_2 - 2| \leq 2$, $|z_3 - 3| \leq 3$, then find the greatest value of $|z_1 + z_2 + z_3|$.



$$|z_1 - 1| \leq 1, |z_2 - 2| \leq 2, |z_3 - 3| \leq 3$$

$$\Rightarrow |z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6|$$

$$\leq |(z_1 - 1)| + |(z_2 - 2)| + |(z_3 - 3)| + |6|$$

$$\leq 1 + 2 + 3 + 6$$

$$\leq 12$$

Hence, the greatest value of $|z_1 + z_2 + z_3|$ is 12



Logarithm of a complex number:

Let $z = x + iy = |z|e^{i\theta}$

$$\therefore \log_e z = \log_e(|z|e^{i\theta})$$

$$= \log_e|z| + \log_e e^{i\theta}$$

$$\Rightarrow \log_e z = \log_e|z| + i\theta$$

$$\Rightarrow \log_e z = \log_e|z| + i(\theta + 2n\pi)$$



$$r = \sqrt{x^2 + y^2} \quad \theta \in (-\pi, \pi]$$



Express the following in $a + ib$ form.

i) $\log_e(1 + i)$

ii) $\log_e(-5)$

$$= \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)$$

$$= \sqrt{2} \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]$$

$$= \sqrt{2} \cdot e^{\frac{i\pi}{4}}$$

$$\log_e z = \log_e |z| + i\theta$$

$$\Rightarrow |z| = \sqrt{2}, \theta = \frac{\pi}{4}$$

$$\log_e(1 + i) = \frac{1}{2} \log_e 2 + i \cdot \left(\frac{\pi}{4} + 2n\pi \right); n \in \mathbb{Z}$$



Express the following in $a + ib$ form.

i) $\log_e(1 + i)$

ii) $\log_e(-5)$

ii) -5

$$= 5(-1)$$

$$= 5 [\cos \pi + i \sin \pi]$$

$$= 5e^{i\pi}$$

$$\log_e z = \log_e |z| + i\theta$$

$$\Rightarrow |z| = 5, \theta = \pi$$

$$\log_e (-5) = \log_e 5 + i \cdot \pi$$

$$= \log_e 5 + i(\pi + 2n\pi); n \in \mathbb{Z}$$



DE MOIVRE'S THEOREM



- If $n \in \mathbb{Z}$ (set of integers), then $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

PROOF:

Let $n \in \mathbb{Z}$

We know, $z = \cos \theta + i \sin \theta = e^{i\theta}$

$$z^n = (\cos \theta + i \sin \theta)^n = (e^{i\theta})^n$$

$$= e^{i(n\theta)}$$

$$\therefore z^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

We know that $|z| = 1$ and $\arg(z) = \theta$ and we have $|z^n| = 1$ and $\arg(z^n) = n\theta$



DE MOIVRE'S THEOREM

- If $n \in \mathbb{Q}$ (set of rational numbers), then $\cos n\theta + i \sin n\theta$ is one of the values of $(\cos \theta + i \sin \theta)^n$

Note:

- $(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta, n \in \mathbb{Z}$ and it is the only solution.
- If $p, q \in \mathbb{Z}$ and $q \neq 0$, then

$$(\cos \theta + i \sin \theta)^{\frac{p}{q}} = \cos\left(\frac{2k\pi+p\theta}{q}\right) + i \sin\left(\frac{2k\pi+p\theta}{q}\right)$$

where $k = 0, 1, 2, \dots, q - 1$



Find all the roots of the equation $z^4 = 1$.

$$\text{Given : } z^4 = 1 \Rightarrow z = (1)^{\frac{1}{4}} \dots (i)$$

$$\Rightarrow z = (1)^{\frac{1}{4}} = (\cos 0 + i \sin 0)^{\frac{1}{4}}$$

$$(\cos \theta + i \sin \theta)^{\frac{p}{q}} = \cos\left(\frac{2k\pi+p\theta}{q}\right) + i \sin\left(\frac{2k\pi+p\theta}{q}\right); k = 0, 1, 2, \dots, q-1$$

Here, $p = 1, q = 4, \theta = 0, k = 0, 1, 2, 3$

$$\therefore z = \cos\left(\frac{2k\pi}{4}\right) + i \sin\left(\frac{2k\pi}{4}\right); k = 0, 1, 2, 3$$

When $k = 0, z = 1$

$$\text{When } k = 1, z = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i$$

$$\text{When } k = 2, z = \cos\pi + i \sin\pi = -1$$

$$\text{When } k = 3, z = \cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2} = -i$$



If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ ($i = \sqrt{-1}$), then $(1 + iz + z^5 + iz^8)^9$ is equal to

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$$z = \frac{\sqrt{3}}{2} + \frac{i}{2} = e^{\frac{i\pi}{6}}$$

$$iz = -\frac{1}{2} + \frac{(\sqrt{3}i)}{2}$$

$$z^5 = e^{\frac{i5\pi}{6}} = \cos \frac{5\pi}{6} + i \sin \left(\frac{5\pi}{6} \right) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$z^8 = e^{\frac{i4\pi}{3}} = \cos \frac{4\pi}{3} + i \sin \left(\frac{4\pi}{3} \right) = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$$

$$iz^8 = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

$$\therefore (1 + iz + z^5 + iz^8)^9 = \left(1 - \frac{1}{2} + \frac{\sqrt{3}i}{2} - \frac{\sqrt{3}}{2} + \frac{i}{2} + \frac{\sqrt{3}}{2} - \frac{i}{2} \right)^9$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}i}{2} \right)^9 = \left(e^{\frac{i\pi}{3}} \right)^9 = e^{i3\pi} = -1$$

A

1

B

-1

C

$(-1 + 2i)^9$

D

0

Session 5

Cube Roots and nth roots of Unity



Properties of Cube Roots of Unity

Integral Power of ω .

$$\omega = \frac{-1+i\sqrt{3}}{2} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$$

Since ω is a cube root of unity, we have $\omega^3 = 1$

$$\Rightarrow (\omega^3)^k = 1^k; k \in \mathbb{Z}$$

So, we can conclude that $\omega^{3k} = 1, k \in \mathbb{Z}$



Properties of Cube Roots of Unity

For example :

i) ω^{18}

$$\omega^{18} = \omega^{3 \cdot 6} = (\omega^3)^6 = 1$$

ii) ω^{10}

$$\omega^{10} = \omega^{12-2} = \frac{\omega^{12}}{\omega^2} = \frac{1}{\omega^2} = \omega$$

iii) ω^{-28}

$$\omega^{-28} = \omega^{-30+2} = \omega^{-30} \cdot \omega^2 = (\omega^3)^{-10} \cdot \omega^2 = 1 \cdot \omega^2 = \omega^2$$

iv) $\omega^{200} + \omega^{198} + \omega^{193}$

$$\begin{aligned}\omega^{200} + \omega^{198} + \omega^{193} &= \omega^{198} \cdot \omega^2 + \omega^{198} + \omega^{192} \cdot \omega \\ &= (\omega^3)^{66} \cdot \omega^2 + (\omega^3)^{66} + (\omega^3)^{64} \cdot \omega = \omega^2 + 1 + \omega = 0\end{aligned}$$



PROPERTY 1 : Sum of cube roots of unity is 0 .

PROPERTY 2 : Product of cube roots of unity is 1 .

PROPERTY 3 : If $1, \omega, \omega^2$ are the cube roots of unity, then the cube root of -1 are $-1, -\omega, -\omega^2$

Proof:

$$(-1)^{\frac{1}{3}} = \sqrt[3]{-1 + 0i}$$

The cube roots of -1 are similar to the roots of the equation $z^3 = -1$.

As we know that $(-1)^3 = -1$ and $\omega^{3n} = 1$

$$\Rightarrow (-\omega)^3 = (-1)^3 \cdot 1 = -1$$

$$\Rightarrow (-\omega^2)^3 = (-1)^3 \cdot (\omega^2)^3 = -1 \cdot 1 = -1$$

Properties of Cube Roots of Unity

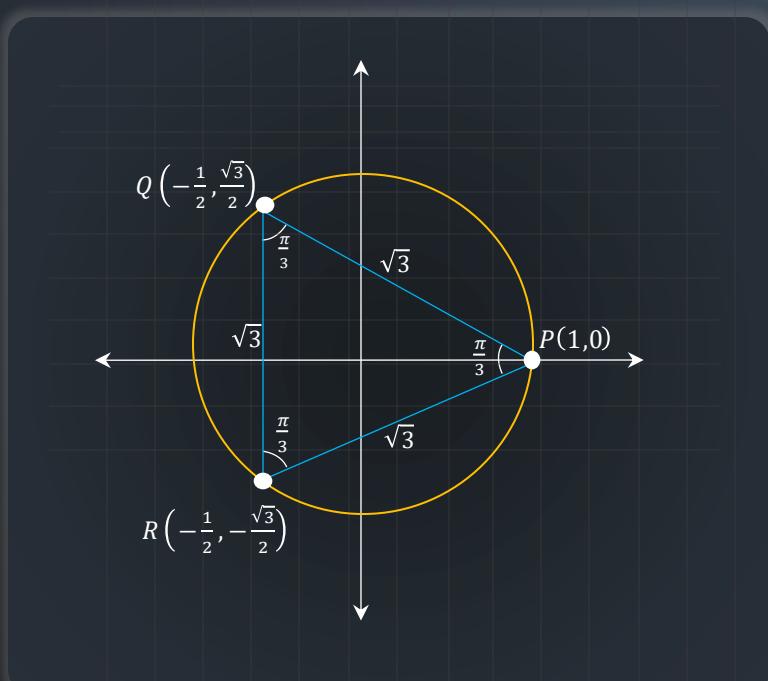
PROPERTY 4 : If we represent the cube roots of unity by points in Argand plane, they form an equilateral triangle with side as $\sqrt{3}$ units.

Also they lie on a circle with unit radius and centre at origin.

$$1 = 1 + 0i \equiv P(1,0)$$

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \equiv Q\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \equiv R\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$



Properties of Cube Roots of Unity

$$\text{PROPERTY 5 : } a^3 + b^3 = (a + b)(a + b\omega)(a + b\omega^2)$$

Note:

If $1, \omega, \omega^2$ are the cube roots of unity, then

$$1 + \omega^k + \omega^{2k} = \begin{cases} 3, & \text{when } k \text{ is a multiple of 3} \\ 0, & \text{when } k \text{ is not a multiple of 3} \end{cases}$$



If α, β are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to

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Given $x^2 - x + 1 = 0$

$$\Rightarrow x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$\Rightarrow -\omega^2 = \frac{1+i\sqrt{3}}{2} = \alpha \text{ (say)}$$

$$\text{and } -\omega = \frac{1-i\sqrt{3}}{2} = \beta \text{ (say)}$$

Now, $\alpha^{101} + \beta^{107}$

$$= (-\omega^2)^{101} + (-\omega)^{107}$$

$$= -\omega^{202} - \omega^{107}$$

$$= -\left((\omega^3)^{67} \cdot \omega + (\omega^3)^{35} \cdot \omega^2\right)$$

$$= -(\omega + \omega^2) = 1$$

A

2

B

-1

C

0

D

1



Key Takeaways

n^{th} Roots of Unity

Let z be an n^{th} root of unity.

$$(e^{i\theta})^{\frac{p}{q}} = e^{i\left(\frac{(2k\pi+p\theta)}{q}\right)}; k = 0, 1, 2, \dots, q-1$$

$$\text{Then } z^n = 1 = e^{i0} \Rightarrow z = (e^{i0})^{\frac{1}{n}}$$

$$(1 + 0i)^{\frac{1}{n}} = (\cos 0 + i \sin 0)^{\frac{1}{n}}$$

$$(\cos \theta + i \sin \theta)^{\frac{p}{q}} = \cos\left(\frac{2k\pi+p\theta}{q}\right) + i \sin\left(\frac{2k\pi+p\theta}{q}\right); k = 0, 1, 2, \dots, q-1$$

$$(1)^{\frac{1}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right); k = 0, 1, 2, \dots, n-1$$

$$(\cos 0 + i \sin 0)^{\frac{1}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right); k = 0, 1, 2, \dots, n-1$$

Here $p = 1, q = n, \theta = 0, k = 0, 1, 2, \dots, n-1$

Key Takeaways

$$(\cos 0 + i \sin 0)^{\frac{1}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right); k = 0, 1, 2, \dots, n-1$$

$$\therefore z = e^{i\left(\frac{2k\pi}{n}\right)}; k = 0, 1, 2, \dots, n-1$$

$$\Rightarrow z = e^{i0}, e^{i\left(\frac{2\pi}{n}\right)}, e^{i\left(\frac{4\pi}{n}\right)}, \dots, e^{i\left(\frac{(n-1)2\pi}{n}\right)}$$

$$\Rightarrow z = 1, e^{i\left(\frac{2\pi}{n}\right)}, e^{i\left(\frac{4\pi}{n}\right)}, \dots, e^{i\left(\frac{(n-1)2\pi}{n}\right)}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \alpha & \alpha^2 & \alpha^{n-1} \end{array}$$

$\Rightarrow 1, \omega, \omega^2$ are cube roots of unity, which are in G.P.

with common ratio ω .

$\Rightarrow z = 1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ are in G.P. with common ratio $\alpha = e^{i\left(\frac{2\pi}{n}\right)} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$

Properties of n^{th} Roots of Unity :

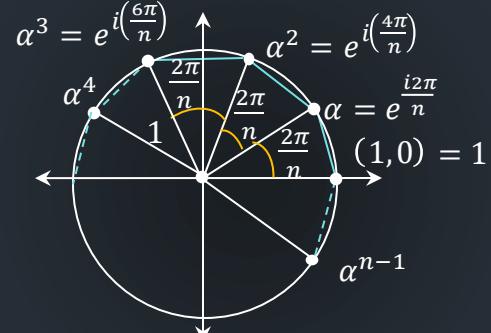
$$z = 1, \alpha, \alpha^2, \dots, \alpha^{n-1} \text{ where } \alpha = e^{i\left(\frac{2\pi}{n}\right)}$$

Property I: $|1| = |\alpha| = |\alpha^2| = |\alpha^3| = \dots = 1$

Property II: If we represent the n^{th} roots of unity by points in Argand plane, they form a n – sided regular polygon with circumcircle of unit radius and centre at origin.

Points equidistant from 1 are conjugates to each other.

Hence, α and α^{n-1}, α^2 and α^{n-2} are conjugates to each other.



Properties of n^{th} Roots of Unity :

$$z = 1, \alpha, \alpha^2, \dots, \alpha^{n-1} \text{ where } \alpha = e^{i\left(\frac{2\pi}{n}\right)}$$

Note: $z^3 = 1 \Rightarrow z^3 - 1 = (z - 1)(z - \omega)(z - \omega^2)$

$$\Rightarrow z \neq 1, (1 + z + z^2) = (z - \omega)(z - \omega^2)$$

Property III: If $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ are roots of $z^n - 1 = 0$

$$\Rightarrow z^n - 1 = (z - 1)(z - \alpha)(z - \alpha^2) \cdots (z - \alpha^{n-1})$$

$$\frac{1(z^n - 1)}{z - 1} = (z - \alpha)(z - \alpha^2)(z - \alpha^3) \cdots (z - \alpha^{n-1}); z \neq 1$$

$$1 + z + z^2 + \cdots + z^{n-1} = (z - \alpha)(z - \alpha^2) \cdots (z - \alpha^{n-1}); z \neq 1$$

Properties of n^{th} Roots of Unity :

$$z = 1, \alpha, \alpha^2, \dots, \alpha^{n-1} \text{ where } \alpha = e^{i\left(\frac{2\pi}{n}\right)}$$

Property IV: If n is even, then ± 1 are the only two real roots of the equation $z^n - 1 = 0$.

Property V: If n is odd, then 1 is the only real root of the equation $z^n - 1 = 0$.

For $n = \text{odd}$, $z^3 = 1$,

Roots are 1, (real root) and ω, ω^2 (complex roots)

Property VI: Sum of the n^{th} roots of unity is zero.

Properties of n^{th} Roots of Unity :

$$z = 1, \alpha, \alpha^2, \dots, \alpha^{n-1} \text{ where } \alpha = e^{i\left(\frac{2\pi}{n}\right)}$$

Property VII: Product of the n^{th} roots of unity is $(-1)^{n-1}$





If α is an imaginary fifth root of unity, then find the value of

$$\log_2 \left| 1 + \alpha + \alpha^2 + \alpha^3 - \frac{1}{\alpha} \right|$$

Solution:

$$\log_2 \left| 1 + \alpha + \alpha^2 + \alpha^3 - \frac{1}{\alpha} \right|$$

$$\text{Here, } \alpha^5 = 1 \quad \therefore \alpha^{-1} = \alpha^4$$

$$\text{Also } 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$$

$$\log_2 \left| 1 + \alpha + \alpha^2 + \alpha^3 - \frac{1}{\alpha} \right| = \log_2 |1 + \alpha + \alpha^2 + \alpha^3 - \alpha^4|$$

$$= \log_2 |- \alpha^4 - \alpha^4|$$

$$= \log_2 |-2\alpha^4| \quad (|\alpha^4| = 1)$$

$$= \log_2 2 = 1$$



$$(1 + \omega - \omega^2)(1 + \omega^2 - \omega^4)(1 + \omega^4 - \omega^8) \cdots 2n \text{ factors} = ?$$

Solution:

$$\begin{aligned}& (1 + \omega - \omega^2)(1 + \omega^2 - \omega^4)(1 + \omega^4 - \omega^8) \cdots 2n \text{ factors} \\&= (-\omega^2 - \omega^2)(-\omega - \omega)(-\omega^2 - \omega^2)(-\omega - \omega)(-\omega^2 - \omega^2) \dots 2n \text{ brackets} \\&= (-2\omega^2)(-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2) \dots 2n \text{ brackets} \\&= (4\omega^3)(4\omega^3)(4\omega^3) \dots n \text{ brackets} \\&= 4^n \times 1 = 4^n\end{aligned}$$



$$\frac{a+bw+cw^2}{aw+bw^2+c} + \frac{aw+bw^2+c}{a+bw+cw^2} \text{ is equal to } \underline{\hspace{2cm}}$$

Solution:

$$\begin{aligned}& \frac{a+bw+cw^2}{aw+bw^2+c} + \frac{aw+bw^2+c}{a+bw+cw^2} \\&= \frac{a+bw+cw^2}{w(a+bw+cw^2)} + \frac{w(a+bw+cw^2)}{a+bw+cw^2} \\&= \omega^2 + \omega = -1\end{aligned}$$

If $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the 5^{th} roots of unity, then

$$i) \prod_{i=1}^4 (2 - \alpha_i) = \quad ii) \sum_{i=1}^4 \frac{1}{2 - \alpha_i}$$

Solution:

Given : $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the 5^{th} roots of unity.

$$\Rightarrow z^5 - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)(z - \alpha_4)$$

$$\Rightarrow (z - \alpha_1)(z - \alpha_2)(z - \alpha_3)(z - \alpha_4) = \frac{z^5 - 1}{z - 1} \text{ where } z = 2$$

$$i) (2 - \alpha_1)(2 - \alpha_2)(2 - \alpha_3)(2 - \alpha_4) = \frac{2^5 - 1}{2 - 1} = 31$$

$$ii) \sum_{i=1}^4 \frac{1}{2 - \alpha_i} = \left(\frac{1}{2 - \alpha_1}\right) + \left(\frac{1}{2 - \alpha_2}\right) + \left(\frac{1}{2 - \alpha_3}\right) + \left(\frac{1}{2 - \alpha_4}\right)$$

$$z^5 - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)(z - \alpha_4)$$

Take log on both sides.

$$\log(z^5 - 1) = \log(z - 1) + \log(z - \alpha_1) + \log(z - \alpha_2) + \log(z - \alpha_3) + \log(z - \alpha_4)$$

If $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the 5^{th} roots of unity, then

$$i) \prod_{i=1}^4 (2 - \alpha_i) = \quad ii) \sum_{i=1}^4 \frac{1}{2 - \alpha_i}$$

Solution:

$$\log(z^5 - 1) = \log(z - 1) + \log(z - \alpha_1) + \log(z - \alpha_2) + \log(z - \alpha_3) + \log(z - \alpha_4)$$

Differentiate with respect to z

$$\Rightarrow \frac{5z^4}{z^5 - 1} = \frac{1}{z-1} + \frac{1}{z-\alpha_1} + \frac{1}{z-\alpha_2} + \frac{1}{z-\alpha_3} + \frac{1}{z-\alpha_4}$$

Put $z = 2$

$$\Rightarrow \frac{5 \cdot 2^4}{2^5 - 1} = 1 + \left(\frac{1}{2-\alpha_1} + \frac{1}{2-\alpha_2} + \frac{1}{2-\alpha_3} + \frac{1}{2-\alpha_4} \right)$$

$$\Rightarrow \frac{40}{31} = \left(\frac{1}{2-\alpha_1} + \frac{1}{2-\alpha_2} + \frac{1}{2-\alpha_3} + \frac{1}{2-\alpha_4} \right)$$

$$\therefore \sum_{i=1}^4 \frac{1}{2 - \alpha_i} = \frac{49}{31}$$



Find the value of $\sum_{k=1}^6 \left(\sin \frac{2k\pi}{7} - i \cos \frac{2k\pi}{7} \right)$



Solution:

$$\text{Let } S = \sum_{k=1}^6 \left(\sin \frac{2k\pi}{7} - i \cos \frac{2k\pi}{7} \right) = \sum_{k=1}^6 \frac{1}{i} \left(i \sin \frac{2k\pi}{7} + i^2 \cos \frac{2k\pi}{7} \right)$$

$$= \frac{1}{i} \sum_{k=1}^6 \left(\cos \frac{2k\pi}{7} + i \sin \frac{2k\pi}{7} \right)$$

$$a = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7} = e^{i\left(\frac{2\pi}{7}\right)}; \quad a^2 = e^{i\left(\frac{4\pi}{7}\right)}; \quad a^3 = e^{i\left(\frac{6\pi}{7}\right)}$$

$$= \frac{1}{i} (a + a^2 + a^3 + a^4 + a^5 + a^6)$$

$$= \frac{1}{i} \times -1 = -\frac{1}{i} \times \frac{i}{i} = -\frac{i}{-1}$$

$$= i$$

Key Takeaways

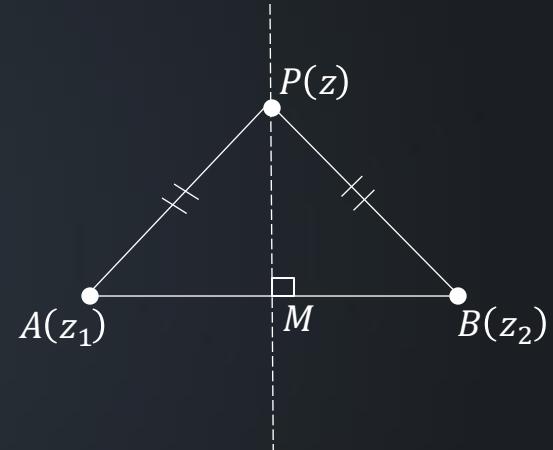
Locus Models Based on Distance (Modulus)

RESULT: $|z - z_1| = |z - z_2|$

NOTE:

$|z_1 - z_2|$ represents the distance between z_1 and z_2

$\Rightarrow z$ lies on the perpendicular bisector of the line segment joining z_1 and z_2



Key Takeaways

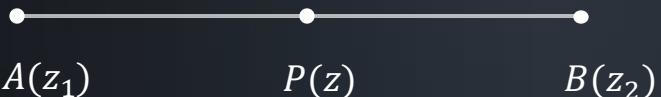
Locus Models Based on Distance (Modulus)

RESULT:

$$|z - z_1| + |z - z_2| = |z_1 - z_2|$$

z lies on the line segment joining z_1 & z_2

$$PA + PB = AB$$



Locus Models Based on Distance (Modulus)

RESULTS:

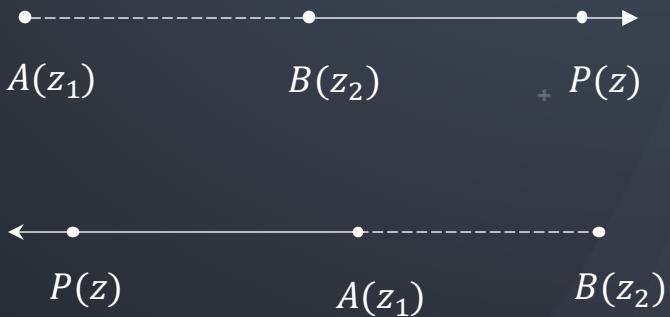
$$||z - z_1| - |z - z_2|| = |z_1 - z_2|$$

z lies on the ray emanating from the point B , if

$$|z - z_1| - |z - z_2| = |z_1 - z_2|$$

z lies on the ray emanating from the point A , if

$$|z - z_2| - |z - z_1| = |z_1 - z_2|$$





The equation $|z - i| = |z - 1|$, $i = \sqrt{-1}$, represents

JEE MAIN 2019

Method 1:

$$\text{Given, } |z - i| = |z - 1|$$

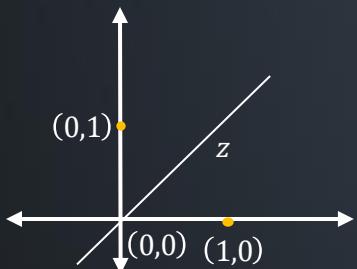
$$\text{Suppose } z = x + iy$$

$$\Rightarrow |x + iy - i| = |x + iy - 1|$$

$$\Rightarrow x^2 + (y - 1)^2 = (x - 1)^2 + (y)^2$$

$$\Rightarrow -2y + 1 = -2x + 1$$

$$\Rightarrow y = x$$



A

A circle of radius is 1 unit

B

A circle of radius is $\frac{1}{2}$ unit

C

The line through the origin with slope -1

D

The line through the origin with slope 1

Which is a line passing through the origin with slope 1



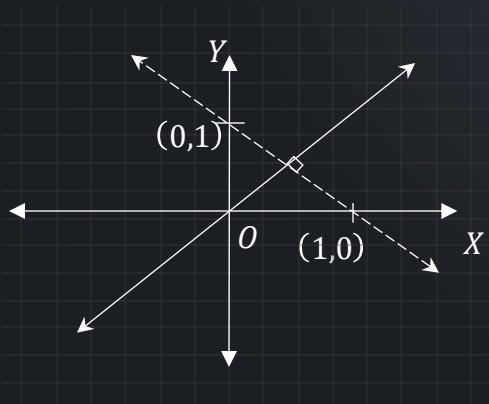
The equation $|z - i| = |z - 1|$, $i = \sqrt{-1}$, represents

Method 2:

$$\text{Given : } |z - i| = |z - 1|$$

$$\text{Let } z_1 = 0 + i \text{ & } z_2 = 1 + 0i$$

z lies on perpendicular bisector of line segment joining z_1 & z_2



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A

A circle of radius is 1 unit

B

A circle of radius is $\frac{1}{2}$ unit

C

The line through the origin with slope -1

D

The line through the origin with slope 1

Session 6

**Rotational Theorem of Complex
Numbers**

Key Takeaways

Equation of Ray:

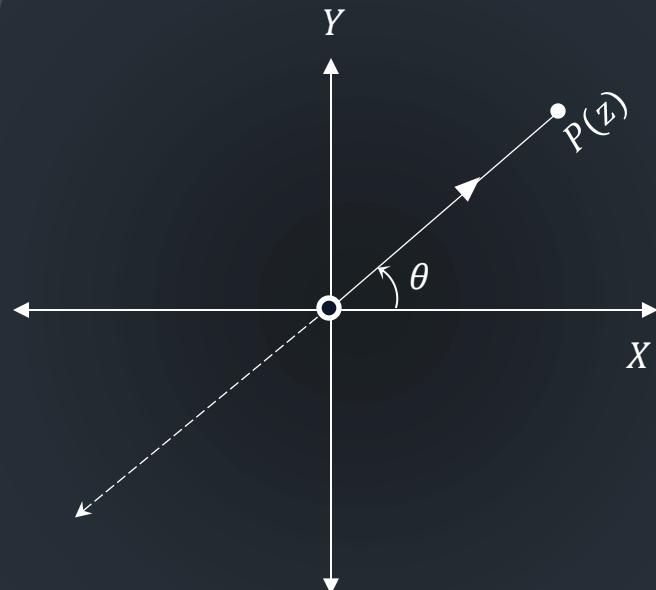
$\arg(z) = \theta$ represents a ray emanating from origin making an angle θ with the positive direction of real axis.

If $z = x + iy$

$$\tan \theta = \frac{y}{x}$$

$$\Rightarrow y = (\tan \theta)x$$

This ray is part of the line $y = (\tan \theta)x$



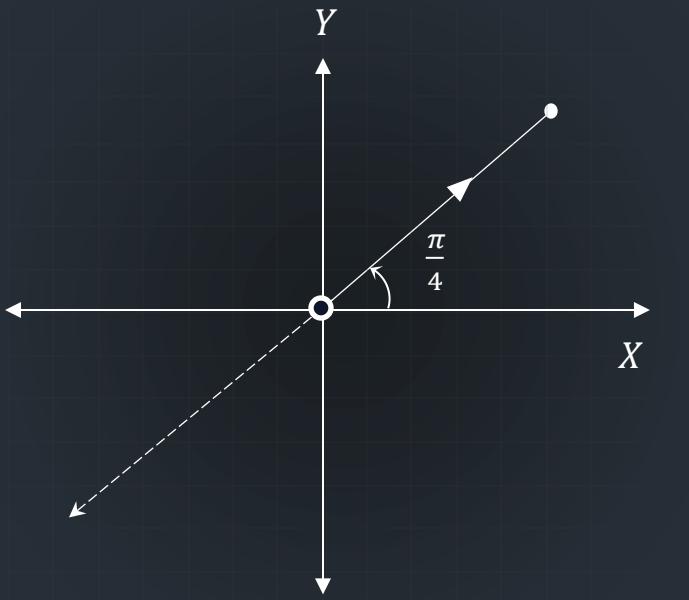
Key Takeaways

Equation of Ray:

$$\arg(z) = \theta$$

$$y = (\tan \theta)x$$

Example : $\arg(z) = \frac{\pi}{4}$



Equation of Ray:

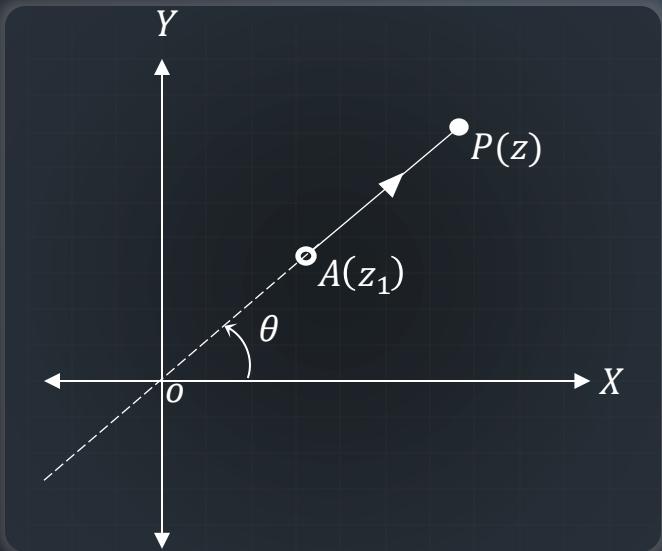
$\arg(z - z_1) = \theta$ represents a ray emanating from z_1 making an angle θ with the positive direction of real axis.

If $z = x + iy$ & $z_1 = x_1 + iy_1$

$$\tan \theta = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow y - y_1 = \tan \theta (x - x_1)$$

This ray is part of the line $y - y_1 = \tan \theta (x - x_1)$



Let $z = x + iy$ be a non-zero complex number such that $z^2 = i|z|^2$, where $i = \sqrt{-1}$, then z lies on

[JEE (MAIN) 2020]

A

The line $y = x$

B

The real axis

C

The imaginary axis

D

The line $y = -x$

Solution:

$$\text{Given : } z = x + iy$$

$$z^2 = i|z|^2$$

$$\Rightarrow (x + iy)^2 = i(x^2 + y^2) \quad |z| = \sqrt{x^2 + y^2}$$

$$\Rightarrow (x^2 - y^2) + i(2xy) = i(x^2 + y^2)$$

Let $z = x + iy$ be a non-zero complex number such that $z^2 = i|z|^2$, where $i = \sqrt{-1}$, then z lies on

Solution:

[JEE (MAIN) 2020]

Comparing the real and imaginary parts

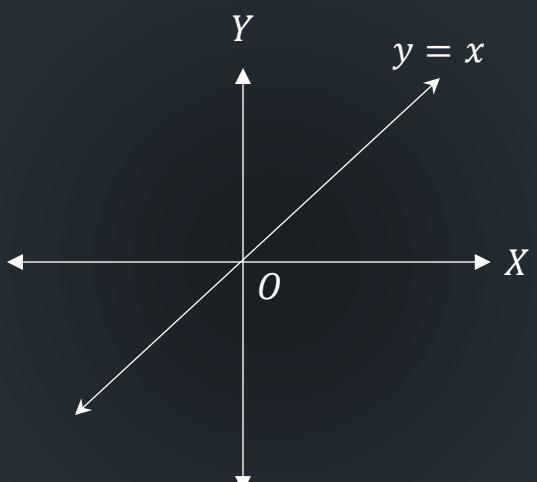
$$x^2 - y^2 = 0 \text{ and } 2xy = x^2 + y^2$$

$$\Rightarrow (x+y)(x-y) = 0 \text{ and } (x-y)^2 = 0$$

$$\Rightarrow (x+y)(x-y) = 0 \text{ and } (x-y)^2 = 0$$

$$\Rightarrow x - y = 0$$

Locus of z is the line $y = x$





If $\operatorname{Re} \left(\frac{z-1}{2z+i} \right) = 1$, where $z = x + iy$, then the point (x, y) lies on

[JEE (MAIN) 2019]

Solution:

$$\text{Given : } z = x + iy$$

$$\operatorname{Re} \left(\frac{z-1}{2z+i} \right) = 1$$

$$\Rightarrow \operatorname{Re} \left(\frac{(x-1) + iy}{2x + (2y+1)i} \times \frac{2x - (2y+1)i}{2x - (2y+1)i} \right) = 1$$

$$\Rightarrow \frac{(x-1)2x + y(2y+1)}{(2x)^2 + (2y+1)^2} = 1$$

$$\Rightarrow 2x^2 + 2y^2 + 2x + 3y + 1 = 0$$

$$\Rightarrow x^2 + y^2 + x + \frac{3}{2}y + \frac{1}{2} = 0$$

A

a circle whose center is at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$

B

a straight line whose slope is $-\frac{2}{3}$

C

a straight line whose slope is $\frac{3}{2}$

D

a circle whose diameter is $\frac{\sqrt{5}}{2}$



If $\operatorname{Re} \left(\frac{z-1}{2z+i} \right) = 1$, where $z = x + iy$, then the point (x, y) lies on

Solution:

$$\Rightarrow x^2 + y^2 + x + \frac{3}{2}y + \frac{1}{2} = 0 \text{ (circle)}$$

$$\text{Center} \equiv (-g, -f) \equiv \left(-\frac{1}{2}, -\frac{3}{4} \right)$$

$$\text{Radius} \equiv \sqrt{g^2 + f^2 + c} \equiv \sqrt{\frac{1}{4} + \frac{9}{16} - \frac{1}{2}} \equiv \sqrt{\frac{5}{4}}$$

$$\text{Diameter} \equiv 2\sqrt{\frac{5}{4}} \equiv \frac{\sqrt{5}}{2}$$

[JEE (MAIN) 2019]

A

a circle whose center is at $\left(-\frac{1}{2}, -\frac{3}{2} \right)$

B

a straight line whose slope is $-\frac{2}{3}$

C

a straight line whose slope is $\frac{3}{2}$

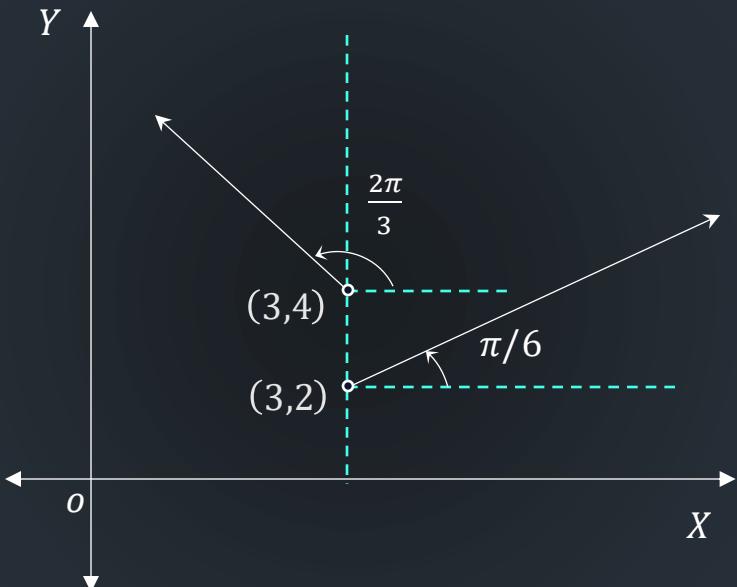
D

a circle whose diameter is $\frac{\sqrt{5}}{2}$



Find the points of intersection of the two rays $\arg(z - 3 - 2i) = \frac{\pi}{6}$ and $\arg(z - 3 - 4i) = \frac{2\pi}{3}$.

Solution:



$$i) \arg(z - 3 - 2i) = \frac{\pi}{6}$$

$$ii) \arg(z - 3 - 4i) = \frac{2\pi}{3}$$

There is no point of intersection for the given rays.

Key Takeaways

Complex Number as a rotating vector in the Argand Plane.

$$\text{Let } z = |z|(\cos \theta + i \sin \theta) = |z|e^{i\theta}$$

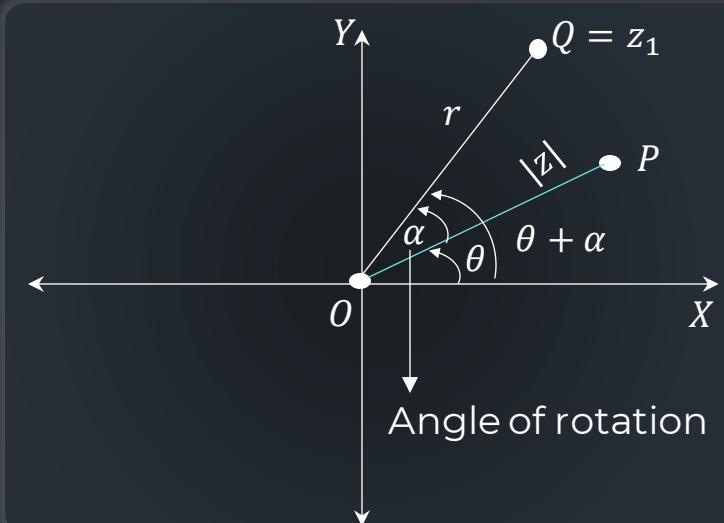
$$OP = |z| \quad \angle POX = \theta$$

Consider the complex number $z_1 = ze^{i\alpha}$

$$\Rightarrow z_1 = |z|e^{i\theta} \cdot e^{i\alpha} \Rightarrow z_1 = |z|e^{i(\theta+\alpha)}$$

$\Rightarrow z_1$ represents a point in Argand Plane where
 $\angle QOX = \theta + \alpha$

\Rightarrow Multiplication of z with $e^{i\alpha}$ rotates the vector \overrightarrow{OP}
 through angle α in anti-clockwise sense.



Key Takeaways

Complex Number as a rotating vector in the Argand Plane:

Consider the complex number $z = re^{i\theta}$ then,

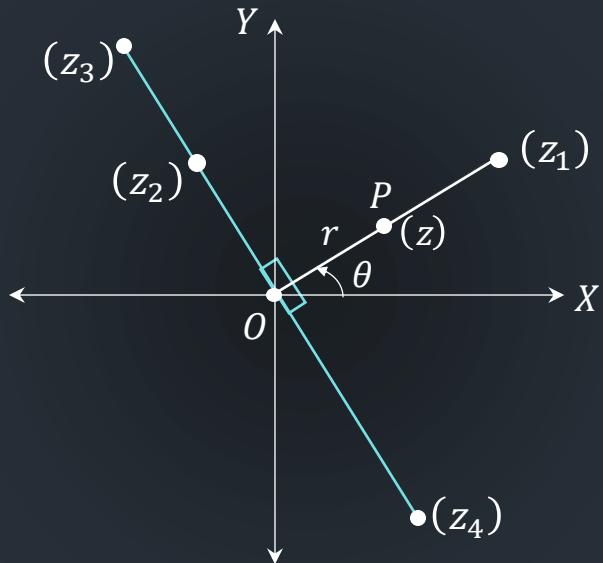
$$1) 2z = z_1 = 2 \cdot re^{i\theta}$$

where $2r = |2z| = 2|z|$ and $\theta = \arg \text{ of } 2z = \arg \text{ of } z$

$$2) z_2 = iz = e^{\frac{i\pi}{2}} \cdot z$$

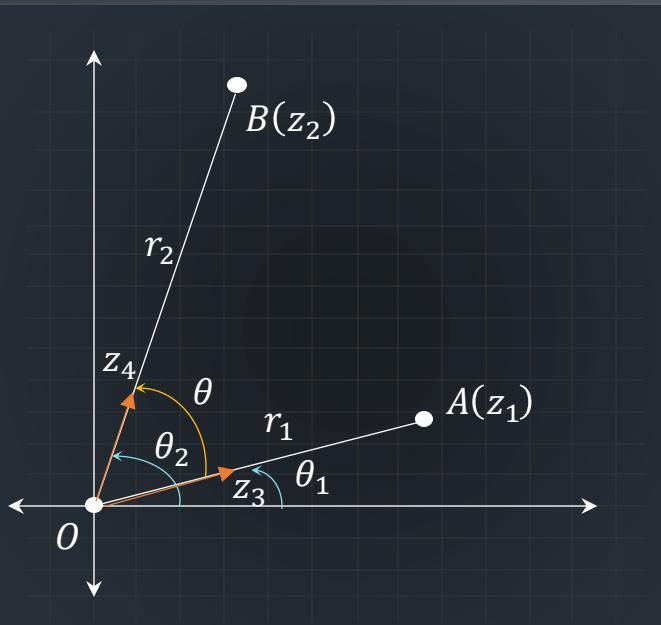
$$3) z_3 = 2iz$$

$$4) z_4 = -2iz$$



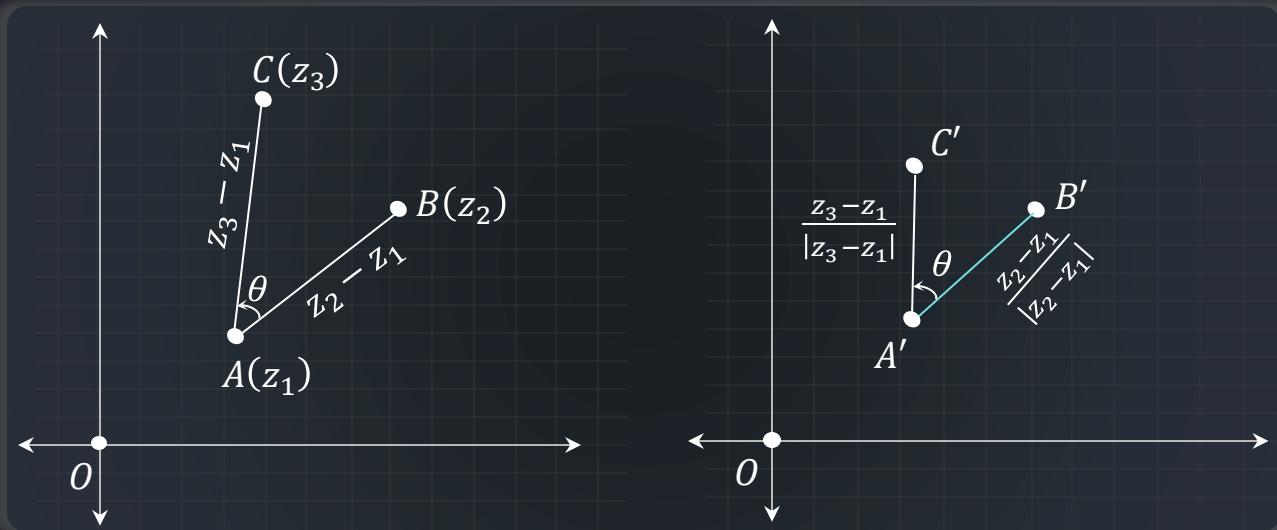
Rotation Theorem (Coni's Formula):

$$\frac{z_2}{z_1} = \frac{|z_2|}{|z_1|} e^{i\theta}$$



Key Takeaways

Rotation Theorem:



Let $A(z_1), B(z_2), C(z_3)$ be three points in Argand plane.

$$\overrightarrow{AB} = z_2 - z_1 \text{ and } \overrightarrow{AC} = z_3 - z_1 \Rightarrow \overrightarrow{AC'} = \overrightarrow{AB'} \cdot e^{i\theta}$$

$$\Rightarrow \frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_2 - z_1}{|z_2 - z_1|} \cdot e^{i\theta}$$

If the area of the triangle whose sides are represented by z , iz and $z + iz$ is 200 sq. units, then $|z| = \underline{\hspace{2cm}}$.

Solution:

$$\text{Let } z = |z|e^{i\theta}$$

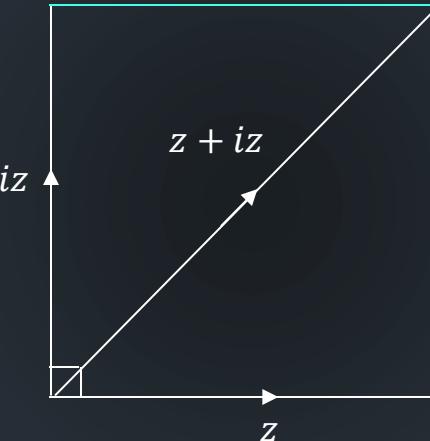
$$iz = |z|e^{i\theta} \cdot e^{i\frac{\pi}{2}}$$

\therefore Angle between z and iz is $\frac{\pi}{2}$.

So, triangle is right angled.

$$\therefore \text{Area} = \frac{1}{2}|z||iz| = 200$$

$$\Rightarrow |z| = 20$$



Key Takeaways

Rotation Theorem:

Let $A(z_1), B(z_2), C(z_3), D(z_4)$ be four points in Argand plane.

$$AB = z_2 - z_1$$

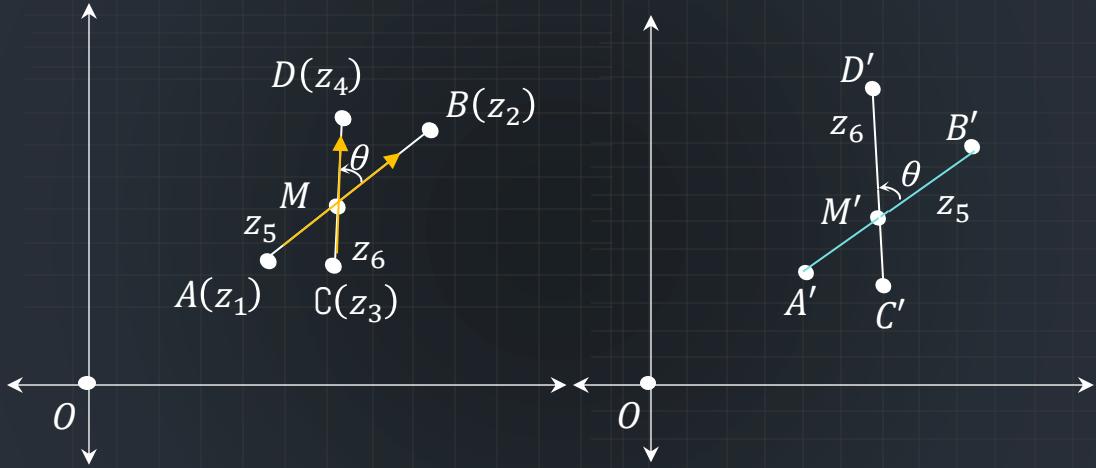
$$CD = z_4 - z_3$$

$$z_6 = z_5 \cdot e^{i\theta}$$

$$\frac{z_4 - z_3}{|z_4 - z_3|} = \left(\frac{z_2 - z_1}{|z_2 - z_1|} \right) \cdot e^{i\theta}$$

$$\Rightarrow \frac{z_4 - z_3}{z_2 - z_1} = \frac{|z_4 - z_3|}{|z_2 - z_1|} \cdot e^{i\theta}$$

$$\text{Here, } z_5 = \frac{z_2 - z_1}{|z_2 - z_1|} \text{ and } z_6 = \frac{z_4 - z_3}{|z_4 - z_3|}$$





If z_1, z_2, z_3 are the vertices of an equilateral triangle and z_0 is its circumcentre, then prove that:

- i) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$
- ii) $3z_0^2 = z_1^2 + z_2^2 + z_3^2$

Let : $A = z_1, B = z_2, C = z_3$

ΔABC is equilateral triangle

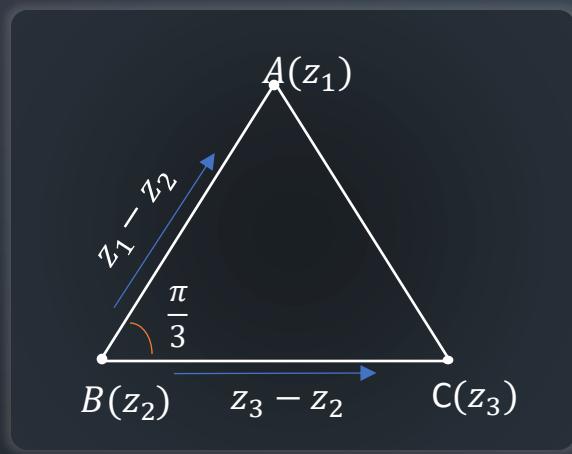
$$AB = BC = CA$$

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{|z_1 - z_2|}{|z_3 - z_2|} e^{\frac{i\pi}{3}} \quad \dots (i)$$

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{\frac{i\pi}{3}} \quad \dots (ii)$$

$$\because |z_1 - z_2| = |z_3 - z_2| = |z_3 - z_1|$$

$$\Rightarrow \frac{z_1 - z_2}{z_3 - z_2} = \frac{z_3 - z_1}{z_2 - z_1} \Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$





If z_1, z_2, z_3 are the vertices of an equilateral triangle and z_0 is its circumcentre, then prove that:

- i) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$
- ii) $3z_0^2 = z_1^2 + z_2^2 + z_3^2$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

Circumcentre, $z_0 = \frac{z_1 + z_2 + z_3}{3}$

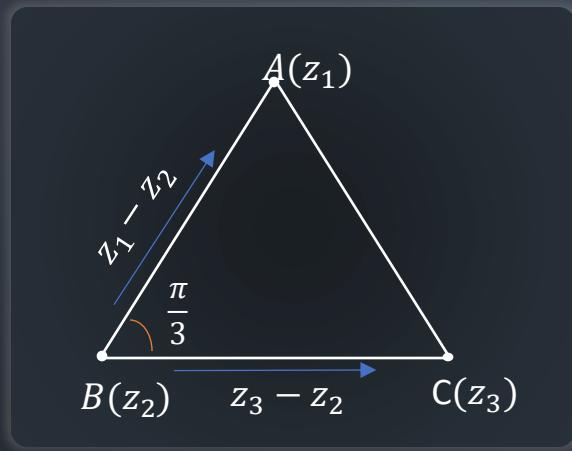
$$\Rightarrow 3z_0 = z_1 + z_2 + z_3$$

$$\Rightarrow 9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1 z_2 + z_2 z_3 + z_3 z_1)$$

$$= z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2)$$

$$= 3(z_1^2 + z_2^2 + z_3^2)$$

$$\Rightarrow 3z_0^2 = z_1^2 + z_2^2 + z_3^2$$



The centre of a regular polygon of n sides is located at the point $z = 0$ and one of its vertex z_1 is known. If z_2 be the vertex adjacent to z_1 , then z_2 is equal to

Let A be the vertex with affix z_1

$$z_1 \cdot e^{i\frac{2\pi}{n}} = z_2$$

or

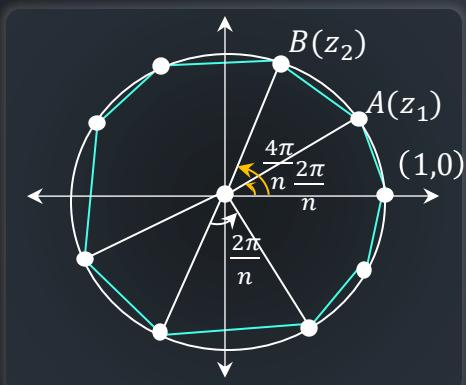
$$z_1 \cdot e^{-\frac{2\pi}{n}} = z_2$$

$$\Rightarrow z_2 = z_1 \left(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \right)$$

or

$$z_2 = z_1 \left(\cos \left(\frac{2\pi}{n} \right) - i \sin \left(\frac{2\pi}{n} \right) \right)$$

$$\Rightarrow z_2 = z_1 \left(\cos \frac{2\pi}{n} \pm i \sin \frac{2\pi}{n} \right)$$



A

$$z_1 \left(\cos \frac{2\pi}{n} \pm i \sin \frac{2\pi}{n} \right)$$

B

$$z_1 \left(\cos \frac{\pi}{n} \pm i \sin \frac{\pi}{n} \right)$$

C

$$z_1 \left(\cos \frac{\pi}{2n} \pm i \sin \frac{\pi}{2n} \right)$$

D

None of these

Session 7

Geometrical Applications of Complex Numbers

Key Takeaways

Circle

$|z| = r$ ($r \in \mathbb{R}^+$) represents a circle whose centre is the origin and radius 'r'

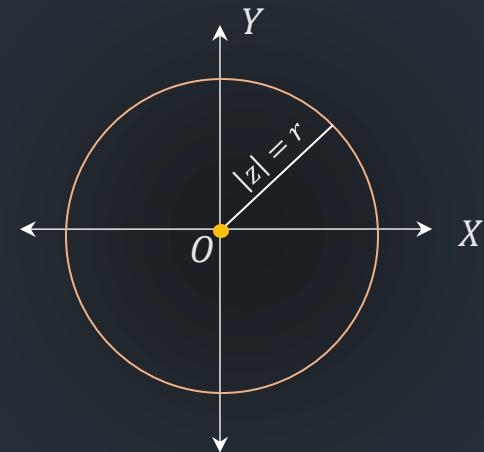
$$z = x + iy$$

$$|z| = r \dots (1)$$

$$\Rightarrow |x + iy| = r$$

$$\Rightarrow \sqrt{x^2 + y^2} = r \Rightarrow x^2 + y^2 = r^2$$

$$\Rightarrow (x - 0)^2 + (y - 0)^2 = r^2 \dots (2) \rightarrow \text{Circle (centre } (0,0) \text{ and radius } (r) \text{)}$$



Key Takeaways

Circle

$|z - z_1| = r$ ($r \in \mathbb{R}^+$) represents a circle whose centre is the z_1 and radius ' r '

$$z = x + iy; z_1 = x_1 + iy_1$$

$$|z - z_1|^2 = r^2$$

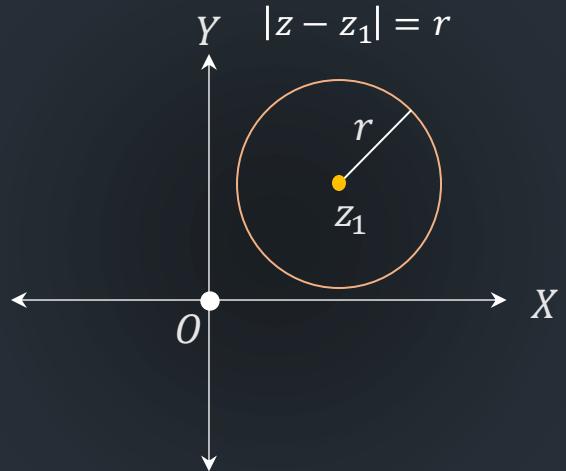
$$(z - z_1)(\bar{z} - \bar{z}_1) = r^2 \quad (\because z\bar{z} = |z|^2)$$

$$\Rightarrow z\bar{z} - z\bar{z}_1 - \bar{z}z_1 + z_1\bar{z}_1 - r^2 = 0$$

$$\Rightarrow |z|^2 - z\bar{z}_1 - \bar{z}z_1 + |z_1|^2 - r^2 = 0$$

$$\Rightarrow |z|^2 - z\bar{z}_1 - \bar{z}z_1 + |z_1|^2 - r^2 = 0$$

$$\Rightarrow z\bar{z} + \bar{a}z + az + b = 0 \longrightarrow \text{General form}$$





Circle

$$|z - z_1| = r$$

$$\Rightarrow |z|^2 - z\bar{z}_1 - \bar{z}z_1 + |z_1|^2 - r^2 = 0$$

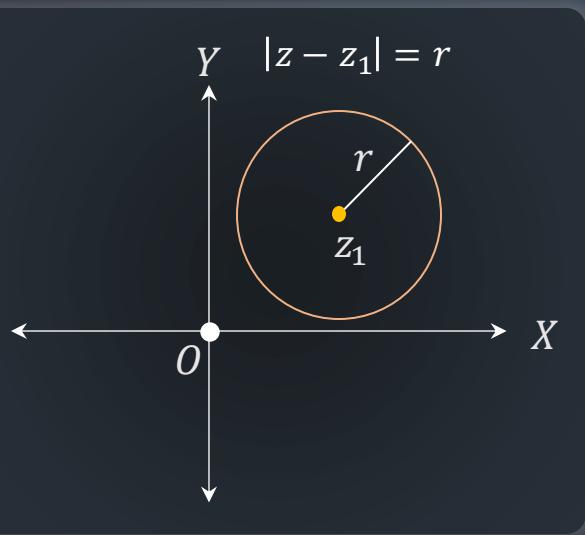
$\Rightarrow z\bar{z} + \bar{a}z + az + b = 0$ \rightarrow General form

whose centre $z_1 = -a$

$$r = \sqrt{|z_1|^2 - b} = \sqrt{|a|^2 - b}$$

Note :

- For real circle : $|a|^2 - b \geq 0$.
- a is allowed to be a complex number,
but b is always a Real number



Let z_1 and z_2 be two complex numbers satisfying $|z_1| = 9$ and $|z_2 - 3 - 4i| = 4$. Then, the minimum value of $|z_1 - z_2|$ is _____

Solution: $z_1 = |z| = 9$

$$z_2 = |z - (3 + 4i)| = 4$$

$$\text{i.e., } |z - z_1| = 4$$

$$C_1 \equiv (0,0), C_2 \equiv (3,4)$$

$$C_1C_2 = \sqrt{3^2 + 4^2}$$

$$\Rightarrow r_1 - r_2 = 5$$

\therefore Two circles touch each other internally

\therefore minimum value of $|z_1 - z_2|$ is 0.

JEE Main (2019)

A

1

B

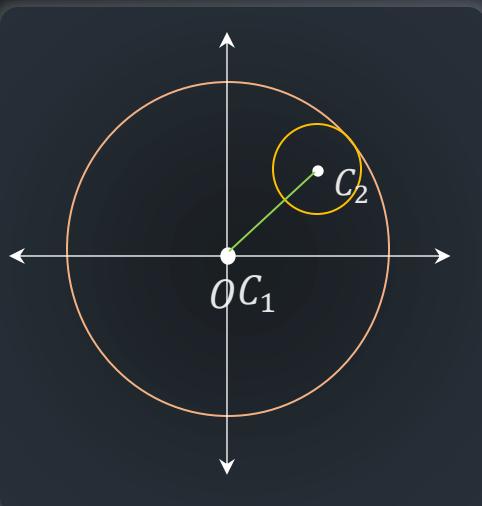
2

C

$\sqrt{2}$

D

0





What is the centre and radius for $z\bar{z} - (3 - 4i)z - (3 + 4i)\bar{z} + 9 = 0$

Solution: As we know that, for

$$\Rightarrow |z|^2 - z\bar{z}_1 - \bar{z}z_1 + |z_1|^2 - r^2 = 0$$

$$\text{centre } z_1 = -a$$

$$r = \sqrt{|z_1|^2 - b} = \sqrt{|a|^2 - b}$$

$$\text{Comparing with } z\bar{z} - (3 - 4i)z - (3 + 4i)\bar{z} + 9 = 0$$

$$\Rightarrow b = 9$$

$$\Rightarrow \text{Centre: } -a = -(-3 - 4i) = 3 + 4i$$

$$\Rightarrow \text{Radius: } \sqrt{|a|^2 - b} = \sqrt{(9 + 16) - 9}$$

$$= \sqrt{16} = 4$$

A

(3 + 4i), 5

B

(3 + 4i), 4

C

(3 - 4i), 4

D

(3 - 4i), 5

Key Takeaways

Equation of an Arc:

Let z_1, z_2 be any two complex numbers.

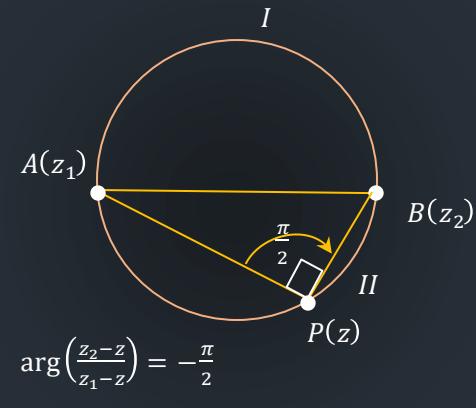
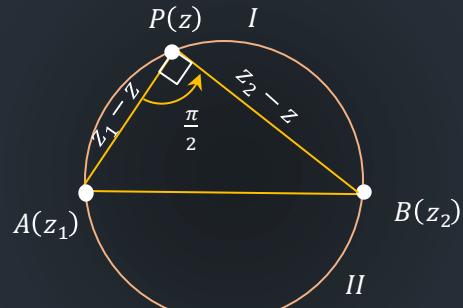
$$\therefore \frac{z_1 - z}{|z_1 - z|} \cdot e^{\frac{i\pi}{2}} = \frac{z_2 - z}{|z_2 - z|}$$

$$\Rightarrow \frac{|z_2 - z|}{|z_1 - z|} \cdot e^{\frac{i\pi}{2}} = \frac{z_2 - z}{z_1 - z}$$

$$\Rightarrow z' = |z'| e^{\frac{i\pi}{2}}; \quad \frac{\pi}{2} \in (-\pi, \pi]$$

$$\Rightarrow \arg(z') = \frac{\pi}{2}$$

$$\Rightarrow \arg\left(\frac{z_2 - z}{z_1 - z}\right) = \frac{\pi}{2}$$



Key Takeaways

Equation of an Arc:

Let z_1, z_2 be any two complex numbers.

$$\arg(z_1 - z) = \theta_1$$

$$\arg(z_2 - z) = \theta_2$$

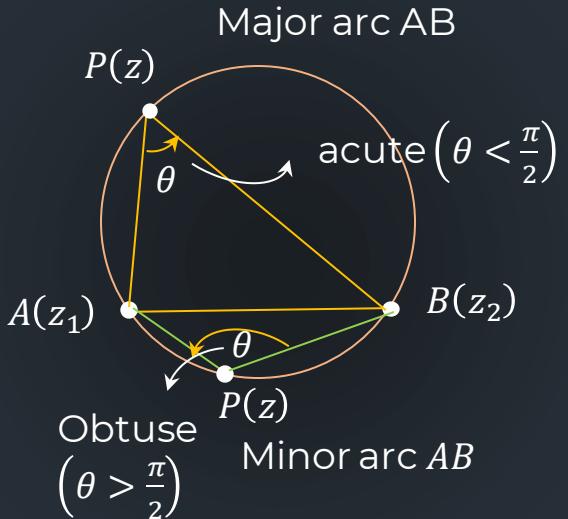
i.e., $\theta_1 + \theta = \theta_2$

$$\Rightarrow \theta_2 - \theta_1 = \theta; \quad \theta \in (-\pi, \pi]$$

$$\Rightarrow \arg(z_2 - z) - \arg(z_1 - z) = \theta$$

$$\Rightarrow \arg\left(\frac{z_2 - z}{z_1 - z}\right) = \theta$$

acute → Major arc
obtuse → Minor arc



Note:

$$\text{For } \arg\left(\frac{z_2-z}{z_1-z}\right) = \theta$$

- Case I : $\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ (Semicircle)

- Case II : $\theta < \frac{\pi}{2}$ (Major Arc)

- Case III : $\theta > \frac{\pi}{2}$ (Minor Arc)



Find the center of the arc represented by $\arg\left[\frac{z - 3i}{z - 2i + 4}\right] = \frac{\pi}{4}$

Solution: $C(z_0)$ = centre of the arc given by $\arg\left[\frac{z - 3i}{z - 2i + 4}\right] = \frac{\pi}{4}$

$$\Rightarrow \arg\left[\frac{z - z_2}{z - z_1}\right] = \frac{\pi}{4} \Rightarrow \arg\left[\frac{z - (0 + 3i)}{z - (-4 + 2i)}\right] = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{4} \rightarrow \text{Acute}$$

$$z_1 = -4 + 2i \equiv A(-4, 2)$$

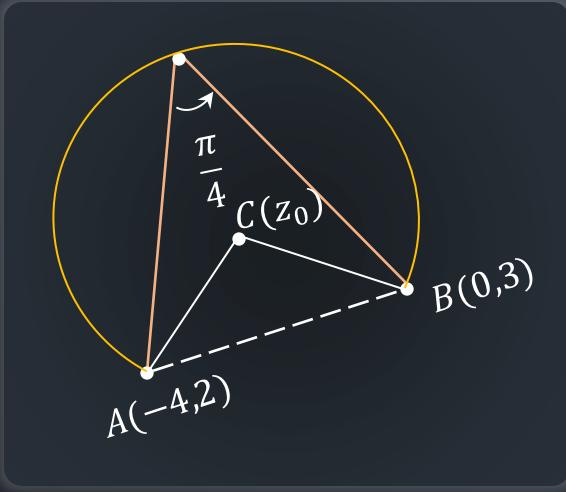
$$z_2 = 3i \equiv B(0, 3)$$

$$\overrightarrow{CA} = -4 + 2i - z_0$$

$$\overrightarrow{CB} = 3i - z_0$$

$$l.e., \frac{-4+2i-z_0}{|\overrightarrow{CA}|} \cdot e^{\frac{i\pi}{2}} = \frac{3i-z_0}{|\overrightarrow{CB}|}$$

$$\Rightarrow \begin{pmatrix} -(3i-z_0) \\ -(4+2i-z_0) \end{pmatrix} = i \Rightarrow \frac{z_0-3i}{z_0+4-2i} = i$$





Find the center of the arc represented by $\arg\left[\frac{z - 3i}{z - 2i + 4}\right] = \frac{\pi}{4}$

Solution: $\frac{z_0 - 3i}{z_0 + 4 - 2i} = i$

$$\Rightarrow z_0 - 3i = z_0 i + 4i + 2$$

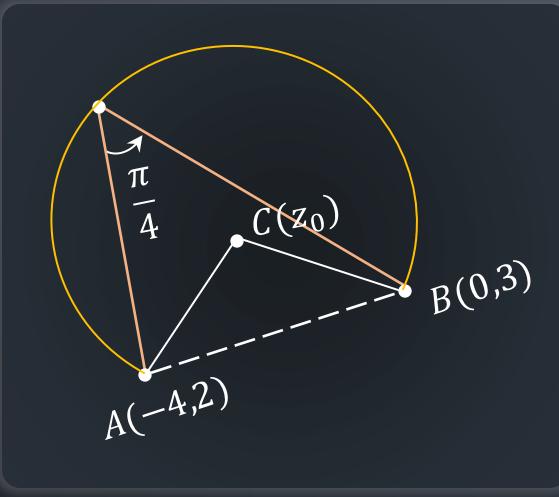
$$\Rightarrow z_0(1 - i) - 7i = 2$$

$$\Rightarrow z_0(1 - i) = 2 + 7i$$

$$\Rightarrow z_0 = \frac{2+7i}{1-i} \times \frac{1+i}{1+i} = \frac{2+2i+7i-7}{1+1}$$

$$= \frac{9i-5}{2} = \frac{1}{2}(-5 + 9i)$$

$$z_0 = \frac{1}{2}(-5 + 9i)$$





Key Takeaways

Straight Line:

Let $ax + by + c = 0$ is a straight line with slope $= -\frac{a}{b}$

Let $z = x + iy$, $\bar{z} = x - iy$

$$\Rightarrow \frac{z+\bar{z}}{2} = x \text{ and } \frac{z-\bar{z}}{2i} = y$$

$$L : a\left(\frac{z+\bar{z}}{2}\right) + b\left(\frac{z-\bar{z}}{2i}\right) + c = 0$$

$$az + a\bar{z} - ibz + ib\bar{z} + 2c = 0$$

$$L : (a - ib)z + (a + ib)\bar{z} + 2c = 0$$

$$\bar{\alpha}z + \alpha\bar{z} + k = 0 \quad \longleftrightarrow \quad ax + by + c = 0$$

where $\alpha = a + ib$, $k = 2c$



Key Takeaways

Straight Line:

$$\bar{\alpha}z + \alpha\bar{z} + k = 0 \quad \longleftrightarrow \quad ax + by + c = 0$$

where $\alpha = a + ib, k = 2c$

Proof: Let $\bar{\alpha} = a - ib$ where $\frac{\alpha + \bar{\alpha}}{2} = a$ and $\frac{\alpha - \bar{\alpha}}{2i} = b$

$$-\frac{a}{b} = -\left(\frac{\alpha + \bar{\alpha}}{2}\right) \times \frac{2i}{(\alpha - \bar{\alpha})} = -\frac{i(\alpha + \bar{\alpha})}{(\alpha - \bar{\alpha})}$$

Note:

- Slope of the line $\bar{\alpha}z + \alpha\bar{z} + k = 0$ is

$$-\frac{a}{b} = -\frac{Re(\alpha)}{Im(\alpha)} = -\frac{(\alpha + \bar{\alpha})i}{(\alpha - \bar{\alpha})}$$

Note :

- Distance of a point $z_1(x_1 + iy_1)$ from the straight line $\bar{a}z + \alpha\bar{z} + k = 0$ is

$$= \frac{|\bar{a}z_1 + \alpha\bar{z}_1 + k|}{2|\alpha|}$$

- Equation of a line parallel to $\bar{a}z + \alpha\bar{z} + k = 0$ is

$$\bar{a}z + \alpha\bar{z} + \lambda = 0 \quad (\lambda \in \mathbb{R})$$

- Equation of a line perpendicular to $\bar{a}z + \alpha\bar{z} + k = 0$ is

$$\bar{a}z - \alpha\bar{z} + i\lambda = 0 \quad (\lambda \in \mathbb{R})$$

All the possible points in the set $S = \left\{ \frac{\alpha+i}{\alpha-i} : \alpha \in \mathbb{R}, i = \sqrt{-1} \right\}$ lies on

Solution:

$$\text{Let, } \frac{\alpha+i}{\alpha-i} = z$$

Taking the modulus on both side

$$\left| \frac{\alpha+i}{\alpha-i} \right| = |z|$$

As α is real,

$$\therefore \frac{\sqrt{\alpha^2+1^2}}{\sqrt{\alpha^2+(-1)^2}} = |z|$$

$$\Rightarrow |z| = 1$$

Hence the points in S lies on the circle with radius one unit

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A

A circle whose radius is $\sqrt{2}$ unit

B

A circle whose radius is 1 unit

C

A Straight line whose slope is 1

D

A Straight line whose slope is -1

A complex number z is said to be unimodular, if $|z| = 1$, suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1 \bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a:

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Solution:

$$\left| \frac{z_1 - 2z_2}{2 - z_1 \bar{z}_2} \right| = 1 \text{ on squaring}$$

$$\frac{z_1 - 2z_2}{2 - z_1 \bar{z}_2} \cdot \frac{(\bar{z}_1 - 2\bar{z}_2)}{2 - \bar{z}_1 z_2} = 1$$

$$\begin{aligned} & \Rightarrow |z_1|^2 - 2\bar{z}_1 z_2 - 2z_1 \bar{z}_2 + 4|z_2|^2 \\ & = 4 - 2\bar{z}_1 z_2 - 2z_1 \bar{z}_2 + |z_1|^2 |z_2|^2 \\ & \Rightarrow |z_1|^2(1 - |z_2|^2) - 4(1 - |z_2|^2) = 0 \\ & \Rightarrow |z_1| = 2 \quad (\because |z_2| \neq 1) \\ & \Rightarrow |z_1| \text{ is a circle of radius } 2. \end{aligned}$$

A

Straight line parallel to x -axis

B

A circle of radius 2 unit

C

Straight line parallel to y -axis

D

A circle of radius $\sqrt{2}$ unit

Key Takeaways

Equation of Ellipse :

Let z_1, z_2 be any two complex numbers.

For locus of z to be an ellipse

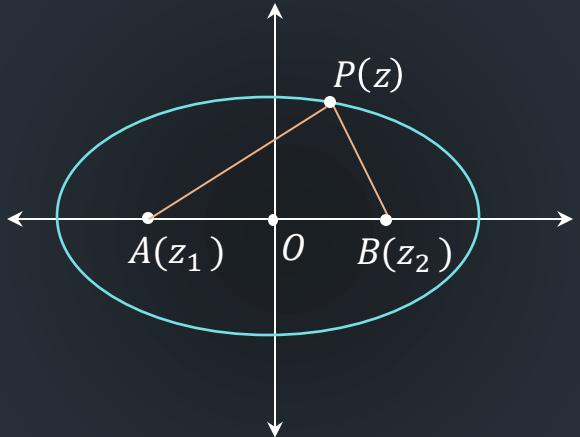
- $|PA| + |PB| = k > |AB|$

$$|z_1 - z| + |z_2 - z| = k > |AB| = 2a$$

- If $|z - z_1| + |z - z_2| = k > |z_1 - z_2|$ then
locus of z is an ellipse whose foci are z_1 and z_2 .

$$|PA + PB| = 2a$$

$$e = \frac{|z_1 - z_2|}{k}$$



Key Takeaways

Equation of Hyperbola :

Let z_1, z_2 be any two complex numbers.

For locus of z to be a hyperbola

- $|PA| - |PB| = k < |AB|$

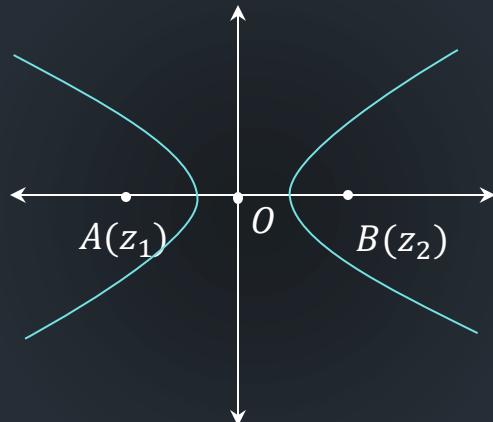
$$|z_1 - z| + |z_2 - z| = k < |AB| = 2a$$

- If $\|z - z_1\| - \|z - z_2\| = k < |z_1 - z_2|$ then

locus of z is a hyperbola whose foci are z_1 and z_2 .

$$|PA - PB| = 2a$$

$$e = \frac{|z_1 - z_2|}{k}$$





If z is a complex number satisfying the equation $|z + i| + |z - i| = 8$, then the maximum value of $|z|$ is_____.

Solution: Given : $|z + i| + |z - i| = 8$

$$\Rightarrow |z - (0 - i)| + |z - (0 + i)| = 8$$

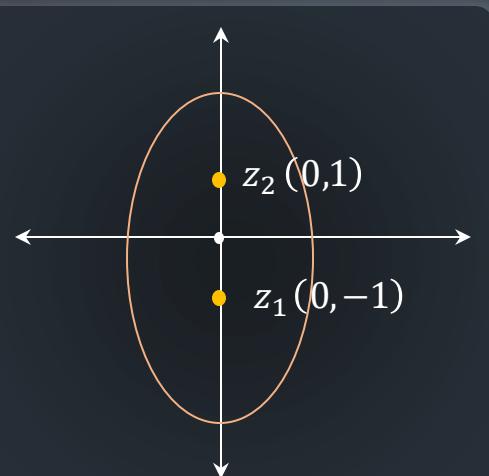
Comparing with $|z - z_1| + |z - z_2| = k$,we have

$$z_1 = -i, \quad z_2 = i, \quad k = 8$$

$$|z_1 - z_2| = |-2i| = 2$$

$$\text{Clearly } k > |z_1 - z_2|$$

Therefore, locus of z is an ellipse with foci z_1 and z_2 .





If z is a complex number satisfying the equation

$|z + 1| + |z + 3| \leq 8$, then find range of $|z - 4|$ is_____.

Solution: Given : $|z - 1| + |z + 3| \leq 8$

$$\Rightarrow |z - (1 + 0, i)| + |z - (-3 + 0, i)| \leq 8$$

$\Rightarrow z$ lies inside or on the ellipse
whose foci are $(1, 0)$ and $(-3, 0)$

Length of major axis $2a = 8 \Rightarrow a = 4$

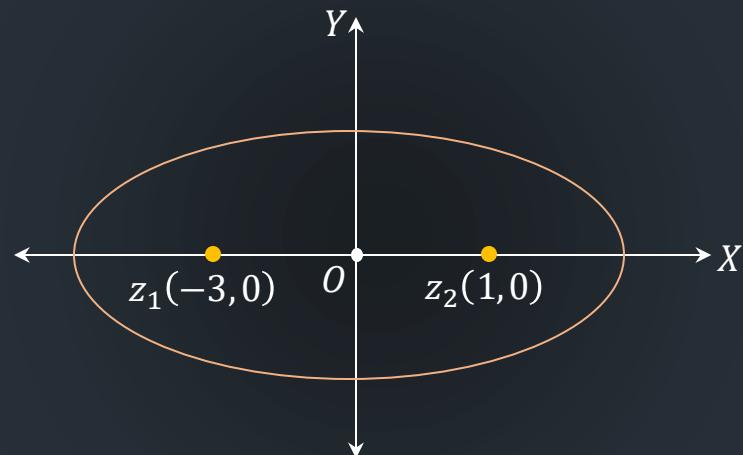
\therefore Vertices $\equiv (-5, 0)$ and $(3, 0)$

$|z - 4|$ represents the distance of $P(4, 0)$
from the ellipse

$\therefore |z - 4|_{min} = PA = 1$ and

$\therefore |z - 4|_{max} = PA' = 9$

Hence, range of $|z - 4|$ is $[1, 9]$





THANK
YOU