Welcome to

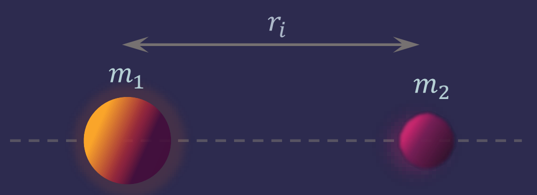
Aakash + BYJU'S LIVE

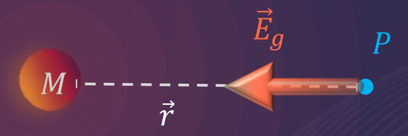
Gravitation

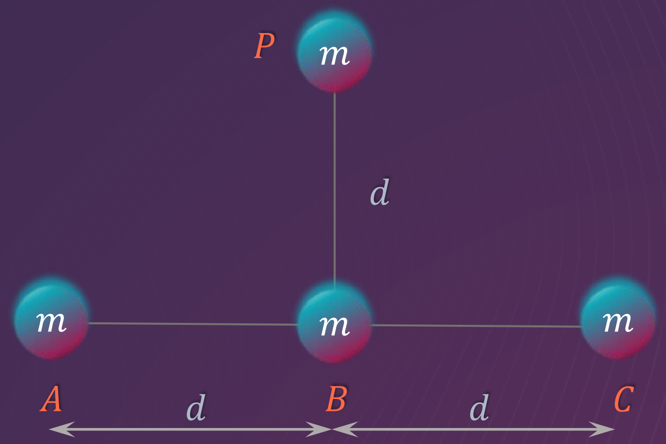
$R d\theta$

$R \sin\theta$

$\vec{E}_g = \dfrac{\vec{F}}{m}$





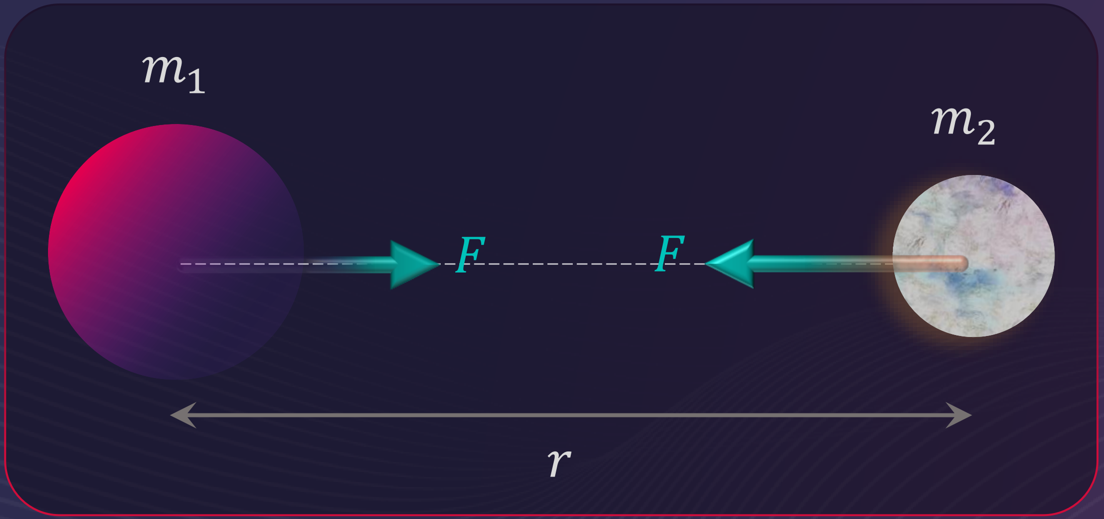


$\vec{E}_g(x) = -\dfrac{GMx}{(x^2+R^2)^{3/2}}\hat{x}$

Newton's Law of Gravitation

According to Newton's law of gravitation, "Every particle in the universe attracts every other particle with a force whose magnitude is:

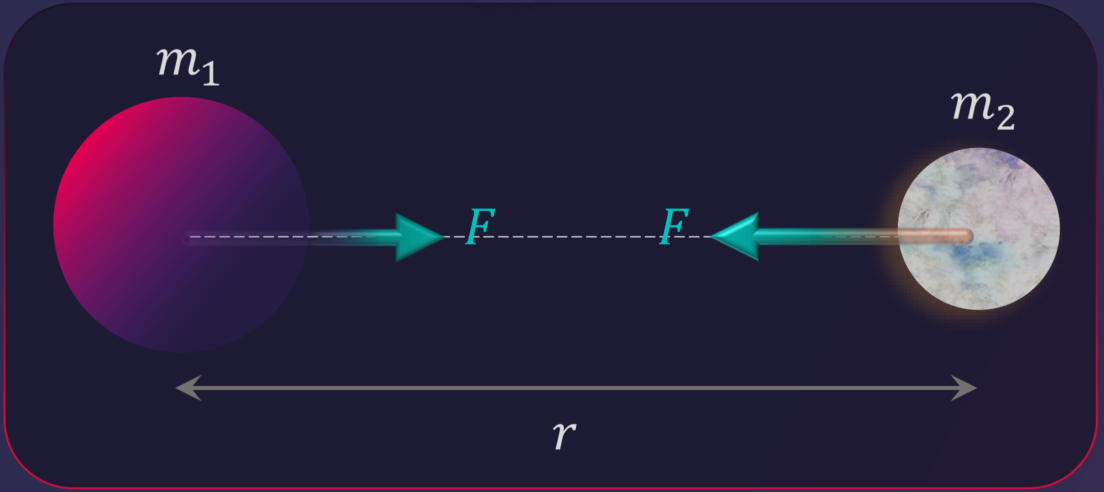
- Directly proportional to the product of their masses $\Rightarrow F_G \propto m_1 m_2$
- Inversley proportional to the square of the distance between their centers $\Rightarrow F_G \propto \frac{1}{r^2}$



$$F_G = \frac{G m_1 m_2}{r^2}$$

G : Universal Gravitational Constant

Properties of Gravitational Force



$$F_G = \frac{G m_1 m_2}{r^2}$$

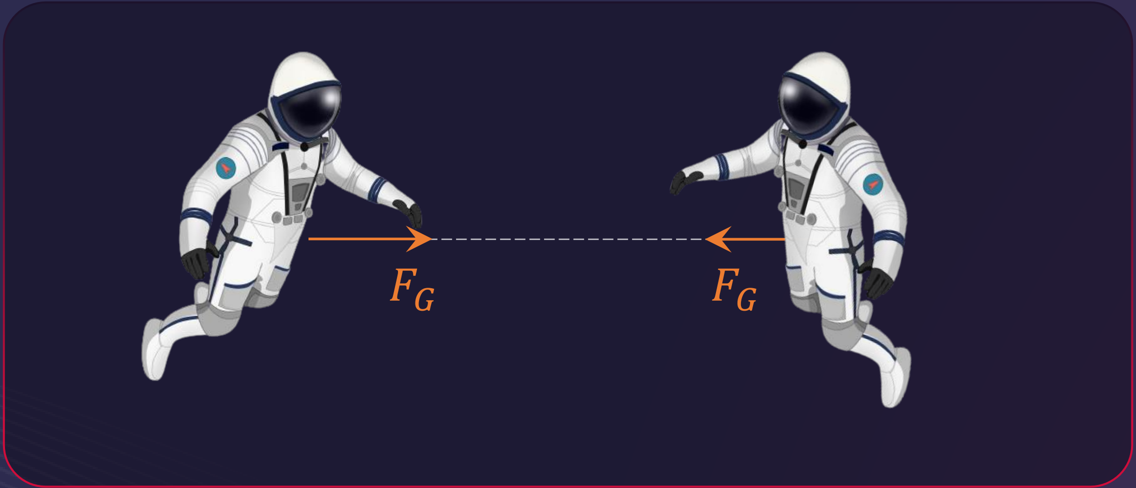
- Acts along the line joining the two masses.
- Always attractive in nature.
- Always acts as an action-reaction pair.
- Is independent of medium.
- Conservative force in nature.
- Obeys inverse square law $\left[F_G \propto \frac{1}{r^2} \right]$.
- It is a central force

NOTE: Central force is defined as the force whose magnitude depends only on distance r from a fixed point and is always directed either away or towards that fixed point.

$$\vec{F} = f(r)\hat{r}$$

After losing contact with their spacecraft, two astronauts are floating in free space where there is no external gravitational force acting on them. Which of the following event will occur eventually?

Solution :



a They will keep floating at the same distance between them.

b They will move towards each other.

c They will move away from each other.

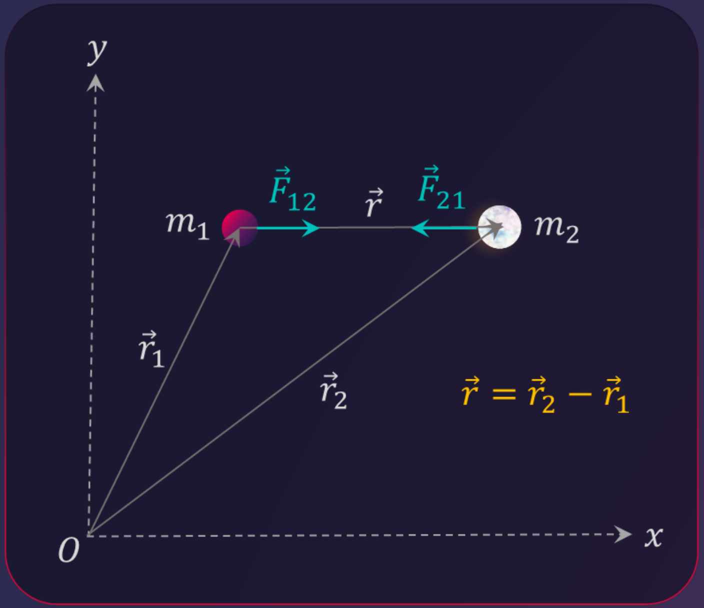
d They will become stationary

No external gravitational force is acting on the astronauts.

The only force present is the gravitational force between themselves.

Since the gravitational force is always attractive, they will accelerate individually, and will move towards each other.

Vector form of Gravitational Force



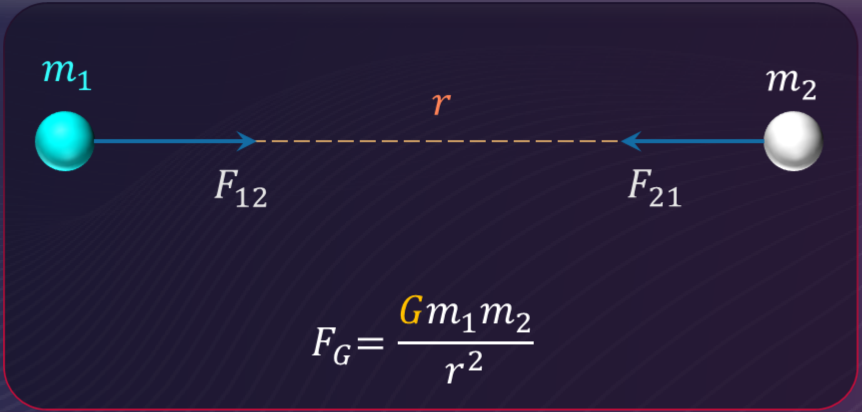
$$\vec{F}_{21} = -G\frac{m_1 m_2}{r^2}\hat{r}$$

$$\vec{F}_{12} = G\frac{m_1 m_2}{r^2}\hat{r}$$

NOTE: $\hat{r}$ is the unit vector along $\vec{r}$.

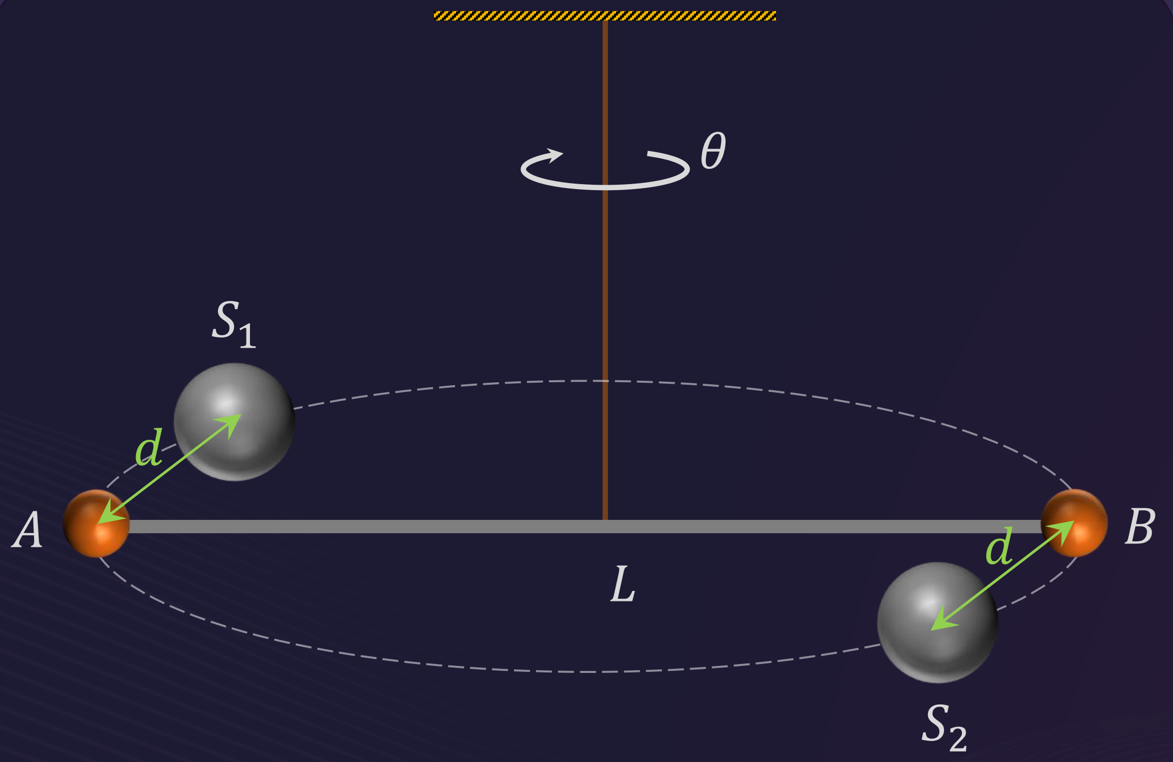
$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

Universal Gravitational Constant



- The numerical value of Universal Gravitational Constant (G) was determined by Cavendish experiment in 1798.

Cavendish Experiment



Cavendish's Torsion Balance

- Torsion arises in the wire due to gravitational attraction b/w m & M.

$$F = \frac{GmM}{d^2}$$

- At Equilibrium, the magnitude of gravitational torque and restoring torque will be equal.

$$\Rightarrow FL = \tau\theta \qquad (\tau = \text{torsional constant})$$

$$\Rightarrow \frac{GMm}{d^2} L = \tau\theta$$

$$\Rightarrow G = \frac{\tau\theta d^2}{MmL}$$

$$\boxed{G = 6.67 \times 10^{-11} N\, m^2/kg^2}$$

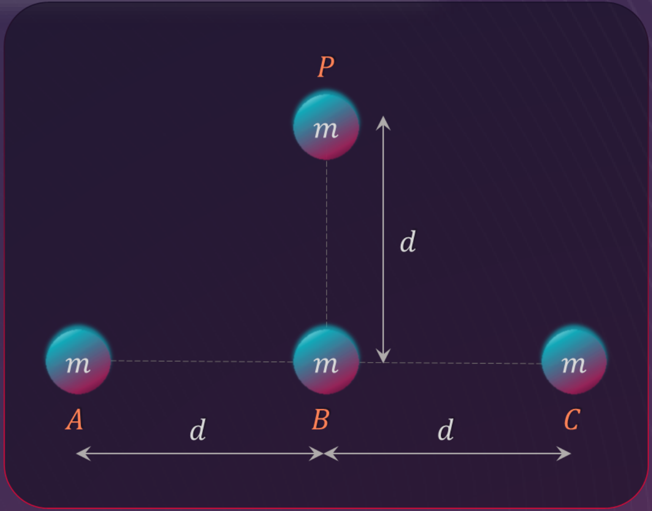
Properties of G

Gravitational constant has the numerical value of gravitational force between two bodies with unit mass separated by a unit distance.

$$F_G = \frac{G m_1 m_2}{r^2} = G \quad [\text{If } m_1 = m_2 = 1\, unit,\ r = 1\, unit]$$

- Its value is independent of mass, size, shape etc. of the bodies.
- Its value is medium independent.
- Value of G is so small that gravitational forces are normally smaller unless one or both masses are large.
- Dimensional formula of G: $M^{-1}L^3T^{-2}$

Three particles A, B and C, each of mass m, are placed on a line with $AB = BC = d$. Find the net gravitational force on a fourth particle P of same mass, placed at a distance d from the particle B on the perpendicular bisector of the line AC.



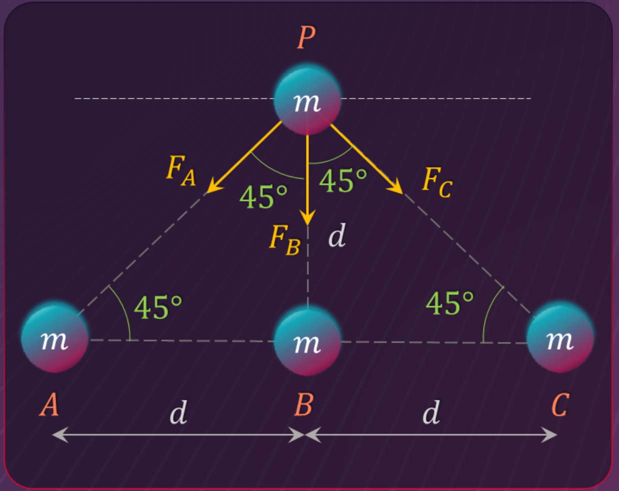
Solution : Gravitational force on a particle P due to A, B and C:

$$F_A = F_C = \frac{Gm^2}{2d^2} \qquad F_B = \frac{Gm^2}{d^2}$$

$$\therefore F_{net} = F_A cos45° + F_B + F_C cos45°$$

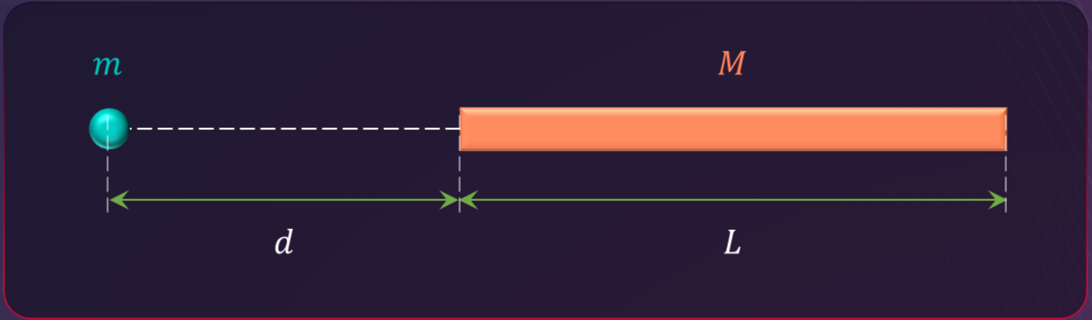
$$\boxed{F_{net} = \frac{Gm^2}{d^2}\left(1 + \frac{1}{\sqrt{2}}\right)}$$

(along PB)

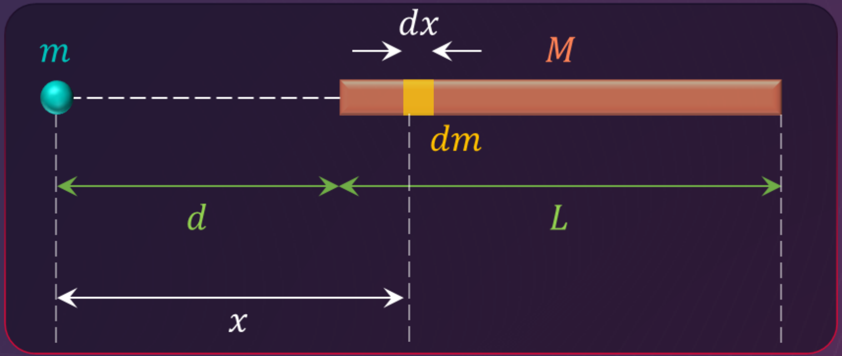


NOTE: Net Gravitational force on a given particle is the vector sum of gravitational force exerted by all individual particles around it. This is also known as Superposition Principle.

A particle of mass m is placed at a distance d from one end of a uniform rod with length L and mass M as shown in the figure. Find the magnitude of the gravitational force on the particle due to the rod.



Solution : Consider an elementary mass dm of length dx at a distance x from m.



$$dm = \left(\frac{M}{L}\right) dx$$

$\therefore$ The gravitational force on m due to dm,

$$dF = \frac{Gm(dm)}{x^2} = \frac{GmM}{Lx^2} dx$$

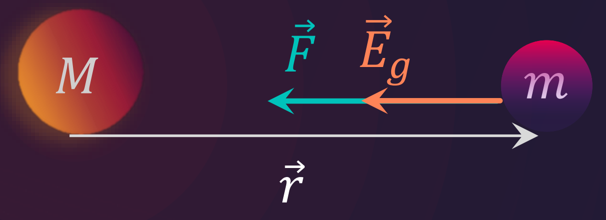
$$\Rightarrow F = \int dF = \frac{GmM}{L} \int_d^{L+d} \frac{dx}{x^2} = \frac{GmM}{L} \left[-\frac{1}{x}\right]_d^{L+d}$$

$$\Rightarrow F = \frac{GmM}{L} \left[-\frac{1}{L+d} + \frac{1}{d}\right]$$

$$\boxed{F = \frac{GmM}{d(L+d)}}$$

Gravitational Field and Field Strength

The region surrounding the mass where other masses experience a gravitational force is known as its gravitational field.



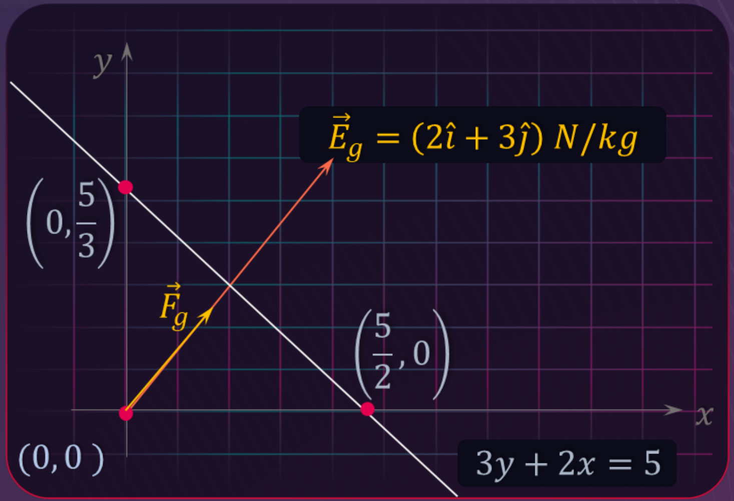
$$\vec{E}_g = \frac{\vec{F}}{m} = -\frac{GM}{|r^2|}(\hat{r}) \Rightarrow \boxed{\left|\vec{E}_g\right| = \frac{GM}{r^2}}$$

Force experienced by a unit mass placed at a point in a gravitational field gives the field strength.

- Vector quantity
- Units: $Newton/kg$ or m/s^2
- Dimension: $[M^0LT^{-2}]$
- Direction: Along the net Gravitational force on the unit mass
- Near earth's surface the field strength is $9.8\ m/s^2$

The gravitational field in a region is given by $\vec{E}_g = (2\hat{\imath} + 3\hat{\jmath})\ N/kg$. Show that no work is done by the gravitational field when a particle is moved on the line $3y + 2x = 5$.

Solution : Gravitational Force on a particle of mass m is,



$$\vec{F}_g = 2m\hat{\imath} + 3m\hat{\jmath}$$

$\Rightarrow$ Slope, $m_1 = \frac{3}{2}$

Particle moves along the line $3y + 2x = 5$

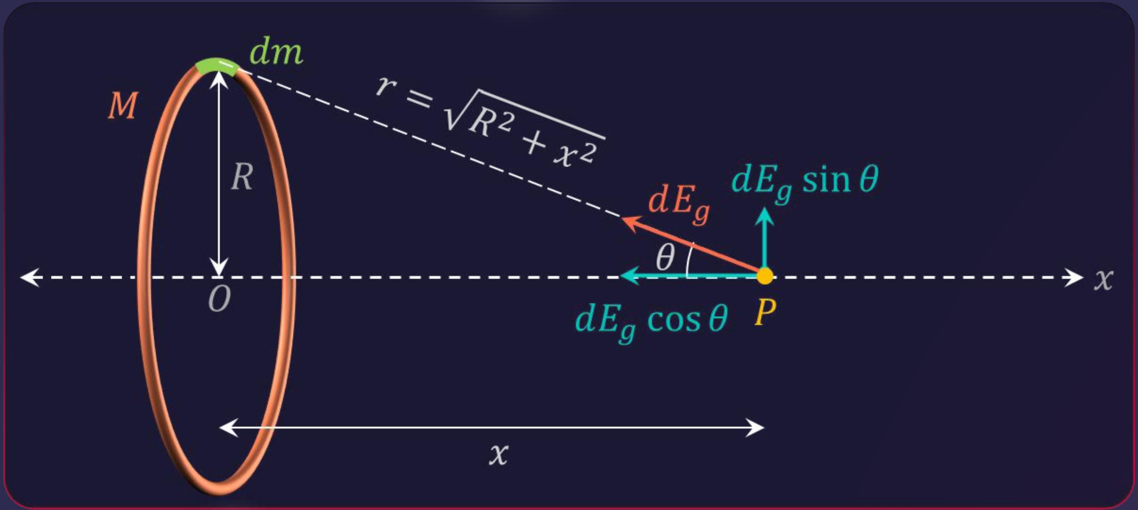
$\Rightarrow$ Slope, $m_2 = -\frac{2}{3}$

Condition of Orthogonality: $m_1 m_2 = -1$

Hence $\vec{F}_g$ & $\vec{s}$ are perpendicular

$\therefore W = \vec{F}_g . \vec{s} = 0$

Gravitational field on the axis of a Ring



$$E_g = \int dE_g cos\theta = \int \frac{Gdm}{r^2} \cdot \frac{x}{r}$$

$$\vec{E}_g(x) = -\frac{GMx}{(x^2+R^2)^{\frac{3}{2}}}\hat{x} = -\frac{GM\cos\theta}{r^2}\hat{x}$$

Variation in g due to Earth's Rotation

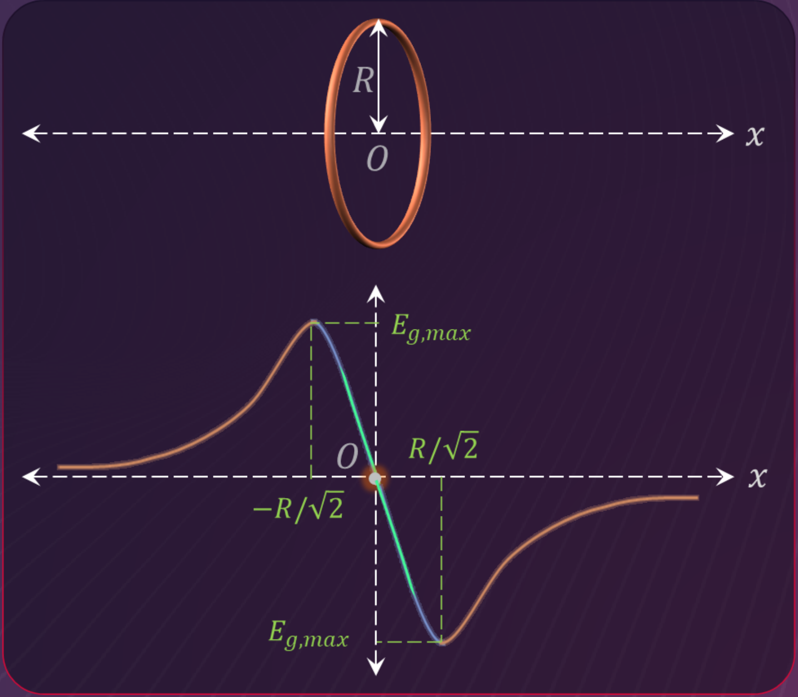
$$|\vec{E}_g(x)| = \frac{GMx}{(x^2+R^2)^{\frac{3}{2}}}$$

NOTE: Gravitational field due to the ring on its axis has the maximum magnitude at $|x| = \frac{R}{\sqrt{2}}$ and its value is $\frac{2GM}{3\sqrt{3}\,R^2}$.

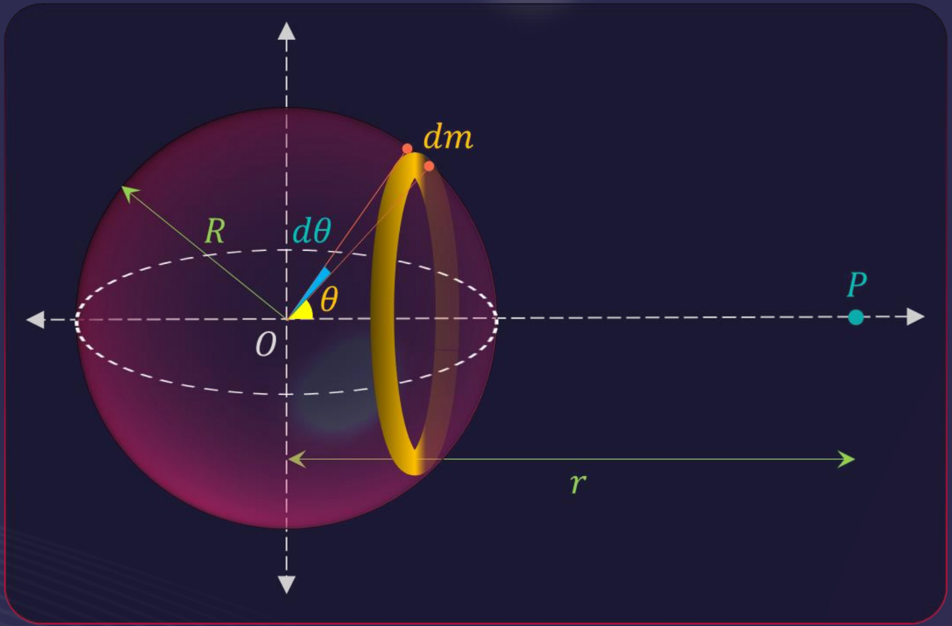
At $x = 0$: $|\vec{E}_g| = 0$

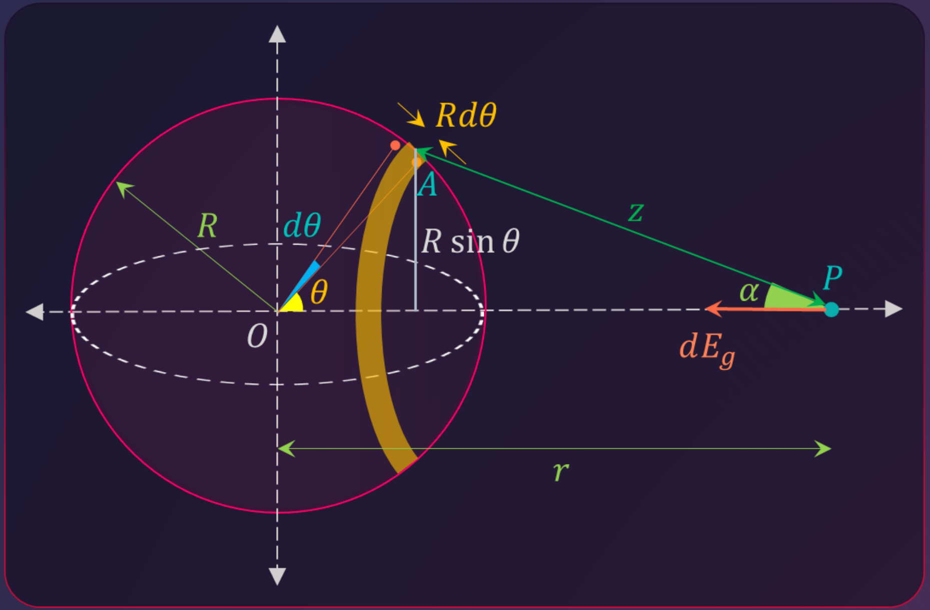
At $|x| \ll R$: $x^2 + R^2 \approx R^2 \Rightarrow |\vec{E}_g| = \frac{GMx}{R^3}$

At $|x| \gg R$: $x^2 + R^2 \approx x^2 \Rightarrow |\vec{E}_g| = \frac{GM}{x^2}$



Gravitational Field due to Spherical Shell





In ΔOAP, By cosine rule

$$R^2 + r^2 - z^2 = 2Rr\cos\theta$$

$$\Rightarrow \frac{z(dz)}{Rr} = \sin\theta d\theta \quad \text{... (2)}$$

And

$$\frac{r^2 + z^2 - R^2}{2rz} = \cos\alpha \quad \text{... (3)}$$

Assuming uniform mass distribution,

$$dm = \frac{M}{4\pi R^2} \times (2\pi R\sin\theta)(Rd\theta)$$

$$\Rightarrow dm = \frac{M}{2}\sin\theta d\theta$$

Field at P due to the ring element:

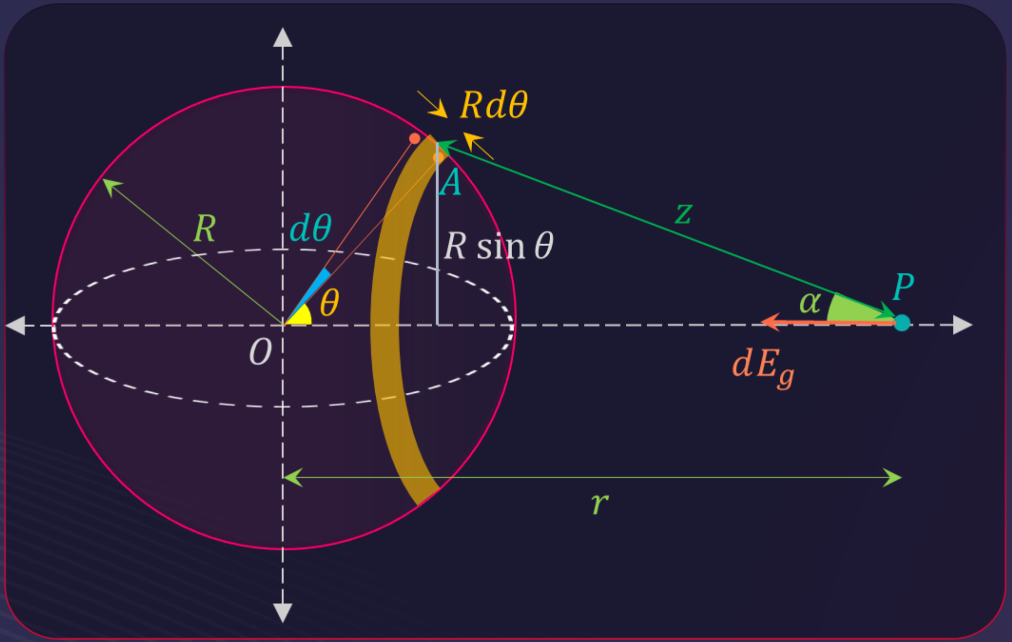
$$\vec{E_g}(x) = -\frac{GM\cos\theta}{r^2}\hat{x}$$

$$dE_g = \frac{GM(\cos\alpha)(\sin\theta)d\theta}{2z^2} \quad \text{... (1)}$$

$$\boxed{dE_g = \frac{GM}{4Rr^2}\left[1 + \frac{r^2 - R^2}{z^2}\right]dz}$$

Gravitational Field due to Spherical Shell

Case 1: P is outside the shell $(r > R)$

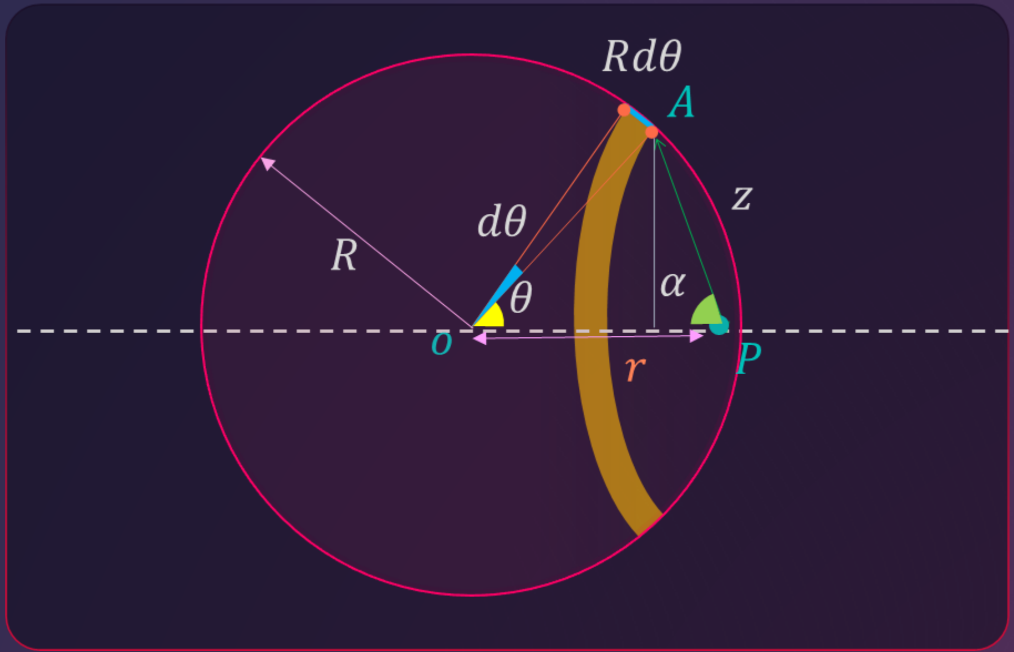


$$E_g = \frac{GM}{4Rr^2}\int_{r-R}^{r+R}\left[1+\frac{(r^2-R^2)}{z^2}\right]dz$$

$$\boxed{\vec{E}_g = -\frac{GM}{r^2}\hat{r}}$$

Note: The gravitational field due a uniform spherical shell can be found out by imagining the whole mass as a point mass of same mass, kept at centre

Case 2: P is inside the shell $(r < R)$



$$E_g = \frac{GM}{4Rr^2}\int_{R-r}^{R+r}\left[1+\frac{(r^2-R^2)}{z^2}\right]dz$$

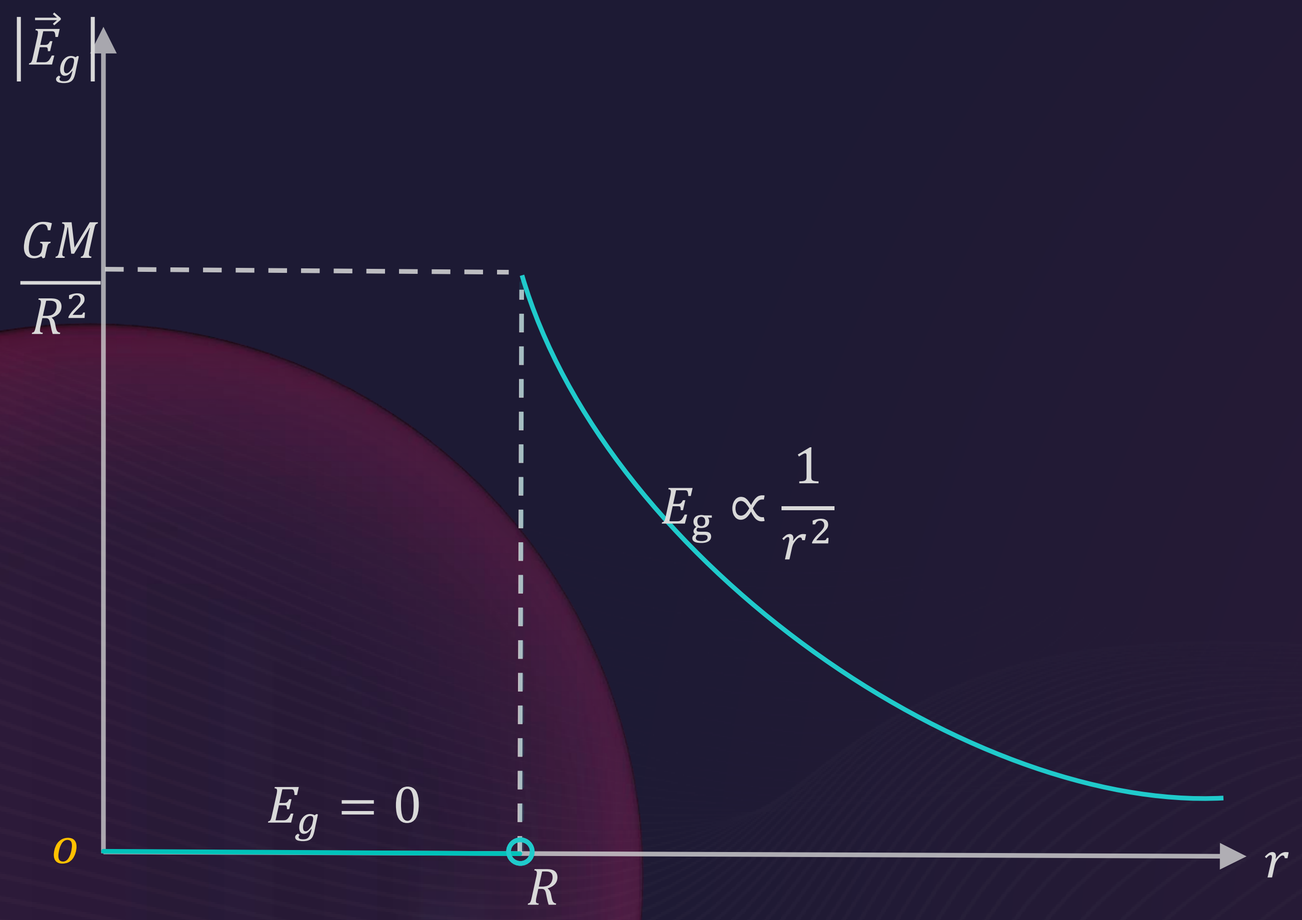
$$E_g = \frac{GM}{4Rr^2}\left[z-\frac{(r^2-R^2)}{z}\right]_{R-r}^{R+r}$$

$$\Rightarrow E_g = \frac{GM}{4Rr^2}\left[z+\frac{(R^2-r^2)}{z}\right]_{R-r}^{R+r}$$

$$E_g = \frac{GM}{4Rr^2}[2R-2R]$$

$$\boxed{E_g = \text{zero}}$$

Gravitational Field due to Spherical Shell



Case 1 : Point outside the shell $(r > R)$

$$\vec{E}_g = -\frac{GM}{r^2}\hat{r}$$

Case 2 : Point inside the sphere $(r < R)$

$$\vec{E}_g = \vec{0}$$

Case 3 : Point on the sphere $(r = R)$

$$\vec{E}_g = -\frac{GM}{R^2}\hat{r}$$

Four identical particles of mass M are located at the corners of a square of side a. What should be their speed, if each of them revolves under the influence of other's gravitational field in a circular orbit circumscribing the square?

Solution :

Net force on B:

$$(\vec{F}_B)_{net} = \vec{F}_{BA} + \vec{F}_{BC} + \vec{F}_{BD}$$

As masses are distributed symmetrically about BD, net gravitational force on B will be only along BD

$$(F_B)_{net} = F_{BD} + F_{BA}\sin 45^0 + F_{BC}\cos 45^0$$

$$(F_B)_{net} = \frac{GM^2}{(\sqrt{2}a)^2} + \frac{GM^2}{a^2}\left(\frac{1}{\sqrt{2}}\right) + \frac{GM^2}{a^2}\left(\frac{1}{\sqrt{2}}\right)$$

$$(F_B)_{net} = \frac{GM^2}{a^2}\left(\frac{1}{2} + \sqrt{2}\right)$$

The direction of the force is towards the center O and it acts as a centripetal force

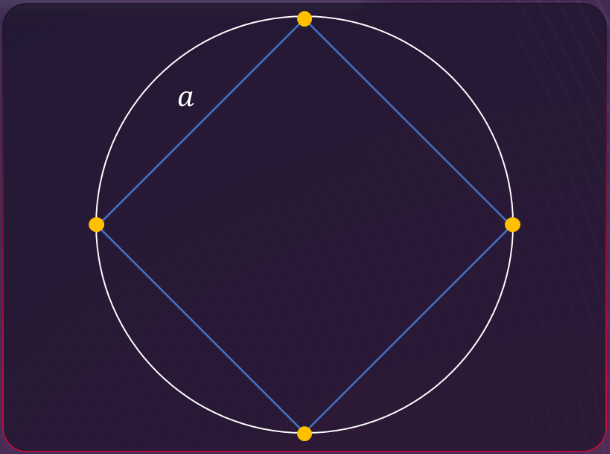
Centripetal force is given by,

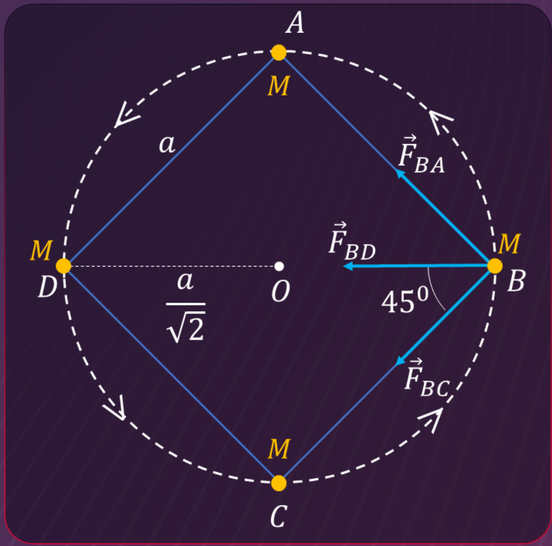
$$F_c = \frac{Mv^2}{r} = \frac{Mv^2}{\frac{a}{\sqrt{2}}} = \frac{\sqrt{2}Mv^2}{a}$$

But, $F_c = (F_B)_{net}$

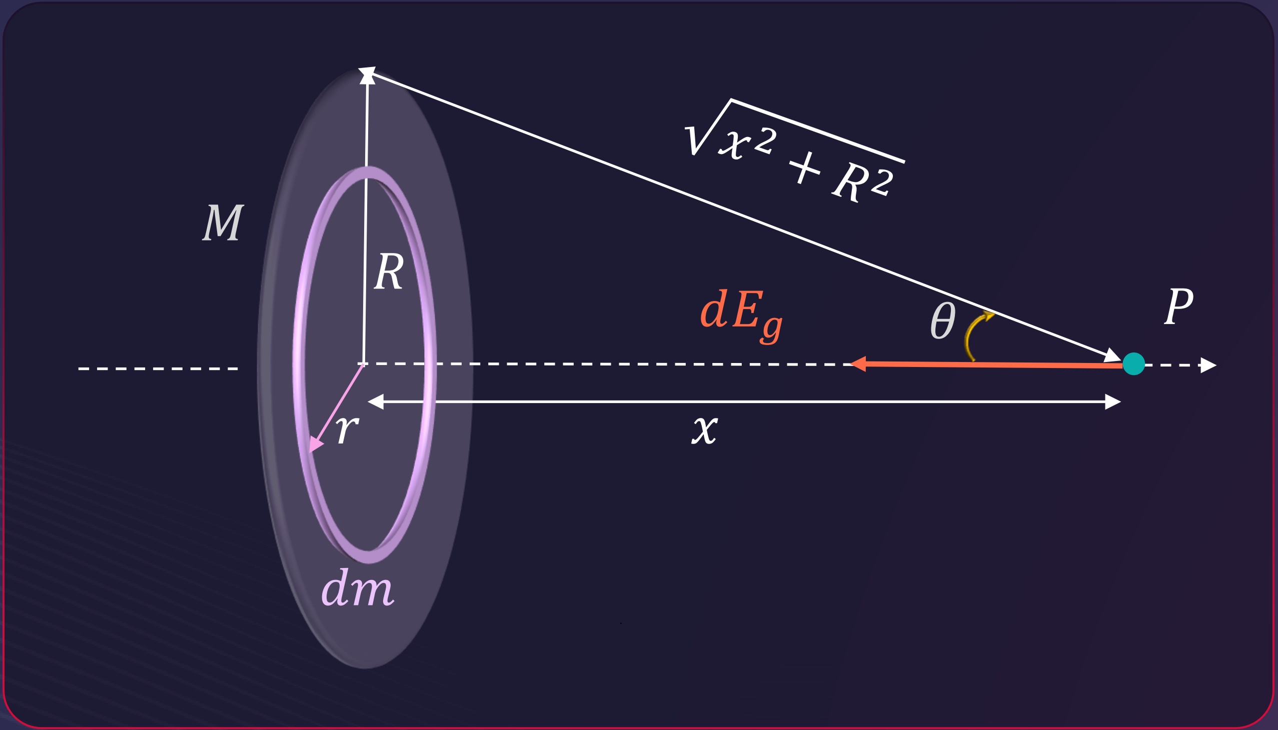
$$v^2 = \frac{GM}{a}\left(1 + \frac{1}{2\sqrt{2}}\right) = \frac{GM}{a}(1.35)$$

$$\boxed{\Rightarrow v = 1.16\sqrt{\frac{GM}{a}}}$$





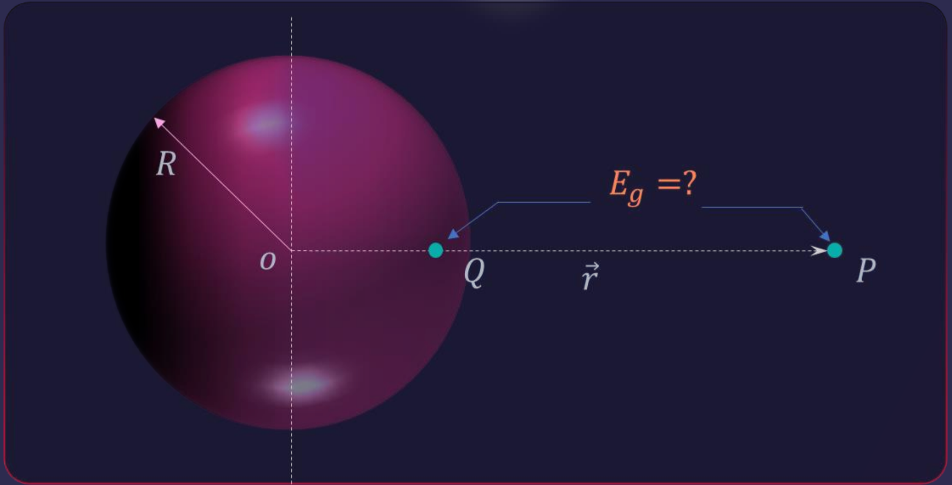
Gravitational Field on the Axis of Disc

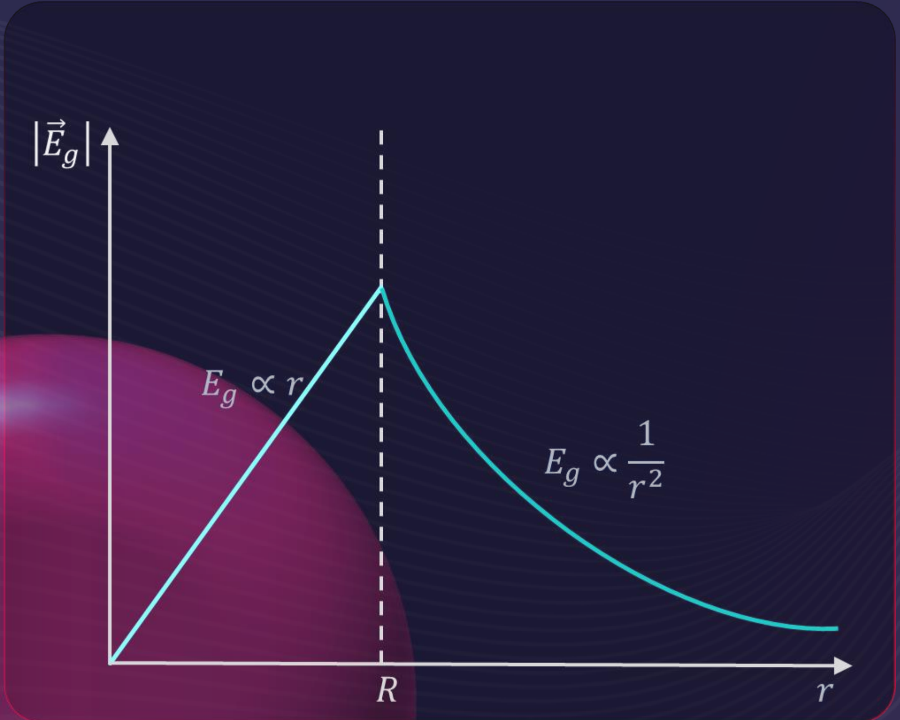


$$\vec{E}_g(x) = -\frac{2GM}{R^2}\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)\hat{x}$$

$$\vec{E}_g(x) = -\frac{2GM}{R^2}(1 - \cos\theta)\hat{x}$$

Field Due to a Uniform Solid Sphere





Case 1 : Point outside the sphere $(r > R)$

$$\vec{E}_g = -\frac{GM}{r^2}\hat{r}$$

Case 2 : Point inside the sphere $(r < R)$

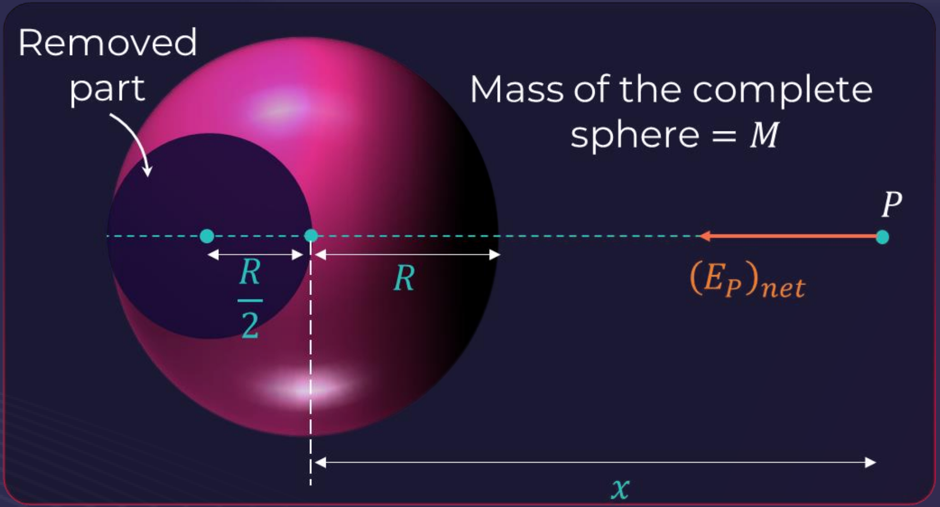
$$\vec{E}_g = -\frac{GMr}{R^3}\hat{r}$$

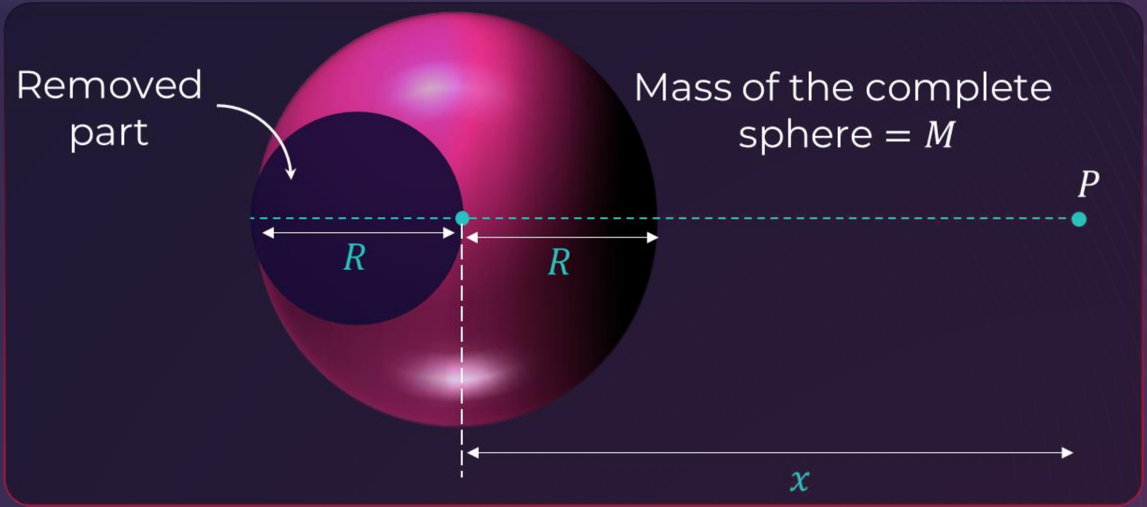
Case 3 : Point on the sphere $(r = R)$

$$\vec{E}_g = -\frac{GM}{R^2}\hat{r}$$

The gravitational field, due to the 'left over' part of a uniform sphere (from which a part as shown has been 'removed out') at a very far off point P, located as shown, would be (nearly)

Solution :





Gravitational field due to remaining part at point P $(E_P)_{net}$ = Gravitational field due to complete sphere $(E_P)_S$ − Gravitational field due to removed part $(E_P)_R$

Mass of the removed part, M':

$$M' = \frac{M}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi\left(\frac{R}{2}\right)^3 = \frac{M}{8}$$

Gravitational field at P:

$$(E_P)_{net} = (E_P)_{Sphere} - (E_P)_{Removed\ part}$$

$$(E_P)_{net} = \frac{GM}{x^2} - \frac{G\frac{M}{8}}{\left(\frac{R}{2}+x\right)^2} = \frac{GM}{x^2} - \frac{GM}{8x^2} \quad \left(\because x \gg \frac{R}{2}\right)$$

$$\boxed{(E_P)_{net} = \frac{7GM}{8x^2}}$$

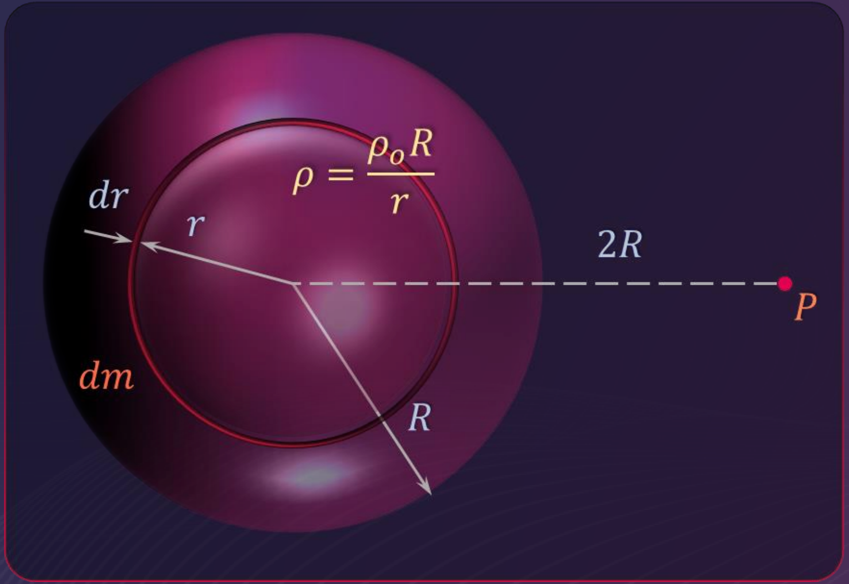
The density of a solid sphere of radius R is given by $\rho = \frac{\rho_o R}{r}$ where ρ_o is the density at the surface and r denotes the distance from the centre. Find the gravitational field due to this sphere at a distance $2R$ from its centre.

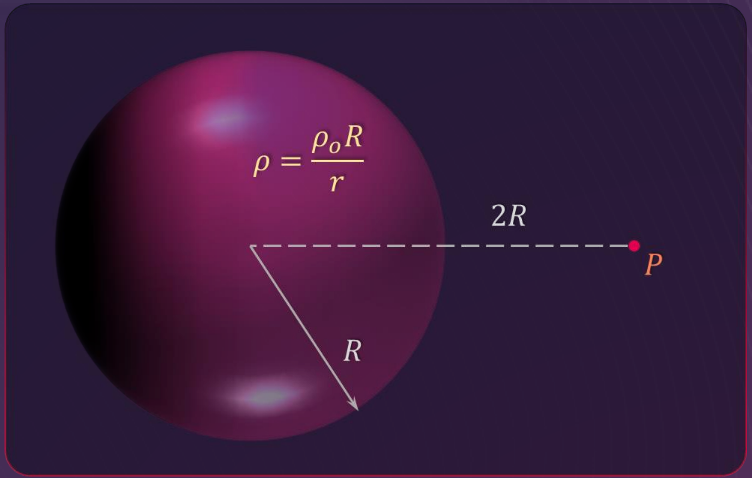
Solution : Gravitational Field at point P

$$E_P = \frac{G}{4R^2}\int_0^R dm$$

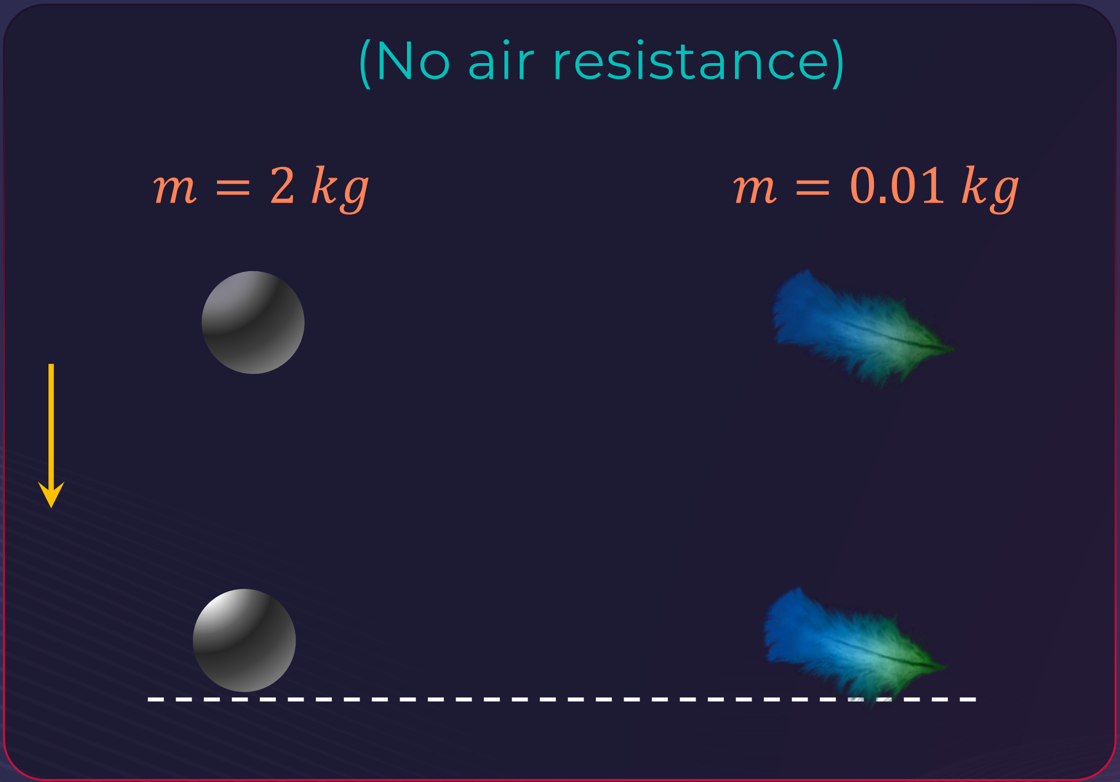
$$M = \int_0^R dm = 2\pi\rho_o R^3$$

$$\boxed{E_P = \frac{\pi G \rho_o R}{2}}$$





Acceleration due to Gravity



Force of attraction on the body of mass m close to the surface of the Earth:

$$|F| = \frac{GMm}{R^2}$$

$$mg = \frac{GMm}{R^2}$$

$$\boxed{g = \frac{GM}{R^2}}$$

g does not depend on the mass of the falling object

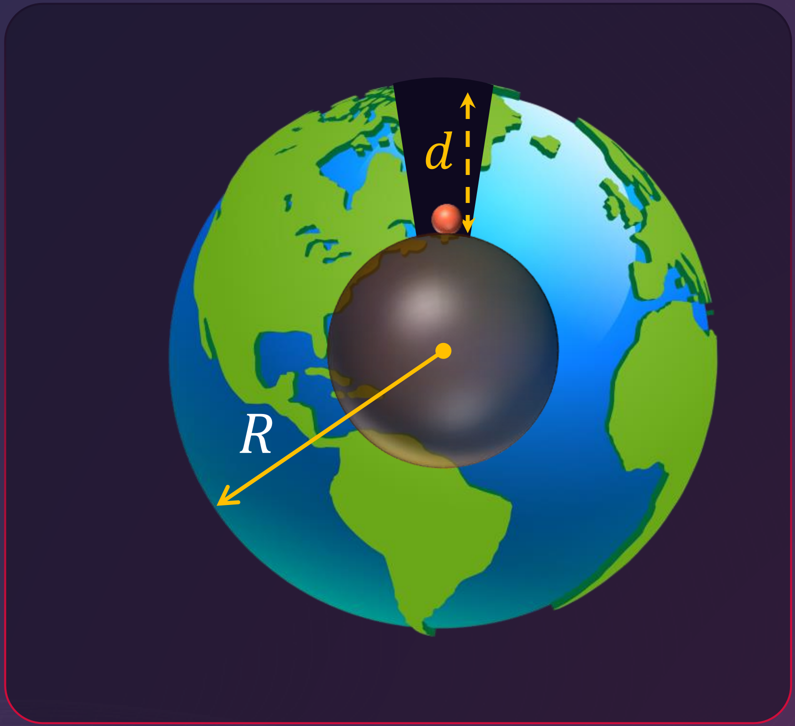
Variation of g with Height & Depth

Due to Height

$$g' = g\left(1 - \frac{2h}{R}\right)$$

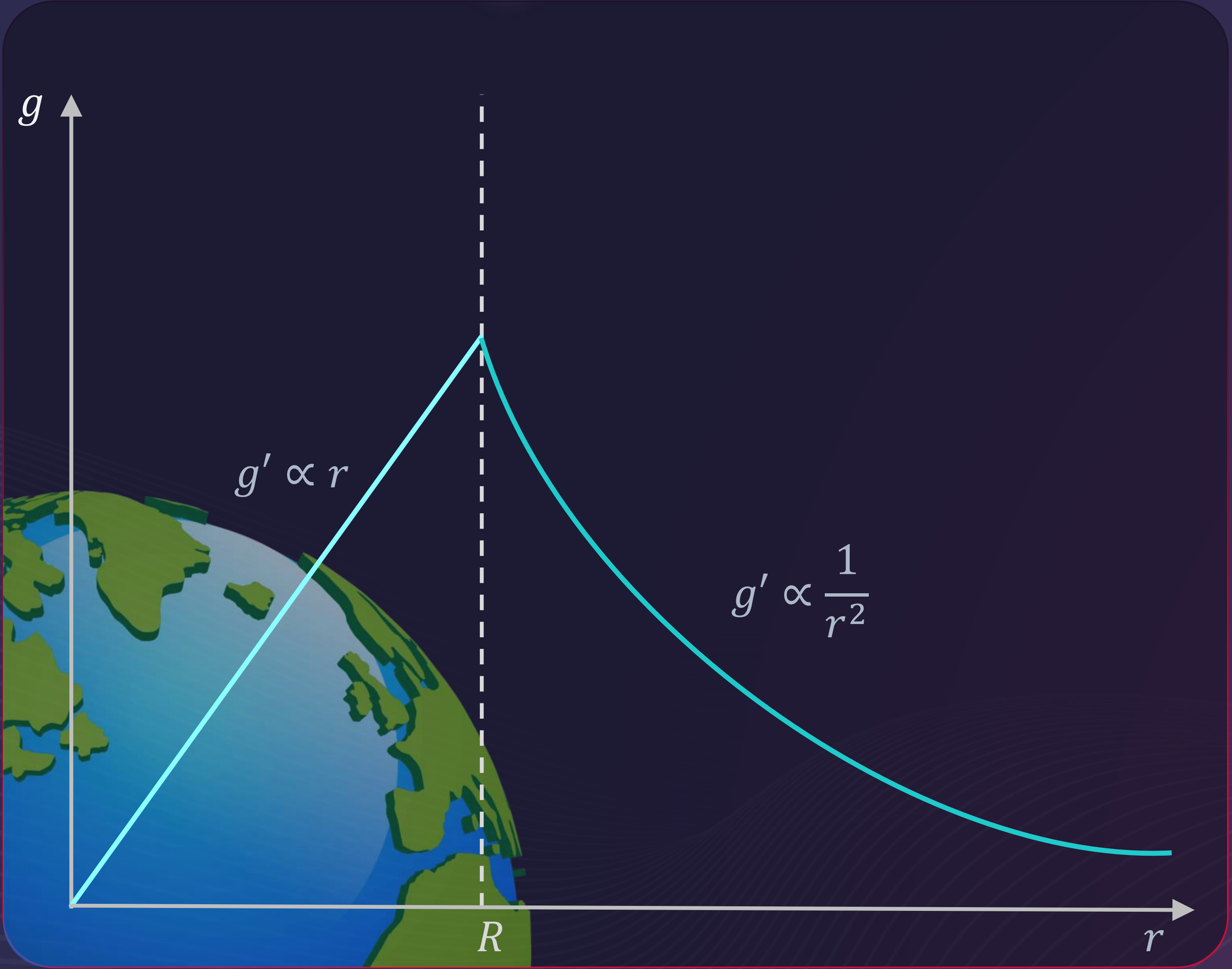
$(h \ll R)$

Due to Depth



$$g' = g\left(1 - \frac{d}{R}\right)$$

Variation of g with r



Case 1 : g' inside the Earth $(r < R)$

$$g' = g\left(1 - \frac{d}{R}\right) = \frac{g(R-d)}{R} = \frac{gr}{R}$$

Case 2 : g' on the Earth $(r = R)$

$$g' = g = \frac{GM}{R^2}$$

Case 3 : g' outside the Earth $(r > R)$

$$g' = \frac{GM}{r^2}$$

Find the ratio of the heights from the earth's surface at which the acceleration due to gravity is $0.99\ g_e$ and $0.64\ g_e$ respectively. Here g_e is the acceleration due to gravity at the surface of the earth.

Solution :

Given : At h_1 $g'_e = 0.99g_e$

At h_2 $g'_e = 0.64g_e$

To Find : Ratio of heights $\left(\frac{h_1}{h_2}\right)$

Solution :

At h_1, $g'_e = 0.99g_e$

$\Rightarrow h_1 \ll R_e$

$$0.99g_e = g_e\left(1 - \frac{2h_1}{R_e}\right)$$

$$\Rightarrow h_1 = \frac{R_e}{200} \qquad \text{----- (1)}$$

At h_2, $g''_e = 0.64g_e$

$\Rightarrow h_2$ is comparable with R_e

$$0.64g_e = \frac{g_e}{\left(1 + \frac{h_2}{R_e}\right)^2}$$

$$\Rightarrow h_2 = \frac{R_e}{4} \qquad \text{----- (2)}$$

From (1) and (2),

$$\boxed{\frac{h_1}{h_2} = \frac{1}{50}}$$

Variation in g due to Earth's Rotation

Acceleration due to gravity for a place at latitude θ is,

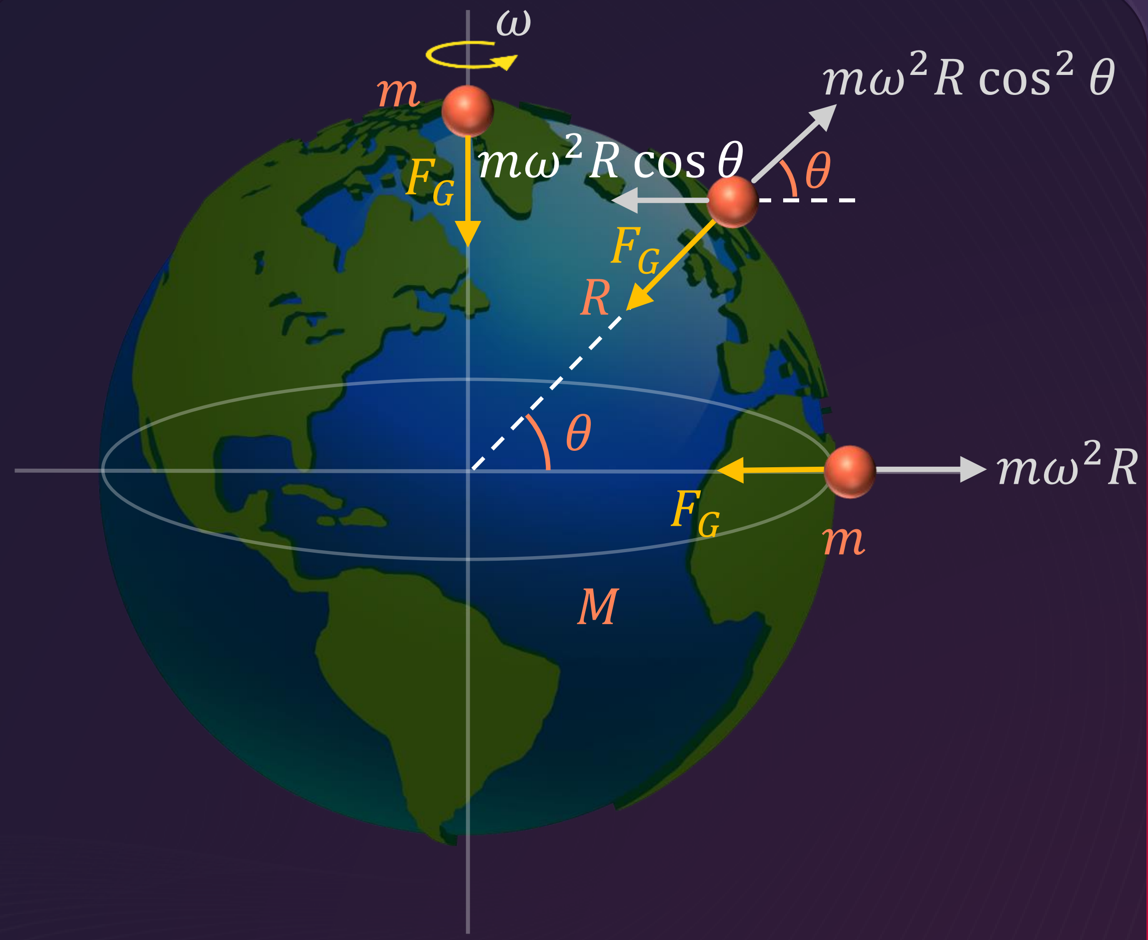
$$\boxed{g' = g - \omega^2 R\cos^2\theta}$$

Case 1: At poles ($\theta = 90°$)

$$\boxed{g' = g}$$

Case 2: At equator ($\theta = 0°$)

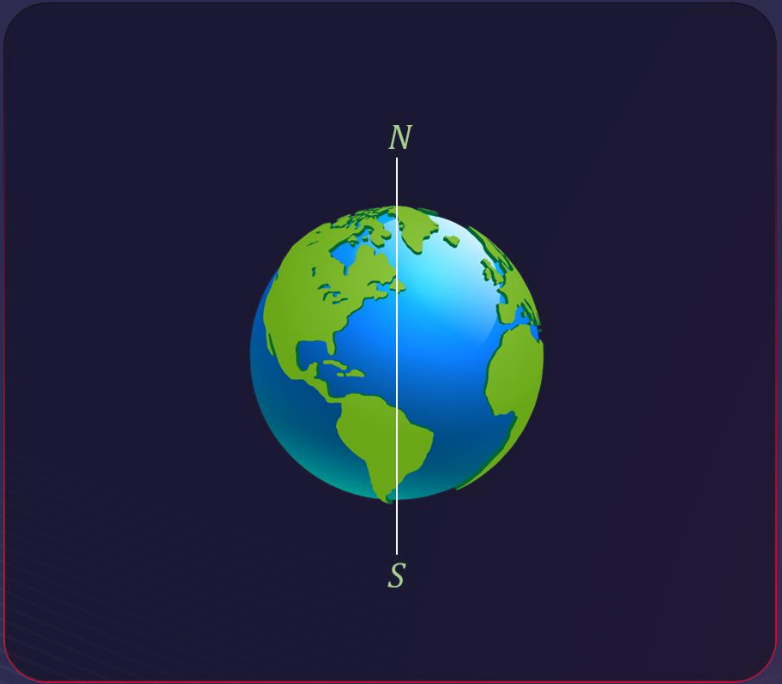
$$\boxed{g' = g - \omega^2 R}$$



$$F_G = \frac{GMm}{R^2}$$

At what rate should the earth rotate so that the apparent g at the equator becomes zero? What will be the length of the day in this situation? $(R = 6400 \times 10^3\ m, g = 10\ ms^{-2})$

Solution :



At equator, $\theta = 0°$

$$g'_{equator} = g - \omega^2 R = 0$$

$$\Rightarrow \omega = \sqrt{\frac{g}{R}}$$

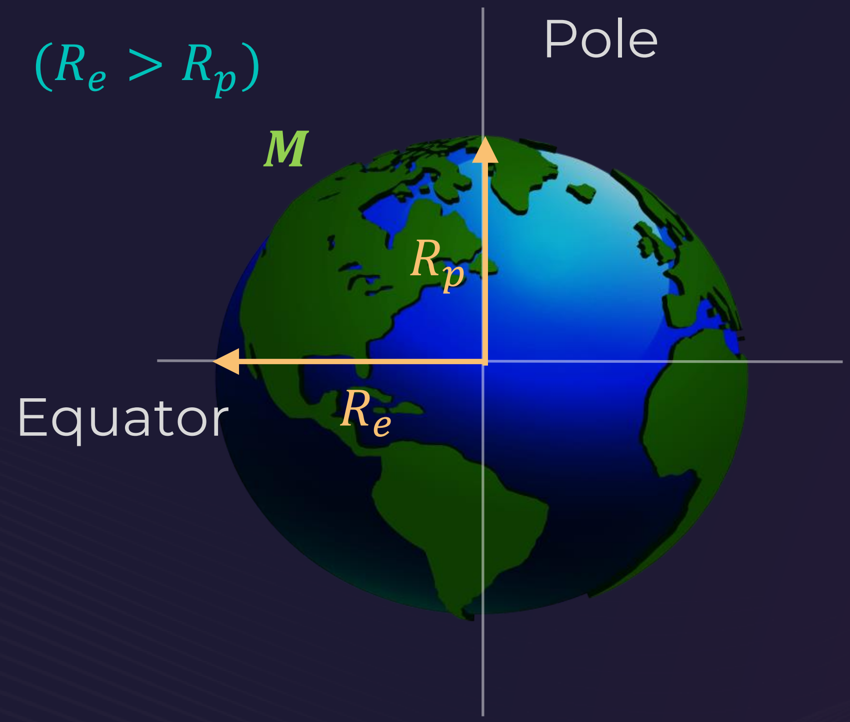
$$\boxed{\omega = 1.25 \times 10^{-3}\ rad\ s^{-1}}$$

Length of the day,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$$

$$\boxed{T \approx 1.4\ hr}$$

Variation in g due to Non Sphericity



Gravitational field at poles

$$g_p = \frac{GM}{R_p^2}$$

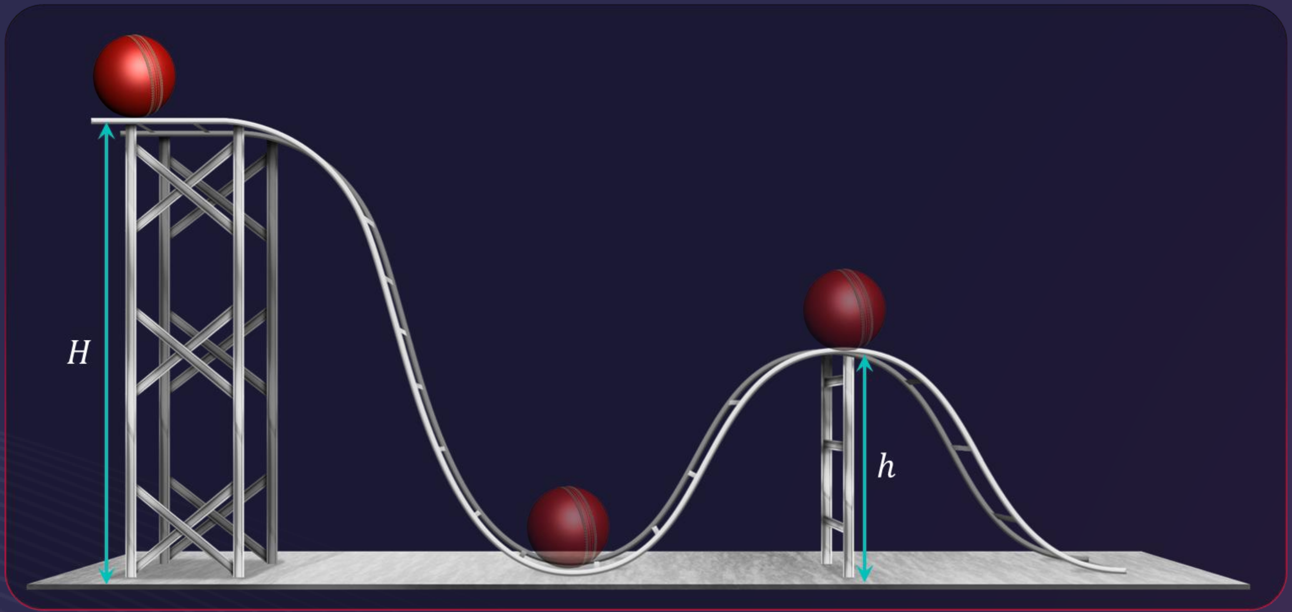
Gravitational field at the equator

$$g_e = \frac{GM}{R_e^2}$$

$$g_p > g_e$$

Gravitational Potential Energy

- The energy possessed or acquired by an object by virtue of its position in the presence of a gravitational field.

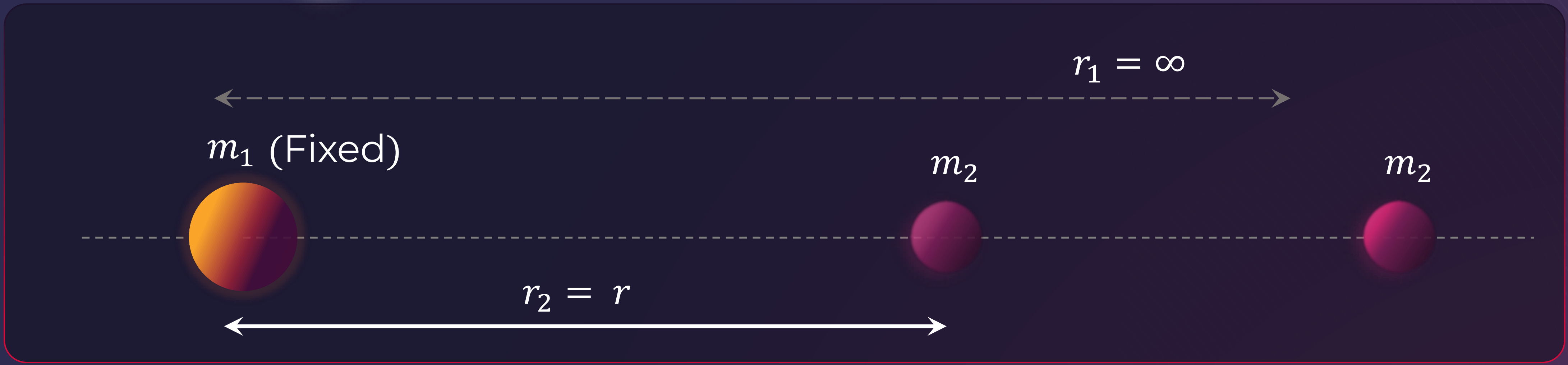


- Negative of the work done by the conservative force as the system changes from initial to final configuration is change in Potential Energy.

Change in potential energy,

$$\Delta U = U_f - U_i = -\left(W_{conservative\ forces}\right)_{i \to f}$$

Gravitational Potential Energy



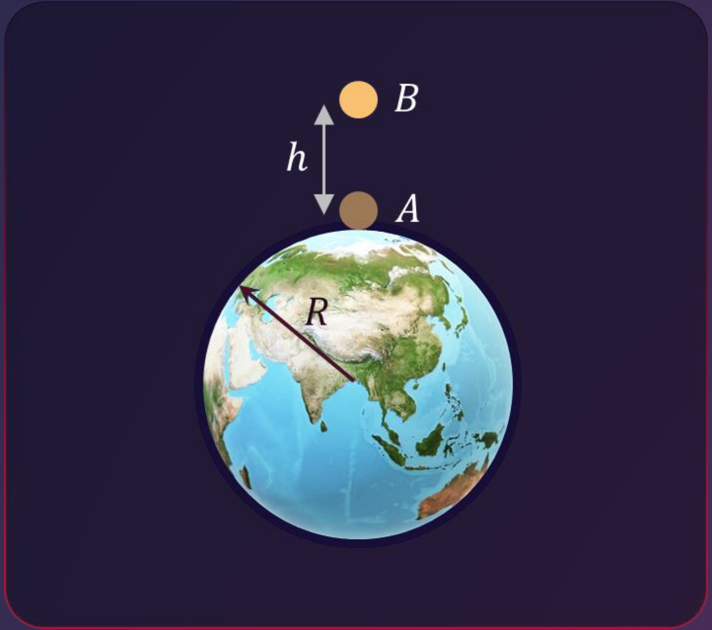
$$\Delta U = U_2 - U_1 = -\int_1^2 \vec{F}.d\vec{r} = -Gm_1m_2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$U(r) - U(\infty) = -Gm_1m_2\left(\frac{1}{r} - \frac{1}{\infty}\right)$$

$$\boxed{U(r) = -\frac{Gm_1m_2}{r}}$$

At the pole of the Earth, a body is imparted an initial velocity v_0 directed vertically upwards. Find the height to which the body will ascend. (Neglect air drag)

Solution : Using the principle of conservation of mechanical energy,



$$K_A + U_A = K_B + U_B$$

$$\Rightarrow \frac{1}{2}mv_0^2 - \frac{GMm}{R} = 0 - \frac{GMm}{R+h}$$

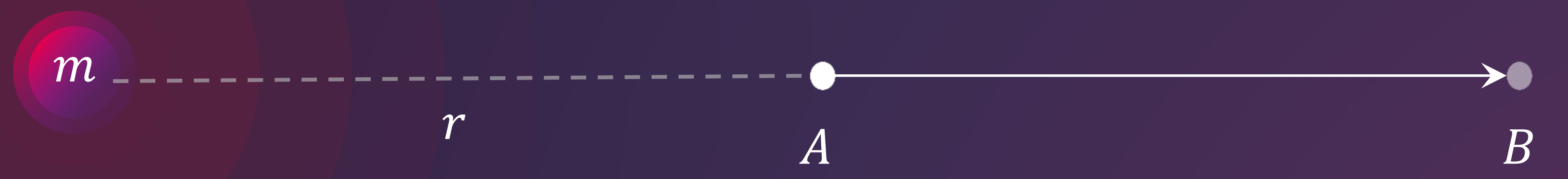
$$\frac{v_0^2}{2} = GM\left[\frac{1}{R} - \frac{1}{R+h}\right] \Rightarrow v_0^2 = 2GM\left[\frac{h}{R^2 + Rh}\right]$$

$$v_0^2(R^2 + Rh) = 2GMh \Rightarrow v_0^2R^2 = h(2GM - Rv_0^2)$$

$$h = \frac{R^2v_0^2}{2GM - Rv_0^2}$$

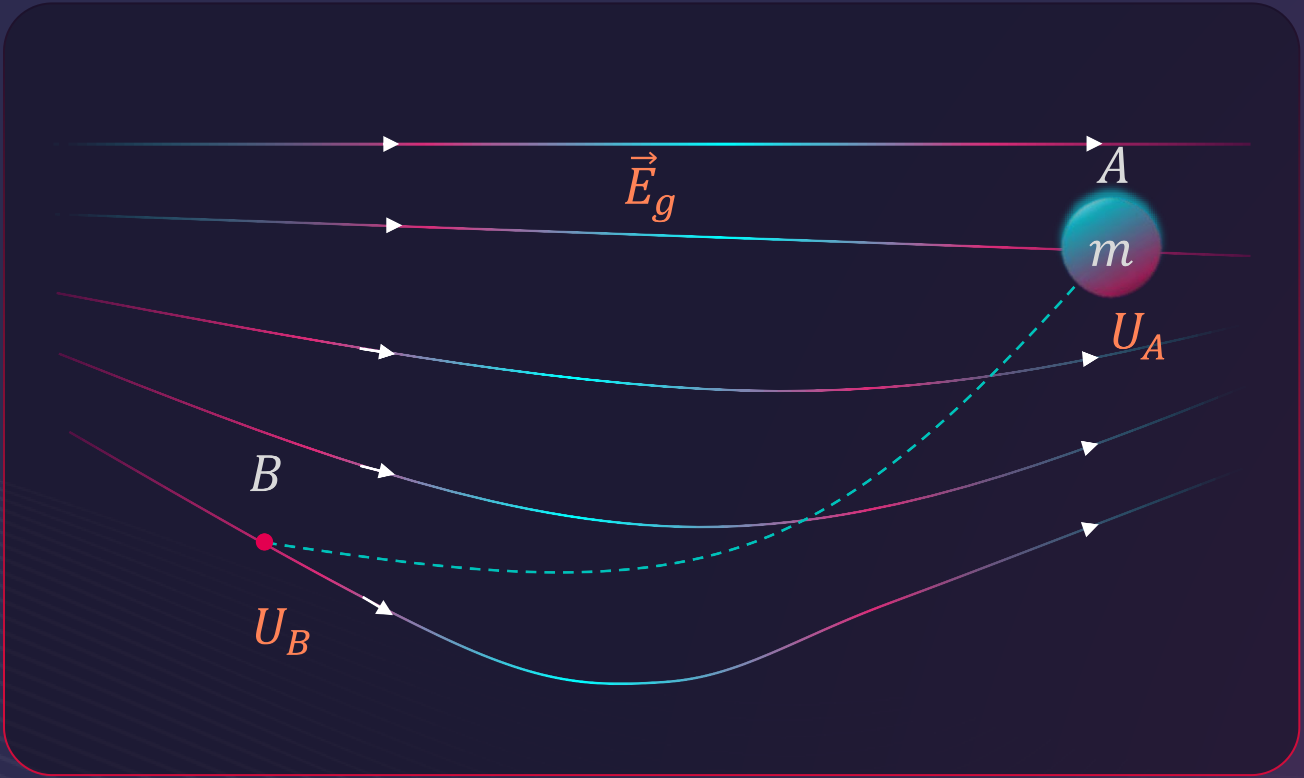
Gravitational Potential Difference

- Gravitational potential difference between two points in a gravitational field is defined as the work done by an external agent in moving a unit mass from the initial point to final point.



$$\Delta V = V_B - V_A = \frac{(\text{Work done by External})_{A \to B}}{\text{Mass}}$$

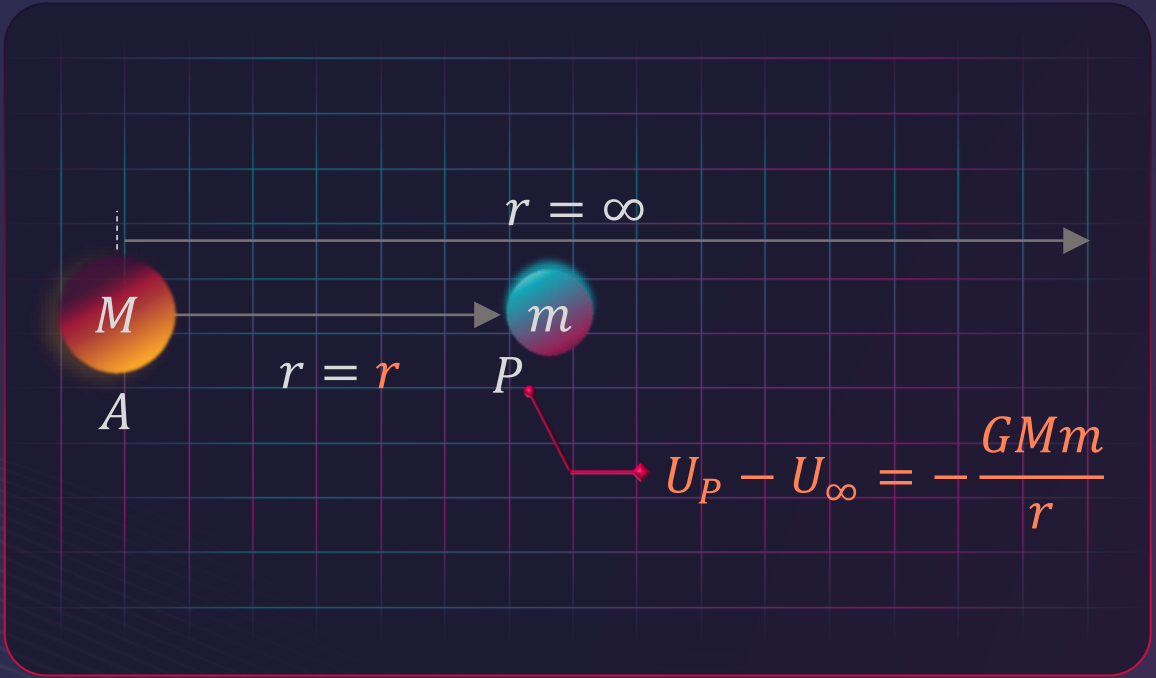
Gravitational Potential



- Change in gravitational potential between points A and B,

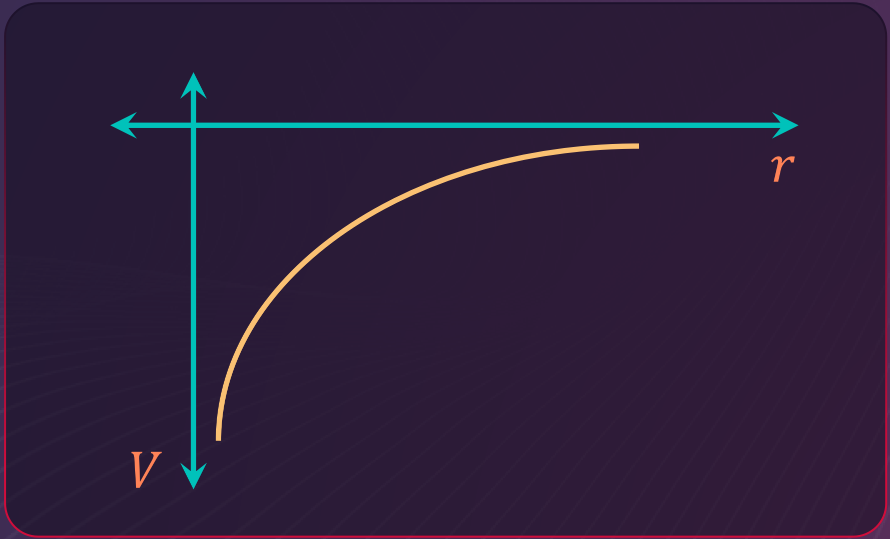
$$\boxed{V_B - V_A = \frac{U_B - U_A}{m}}$$

Potential due to a Point Mass



- Gravitational potential due to a mass M at distance r is

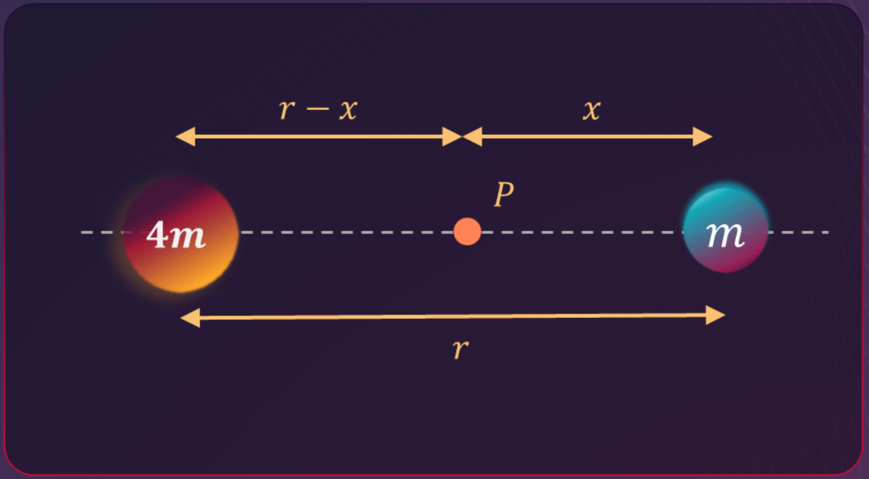
$$V(r) = -\frac{GM}{r}$$



Two bodies of masses m and $4m$ are placed at a distance r from each other. Find the gravitational potential at a point on the line joining them where the gravitational field is zero.

Solution :

Let the gravitational field be zero at the point P located at a distance x from m.



$$\therefore \frac{Gm}{x^2} = \frac{G(4m)}{(r-x)^2}$$

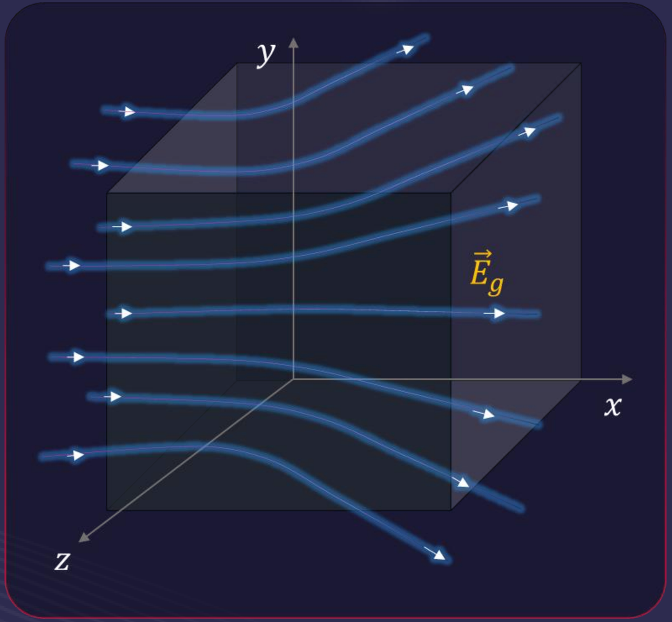
$$\Rightarrow \frac{1}{x} = \frac{2}{r-x}$$

$$\Rightarrow x = \frac{r}{3}$$

Gravitational potential at P is,

$$V = -\frac{Gm}{\left(\frac{r}{3}\right)} - \frac{4Gm}{\left(\frac{2r}{3}\right)} = -\frac{9Gm}{r}$$

Relation b/w Gravitational Field & Potential



$$\boxed{dV = -\vec{E}_g . d\vec{r}}$$

If $\vec{E}_g = E_g(x)\hat{\imath} + E_g(y)\hat{\jmath} + E_g(z)\hat{k}$

& $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$

$$\boxed{dV = -E_g(x)dx - E_g(y)dy - E_g(z)dz}$$

$$\boxed{\int_{V_1}^{V_2} dV = -\int_{r_1}^{r_2} \vec{E}_g . d\vec{r}}$$

- If potential function in a $3D$ space is given as : $V = f(x, y, z)$

  Then, the gravitational field component

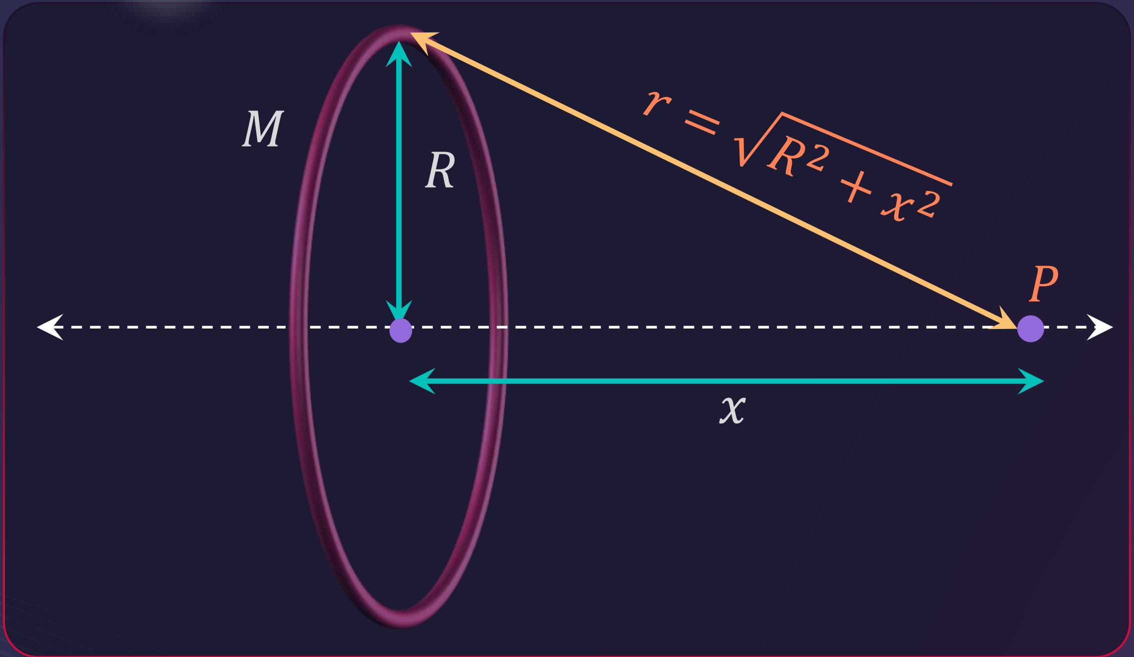
  along x - direction: $E_g(x) = -\dfrac{\partial V}{\partial x}$

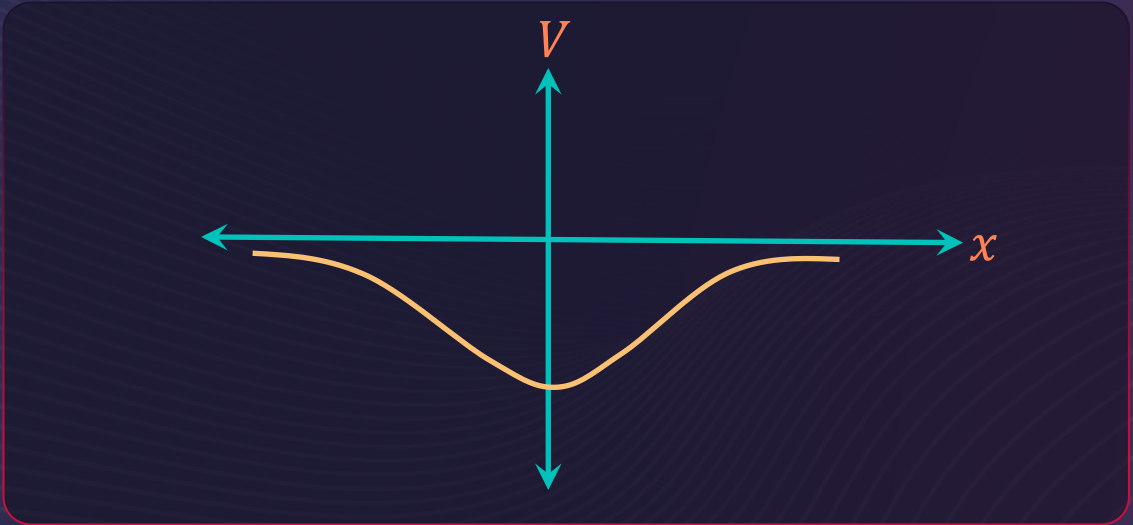
  along y - direction: $E_g(y) = -\dfrac{\partial V}{\partial y}$

  along z - direction: $E_g(z) = -\dfrac{\partial V}{\partial z}$

$$\boxed{\vec{E}_g = -\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath} - \frac{\partial V}{\partial z}\hat{k}}$$

Potential due to a Uniform Ring at an Axial Point





$$\boxed{V_P = -\frac{GM}{\sqrt{R^2 + x^2}}}$$

$$\vec{E}_g = -\frac{\partial V}{\partial x}\hat{\imath}$$

$$\Rightarrow \vec{E}_g = -\frac{\partial}{\partial x}\left(-\frac{GM}{\sqrt{R^2 + x^2}}\right)\hat{\imath}$$

$$\boxed{\vec{E}_g = -\frac{GMx}{(R^2 + x^2)^{3/2}}\hat{\imath}}$$

A uniform circular ring of mass M and radius R is placed in yz plane with centre at origin. A particle of mass m is released from rest at a point $x = 2R$. Find the speed with which it will pass through the centre of ring.

Solution :

Applying mechanical energy conservation,

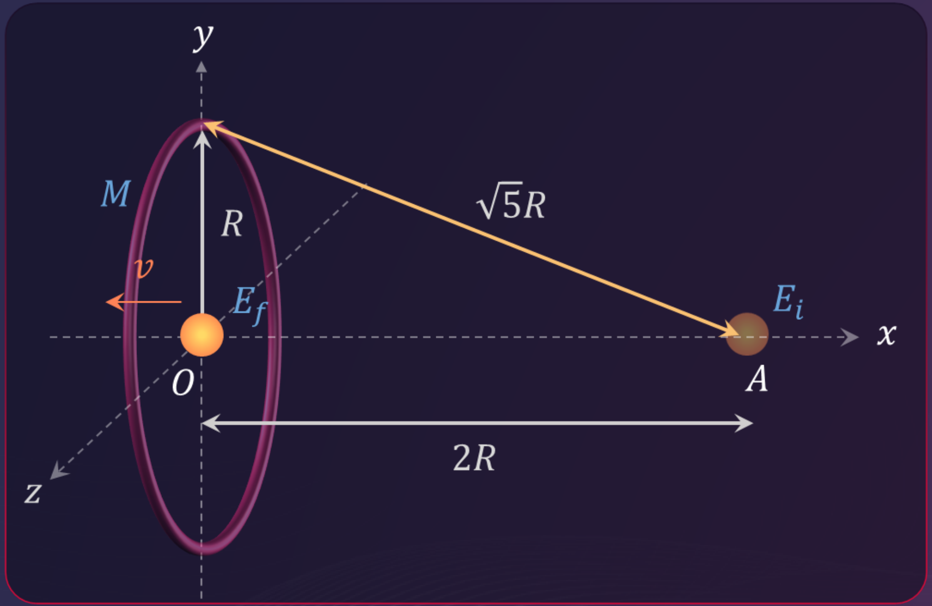
$$K_A + U_A = K_O + U_O$$

$$K_O = U_A - U_O = m(V_A - V_O)$$

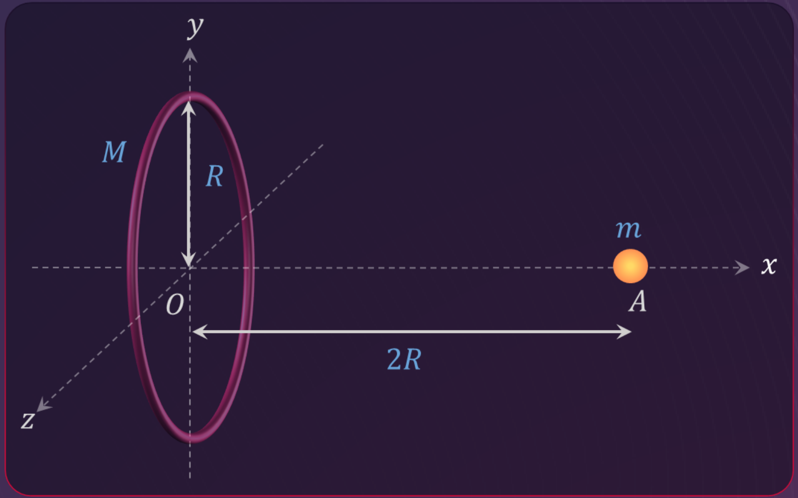
$$\frac{1}{2}mv^2 = m\left[-\frac{GM}{\sqrt{5}R} - \left(-\frac{GM}{R}\right)\right]$$

$$v^2 = \frac{2GM}{R}\left[1 - \frac{1}{\sqrt{5}}\right]$$

$$\boxed{v = \left[\frac{2(\sqrt{5}-1)GM}{\sqrt{5}R}\right]^{1/2}}$$

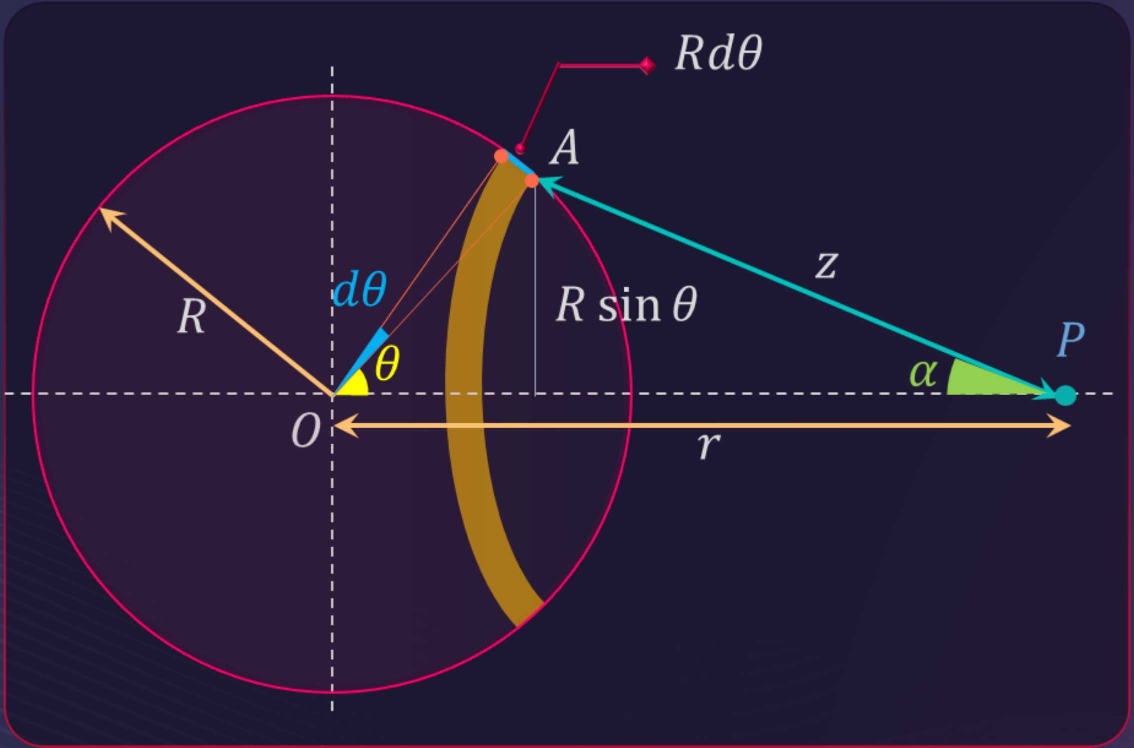


$$V_O = -\frac{GM}{R} \qquad V_A = -\frac{GM}{\sqrt{5}R}$$



Potential due to a Uniform Thin Spherical Shell

To find potential due to the shell, we can divide the shell into multiple small ring elements.



$$dm = \frac{M}{4\pi R^2} \times (2\pi R\sin\theta)(Rd\theta)$$

$$\boxed{dm = \frac{M}{2}\sin\theta\,(d\theta)}$$

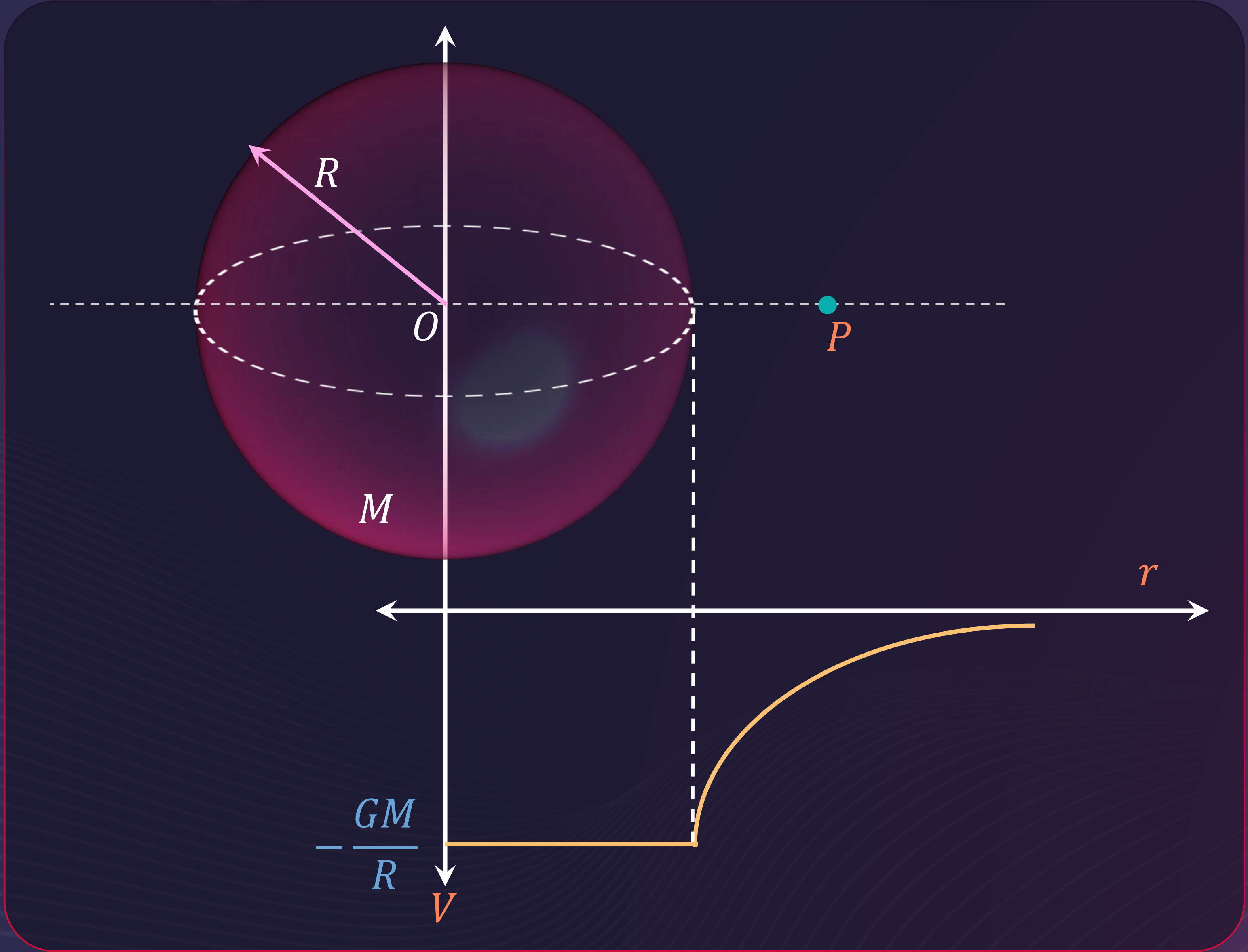
$$dV = -\frac{Gdm}{z} = -\frac{GM\sin\theta}{2}\frac{d\theta}{z}$$

$$\boxed{dV = -\frac{GM}{2Rr}dz}$$

In $\Delta OAP, R^2 + r^2 - z^2 = 2Rr\cos\theta$

Differentiating and solving, $\frac{z}{Rr}dz = \sin\theta\, d\theta$

Potential due to a Uniform Thin Spherical Shell



Case 1: P is outside the shell $(r > R)$

$$V = -\frac{GM}{r}$$

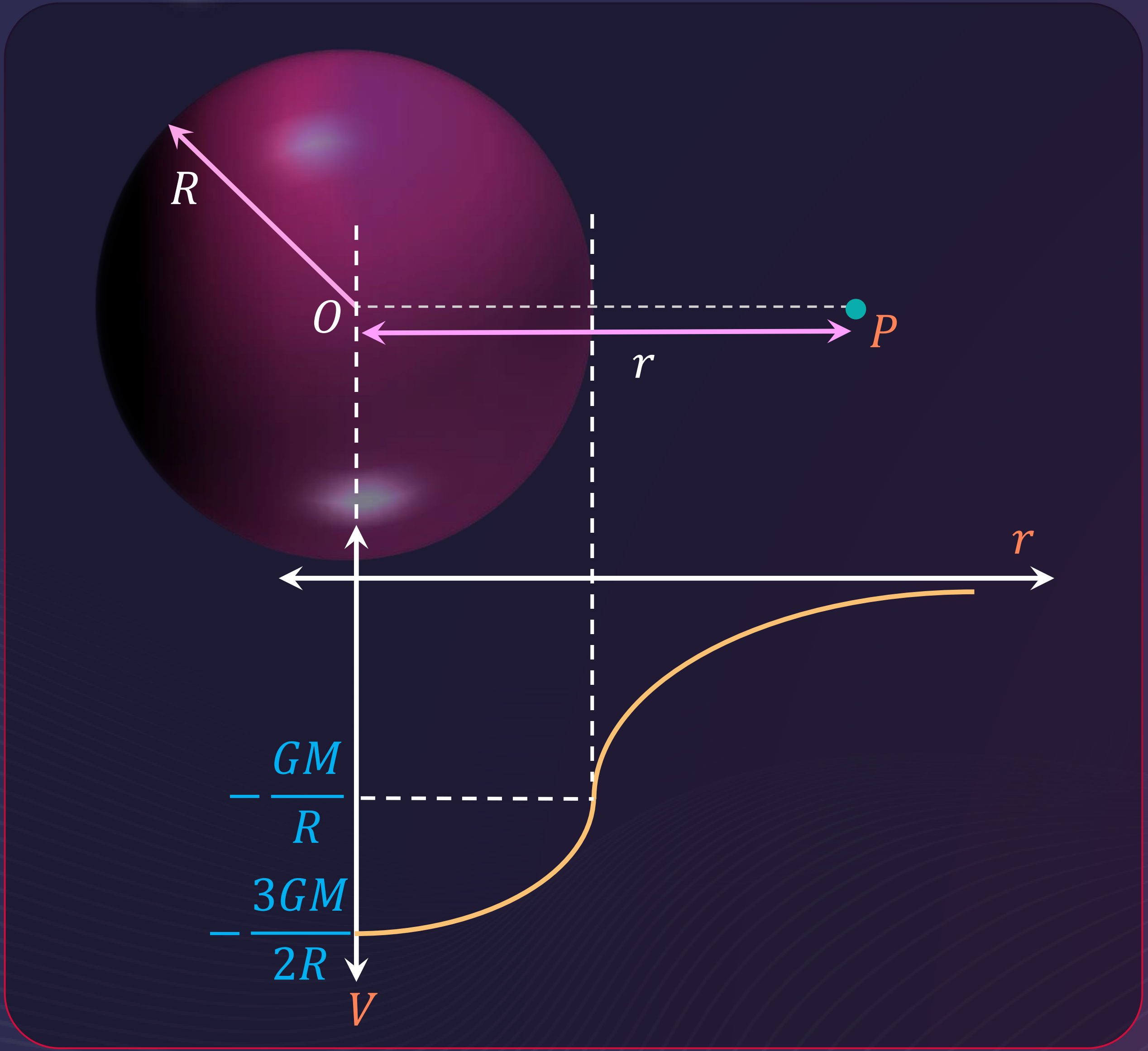
Case 2: P is at the surface of shell $(r = R)$

$$V = -\frac{GM}{R}$$

Case 3: P is inside the shell $(r < R)$

$$V = -\frac{GM}{R}$$

Potential due to a Uniform Solid Sphere



Case 1: Potential at an External Point $(r > R)$

$$V = -\frac{GM}{r}$$

Case 2: Potential at the surface $(r = R)$

$$V = -\frac{GM}{R}$$

Case 3: Potential at an Internal Point $(r < R)$

$$V = -\frac{GM}{2R^3}(3R^2 - r^2)$$

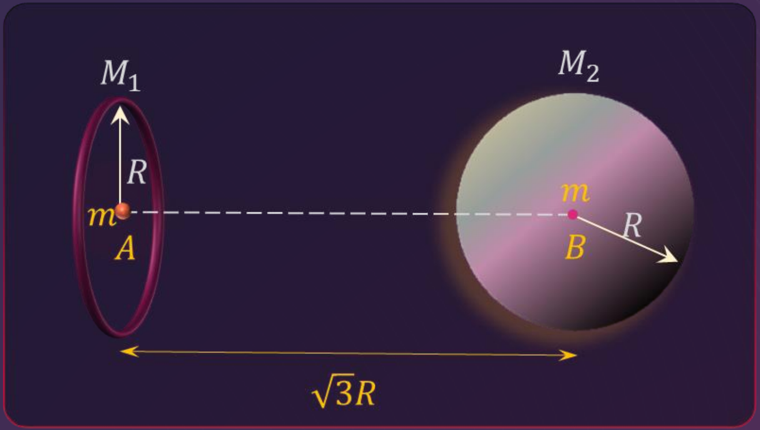
A uniform ring of mass M_1 and a uniform solid sphere of mass M_2 are separated by a distance $\sqrt{3}R$. Imagine that a small object of mass m is displaced from A to B. Find the work done by the gravitational force.

Solution :

At A:

$$V_A = (V_A)_{\text{Due to Ring}} + (V_A)_{\text{Due to solid sphere}}$$

$$V_A = -\frac{GM_1}{R} - \frac{GM_2}{\sqrt{3}R}$$



$$m(V_B - V_A) = U_B - U_A = -W_{conservative}$$

At B:

$$V_B = (V_B)_{\text{Due to Ring}} + (V_B)_{\text{Due to solid sphere}}$$

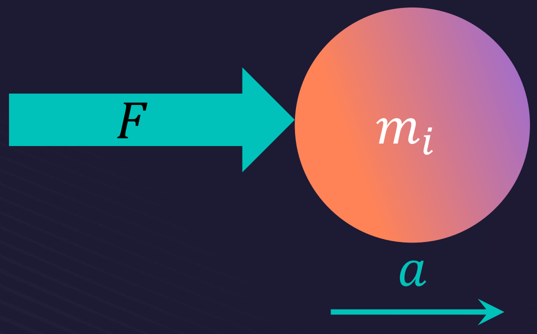
$$V_B = -\frac{GM_1}{2R} - \frac{3GM_2}{2R}$$

$$W_g = -\Delta U = -m(V_B - V_A)$$

$$\boxed{\Rightarrow W = \frac{Gm}{R}\left(-\frac{M_1}{2} + M_2\left(\frac{3}{2} - \frac{1}{\sqrt{3}}\right)\right)}$$

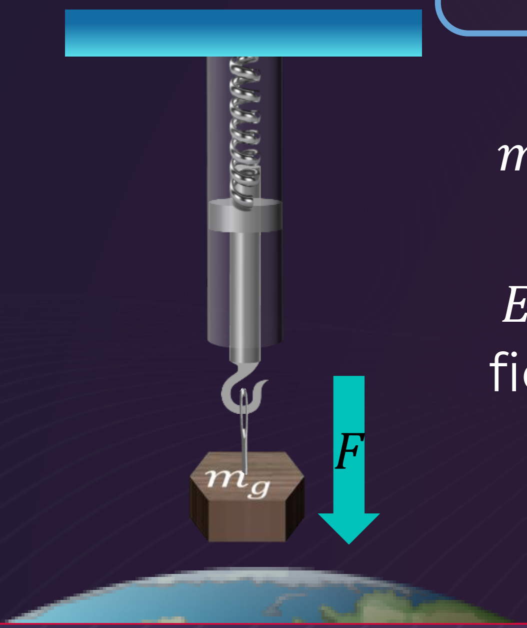
Inertial and Gravitational mass

- **Inertial mass**: It is a measure of an object's resistance to acceleration when a force is applied.



$$\boxed{m_i = \frac{F}{a}}$$

- **Gravitational mass**: It is determined by the strength of the gravitational force experienced by a body in the gravitational field.

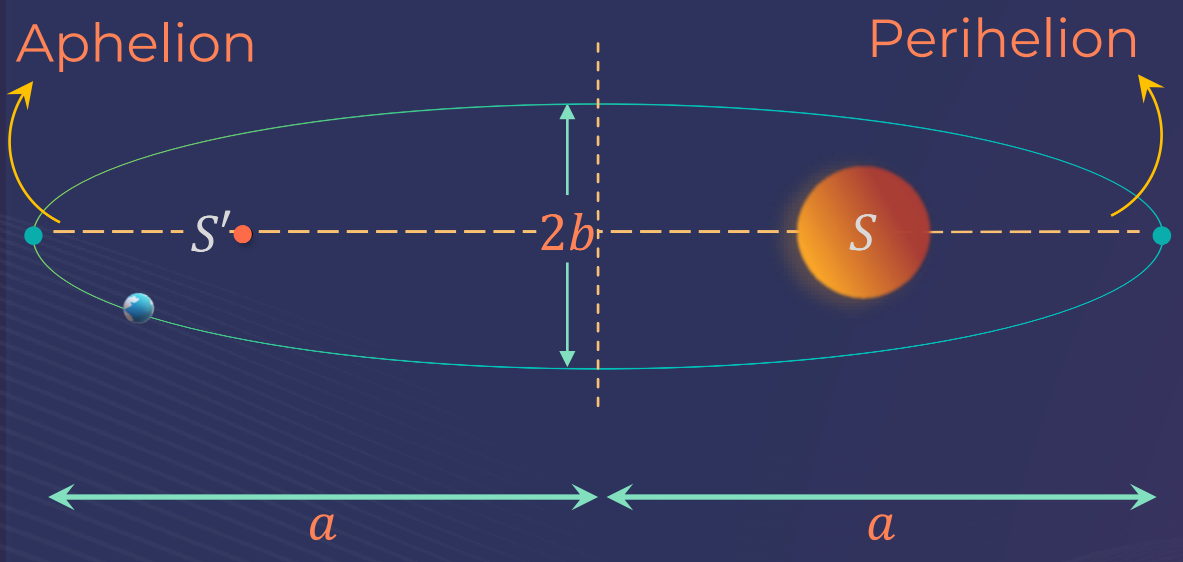


$$\boxed{m_g = \frac{F_g R^2}{GM}}$$

$$m_g = \frac{F_g}{E_g}$$

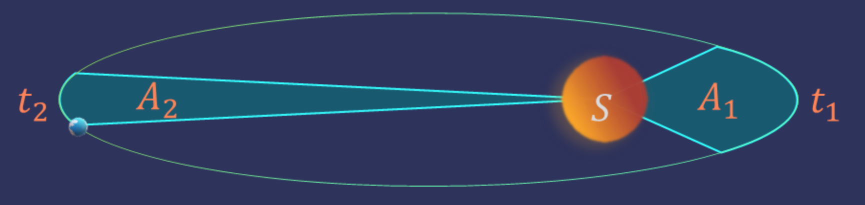
E_g = Gravitational field strength

Kepler's 1st Law : Law of Elliptical Orbits

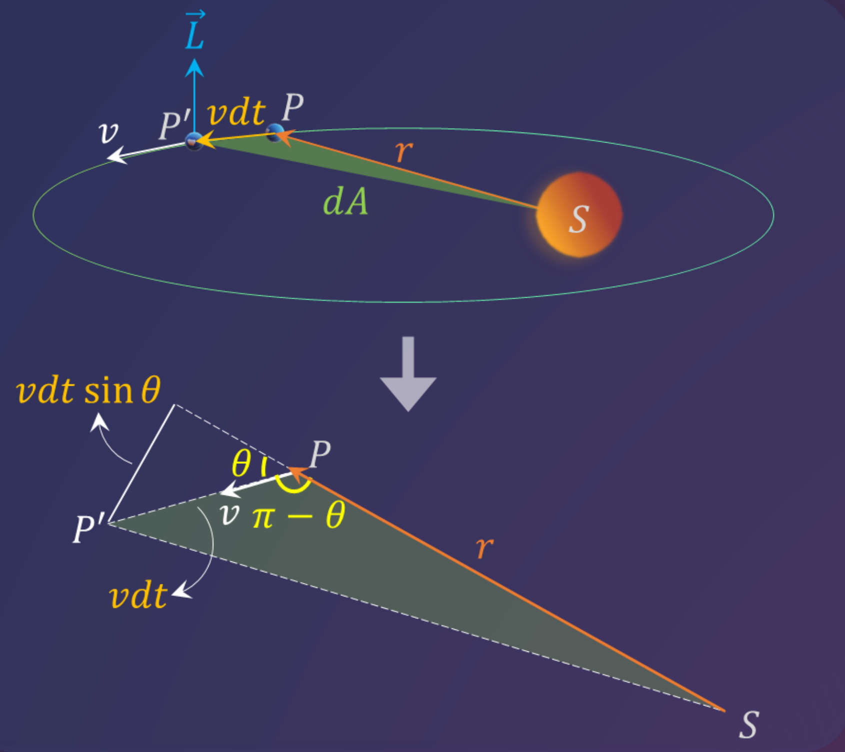


- Perihelion & Aphelion are the closest & farthest points of approach.
- The separation velocity is zero at these points or the entire velocity is perpendicular to the line joining the Sun & Planet.

Kepler's 2^{nd} Law : Law of Areas



If $t_1 = t_2, A_1 = A_2$



- Radius vector joining the Sun to any planet sweeps out equal areas in equal intervals of time.
- Areal velocity $\left(\frac{dA}{dt}\right)$ remains constant.

$$\boxed{\frac{A_1}{t_1} = \frac{A_2}{t_2}}$$

θ- Instantaneous angle between $\vec{v}$ and $\vec{r}$

Area swept by the radius vector,

$$dA \approx Area(\Delta SPP')$$

$$\Rightarrow dA \approx \frac{1}{2}(r)[(vdt)\sin(\pi-\theta)] \quad \left[\because A(\Delta) = \frac{1}{2}bh\right]$$

$$\Rightarrow dA = \frac{rv}{2}dt\sin\theta$$

$$\Rightarrow \frac{dA}{dt} = \frac{rv}{2}\sin\theta$$

$$\frac{dA}{dt} = \frac{rv}{2}\sin\theta \times \frac{m}{m}$$

$$\Rightarrow \frac{dA}{dt} = \frac{mvr\sin\theta}{2m} = \frac{L}{2m}$$

$$\Rightarrow \frac{dA}{dt} = \frac{L}{2m} = constant$$

$$\left[\because \frac{dA}{dt} = constant\right]$$

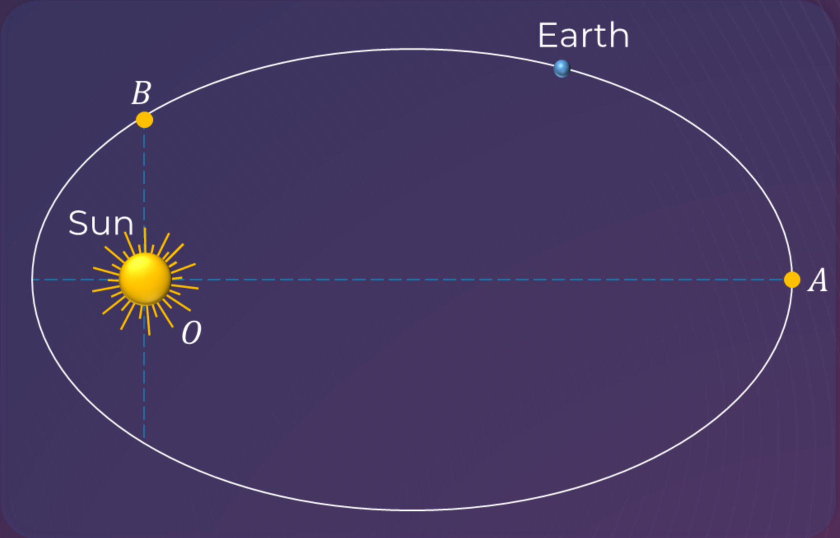
$$\Rightarrow L = constant$$

$$[\because 2m = constant]$$

$\Rightarrow$ Angular momentum of the system remains conserved.

The earth moves around the sun in an elliptical orbit as shown in the figure. The ratio $\frac{OA}{OB} = x$. The ratio of the speed of the earth at B to that at A is

Solution : Given: $\frac{OA}{OB} = x$



Using the conservation of angular momentum,

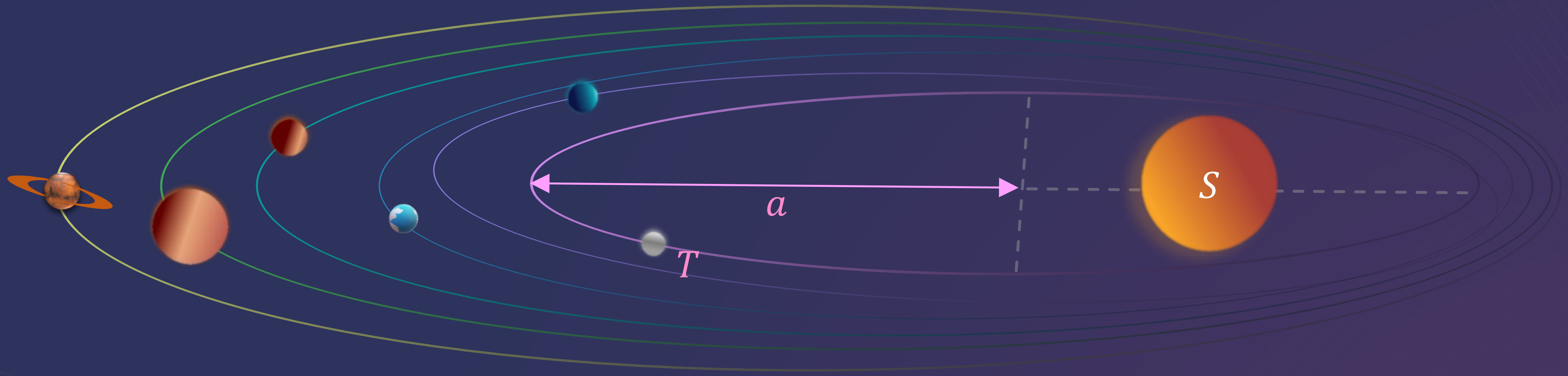
$mvr =$ constant

$\Rightarrow mv_A r_A = mv_B r_B$

$\Rightarrow \frac{v_B}{v_A} = \frac{r_A}{r_B} = \frac{OA}{OB} = x$

$$\boxed{\frac{v_B}{v_A} = x}$$

Kepler's 3rd Law : Harmonic Law/Law of Time Periods



Law states that the square of the planet's time period of revolution is directly proportional to the cube of semi-major axis length of its orbit.

$$\boxed{\frac{T^2}{a^3} = constant}$$

T – Time period of revolution
a – Semi-major axis length of the orbit

NOTE: This is true for a planet – satellite system as well.

The time period of a satellite of the earth is $5h$. If the separation between the earth and the satellite is increased to four times the previous value, the new time period will become

a) $10\ h$

b) $80\ h$

c) $40\ h$

d) $20\ h$

Solution : Given: $T_1 = 5h, \quad R_2 = 4R_1$

To find: T_2

According to Kepler's third law,

$$T^2 \propto R^3$$

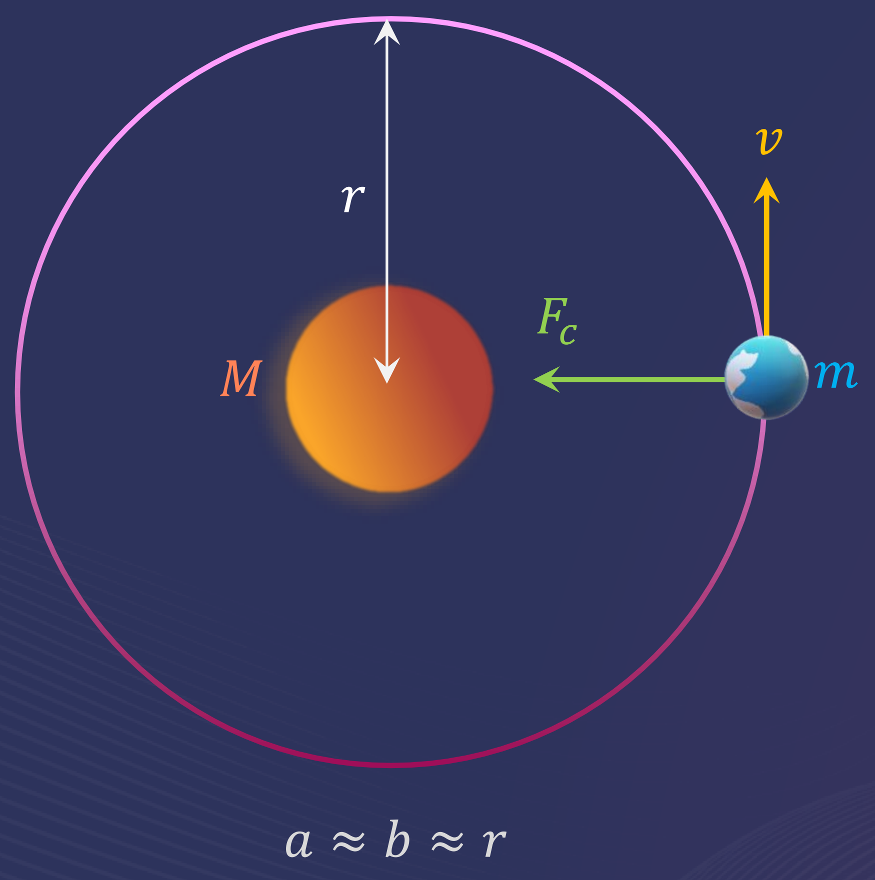
$$\Rightarrow \left(\frac{T_1}{T_2}\right)^2 = \left(\frac{R_1}{R_2}\right)^3$$

$$\Rightarrow \left(\frac{T_1}{T_2}\right) = \left(\frac{R_1}{R_2}\right)^{\frac{3}{2}} = \left(\frac{1}{4}\right)^{3/2}$$

$$\Rightarrow T_2 = 8 \times T_1 = 40\ h$$

$$\boxed{T_2 = 40\ h}$$

Circular Orbits



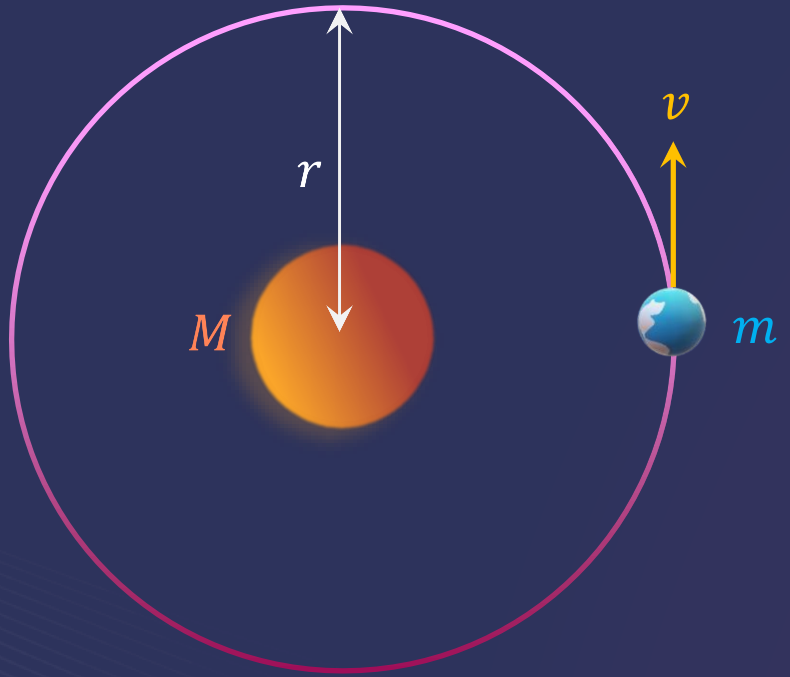
- Since semi-major axis & semi-minor axis are equal, the planet performs uniform circular motion about the Sun.
- Gravitational Force (F_G) provides the necessary Centripetal Force (F_c).

$$\Rightarrow \frac{GMm}{r^2} = \frac{mv^2}{r}$$

$\therefore$ Orbital Speed, $\boxed{v = \sqrt{\frac{GM}{r}}}$

NOTE: Since $M \gg m$, we consider the Sun as an inertial frame of reference

Summary



$$K = \frac{GMm}{2r}$$

$$U = -\frac{GMm}{r}$$

$$E = -\frac{GMm}{2r}$$

- Time period(T): Time taken for completing one revolution.

$$T = \frac{2\pi r}{v} \quad \text{OR} \quad T^2 = \frac{4\pi^2}{GM} r^3$$

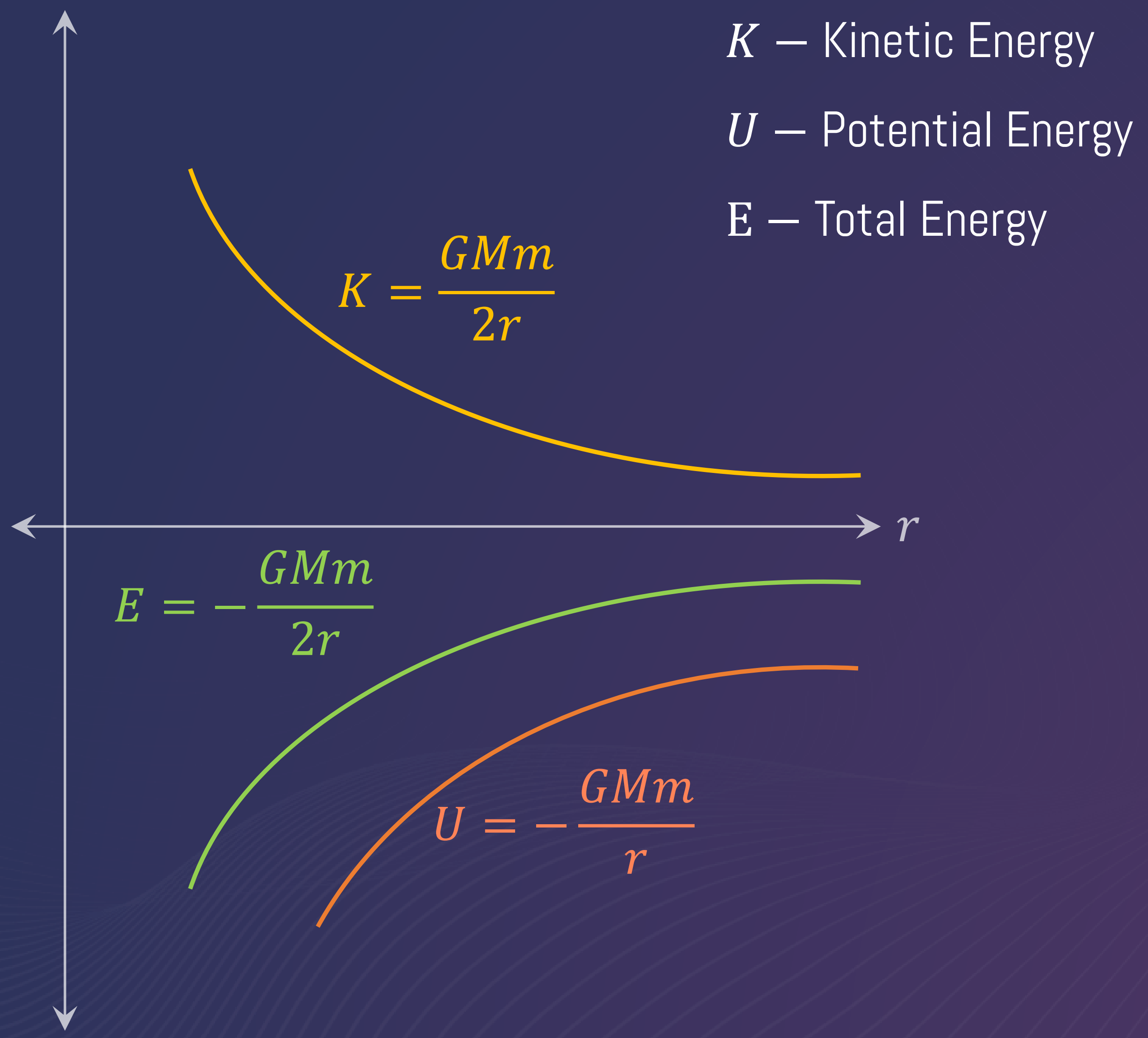
- Total Energy of planet is,

$$E = -K = \frac{U}{2} = -\frac{GMm}{2r}$$

- Binding Energy for an object in orbit is,

$$B.E = |E|$$

Variation of Energy with Distance



A $400\ kg$ satellite is in a circular orbit of radius $2R_E$ about the Earth. How much energy is required to transfer it to a circular orbit of radius $4R_E$? [$M_E = 6 \times 10^{24}\ kg, R_E = 6.4 \times 10^6\ m$]

Solution :

Given: $m = 400\ kg, M_E = 6 \times 10^{24}\ kg, R_E = 6.4 \times 10^6\ m$

Energy required to transfer the satellite from $2R_E$ to $4R_E$,

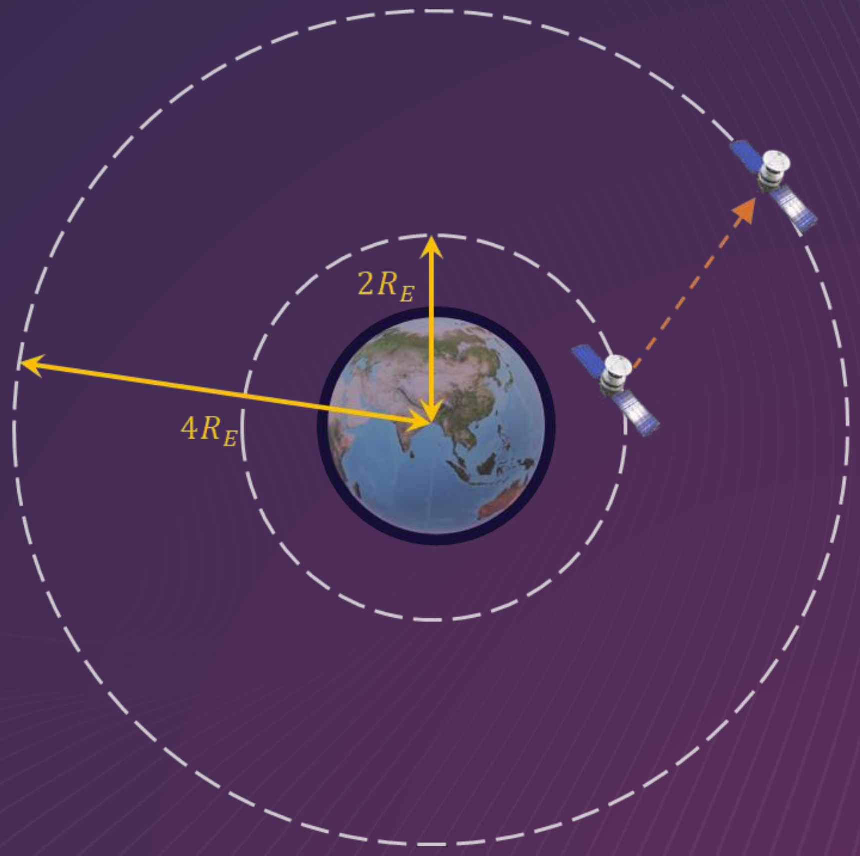
$$E = E_f - E_i$$

$$= \left(-\frac{GMm}{2r_2}\right) - \left(-\frac{GMm}{2r_1}\right)$$

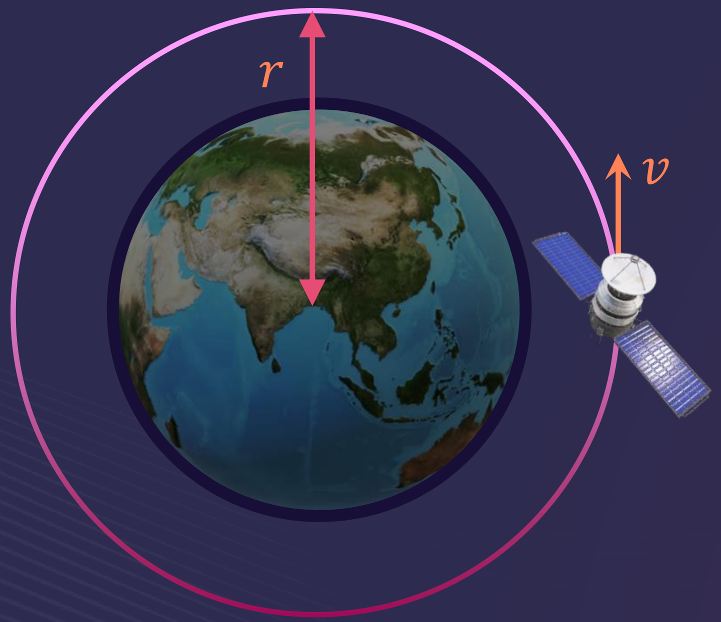
$$= \left(-\frac{GMm}{8R_E}\right) - \left(-\frac{GMm}{4R_E}\right) = \frac{GMm}{8R_E}$$

$$= \frac{6.64 \times 10^{-11} \times 6 \times 10^{24} \times 400}{8 \times 6.4 \times 10^6}$$

$$\boxed{E = 3.13 \times 10^9\ J}$$



Satellites



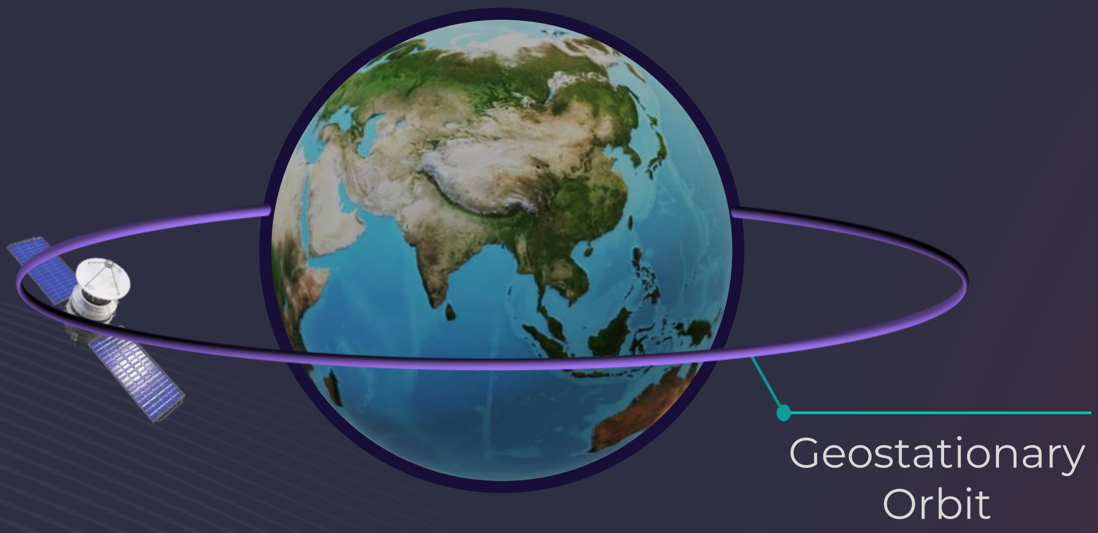
- For a satellite orbiting very close to Earth $(r \approx R)$.

Orbital speed, $$v = \sqrt{\frac{GM}{r}} \approx 7.9\ km/s$$

Time period, $$T = \frac{2\pi R}{v} \approx 84.6\ minutes$$

Total energy, $$E \approx -\frac{GMm}{2R}$$

Geostationary Satellites



$\boxed{T = 24\ hr}$

- For Geostationary satellites,

$$(\omega_{satellite})_{revolution} = (\omega_{earth})_{rotation}$$

Geostationary radius, $\boxed{r \approx 4.2 \times 10^4\ km}$

Orbital velocity, $\boxed{v = \sqrt{\dfrac{GM}{r}} \approx 3.1\ km/s}$

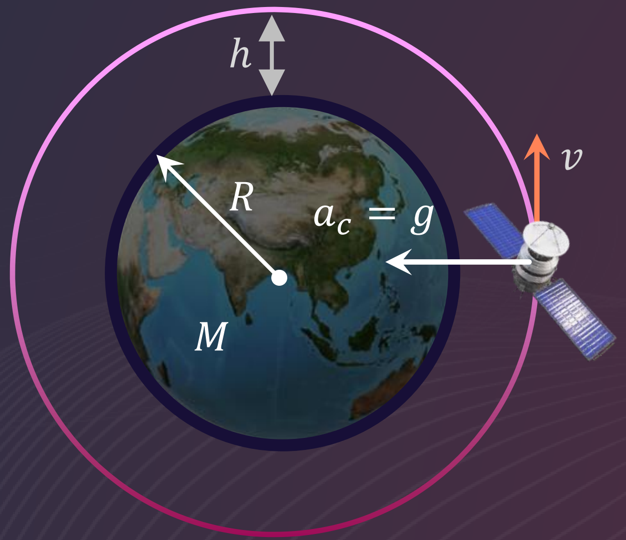
Polar Satellites

- Orbits of these satellites are perpendicular to equator.
- Time period = 2 to 3 hr.
- Height = 200 to 1,000 km.

Weightlessness

- It is the complete or near-complete absence of the sensation of weight. (free-fall)

- Bodies inside satellite experience weightlessness since the satellite has an acceleration towards the Centre of the earth, which is exactly equal to the gravitational acceleration at that point.



The mean radius of earth is R. Its angular speed about its own axis is ω and the acceleration due to gravity at earth's surface is g. Find the radius of the orbit of a geostationary satellite as a function of ω, g & R.

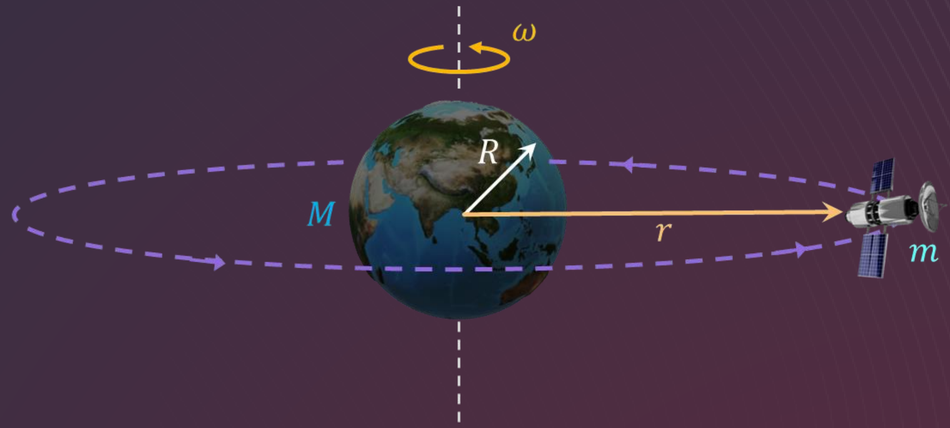
Solution : Centripetal force = Gravitational force

$$\frac{m(r\omega)^2}{r} = \frac{GMm}{r^2} \qquad (\because v = r\omega)$$

$$r^3 = \frac{GM}{\omega^2} = \left[\frac{GM}{R^2}\right]\frac{R^2}{\omega^2}$$

$$r^3 = \frac{gR^2}{\omega^2} \qquad \left(\because g = \frac{GM}{R^2}\right)$$

$$\boxed{r = \left(\frac{gR^2}{\omega^2}\right)^{\frac{1}{3}}}$$



Escape Speed

- Escape speed is the minimum speed required by a body to be projected with to overcome the gravitational pull of the earth.

$$\frac{1}{2}mu^2 + \left(-\frac{GMm}{R}\right) = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R+h}\right)$$

For $h = \infty$, $\frac{1}{2}mv^2 \geq 0$

$$\frac{1}{2}mu^2 + \left(-\frac{GMm}{R}\right) \geq 0$$

$$\boxed{u \geq \sqrt{\frac{2GM}{R}}}$$

A satellite is revolving in a circular orbit at a height h from the Earth's surface (radius of earth R, $h << R$). The minimum increase in its orbital velocity required, so that the satellite could escape from the Earth's gravitational field, is close to (Neglect the effect of atmosphere)

Solution : Extra velocity required to escape the gravitational field,

$$\Delta v = v_{escape} - v_{orbital}$$

$$= \sqrt{\frac{2GM}{R}} - \sqrt{\frac{GM}{R}}$$

$$= \sqrt{2gR} - \sqrt{gR} \qquad \left(\because g = \frac{GM}{R^2}\right)$$

$$\boxed{\Delta v = \sqrt{gR}(\sqrt{2} - 1)}$$

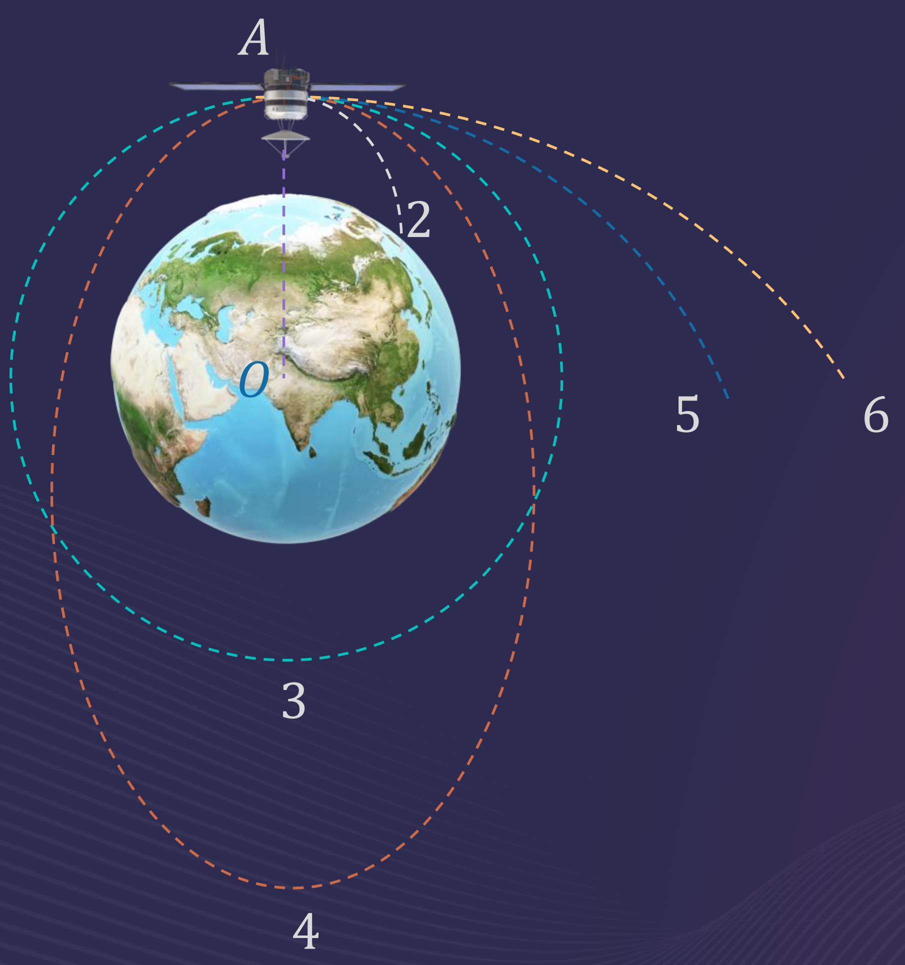
a) $\sqrt{2gR}$

b) $\sqrt{gR/2}$

c) $\sqrt{gR}$

d) $\sqrt{gR}(\sqrt{2} - 1)$

Trajectories for Different Launching Speeds



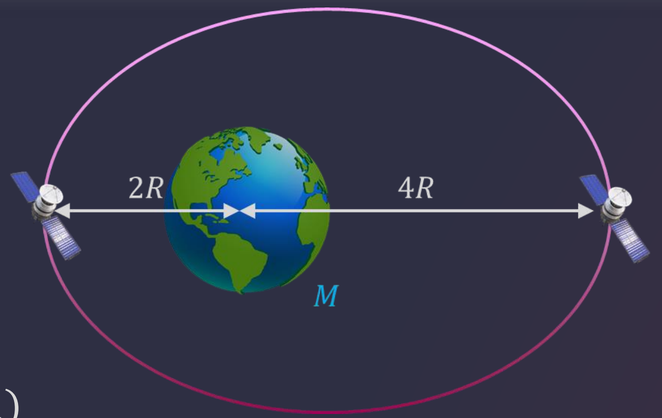
- Case 1: If $v = 0$, then path is a straight line joining AO.
- Case 2: If $0 < v < v_0$, then path is ellipse.
- Case 3: If $v = v_0$, then path is a circle with O as centre.
- Case 4: If $v_0 < v < v_e$, then path is ellipse.
- Case 5: If $v = v_e$, then path is parabola.
- Case 6: If $v > v_e$, then path is hyperbola.

The minimum and maximum distance of a satellite from the center of the earth are $2R$ and $4R$, respectively, where R is the radius of earth and M is the mass of the earth. Find

a) Minimum & Maximum Speed.

b) Radius of curvature at the point of minimum distance of separation.

Solution :



$$L_P = L_A$$

$$\Rightarrow mv_P r_P = mv_A r_A$$

$$\Rightarrow v_P(2R) = v_A(4R)$$

$$\Rightarrow v_P = 2v_A \qquad \dots (1)$$

a)

$$K_A + U_A = K_P + U_P$$

$$\Rightarrow \frac{1}{2}mv_A^2 - \frac{GMm}{4R} = \frac{1}{2}mv_P^2 - \frac{GMm}{2R}$$

$$\Rightarrow \frac{v_A^2}{2} = \frac{v_P^2}{2} - \frac{GM}{4R}$$

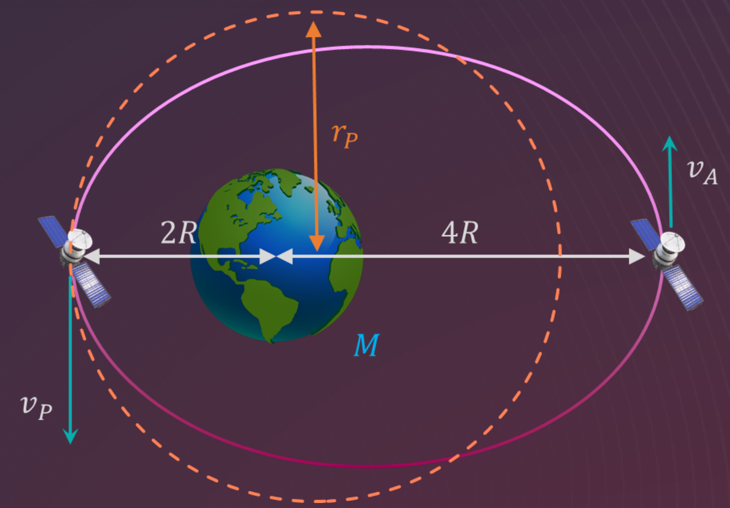
$$\Rightarrow v_A^2 = v_P^2 - \frac{GM}{2R} \qquad \dots (2)$$

$$v_P = 2v_A \qquad \dots\dots (1)$$

$$v_A^2 = v_P^2 - \frac{GM}{2R} \qquad \dots\dots (2)$$

$$v_A^2 = 4v_A^2 - \frac{GM}{2R}$$

$$\boxed{v_A = \sqrt{\frac{GM}{6R}}, \quad v_P = 2\sqrt{\frac{GM}{6R}}}$$



b)

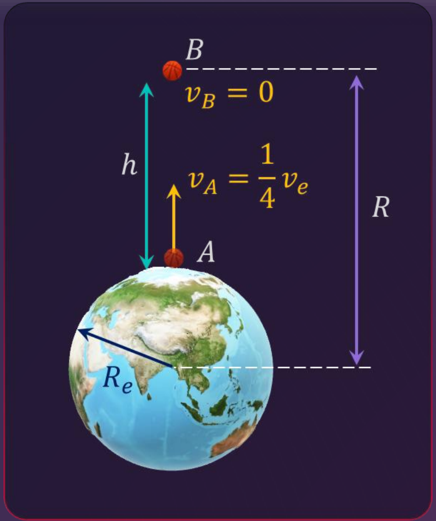
$$\sum \vec{F}_c = m\vec{a}_c$$

$$\frac{GM\cancel{m}}{(2R)^2} = \frac{\cancel{m}v_P^2}{r_P}$$

$$\boxed{r_P = \frac{8R}{3}} \qquad \left(\because v_P = 2\sqrt{\frac{GM}{6R}}\right)$$

A particle is projected vertically upwards from the surface of the Earth(radius R_e) with a speed equal to one fourth of escape speed(v_e).What is the maximum height attained by it above the surface of the Earth?

Solution : From the principle of conservation of mechanical energy,



$$K_A + U_A = K_B + U_B$$

$$\Rightarrow \frac{1}{2} m v_A^2 - \frac{GMm}{R_e} = \frac{1}{2} m(0)^2 - \frac{GMm}{R}$$

$$\Rightarrow \frac{1}{2} m \times \frac{1}{16}\left(\frac{2GM}{R_e}\right) = \frac{GMm}{R_e} - \frac{GMm}{R} \quad \left(\because v_A = \frac{1}{4} v_e = \frac{1}{4}\sqrt{\frac{2GM}{R_e}}\right)$$

$$\Rightarrow R = \frac{16}{15} R_e$$

$$\Rightarrow h = R - R_e \Rightarrow \boxed{h = \frac{R_e}{15}}$$

a) $\frac{16}{15} R_e$

b) $\frac{4}{15} R_e$

c) $\frac{R_e}{15}$

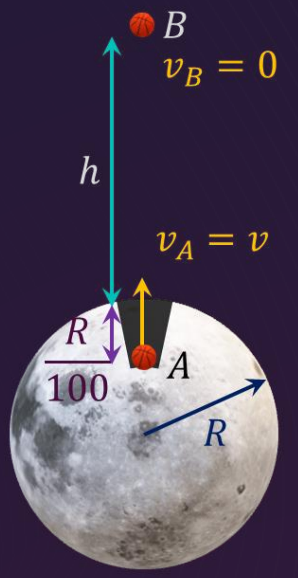
d) None of these

There is a crater of depth $\frac{R}{100}$ on the surface of the moon (radius R). A projectile is fired vertically upward from the crater with velocity, which is equal to the escape velocity v from the surface of the moon. Find the maximum height attained by the projectile.

Solution :

The projectile is fired from A with velocity v equal to escape velocity on surface of moon

$$v_A = v = \sqrt{\frac{2GM}{R}}$$



From the Conservation of mechanical Energy,

$$K_A + U_A = K_B + U_B$$

$$\Rightarrow \frac{1}{2}mv_A^2 - \frac{1}{2}m(0)^2 = m(V_B - V_A)$$

Considering moon to be a uniform solid sphere

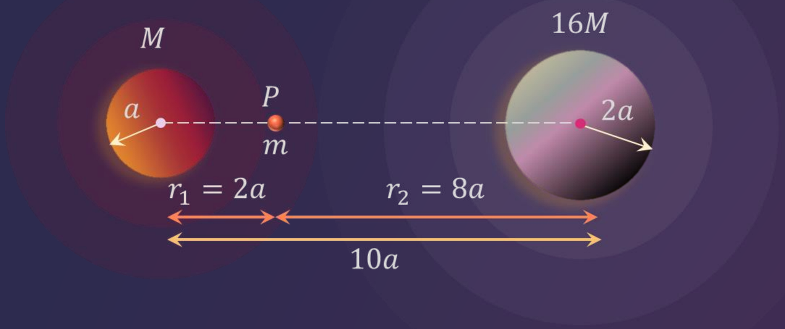
$$\Rightarrow \frac{v_A^2}{2} = \frac{2GM}{2R} = -\frac{GM}{R+h} - \left[-\frac{GM}{R^3}\left(1.5R^2 - 0.5\left(R - \frac{R}{100}\right)^2\right)\right] \quad \because v_A = v = \sqrt{\frac{2GM}{R}}$$

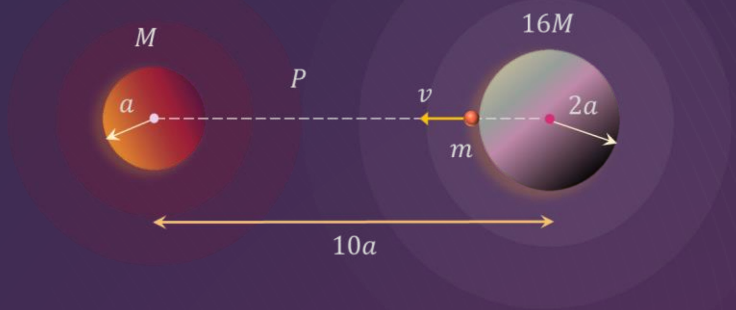
$$\Rightarrow \frac{1}{R} = -\frac{1}{R+h} + \frac{3}{2R} - \frac{1}{2R}\left(\frac{99}{100}\right)^2$$

$$R + h = 2R \times \frac{10000}{10000 - 9801} \approx 100.5R \quad \boxed{\Rightarrow h \approx 99.5R}$$

Distance between the centers of two stars is $10a$. The masses of these stars are M and $16M$ and their radii a and $2a$, respectively. A body of mass m is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of G, M and a.

Solution : Body will reach the smaller sphere when it crosses point P.





$$\frac{GM}{x^2} = \frac{G(16M)}{(10a - x)^2}$$

$$x = 2a \Rightarrow r_1 = 2a \;, r_2 = 8a$$

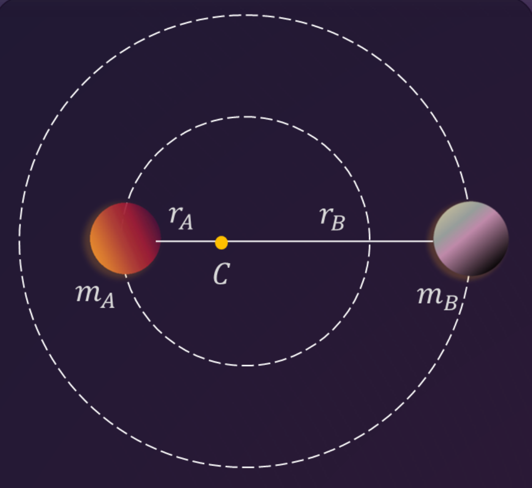
From the principle of conservation of mechanical energy,

$$\frac{1}{2}mv_{min}^2 + \left[-\frac{G(M)m}{8a} - \frac{G(16M)m}{2a}\right] = 0 + \left[-\frac{G(M)m}{2a} - \frac{G(16M)m}{8a}\right]$$

$$\boxed{v_{min} = \frac{3\sqrt{5}}{2}\sqrt{\frac{GM}{a}}}$$

A system of binary stars of masses m_A and m_B are moving in concentric circular orbits of radii r_A and r_B respectively. If T_A and T_B are their time periods respectively then,

Solution : When both are stars are farthest from each other, the gravitational force on one star due to the other is equal to centripetal force.



C – center of mass

$$F_g = F_C = mr\omega^2$$

$$\frac{Gm_Am_B}{(r_A + r_B)^2} = \frac{m_Ar_A4\pi^2}{T_A^2} = \frac{m_Br_B4\pi^2}{T_B^2}$$

The displacement for center of mass C for the system of stars will be zero

$$m_Ar_A = m_Br_B$$

$$\boxed{\Rightarrow T_A = T_B}$$

a $\frac{T_A}{T_B} = \left(\frac{r_A}{r_B}\right)^{\frac{3}{2}}$

b $T_A > T_B \ (if\ r_A > r_B)$

c $T_A > T_B \ (if\ m_A > m_B)$

d $T_A = T_B$