Welcome to

Aakash + BYJU'S NOTES

Laws of Motion

Mechanics

Dynamics (Cause)

Kinematics (Effect)

What is Force?

Force - A push or pull that changes or tends to change the state of motion.

State of motion

– Defined by velocity

– Zero, uniform or non uniform (accelerated)

Can change the direction of motion of a body

Can change the shape of a body

Force

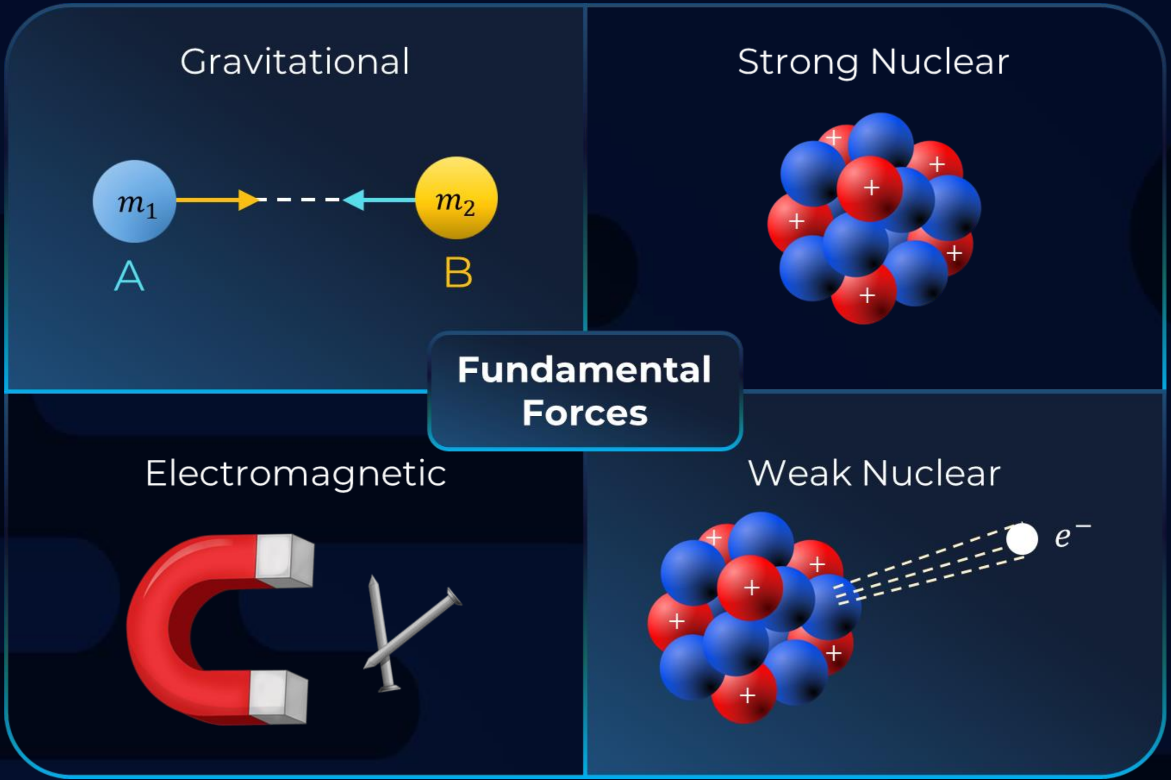
Can change the velocity of a body

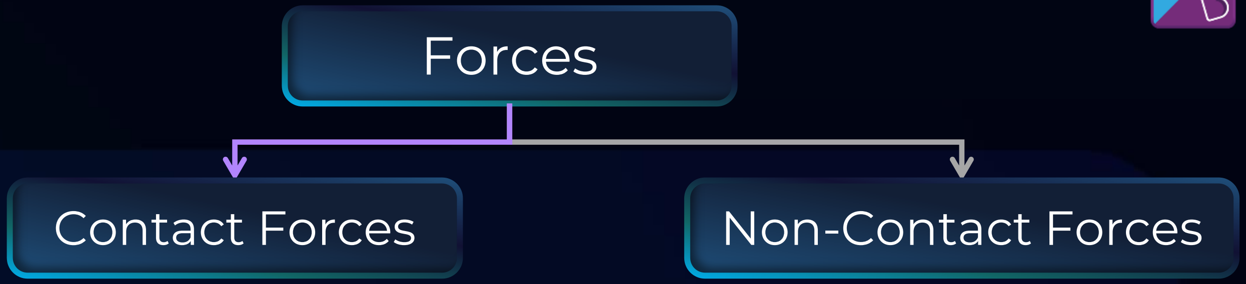
SI Unit : newton (N)

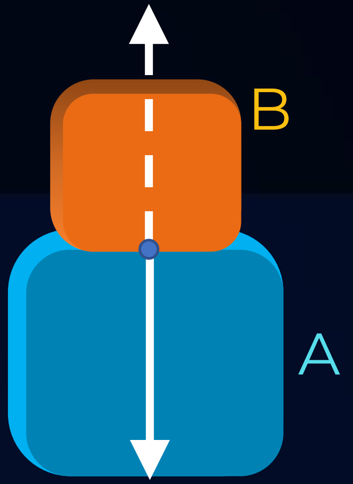
Dimension: $[M^1L^1T^{-2}]$

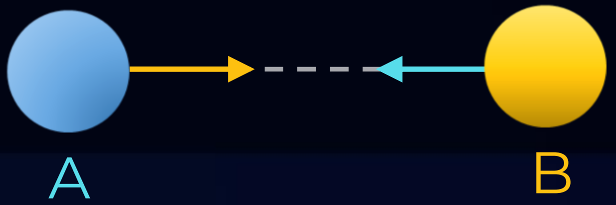
Force is a vector quantity.

Fundamental Forces in Nature



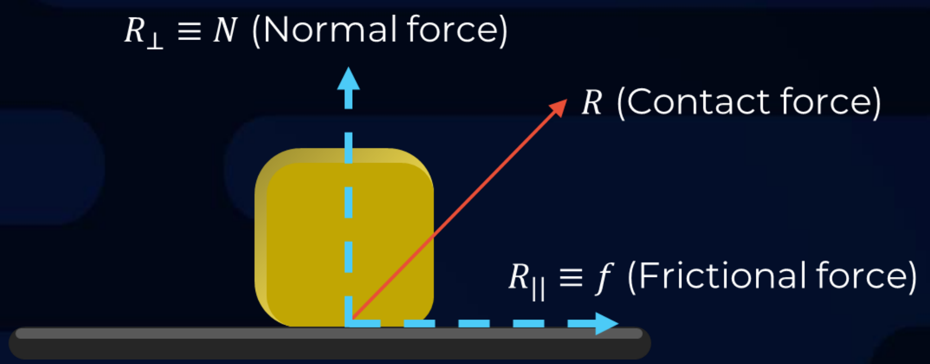






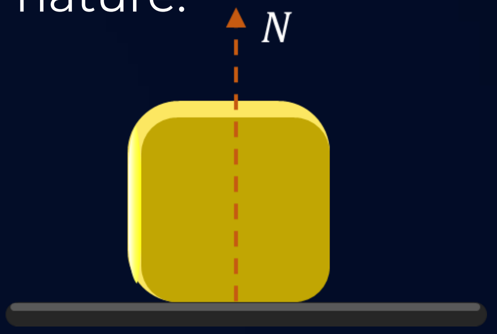
Contact Forces

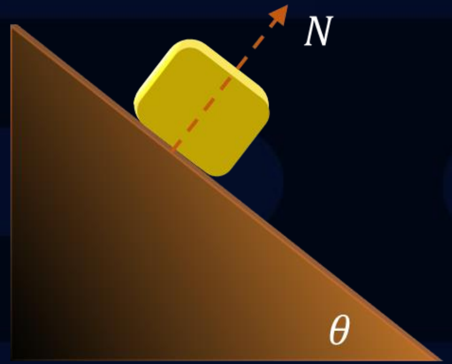
- Experienced by the bodies in contact
- Along the Surface – Frictional force
- Perpendicular to Surface – Normal force
- Electromagnetic in nature



Normal Reaction

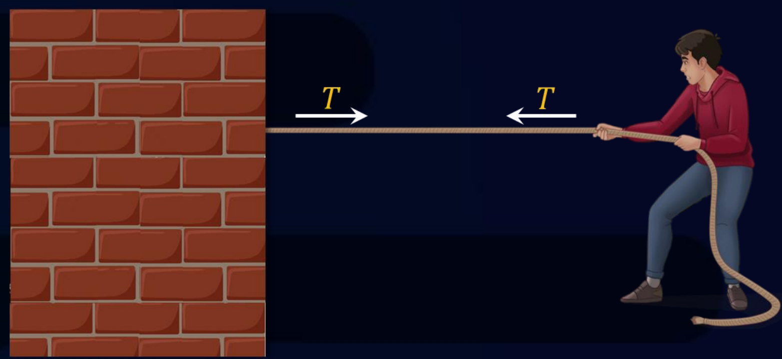
- Normal reaction is always Normal (90 °) to the contact surface or Interface.
- Normal reaction is always pushing in nature.





Tension

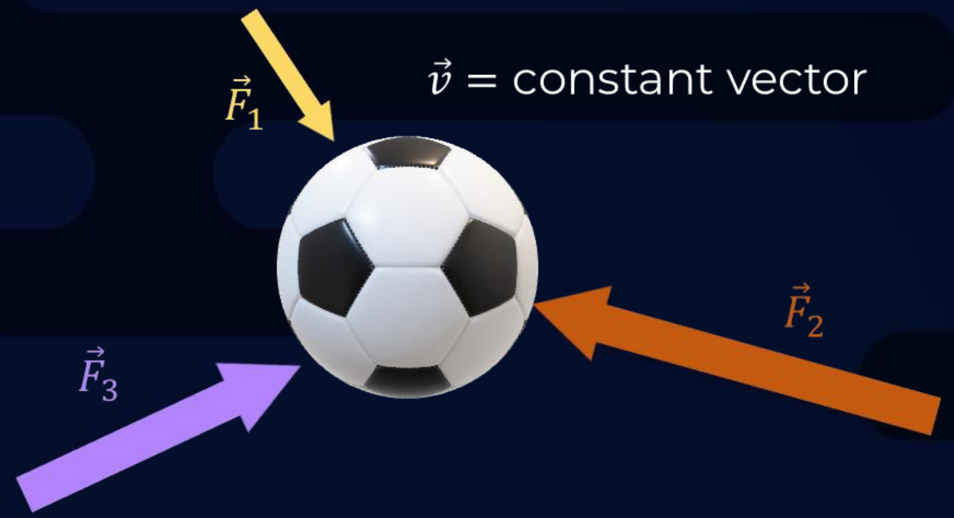
- Tension force develops in the string when it is pulled on both ends and it resists against change in its length.
- Tension always acts away from the tied ends.
- Tension is always pulling in nature.
- Tension force remains same throughout the ideal massless string.



Newton's First Law of Motion

An object will remain in a state of rest or of uniform motion in a straight line unless compelled to change its state by the action of an external force.

If the vector sum of all the forces acting on a body is zero, it remains unaccelerated.



$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 \Rightarrow \vec{a} = 0$$

- Law of Inertia: A body continues to be in its state of rest or of uniform motion along a straight line unless an external force is applied.

Inertia

of Rest

of Motion

of Direction

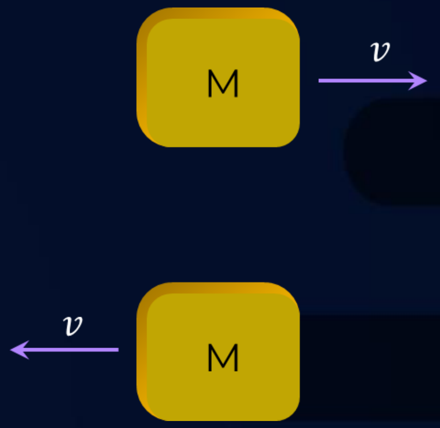
Newton’s Second Law of Motion

- The rate of change of momentum of an object equals the net external force and is in the direction of net force applied.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

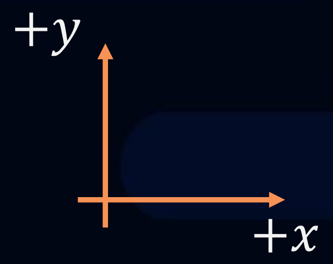
[Assuming mass, $m =$ constant]

Linear Momentum



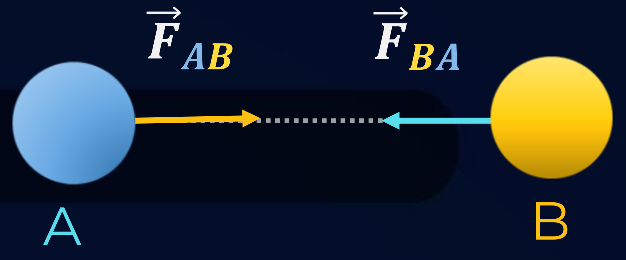
$\vec{p} = +mv\,\hat{\imath}$

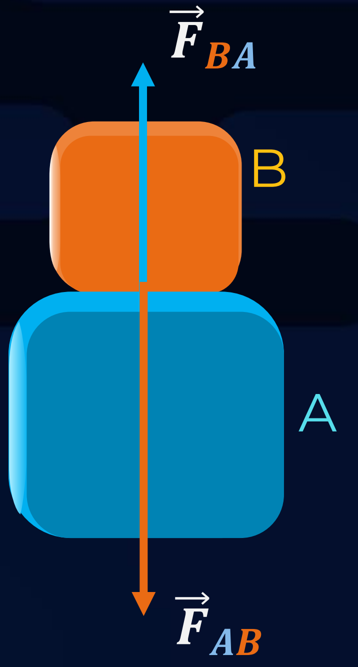
$\vec{p} = -mv\,\hat{\imath}$



Newton's Third Law of Motion

The two forces discussed while explaining Newton's third law constitute an Action - Reaction pair.

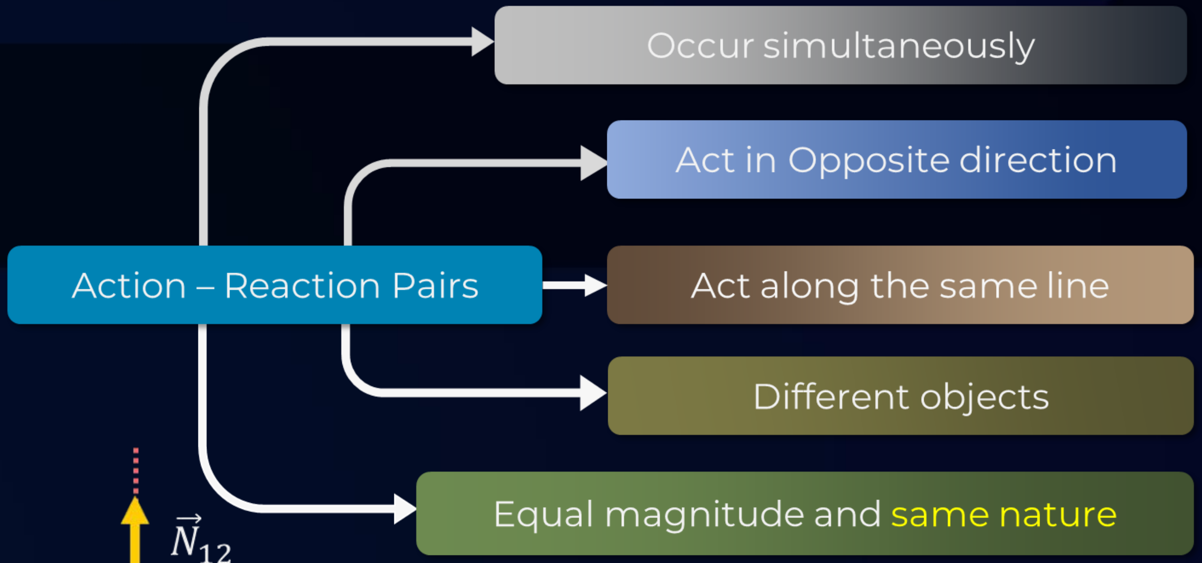




$\vec{F}_{BA}$ = Force on B by A

$\vec{F}_{AB}$ = Force on A by B

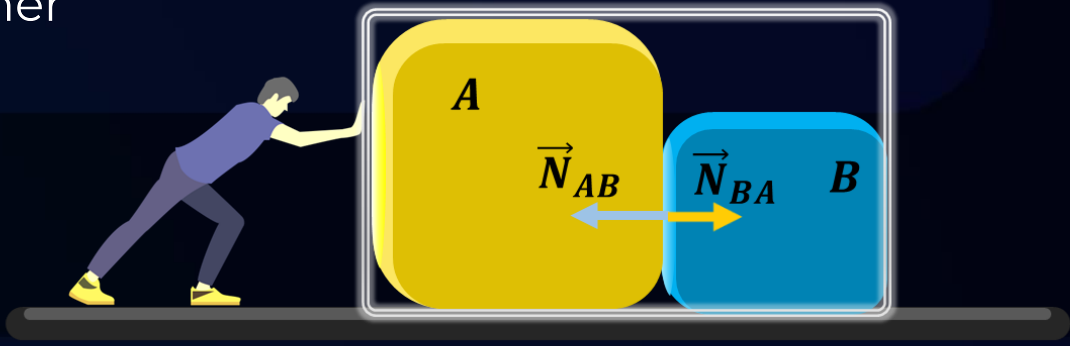
Conditions for Action – Reaction pair.



System

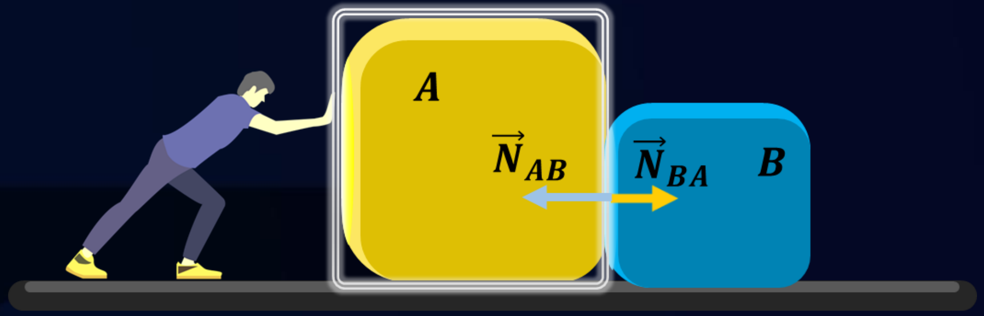
- Two or more than two objects that interact with each other form a system.
- While applying Newton's second law, make sure that all parts of the system chosen have same acceleration.
- The surroundings are everything else.
- Universe = System + Surroundings.
- Action-reaction pairs existing within the system are treated as internal forces.

SYSTEM-1



N_{AB} & N_{BA} are internal forces

SYSTEM-2

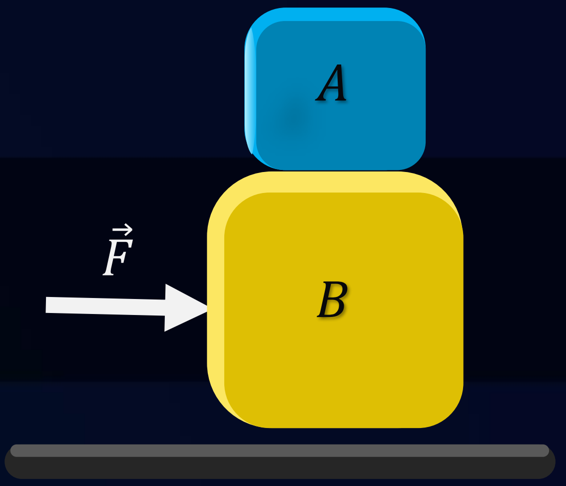


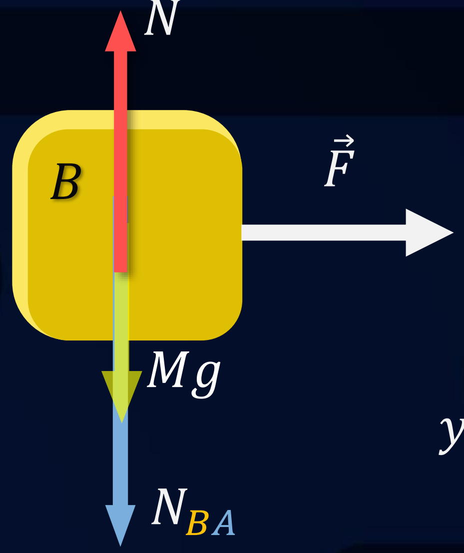
N_{AB} is external,
N_{BA} is not in System

- Note : All surfaces considered are smooth.

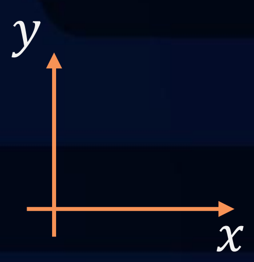
Steps to Draw FBD

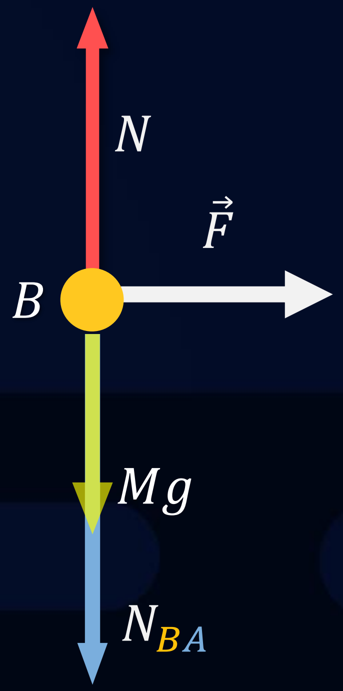
- Identify the system.
- Draw all the external forces acting on the system. (block B of mass M)
- Choose axes and resolve the forces along them.



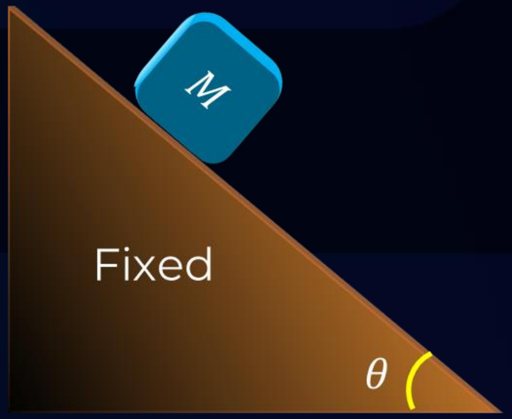


OR





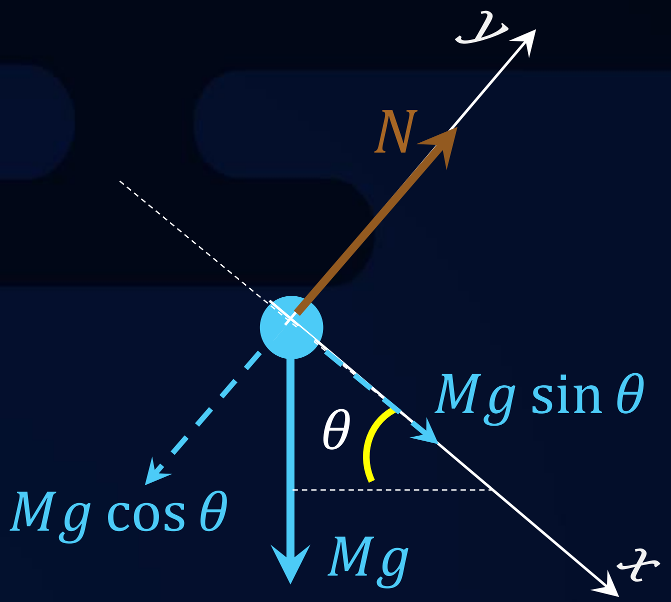
A block of mass M is placed on a wedge which is fixed on the ground as shown in the figure. Draw the free body diagram of the block.

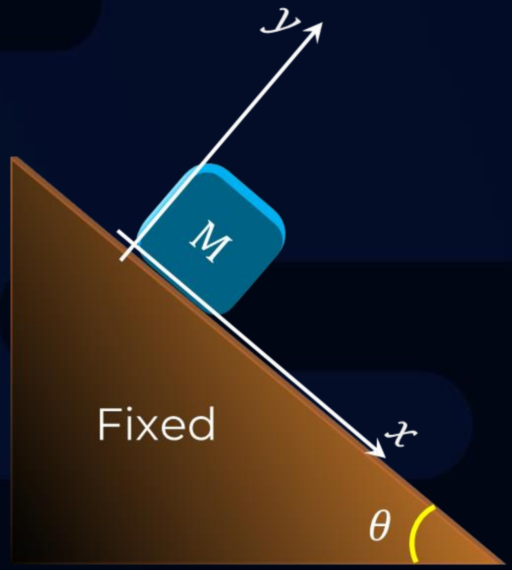


Solution :

- Identify the system.
- Draw all the external forces acting on the system.
- Choose axes and resolve the forces along them.

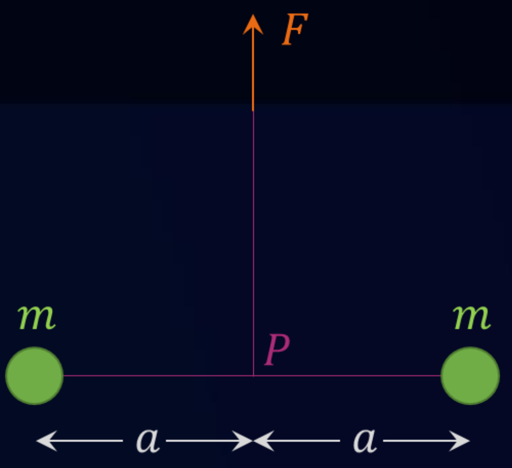
- Note : All surfaces considered are smooth.





Two particles of mass m each are tied at the ends of a light string of length $2a$. The whole system is kept on a frictionless horizontal surface with the string held tight so that each mass is at a distance a from the centre P as shown. Now, the midpoint of the string is pulled vertically upwards with a small but constant force F. As a result, the particles move towards each other on the surface. The magnitude of acceleration, when the separation between them becomes $2x$, is

(JEE 2007)



Solution :

From the FBD (point P)

$$F = T\cos\theta + T\cos\theta$$

$$= 2T\cos\theta$$

$$\Rightarrow T = \frac{F}{2}\sec\theta$$

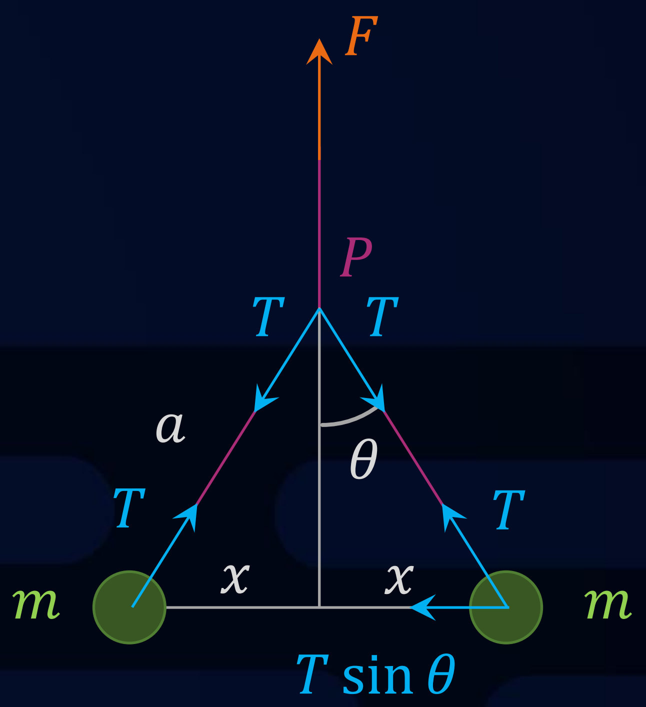
The acceleration of each particle is,

$$a = \frac{T\sin\theta}{m}$$

$$= \frac{F}{2}\sec\theta \times \frac{\sin\theta}{m}$$

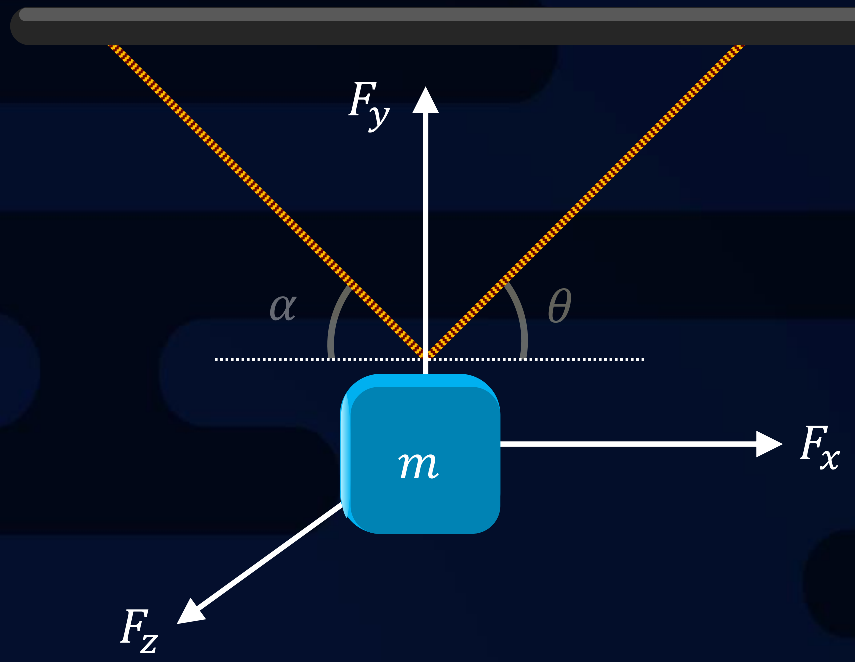
$$= \frac{F\tan\theta}{2m}$$

$$a = \frac{F}{2m}\frac{x}{\sqrt{a^2 - x^2}}$$



Equilibrium

When net force acting on the system is zero, the object is said to be in a state of equilibrium



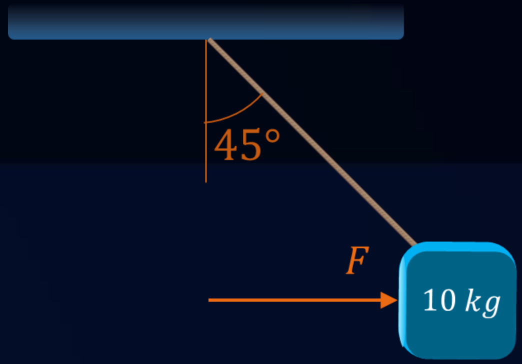
$$\sum \left(\vec{F}_{ext}\right)_{sys} = 0$$

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum F_z = 0$$

A mass of $10\ kg$ is suspended vertically by a rope from the roof. When a horizontal force is applied on the mass, the rope deviated at an angle of $45°$ at the roof point. If the suspended mass is at equilibrium, the magnitude of the force applied is



Solution:

For vertical equilibrium,

$$T \cos 45° = 100\ N$$

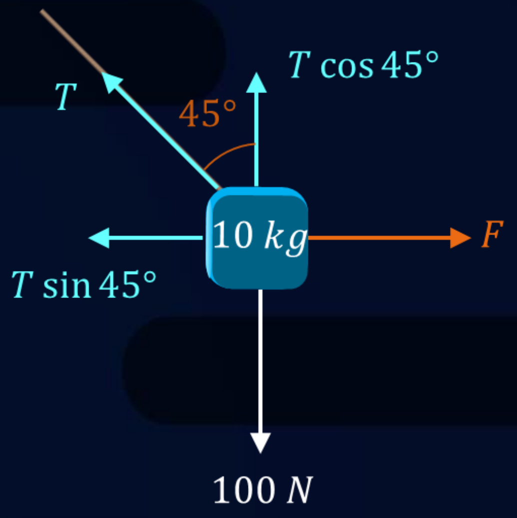
$$T = 100\sqrt{2}\ N$$

For horizontal equilibrium,

$$F = T \sin 45° = 100\sqrt{2} \times \frac{1}{\sqrt{2}}$$

$$F = 100\ N$$

Free Body Diagram



a. $70\ N$

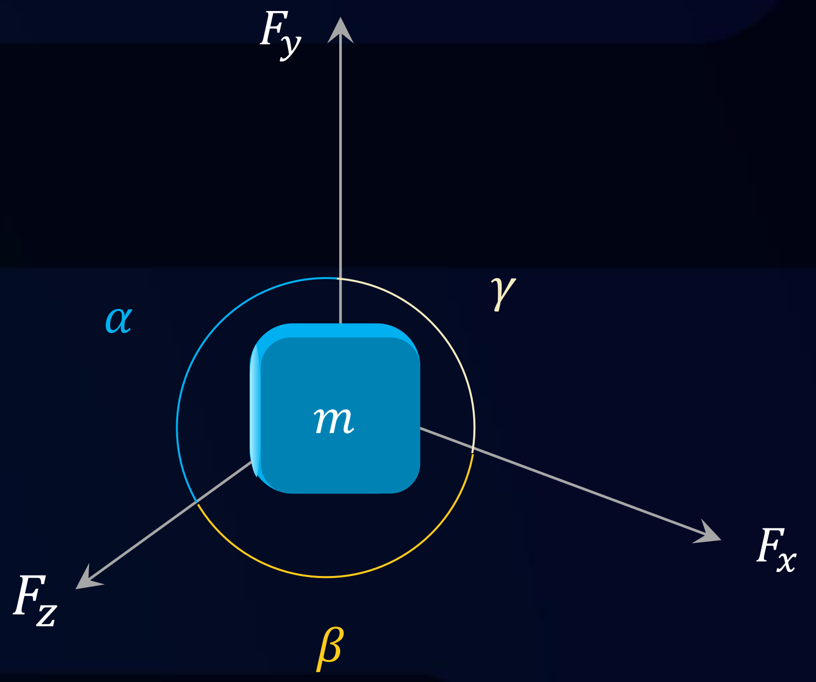
b. $200\ N$

c. $100\ N$

d. $140\ N$

Lami's Theorem

$$\frac{F_x}{\sin\alpha} = \frac{F_y}{\sin\beta} = \frac{F_z}{\sin\gamma}$$



Conditions to apply Lami's Theorem :

- ✓ There should exist only three forces.
- ✓ Forces are to be coplanar and should remain concurrent.
- ✓ The body should remain in equilibrium.

Linear momentum

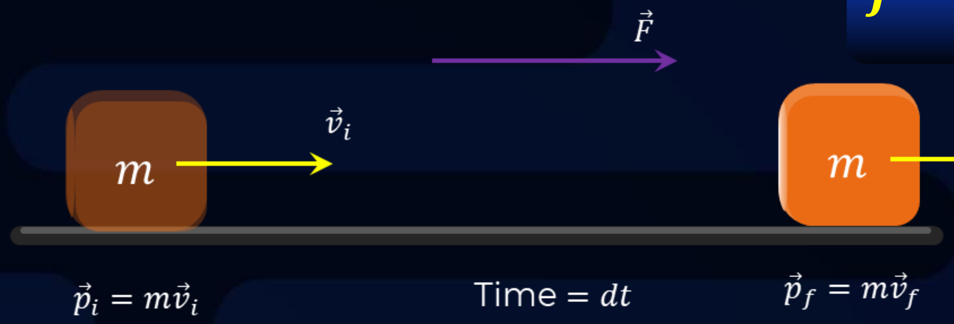
Momentum of a body is defined to be the product of its mass m and velocity v.

- $\vec{p} = m\vec{v}$
- It is a vector quantity.
- SI unit is $kg\ m/s$.

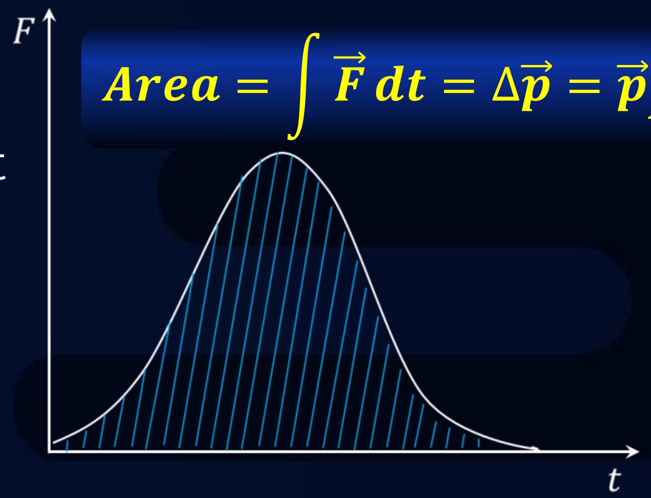
Impulse

The change in momentum suffered by a system due to the action of an external force($\vec{F}$) for an infinitesimally small amount of time is the impulse ($\vec{J}$) given by the $\vec{F}$ to the system.

$$\vec{J} = \int \vec{F}\, dt = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$



$$Area = \int \vec{F}\, dt = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$



Impulse imparted to an object in a given time interval is the area under the curve of force vs. time graph($F - t$) graph in the same interval.

Impulse Force

Force which can change the momentum appreciably even in an infinitesimal time.

Impulsive force

- Acting for a short duration
- +
- Measurable change in momentum

Conservation of Linear momentum

If : The external forces acting on the system add up to zero in a certain direction.

$$\left(\sum \vec{F}_{ext}\right)_{sys} = 0$$

Then : The linear momentum of the system remains constant in that direction.

$$(\vec{p}_{initial})_{sys} = (\vec{p}_{final})_{sys}$$

Conservation of linear momentum

If $\left(\sum \vec{F}_{ext}\right)_x = 0$

$\Longrightarrow (\vec{p}_i)_x = (\vec{p}_f)_x$

Linear momentum is conserved along $x - axis$

If $\left(\sum \vec{F}_{ext}\right)_y = 0$

$\Longrightarrow (\vec{p}_i)_y = (\vec{p}_f)_y$

Linear momentum is conserved along $y - axis$

If $\left(\sum \vec{F}_{ext}\right)_z = 0$

$\Longrightarrow (\vec{p}_i)_z = (\vec{p}_f)_z$

Linear momentum is conserved along $z - axis$

A $4\ kg$ gun fires a $6\ g$ bullet with speed of $500\ ms^{-1}$. Find the ratio of the distance the gun moves backward while bullet is in the barrel to the distance the bullet moves forward during the same time?

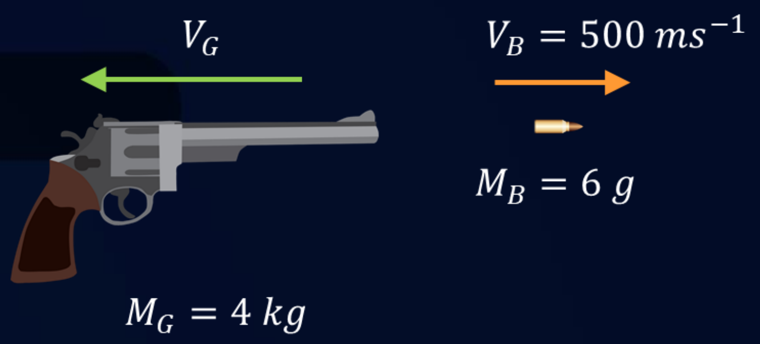
Solution: $M_G = 4\ kg, M_B = 6g$

Applying the principle of conservation of linear momentum,

$$M_G\vec{V}_G + M_B\vec{V}_B = 0$$

$$\Rightarrow M_G\vec{V}_G = -M_B\vec{V}_B$$

$$\Rightarrow \vec{V}_G = -\frac{M_B}{M_G}\vec{V}_B$$



$$\Rightarrow \frac{\overrightarrow{\Delta x_G}}{\Delta t} = -\frac{M_B}{M_G}\frac{\overrightarrow{\Delta x_B}}{\Delta t} \Rightarrow \left|\frac{\Delta x_G}{\Delta t}\right| = \left|-\frac{M_B}{M_G}\frac{\Delta x_B}{\Delta t}\right|$$

$$\Rightarrow \left|\frac{\Delta x_G}{\Delta x_B}\right| = \frac{M_B}{M_G} = \frac{6}{4000} \approx \frac{1}{666}$$

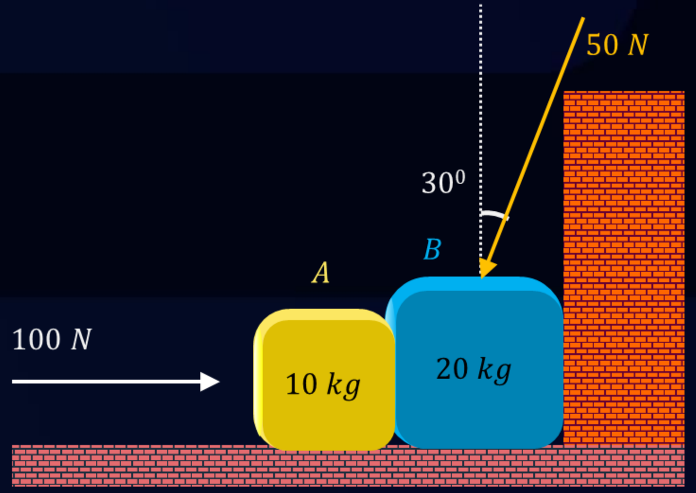
a. 1/600

b. 1/666

c. 2/327

d. None of these

Two blocks are kept in contact as shown in the figure. Find the forces exerted by surfaces (floor and wall) on blocks and the contact force between the two blocks. (consider $g = 10\ m/s^2$). Consider all surfaces to be smooth



Smooth surface

Solution: $m_A = 10\ kg, \quad m_B = 20\ kg$

For $m_A = 10\ kg$

$$\sum \vec{F}_x = 0$$

$$\boxed{N_{AB} = 100\ N = N_{BA}}$$

$$\sum \vec{F}_y = 0$$

$$\boxed{N_A = m_A g = 100\ N}$$

For $m_A = 20\ kg$

$$\sum \vec{F}_x = 0$$

$$N_{wall} + 25 = 100\ N$$

$$\boxed{N_{wall} = 75\ N}$$

$$\sum \vec{F}_y = 0$$

$$N_B = 200 + 25\sqrt{3}$$

$$\boxed{N_B \approx 243\ N}$$

Free Body Diagram

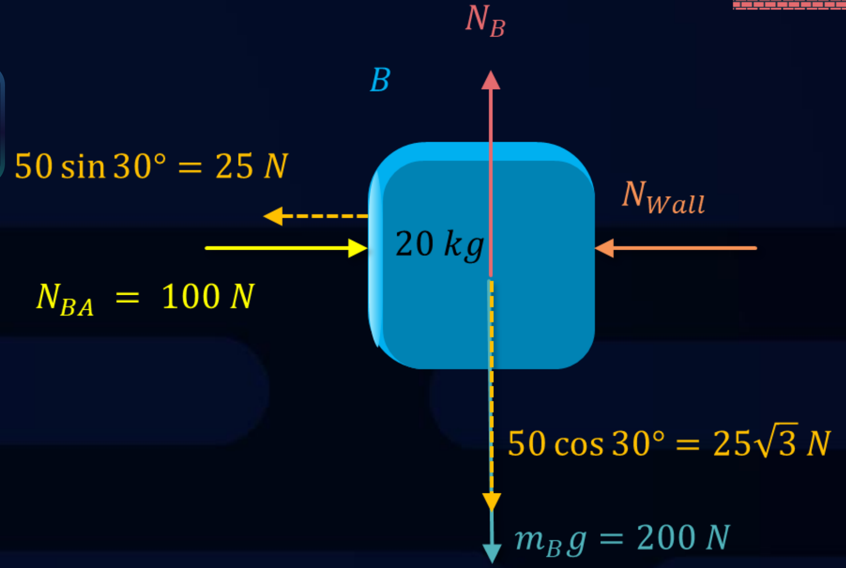
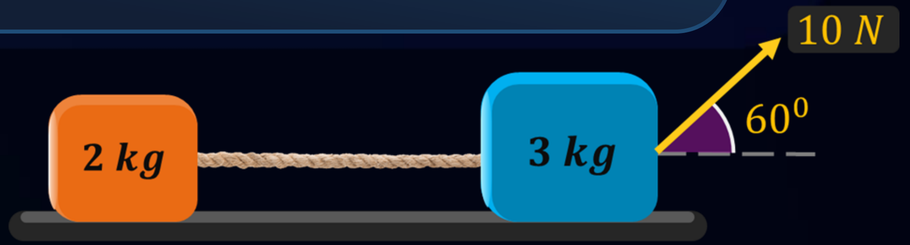


Figure shows two blocks connected by a light inextensible string. A pulling force of 10 N is applied on the bigger block at 60^0 with horizontal. The tension in the string connecting the two masses is (Assume all surfaces to be smooth)



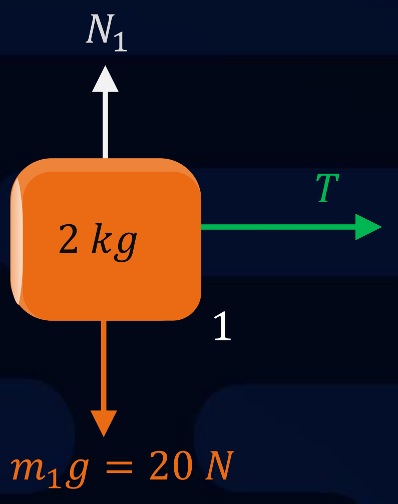
a. 5 N

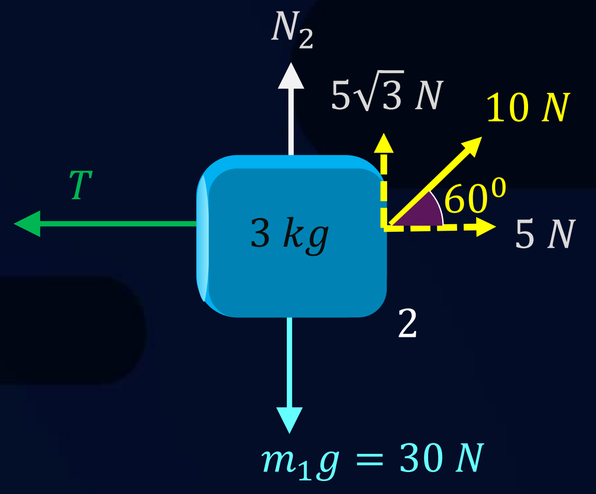
b. 2 N

c. 1 N

d. 3 N

$$\Sigma\vec{F}_x = m\vec{a}_x$$
$$\Sigma\vec{F}_y = m\vec{a}_y$$





$\Sigma\vec{F}_x = m\vec{a}_x$

$\Rightarrow T = (2)(a)$ —— (1)

$\Sigma\vec{F}_y = m\vec{a}_y$

$\Rightarrow N_1 = 20$ —— (2)

$\Sigma\vec{F}_x = m\vec{a}_x$

$\Rightarrow 5 - T = (3)(a)$ —— (3)

$\Sigma\vec{F}_y = m\vec{a}_y$

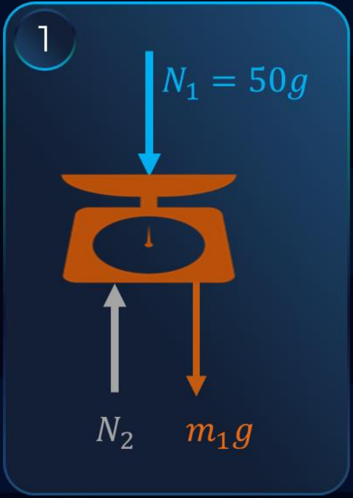
$\Rightarrow N_2 + 5\sqrt{3} = 30$ —— (4)

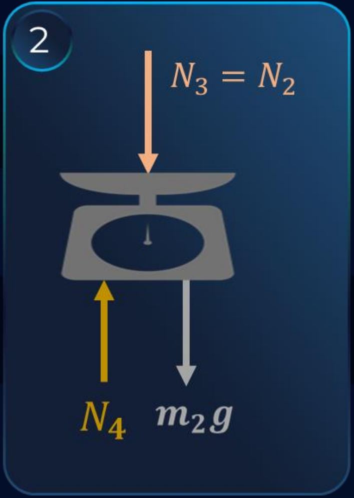
From (1) & (3) : $\boxed{T = 2\ N}$

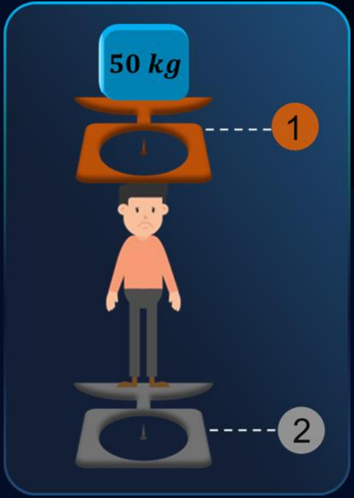
Weighing Machine

- Weighing machine does not measure weight.
- It measures the force with which it is being pressed.
- It measures the normal reaction on the weighing machine at that instant.
- Recalibration option helps to remove the weight of basket/plate.

A man of mass 60 kg is standing on a weighing machine 2 of mass 5 kg placed on ground. Another similar weighing machine is placed over the man's head. A block of mass 50 kg is put on the weighing machine 1. Calculate the readings of weighing machines 1 and 2. $(g = 10\ m/s^2)$







a. 50 kg, 115 kg

b. 45 kg, 115 kg

c. 45 kg, 120 kg

d. 55 kg, 120 kg

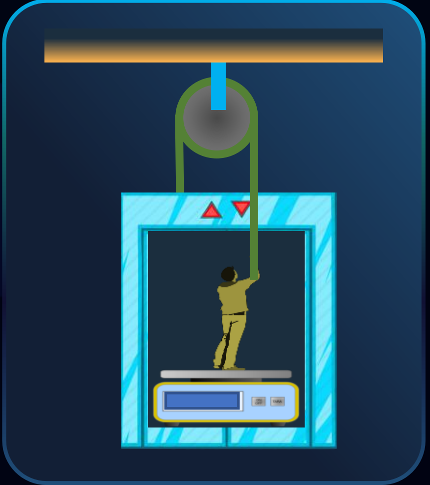
$$N_1 = 50g = 50 \times 10 = 500\ N$$

Reading 1 $= N_1 = 500\ N = 50\ kg$

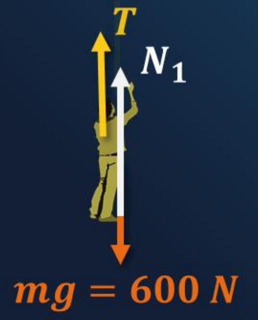
$$N_3 = (50 + 5 + 60)g = 115 \times 10\ N = 1150\ N$$

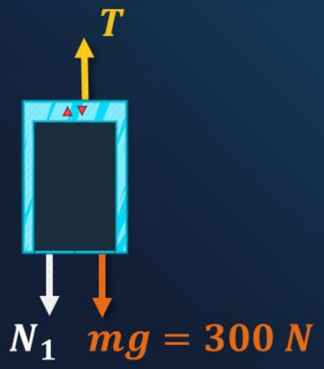
Reading 2 $= N_3 = 1150\ N = 115\ kg$

A man of mass 60 kg is standing on a light weighing machine kept in a box of mass 30 kg. The box is hanging from a pulley fixed to the ceiling through a light rope, the other end of which is held by the man himself. If the man manages to keep the box at rest, What is the reading shown by the machine?



Through mass-less bodies force transmits unadulterated or unchanged.





a. $30\ kg$

b. $15\ kg$

c. $45\ kg$

d. $60\ kg$

As the man is in equilibrium ,

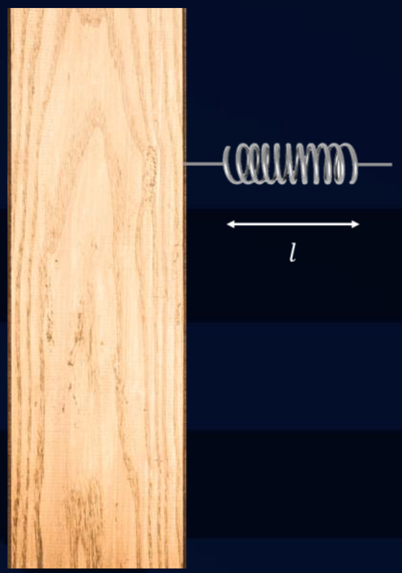
$T + N_1 = mg \Rightarrow T + N_1 = 600\ N$ ------ (1)

As the lift is in equilibrium ,

$T = N_1 + 300\ N$ ------ (2)

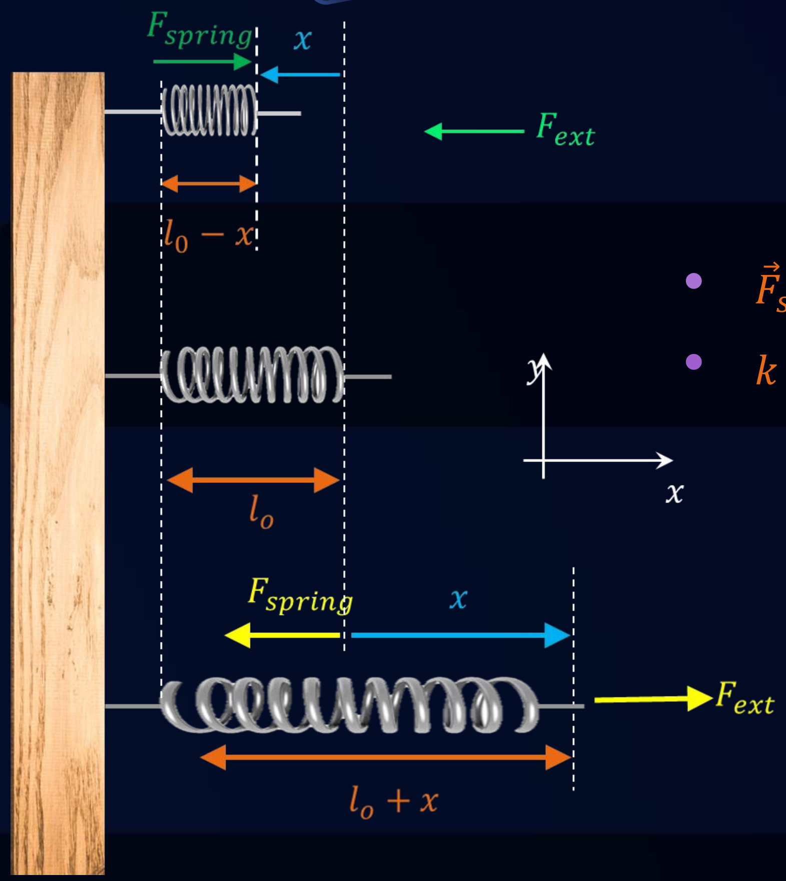
Solving (1) & (2), we get $N_1 = 150\ N = 15\ kg$

Spring



- An ideal spring is an ideal string wound up in the form of a helix.
- Unlike an inextensible string, the spring's length can change.
- Both strings and springs are assumed to be massless in our discussion.

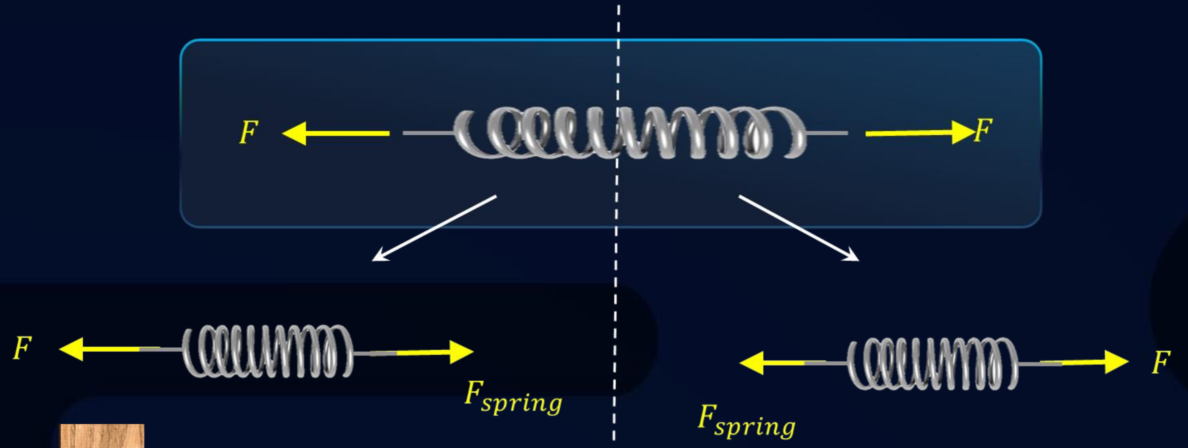
Spring Force



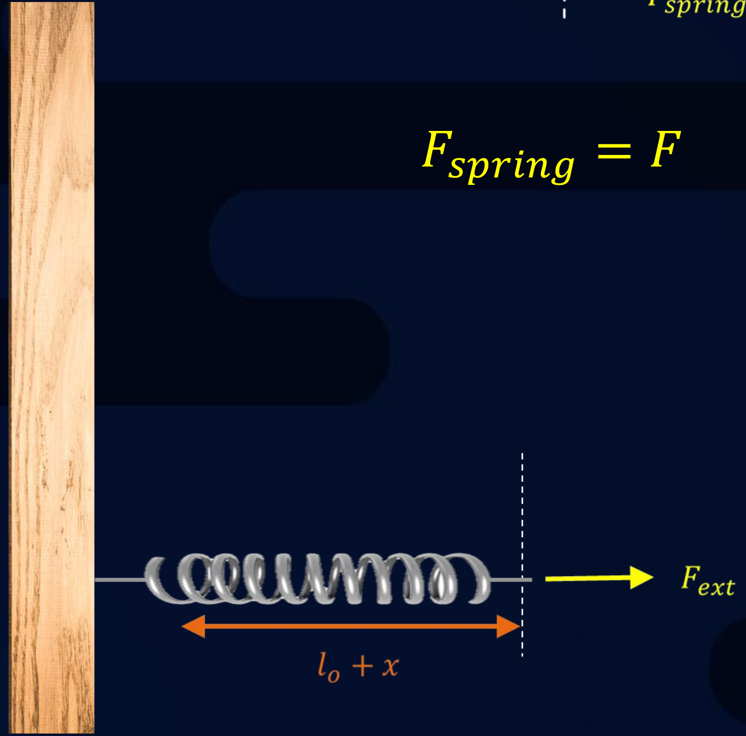
- $\vec{F}_{spring} = -k\vec{x}$
- k is called spring constant.

- A spring resists any kind of deformation with a restoring force.
- Hooke's Law: $F \propto -x$, where x is the change in length of the spring.

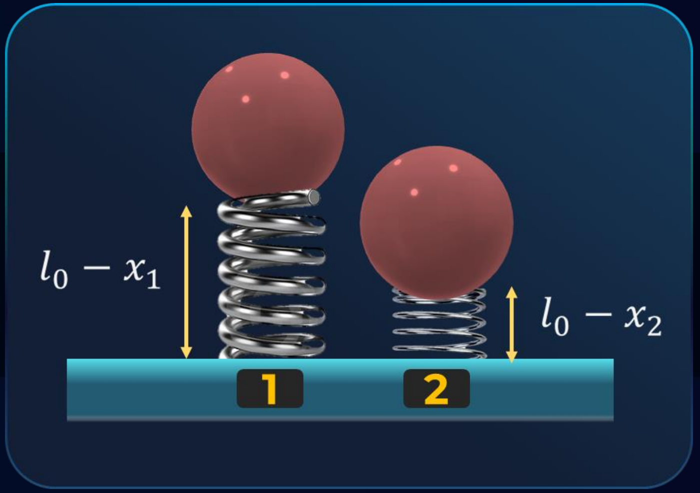
FBD of Spring



$$F_{spring} = F$$



Spring Constant



Stiffness of the spring signifies how difficult it is to deform the spring.

It is measure of inertia of spring to resist any kind of extension or compression in it.

For same force,

$$k \propto \frac{1}{x}$$

$x_2 > x_1 \rightarrow k_1 > k_2$

A particle of mass $0.3\ kg$ is subjected to a force $F = -kx$ with $k = 15\ \text{N}/m$. What will be its initial acceleration, if it is released from a point $x = 20\ cm$?

Solution:

$$x = 20\ cm = \frac{20}{100}\ m = 0.2\ m$$

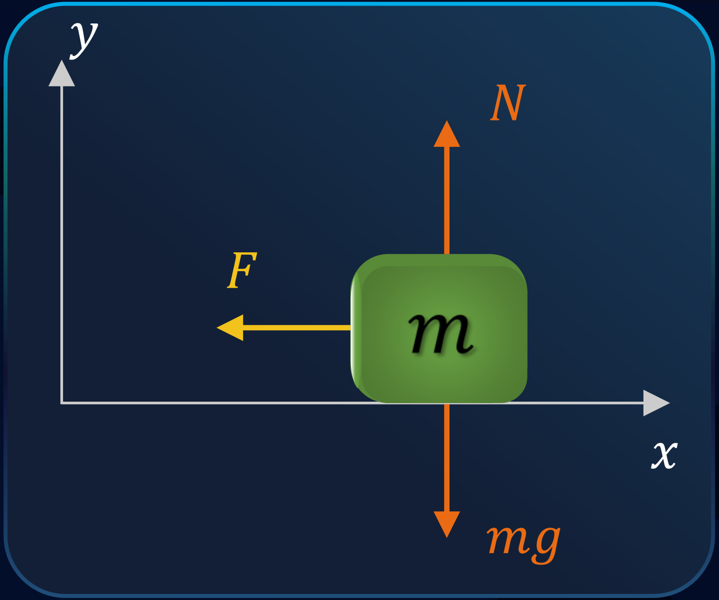
$$F = kx$$

$$ma = kx$$

$$0.3 \times a = 15 \times 0.2$$

$$a = 10\ m/s^2$$

$$\boxed{a = 10\ m/s^2}$$



a. $15\ m/s^2$

b. $20\ m/s^2$

c. $10\ m/s^2$

d. $3\ m/s^2$

Two blocks A and B of masses $2m$ and m respectively, are connected by a massless and inextensible string. The whole system is suspended by a massless spring as shown in the figure. The magnitude of acceleration of A and B immediately after the string is cut, are respectively:

(JEE 2006)

a. $g, \frac{g}{2}$

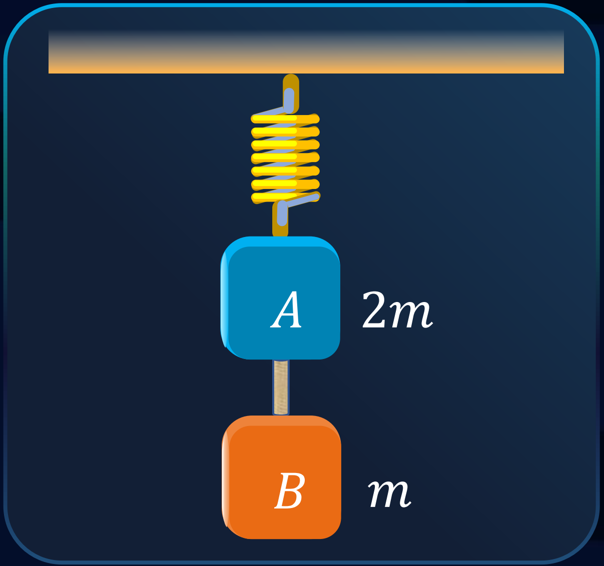
b. $\frac{g}{2}, g$

c. g, g

d. $\frac{g}{2}, \frac{g}{2}$

Solution:

Before string is cut:



FBD of A

$kx = T + 2mg$

$kx = 3mg$

$2ma_A = kx - 2mg$

$2ma_A = 3mg - 2mg$

$a_A = \frac{g}{2}$ ↑

FBD of B

$T = mg$

$ma_B = mg$

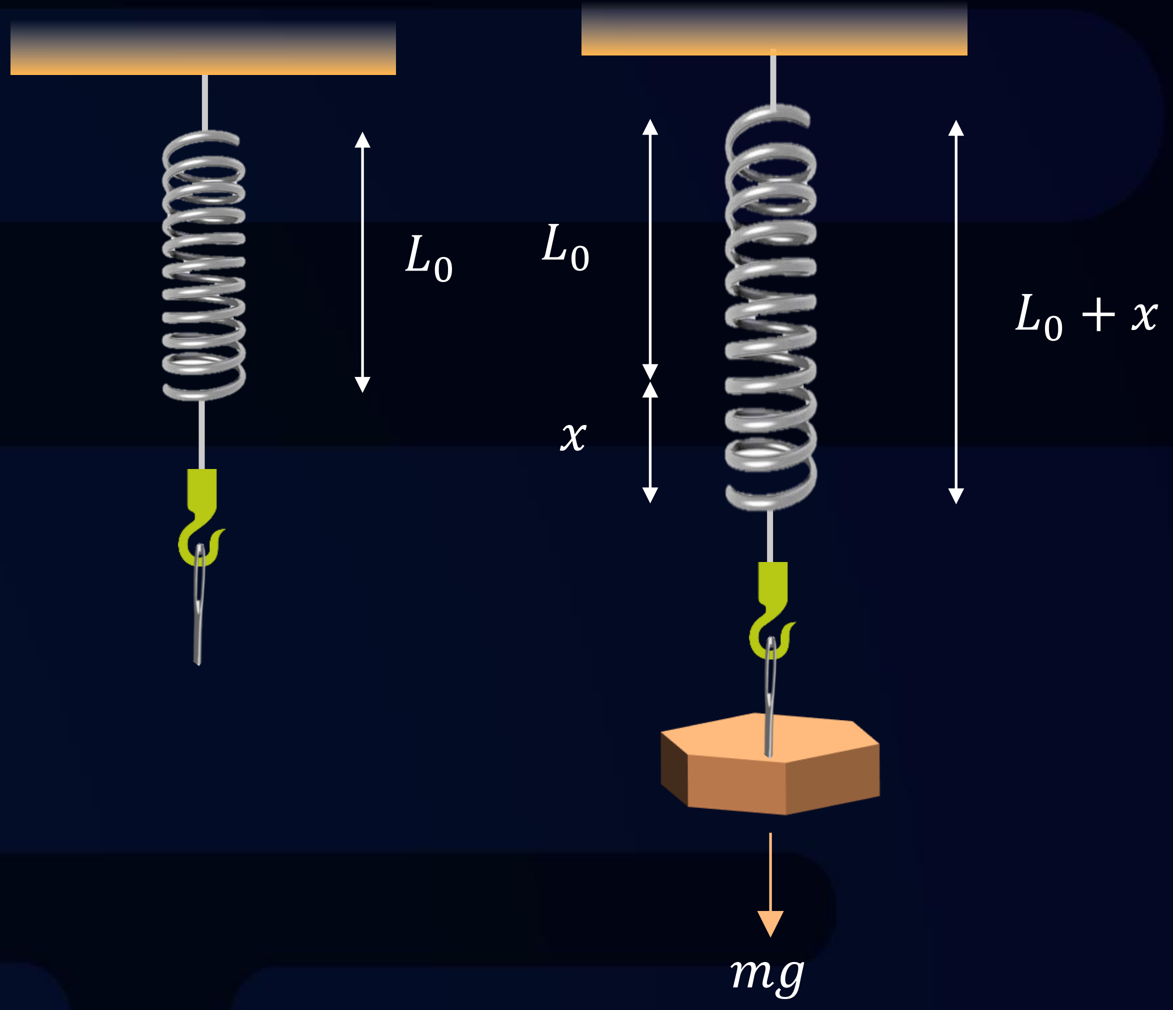
$a_B = g$ ↓

Spring Balance

The reading of spring balance is the tensile force on its spring.

$$F_{spring} = T = kx$$

Spring balance reading gives the tensile force in itself.



A block of mass 20 kg is suspended through two light spring balances as shown in figure. Calculate the
i) Reading of spring balance 1
ii) Reading of spring balance 2

Solution:

$$T = mg$$

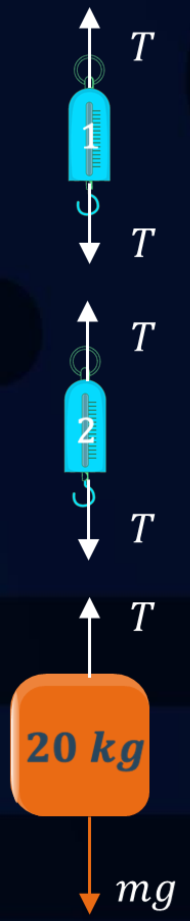
$$T = 20 \times 10$$

$$T = 200\ N$$

$$T = 20\ kg$$

As the springs are mass-less,

Tension will be same throughout.



Reading 1 = Reading 2 = 20 kg

a. $10\ kg, 10\ kg$

b. $20\ kg, 20\ kg$

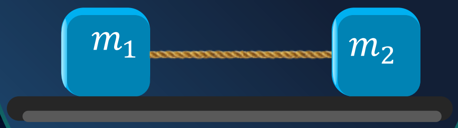
c. $10\ kg, 20\ kg$

d. $15\ kg, 5\ kg$

Constrained Motion

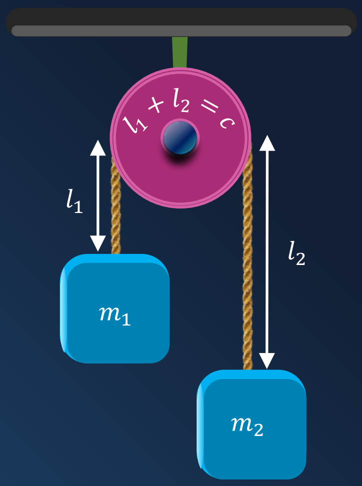
An object is forced to move in a restricted way.

String Constraints



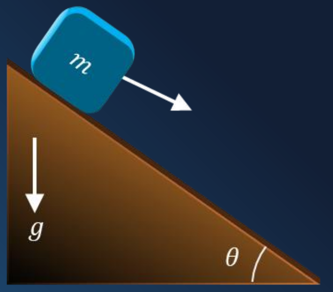
For an inextensible string, velocity components of the endpoints are equal along the string.

Ideal Pulley: massless and frictionless



$$\vec{v}_P = \frac{\vec{v}_1 + \vec{v}_2}{2} \qquad \vec{a}_P = \frac{\vec{a}_1 + \vec{a}_2}{2}$$

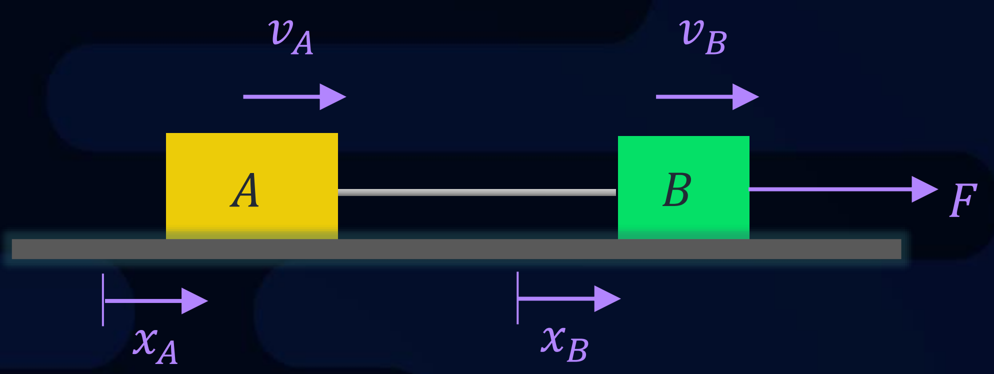
Wedge Constraints



When two bodies are in contact, their velocity components perpendicular to the interface are equal.

Constraint Relations

Mathematical relations by which motion of two or more bodies within a system are related.

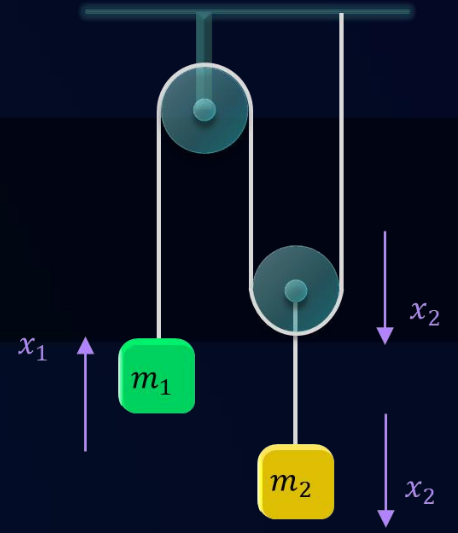


Since the string is inextensible, $|\vec{x}_A| = |\vec{x}_B|$

$$\frac{dx_A}{dt} = \frac{dx_B}{dt} \Rightarrow |\vec{v}_A| = |\vec{v}_B|$$

$$\frac{dv_A}{dt} = \frac{dv_B}{dt} \Rightarrow |\vec{a}_A| = |\vec{a}_B|$$

Pulley block system



- Constraint Relations: $|\vec{x}_1| = |2\vec{x}_2|$

$$|\vec{v}_1| = |2\vec{v}_2|$$

$$|\vec{a}_1| = |\overrightarrow{2a_2}|$$

Steps for Setting up String Constraint Equation

- Step 1: Identify all the objects and number of strings in the problem.
- Step 2: Assume variables and directions to represent the parameters of motion such as displacement, velocity, acceleration etc.
- Step 3: Identify a single string and divide it into different linear sections and write the equation.

$$l_1 + l_2 = constant$$

- Step 4: Differentiate with respect to time.

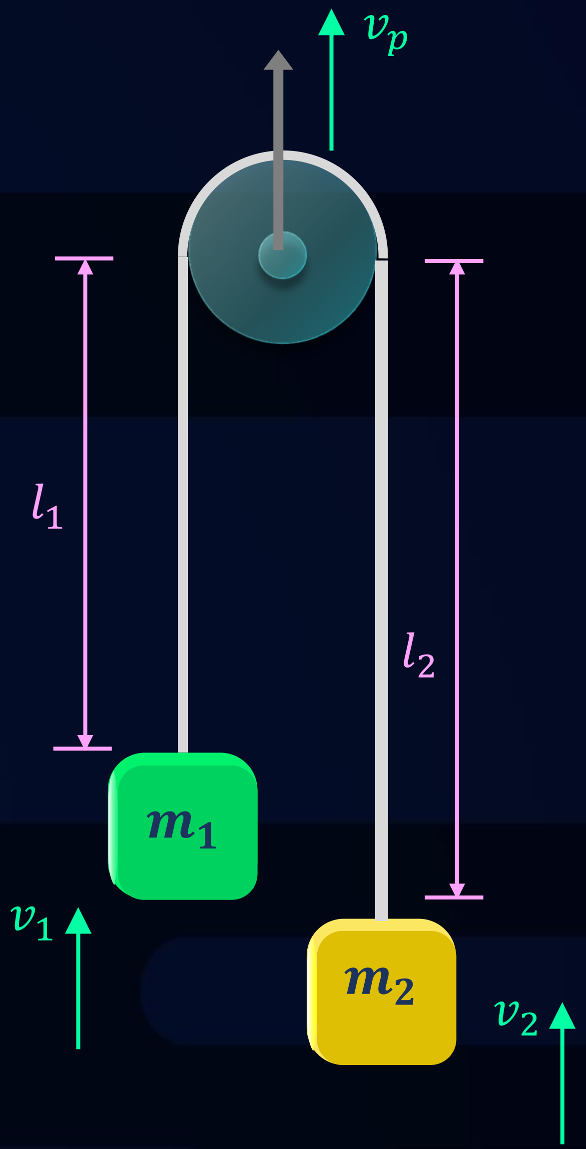
$$\dot{l}_1 + \dot{l}_2 = 0$$

- Step 5: Write differentials in terms of velocities and obtain the constraint equations.

$$\dot{l}_1 + \dot{l}_2 = 0$$

$$(\vec{v_p} - \vec{v_1}) + (\vec{v_p} - \vec{v_2}) = 0$$

- Step 6: Repeat all steps for each string.

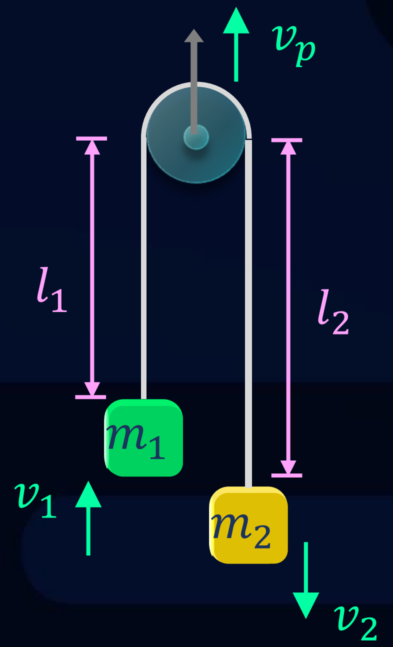


$$\vec{v}_P = \frac{\vec{v}_1 + \vec{v}_2}{2}$$

Similarly,

$$\vec{a}_P = \frac{\vec{a}_1 + \vec{a}_2}{2}$$

Case 2 – String Constraint



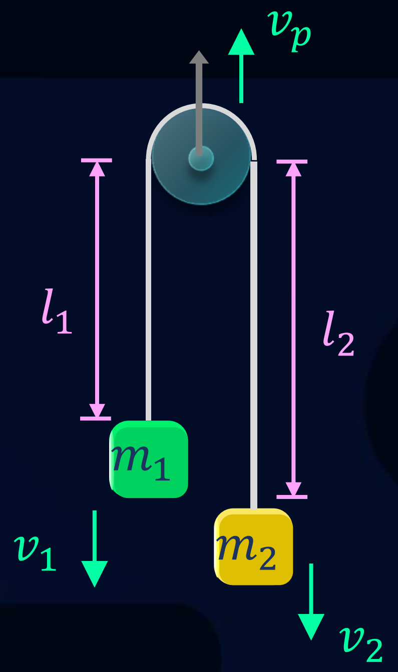
Case 2:

$$\dot{l}_1 + \dot{l}_2 = 0$$

$$(\vec{v}_p - \vec{v}_1) + (\vec{v}_p + \vec{v}_2) = 0$$

$$\boxed{\vec{v}_P = \frac{\vec{v}_1 - \vec{v}_2}{2}}$$

Case 3 – String Constraint



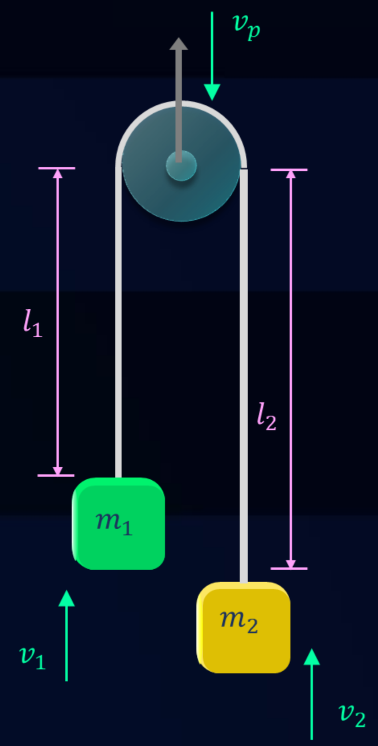
Case 3:

$$\dot{l}_1 + \dot{l}_2 = 0$$

$$(\vec{v}_1 - \vec{v}_p) + (\vec{v}_2 + \vec{v}_p) = 0$$

$$\boxed{\vec{v}_P = \frac{-(\vec{v}_1 + \vec{v}_2)}{2}}$$

Case 4 – String Constraint



For the frame of pulley,

$$(v_1 + v_p) = -(v_2 + v_p)$$

$$\boxed{\vec{v}_P = \frac{\vec{v}_1 + \vec{v}_2}{2}}$$

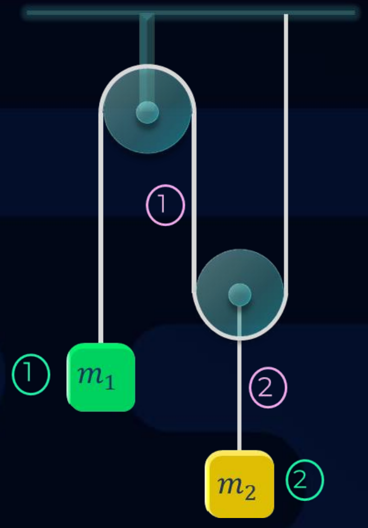
Case 4:

$$\dot{l}_1 + \dot{l}_2 = 0$$

$$\boxed{\vec{v}_P = \frac{-(\vec{v}_1 + \vec{v}_2)}{2}}$$

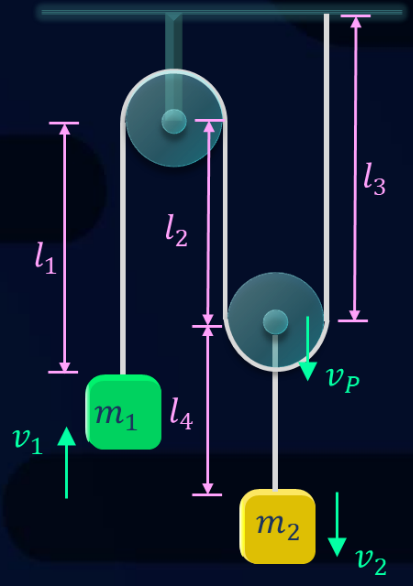
Find the constraint relations of the pulley block system shown in the diagram?

Step 1: Identify all the objects and number of strings in the problem.



Step 2: Assume variables and directions to represent the parameters of motion such as displacement, velocity, acceleration etc.

Step 3: Identify a single string and divide it into different linear sections and write the equation.

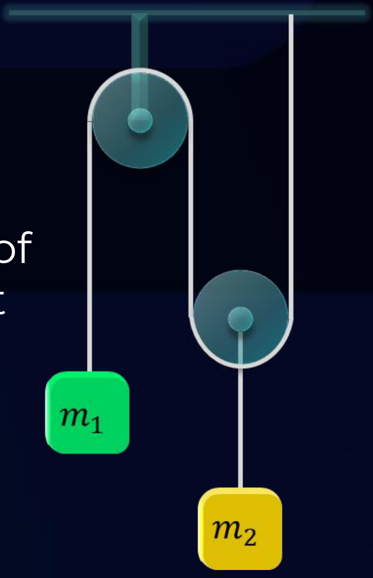


$$\ell_1 + \ell_2 + \ell_3 = constant$$

$$\ell_4 = constant$$

Step 4: Select one string & differentiate with respect to time.

$$\frac{d\ell_1}{dt} + \frac{d\ell_2}{dt} + \frac{dl_3}{dt} = 0$$



Step 5: Write differentials in terms of velocities and obtain the constraint equations.

$$\frac{d\ell_1}{dt} + \frac{d\ell_2}{dt} + \frac{dl_3}{dt} = 0$$

$$-v_1 + v_p + v_p = 0$$

$$v_1 = 2v_p \longrightarrow (1)$$

Step 6: Repeat all steps for each string.

$$\ell_4 = constant$$

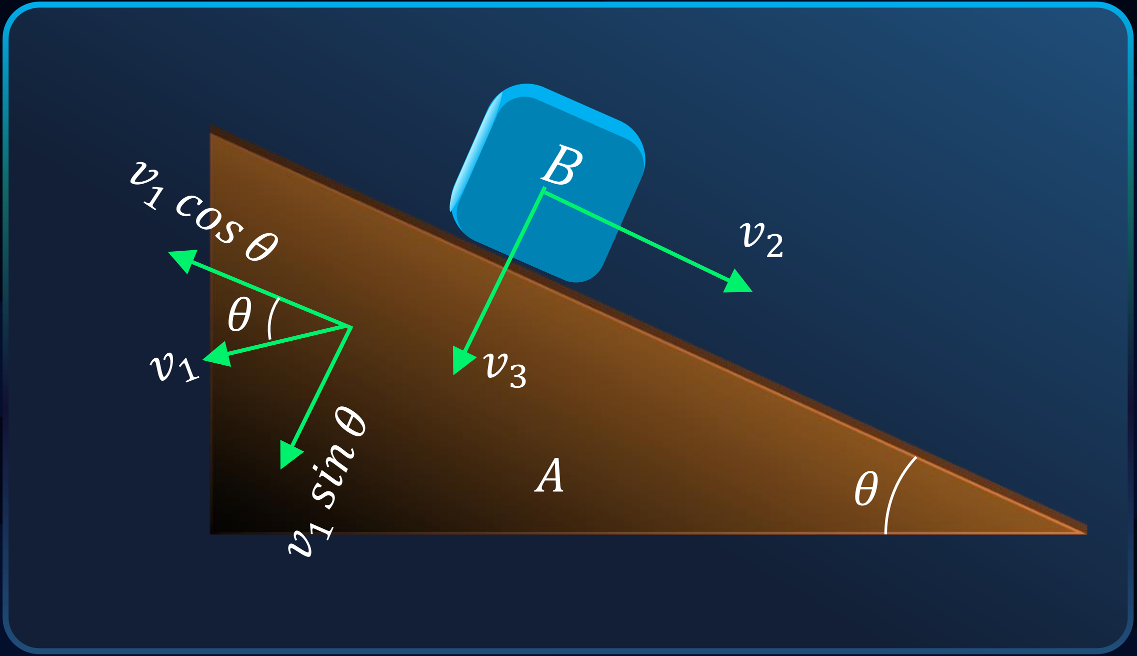
$$\frac{dl_4}{dt} = 0$$

$$v_p = v_2 \longrightarrow (2)$$

From (1) & (2), $\boxed{v_1 = 2v_2}$ Similarly, $\boxed{x_1 = 2x_2 \quad a_1 = 2a_2}$

Wedge Constraint

- When two bodies are in contact as shown, their velocity components perpendicular to the interface are equal.



Assumptions:

a. Objects are rigid.

b. The block and the wedge are always in contact.

- Velocity of separation perpendicular to the interface is zero.

$$v_3 = v_1 \sin\theta$$

A rod of mass $2\,m$ moves vertically downward on the surface of wedge of mass m as shown in figure. Find the relation between velocity of rod and that of the wedge at any instant.

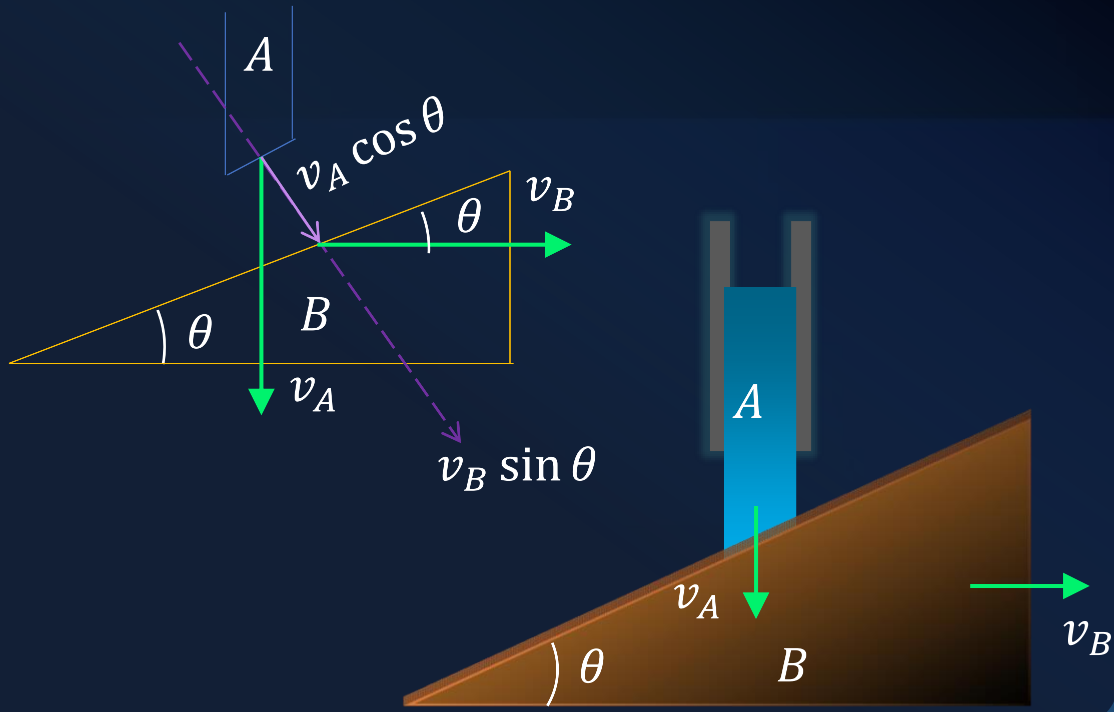
Given: $m_A = 2m,\ m_B = m$

To find: The relation between v_A and v_B

Solution: The velocity components perpendicular to the interface are equal.

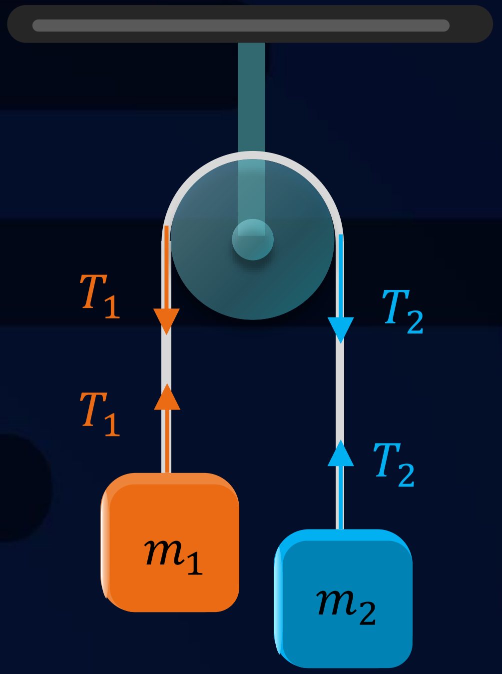
$$v_A \cos\theta = v_B \sin\theta$$

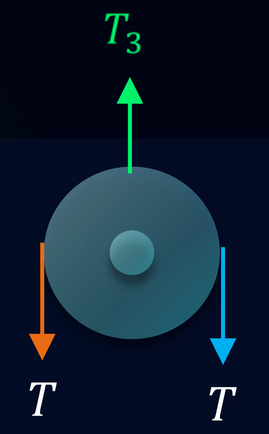
$$v_A = v_B \tan\theta$$



Ideal Pulley

- An ideal pulley is light (massless) and frictionless.
- Net force on any massless body is always zero.



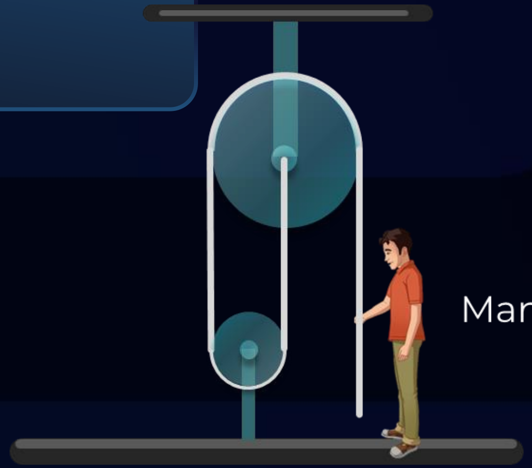


FBD of a pulley

Since pulley is frictionless $T_1 = T_2 = T$

Since pulley is massless $T_3 = 2T$

What force must man exert on rope to keep platform in equilibrium? (Take $g = 10\ m/s^2$)



Solution: $m_1 = 50\ kg,\ m_2 = 40\ kg$

Since pulley is massless $T + T + T = F$

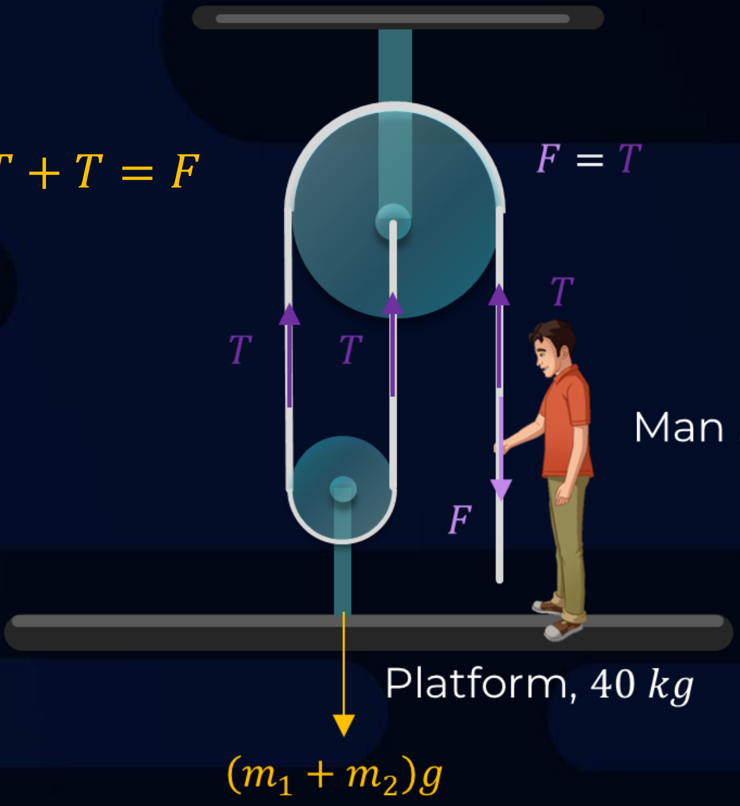
$$T + T + T = (m_1 + m_2)g$$

$$= (50 + 40) \times 10$$

$$3T = 900$$

$$T = F = 300\ N$$

FBD



a. $100\ N$

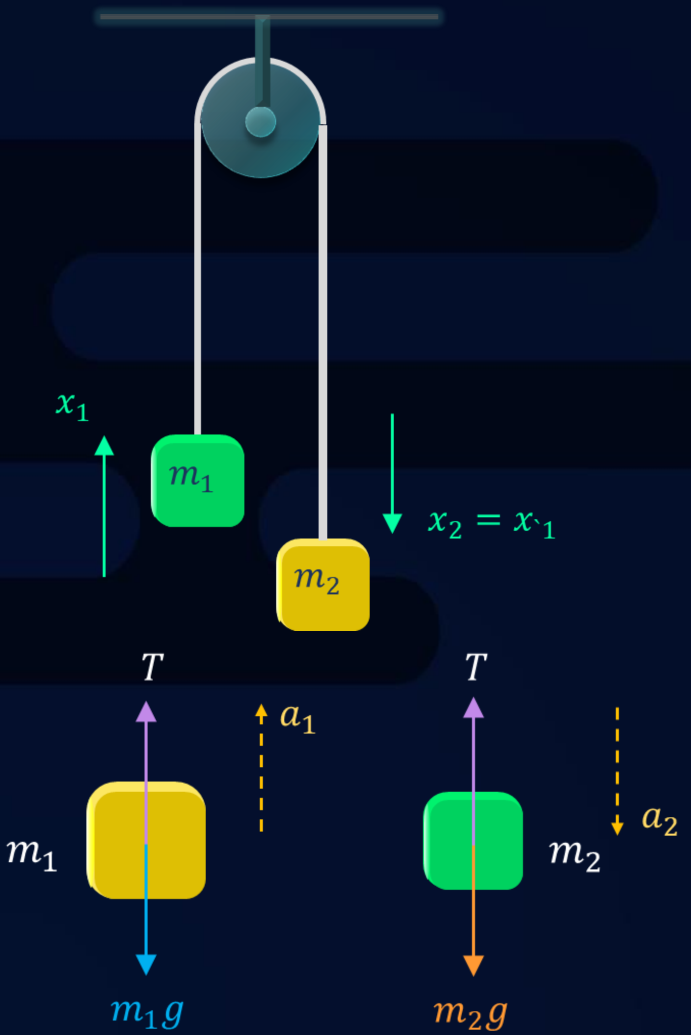
b. $200\ N$

c. $300\ N$

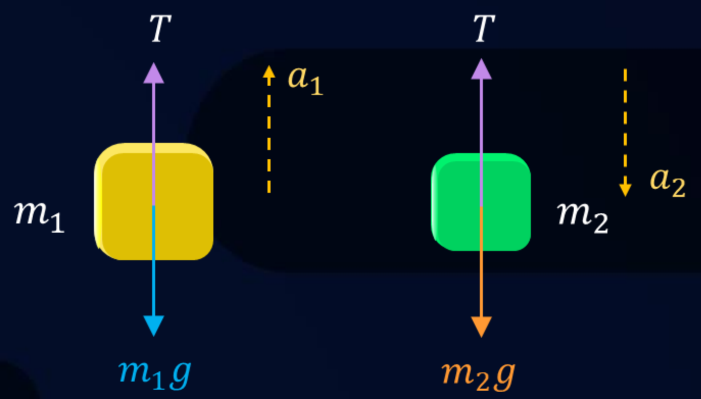
d. $500\ N$

Steps to solve a simple pulley problem

Step 1: Assume direction of accelerations and draw FBD for all masses. If required, resolve the forces along the axes.



Step 2: Write equations using Newton's second law along each axis for all blocks.



$$T - m_1 g = m_1 a_1$$

$$m_2 g - T = m_2 a_2$$

Step 3: Write the constraint equations.

$$a_1 = a_2$$

Step 4: Solve the obtained equations.

$$T - m_1 g = m_1 a_1 \quad \text{-------------} \quad (1)$$

$$m_2 g - T = m_2 a_2 \quad \text{-------------} \quad (2)$$

$$a_1 = a_2 = a \quad \text{-------------} \quad (3)$$

$$\boxed{a = \left[\frac{m_2 - m_1}{m_2 + m_1}\right] g}$$

$$\boxed{T = \left[\frac{2 m_1 m_2}{m_2 + m_1}\right] g}$$

Derive the constraint relationship between the velocities of three blocks.

Method:

1. Identify the strings, blocks, and pulleys.
2. Assign variables to movables & static parts.
3. Use the constraint relationship to solve.

Solution :

$l_1 + l_2 =$ constant

On differentiating, $\dot{l}_1 + \dot{l}_2 = 0$

$-v_1 + v_P = 0$

$v_1 = v_P \ldots (1)$

$l_3 + l_4 =$ constant

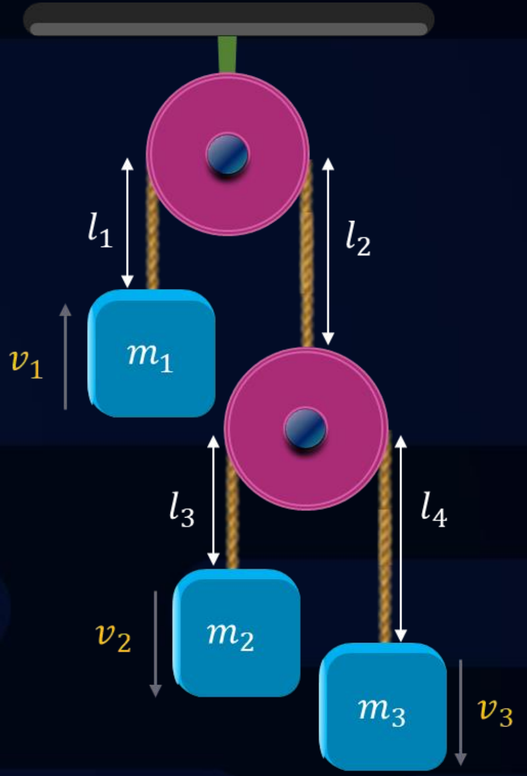
$\dot{l}_3 + \dot{l}_4 = 0$

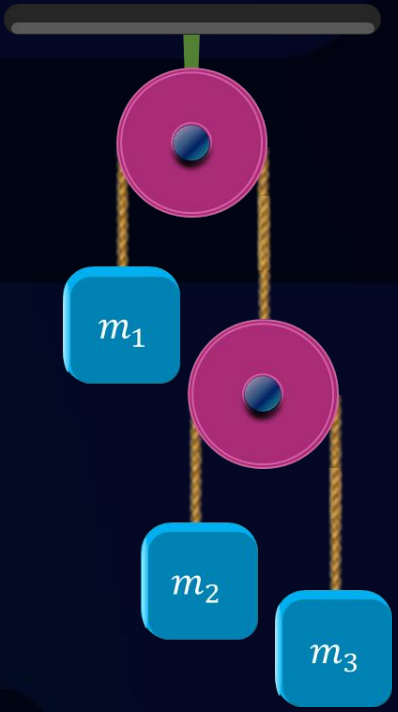
$(v_2 - v_p) + (v_3 - v_P) = 0$

$$v_P = \frac{(v_2 + v_3)}{2} \ldots (2)$$

Using $eq(1)$ & (2)

$$\boxed{v_1 = \frac{(v_2 + v_3)}{2}}$$





Find velocity of end A at the instant when rod is making an angle θ with horizontal.

Solution:

$$x^2 + y^2 = l^2$$

On differentiating w.r.t time,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$-\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \qquad \left(\frac{dx}{dt} = v_B, \frac{dy}{dt} = -v_A, \frac{x}{y} = cot\theta\right)$$

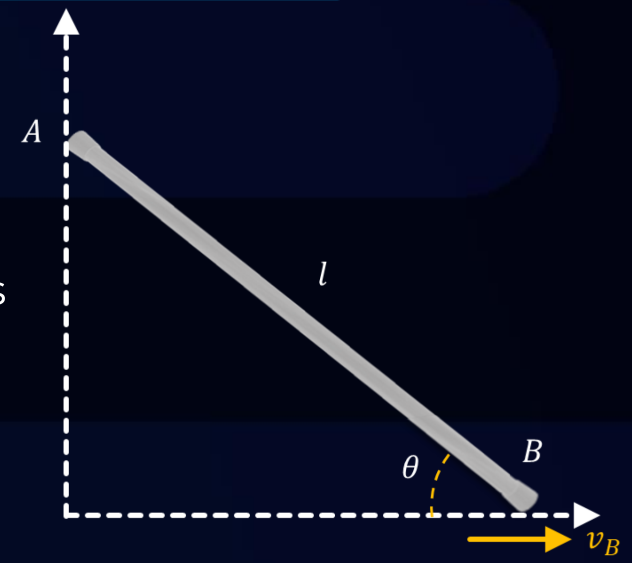
$$\boxed{v_A = v_B \cot\theta}$$

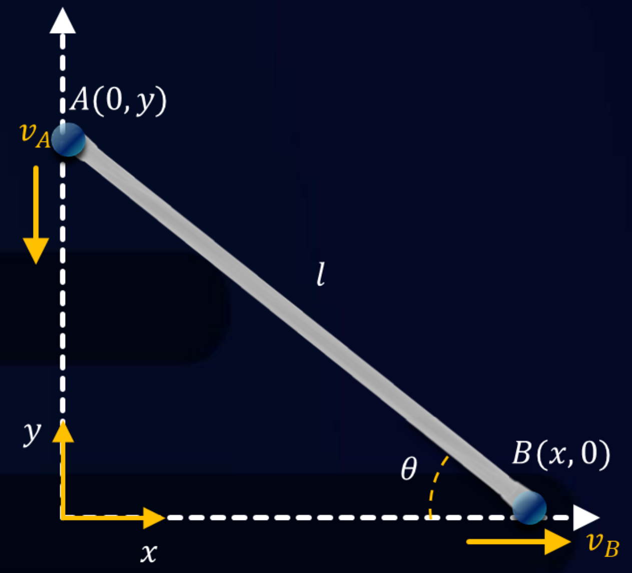
Alternative Solution:

If separation between two points is constant, then their velocity components must be equal along the line joining them.

$$v_A sin\theta = v_B \cos\theta$$

$$\boxed{v_A = v_B \cot\theta}$$





In the device shown block A is given acceleration $12t$ and block B is given acceleration $3\ m/s^2$, both towards right. Both the blocks start from rest. Find the time when block C comes to rest.

Solution: $l_1 + l_2 + l_3 + l_4 = \text{constant}$

$$\ddot{l}_1 + \ddot{l}_2 + \ddot{l}_3 + \ddot{l}_4 = 0$$

$$-a_A - a_C - a_C + a_B = 0$$

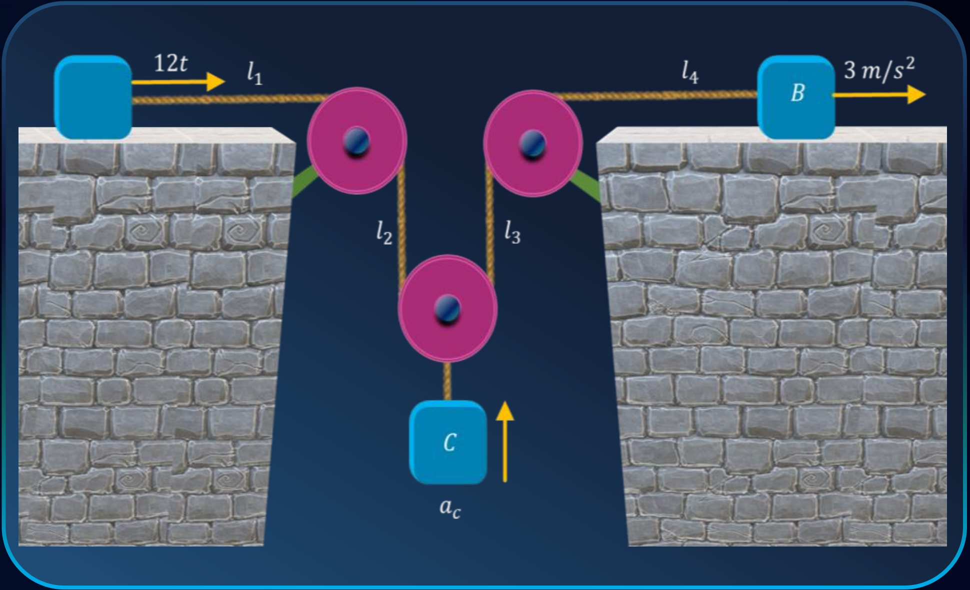
$$2a_C = a_B - a_A$$

$$a_C = \frac{3}{2}(1 - 4t) = \frac{dv}{dt}$$

$$\int_0^0 dv = \int_0^t \frac{3}{2}(1 - 4t)\,dt$$

$$0 = \frac{3}{2}(t - 2t^2) \Rightarrow t = 0 \text{ or } t = \frac{1}{2}$$

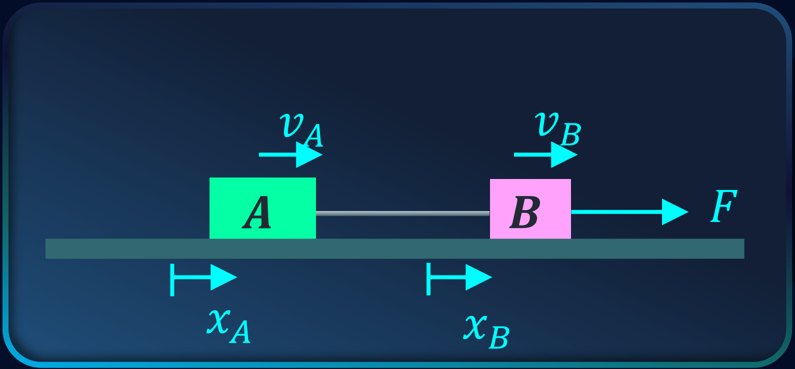
$$t = 0.5\ s$$



Newton's Laws on a System

- Case 1: When the bodies in the system have same acceleration

For inextensible string,



$$\vec{x}_A = \vec{x}_B$$

$$\Rightarrow \frac{d\vec{x}_A}{dt} = \frac{d\vec{x}_B}{dt} \Rightarrow \vec{v}_A = \vec{v}_B$$

$$\Rightarrow \frac{d\vec{v}_A}{dt} = \frac{d\vec{v}_B}{dt} \Rightarrow \vec{a}_A = \vec{a}_B$$

- It is mathematically advantageous to select the bodies having same acceleration as a system.

For block A, B and string as a system,

$$\left(\sum \vec{F}_{ext}\right)_{sys,\, x} = \left(m_{sys}\vec{a}_{sys}\right)_x$$

$$\Rightarrow \vec{F} = (m_A + m_B)\vec{a} \quad \Rightarrow \boxed{\vec{a} = \frac{\vec{F}}{m_A + m_B}}$$

- Case 2: When the bodies in the system have different acceleration

For a system with masses as per table,

$$\left(\sum \vec{F}_{ext}\right)_{sys,\, x} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots$$

Mass	Acceleration
m_1	$\vec{a}_1$
m_2	$\vec{a}_2$
m_3	$\vec{a}_3$
...	...

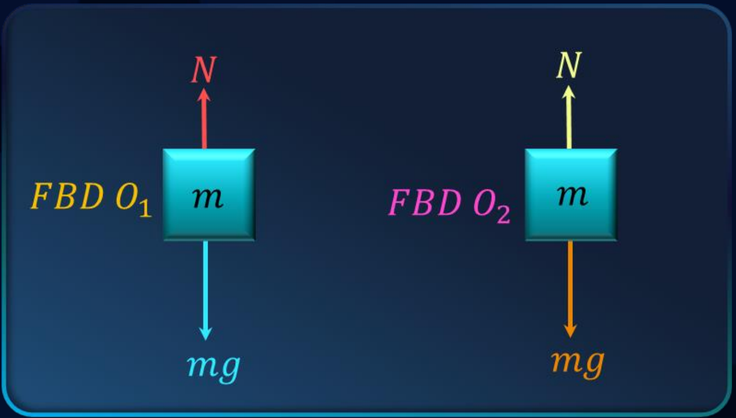
Frame of Reference

- A frame of reference (or reference frame) is an abstract coordinate system whose origin, orientation, and scale are specified by a set of reference points whose position is identified both mathematically and physically.

Inertial Frame of Reference	Non-Inertial Frame of Reference
Newton's laws are valid.	Newton's laws are not valid.
The observer should be non-accelerating.	The observer has non-zero acceleration.

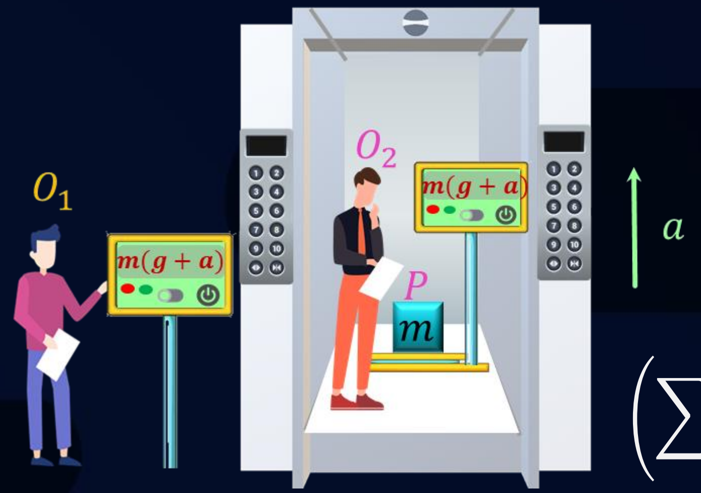
Frame of Reference

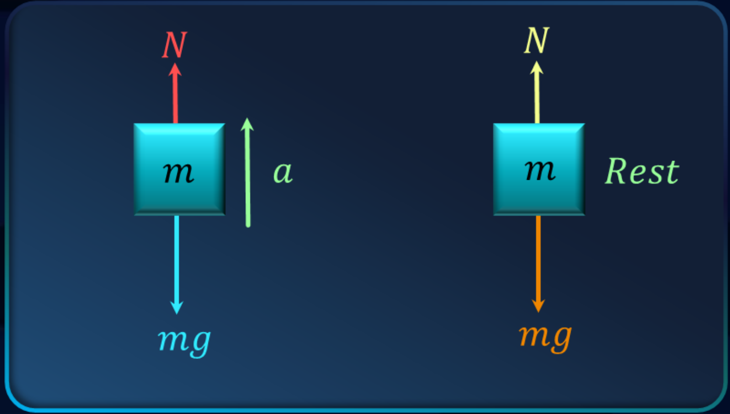
- Case 1: When Lift is at rest



$N = mg$ $\quad$ $N = mg$

- Case 2: When Lift is accelerating upwards





$N - mg = ma$ $\quad$ $N = mg$

$\Rightarrow N = m(g+a)$

Newton's law seem to fail.

For a point P on the block,

$$\vec{a}_{P,elevator} = \vec{a}_{P,G} - \vec{a}_{elevator,G}$$

$$\Rightarrow m\vec{a}_{P,elevator} = m\vec{a}_{P,G} - m\vec{a}_{elevator,G}$$

$$\left(\sum \vec{F}_P\right)_{elevator} = \left(\sum \vec{F}_P\right)_{Ground} + \underbrace{(-m\vec{a}_{elevator,\,G})}$$

Does not make a physical sense

Non-Inertial Frame of Reference

- The frames of reference which are accelerating w.r.t. ground are known as Non-inertial frames of reference.
- Newton's law seem to fail in non-inertial frame.

Example: In an accelerating lift, the weight of a body seems to be changed.

For non-inertial frames,

$$\left(\sum \vec{F}_{ext}\right)_{Frame} = \left(\sum \vec{F}_{ext}\right)_{Ground} + \left(-m\vec{a}_{Frame,\,G}\right)$$

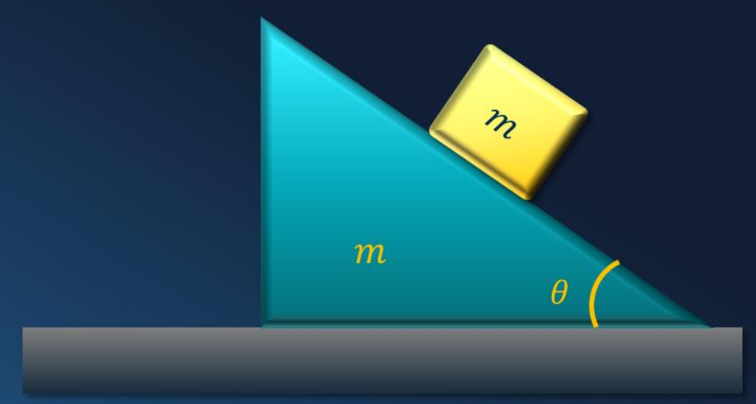
Pseudo Force

- The pseudo forces are also called inertial forces although their need arises because of the use of non-inertial frames.
- It's a fictitious force i.e., it will not have a reaction pair.
- It always acts in the direction opposite to the acceleration of the frame of reference.

$$\boxed{\vec{F}_{pseudo} = -m_{sys}\vec{a}_{non-inertial\ frame}}$$

$$\left(\sum \vec{F}_{ext}\right)_{Frame} = \left(\sum \vec{F}_{ext}\right)_{Ground} + \vec{F}_{pseudo}$$

A block of mass m resting on a wedge of angle θ as shown in the figure. The wedge is given an acceleration a. What is the minimum value of a so that the mass m falls freely?



Solution:

The block will fall freely when there is only gravitational force acting on it. Thus, normal force between wedge and block is zero.

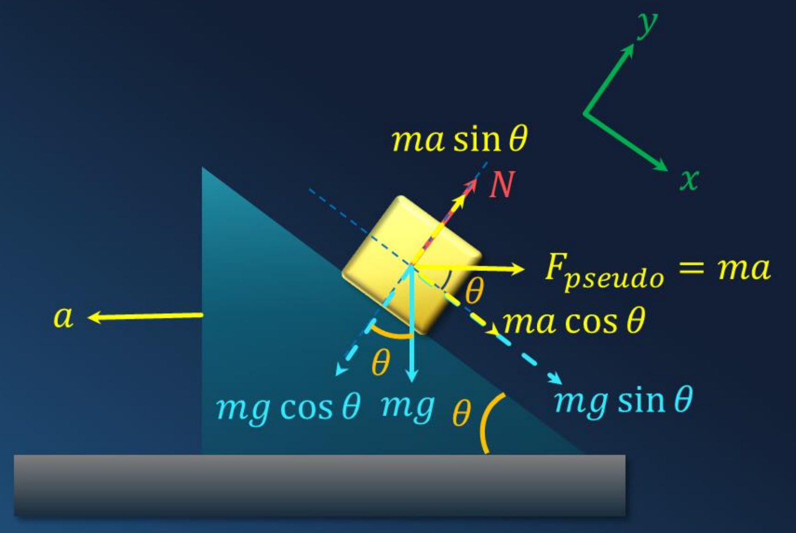
In wedge frame,

$$\sum \vec{F}_y = m\vec{a}_y$$

$$N + ma\sin\theta = mg\cos\theta$$

$$\Rightarrow a = \frac{g\cos\theta}{\sin\theta} \qquad [\because N = 0]$$

$$\Rightarrow \boxed{a = g\cot\theta}$$



?

Having gone through a plank of thickness h, a bullet changed its velocity from v_0 to v. Find the time of motion of the bullet in the plank, assuming the resistance force to be proportional to the square of the velocity.

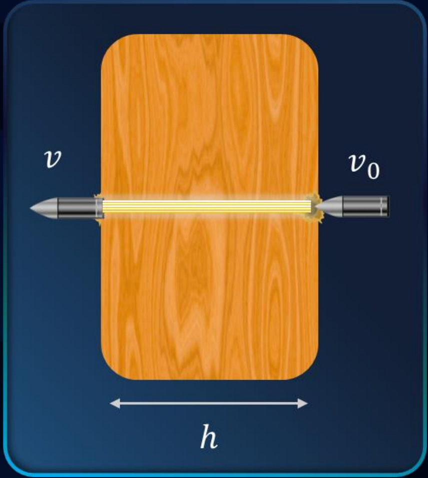
Solution:

Resistive force: $F_R = -kv^2$

Let the acceleration of bullet in plank be a.

$\Rightarrow F_R = -kv^2 = ma \quad [m = \text{Mass of the Bullet}]$

$$\Rightarrow a = -\frac{kv^2}{m}$$



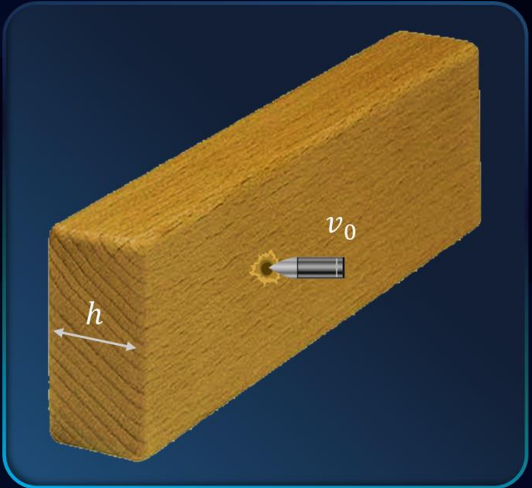
Using Definition of acceleration,

$$a = \frac{dv}{dt} = -\frac{kv^2}{m}$$

$$\Rightarrow \int_{v_0}^{v} \frac{dv}{v^2} = -\frac{k}{m}\int_0^t dt \quad \Rightarrow \left[-\frac{1}{v}\right]_{v_0}^{v} = -\frac{k}{m}[t]_0^t$$

$$\boxed{t = \frac{m(v_0 - v)}{kvv_0} \quad \dots.(1)}$$

Value of constant k:



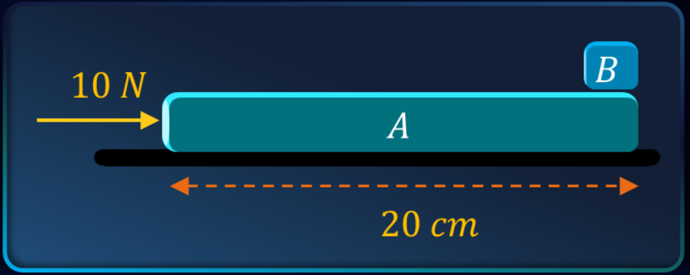
$$a = v\frac{dv}{dx} = -\frac{kv^2}{m}$$

$$\Rightarrow \int_{v_0}^{v} \frac{dv}{v} = -\frac{k}{m}\int_0^h dx \Rightarrow [\ln v]_{v_0}^{v} = -\frac{k}{m}[x]_0^h \quad \Rightarrow \ln\frac{v_0}{v} = \frac{kh}{m}$$

Putting this value of k in equation (1),

$$t = \frac{m(v_0 - v)}{kvv_0} \quad \Rightarrow \boxed{t = \frac{h(v_0 - v)}{vv_0 \ln\frac{v_0}{v}}}$$

A small block B is placed on another block of A of mass $5\ kg$ and length $20\ cm$. Initially the block B is near the right end of block A. A constant horizontal force of $10\ N$ is applied to block A. All the surfaces are assumed to be frictionless. Find the time elapsed before the block B separates from A.



a. $1/\sqrt{5}\ s$

b. $1/\sqrt{2}\ s$

c. $1/\sqrt{10}\ s$

d. None of these

Solution:

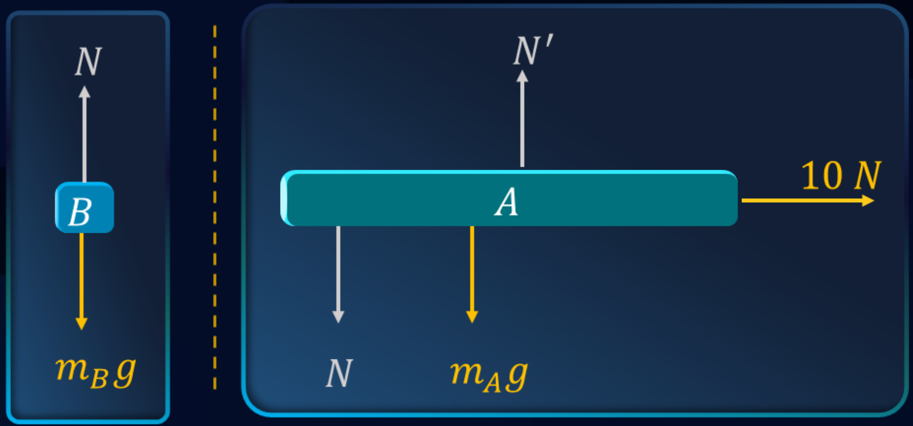
Let the acceleration of block A be a.

For block A, $\sum \vec{F_x} = m\vec{a_x}$

$$\Rightarrow 10 = 5a$$

$$\Rightarrow a = 2\ m/s^2$$

Free Body Diagram



$N = m_B g$

$N' = m_A g + N$

Using second equation of motion,

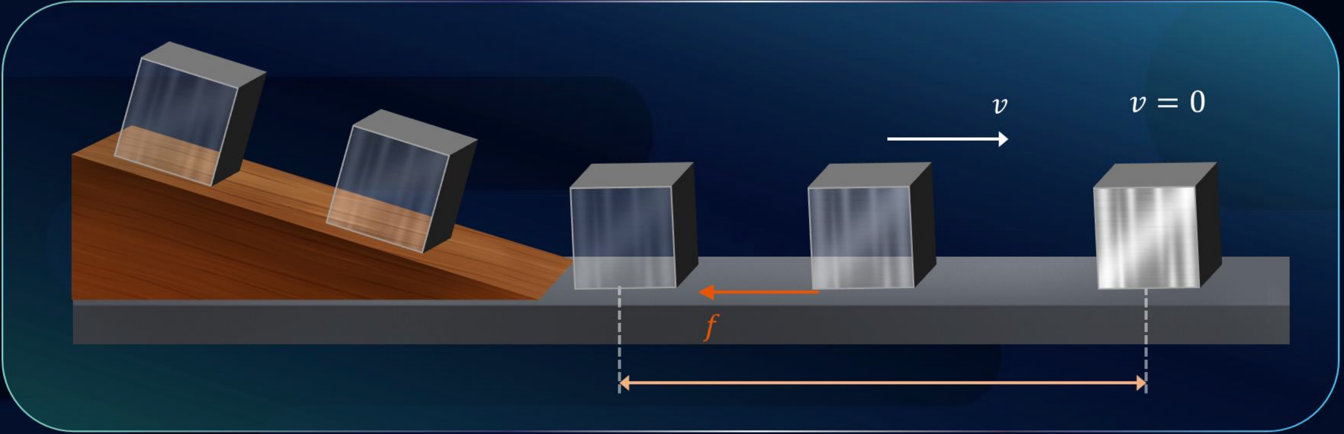
$$x = ut + \frac{1}{2}at^2$$

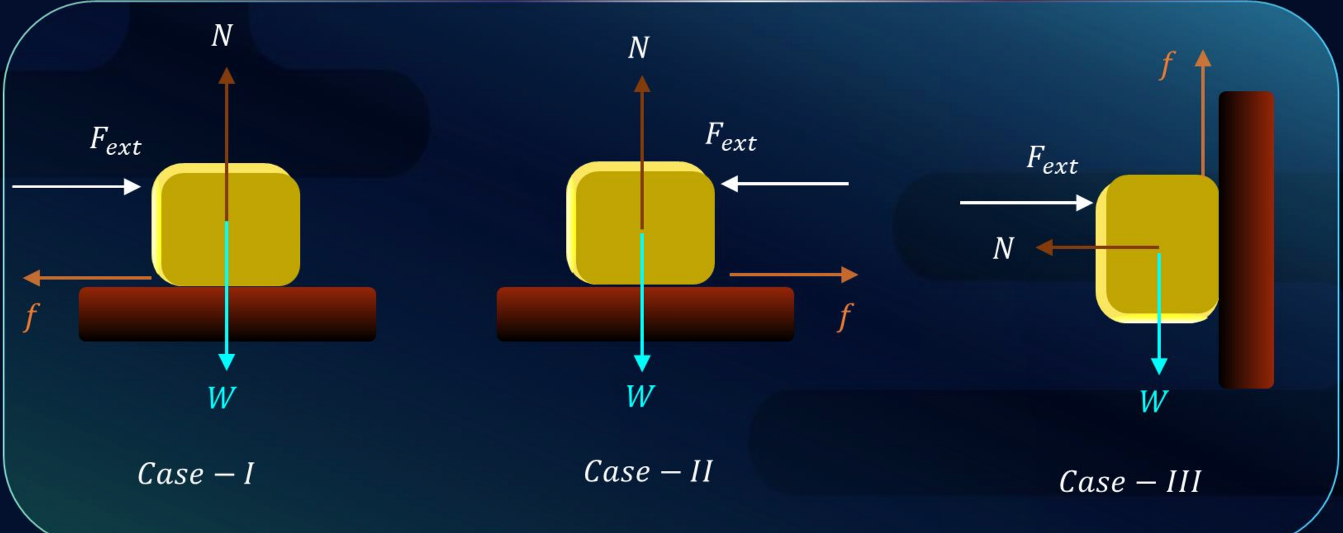
$$\frac{1}{5} = + \frac{1}{2} \times 2 \times t^2$$

$$\boxed{t = \frac{1}{\sqrt{5}}\ s}$$

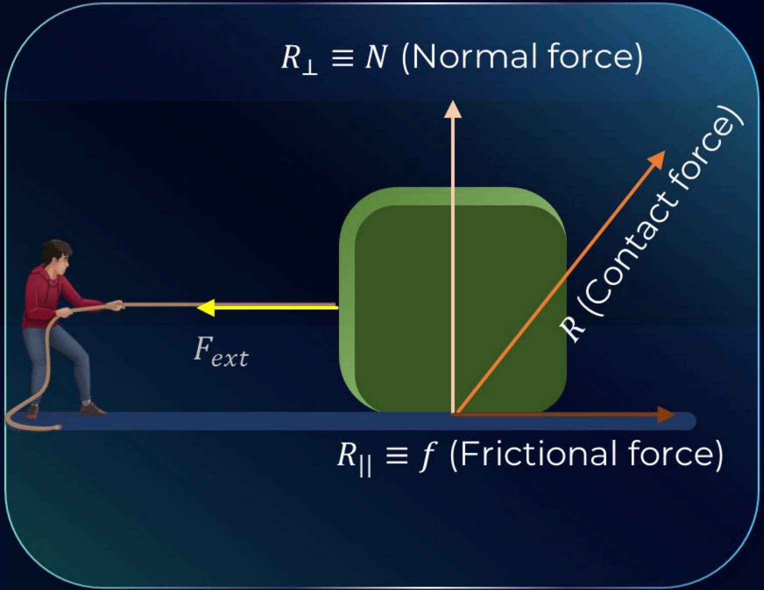
Friction

It is a force related to two surfaces in contact, which opposes the relative motion between them.



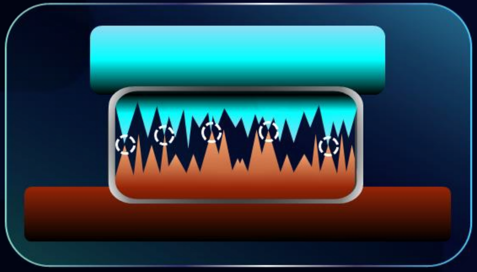


Macroscopic View



Microscopic View

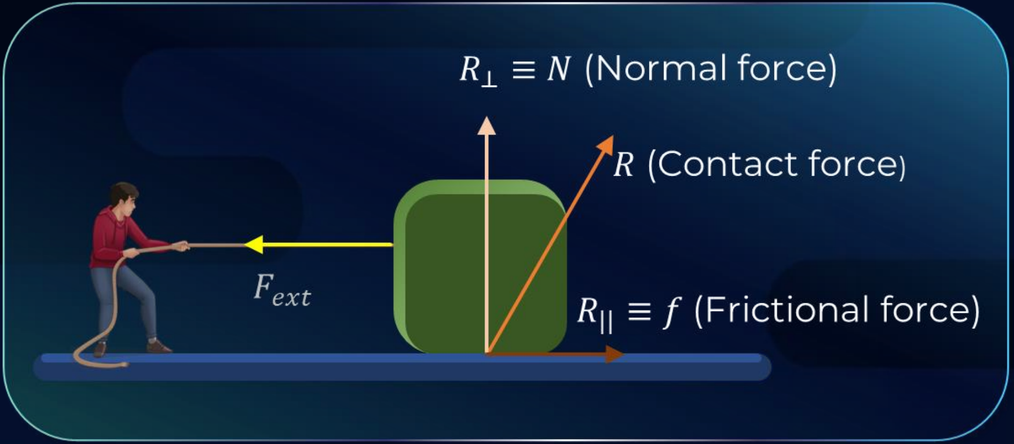
Interlocking



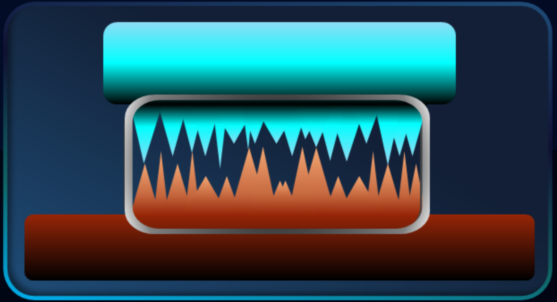
Cold-Welding

Macroscopic View

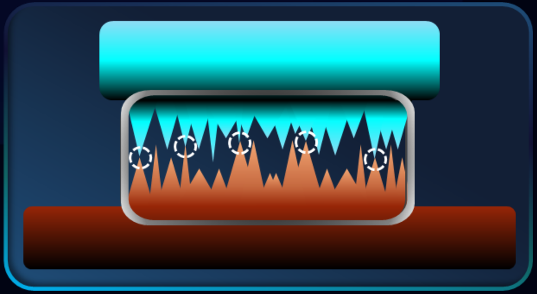
- When two bodies are brought in contact, electromagnetic forces act between the microscopic particles between the surfaces of the bodies.
- As a result, each body exerts a contact force on the other.



Microscopic View



Interlocking



Cold-Welding

- Every surface has irregularities.
- The interlocking between irregularities gives rise to friction.

- Two surfaces having a very few area in contact implies a very less effective contact area and increased pressure.
- Bonds formed due to this high pressure opposes relative motion, which leads to friction.

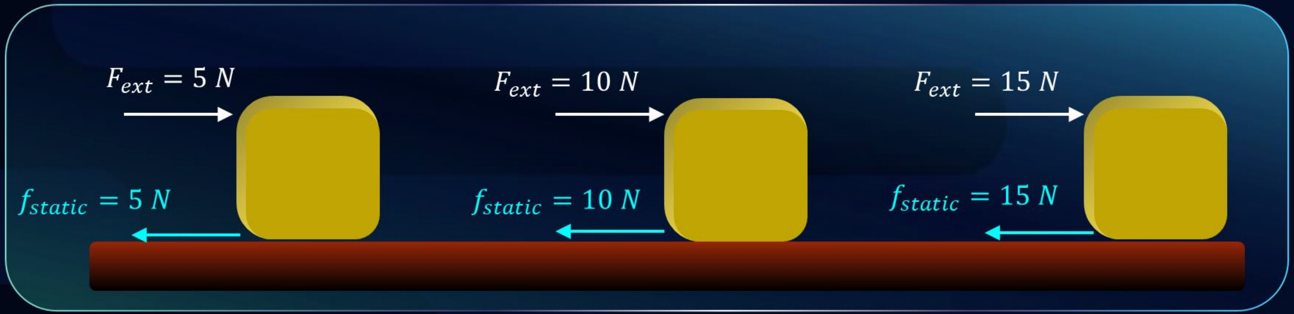
Classification of Friction

Friction

- Static Friction
  - Friction between two objects when there is a tendency of relative motion between them.
- Kinetic Friction
  - Friction between two objects when there is a relative motion between them.

Static Friction

It is a variable resistive force which is equal and opposite to external force until it surpasses the threshold of motion when the slipping starts.

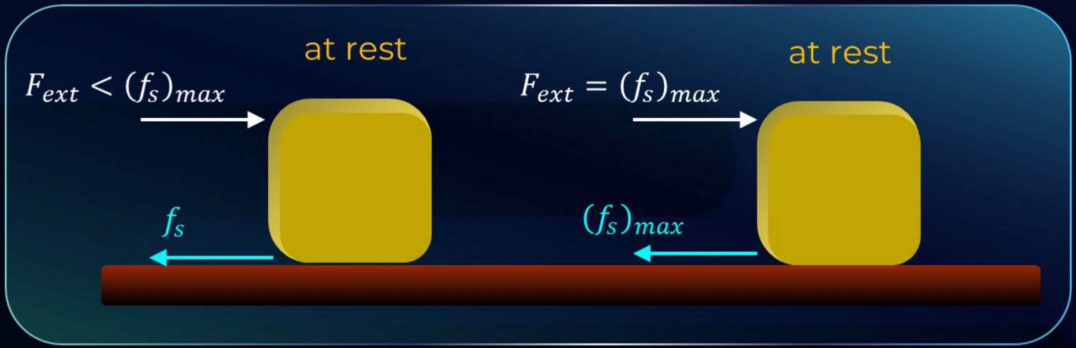


Limiting Friction

- It is the maximum possible frictional force between two surfaces just before sliding begins.
- If $F_{ext} > (f_s)_{max}$ then the slipping/relative motion starts.

$$(f_s)_{max} \propto N$$

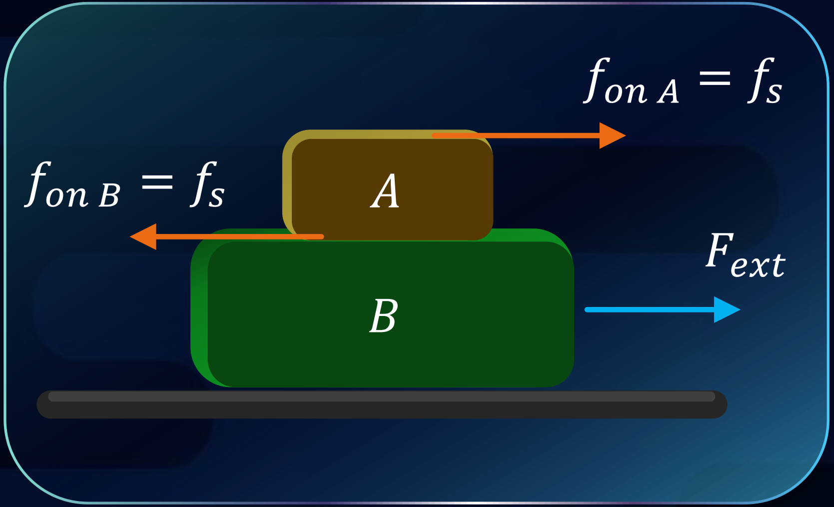
$$(f_s)_{max} = \mu_s N$$



Direction of Static Friction

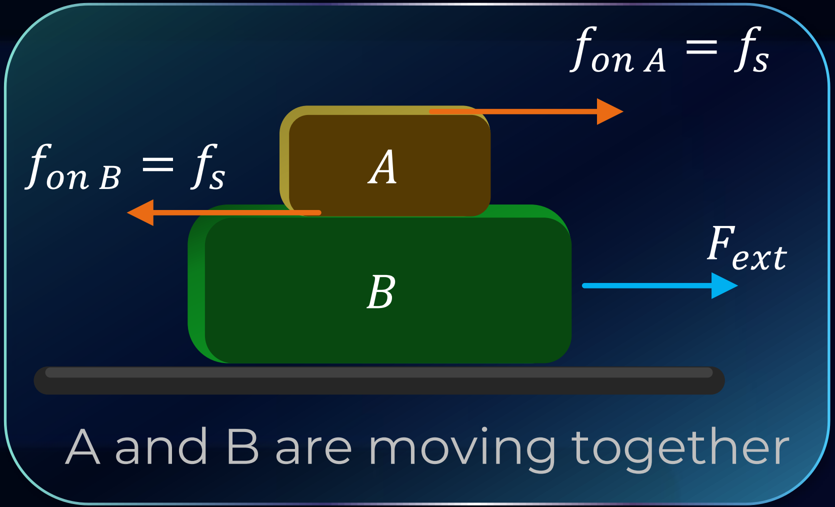
Method 1:

- Static friction acts to prevent slipping.



Method 2:

- Self adjusting nature of force.

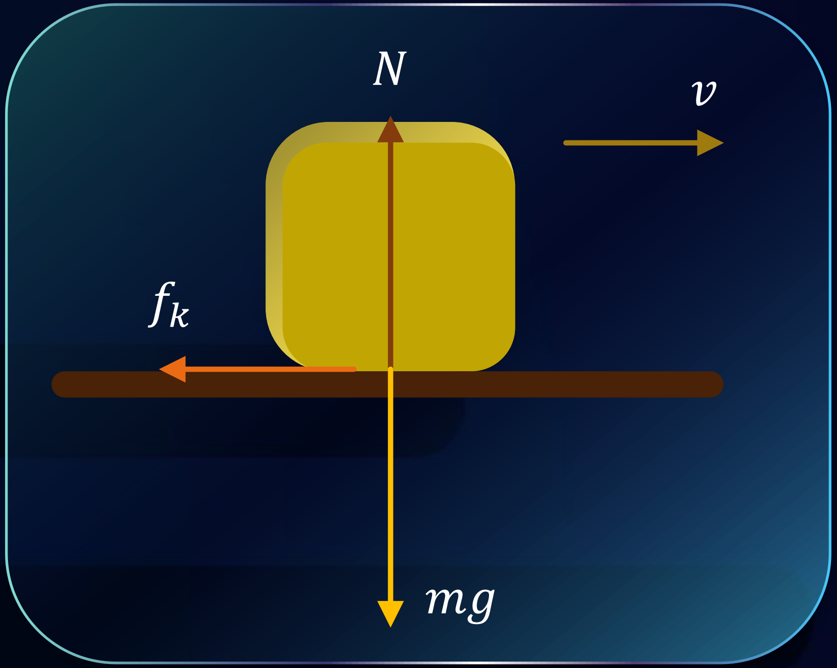


Kinetic Friction

- It is the friction between two surfaces having relative motion between them.
- Represented by f_k
- The magnitude of the kinetic friction f_k is proportional to the normal force N acting between the surfaces.

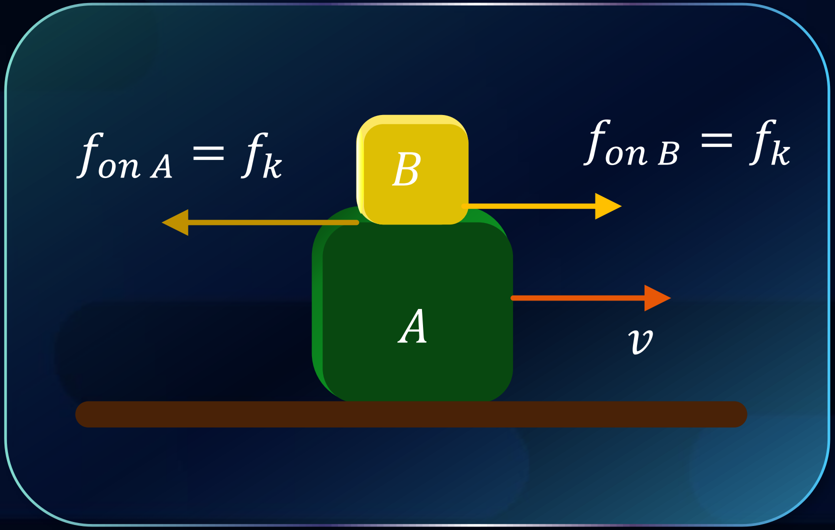
$$f_k \propto N$$

$$f_k = \mu_k N$$

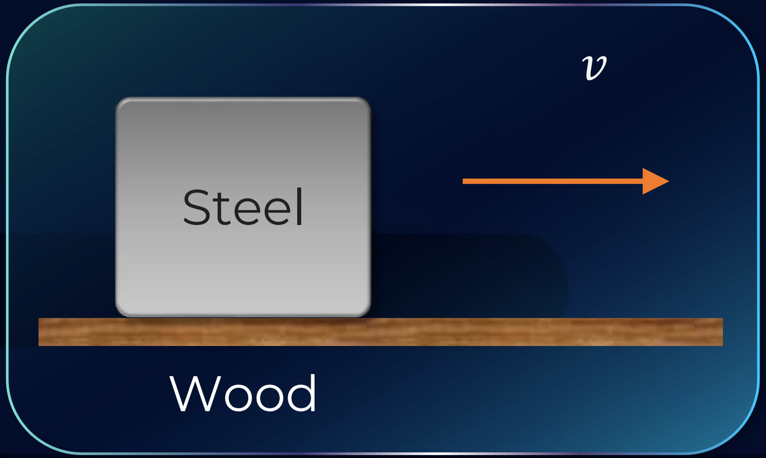


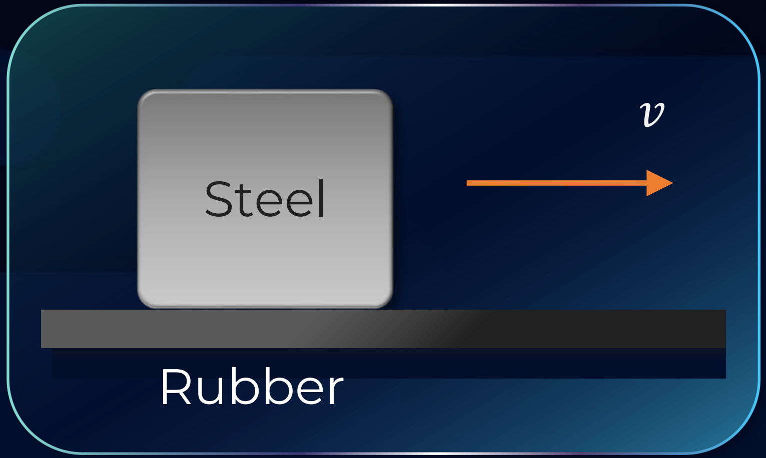
Direction of Kinetic Friction

- For an object the direction of kinetic friction is opposite to its relative velocity with respect to the other object in contact.
- The directions of kinetic friction acting on the bodies are mutually opposite (Newton's third law of motion).



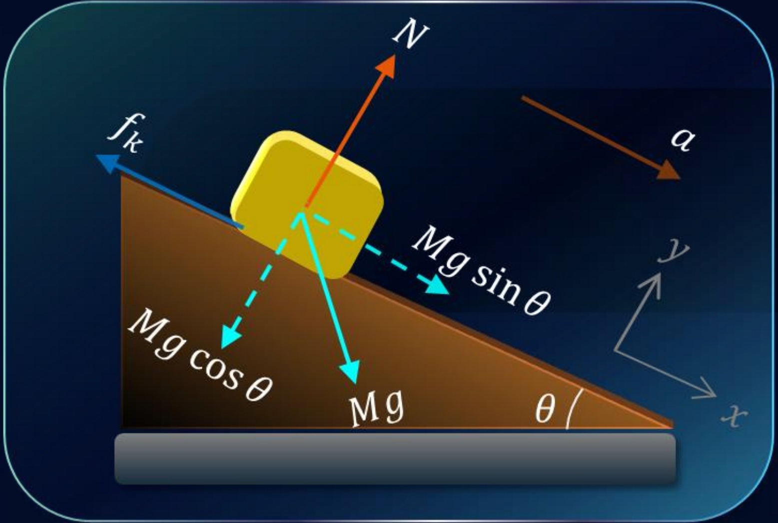
Properties of Coefficient of Friction (μ)

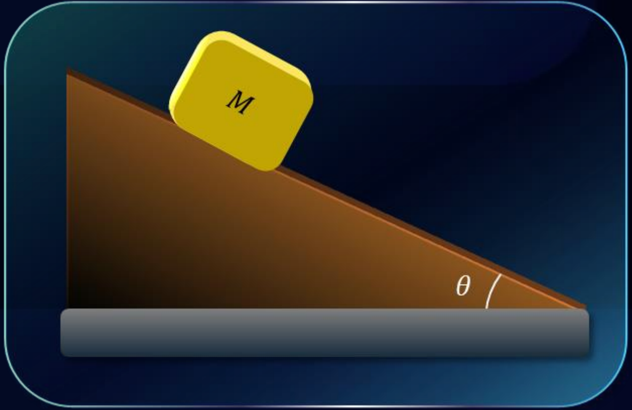




- μ depends on the nature of material of the surfaces in contact.
- μ is independent of the area of the surface in contact.
- μ is independent of the relative speed of the surfaces in contact.

A block of mass M is sliding down on a fixed wedge which is making an angle θ with the horizontal. The coefficient of friction between the block and the wedge is μ. Find the magnitude and direction of the friction on the block and its acceleration.





Solution:

Kinetic friction is acting between the block and the wedge

In X direction, $Mg\sin\theta - f_k = Ma$

In Y direction, $Mg\cos\theta = N$

$$f_k = \mu N$$

$$\vec{f}_k = -\mu Mg\cos\theta\,\hat{\imath}$$

$$Mg\sin\theta - \mu Mg\cos\theta = Ma$$

$$\vec{a} = (g\sin\theta - \mu g\cos\theta)\hat{\imath}$$

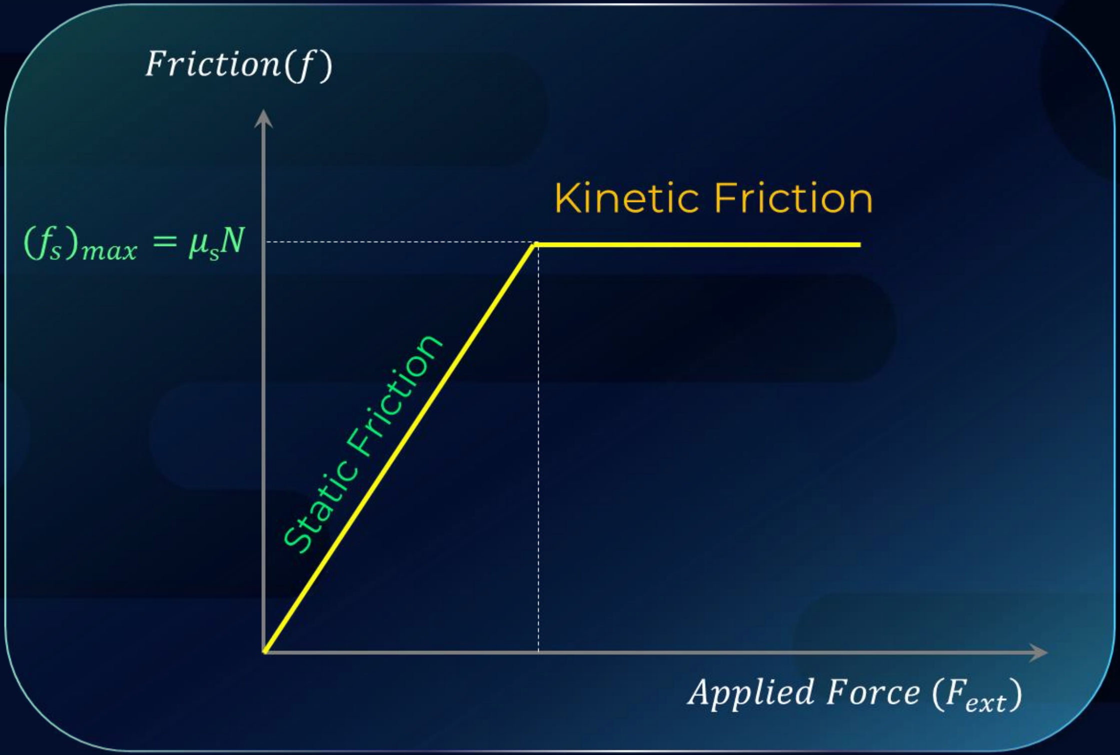
a. $\vec{f} = -(\mu Mg\cos\theta)\hat{\imath}$ $\vec{a} = (g\sin\theta - \mu g\cos\theta)\hat{\imath}$

b. $\vec{f} = (\mu Mg\cos\theta)\hat{\imath}$ $\vec{a} = (g\sin\theta + \mu g\cos\theta)\hat{\imath}$

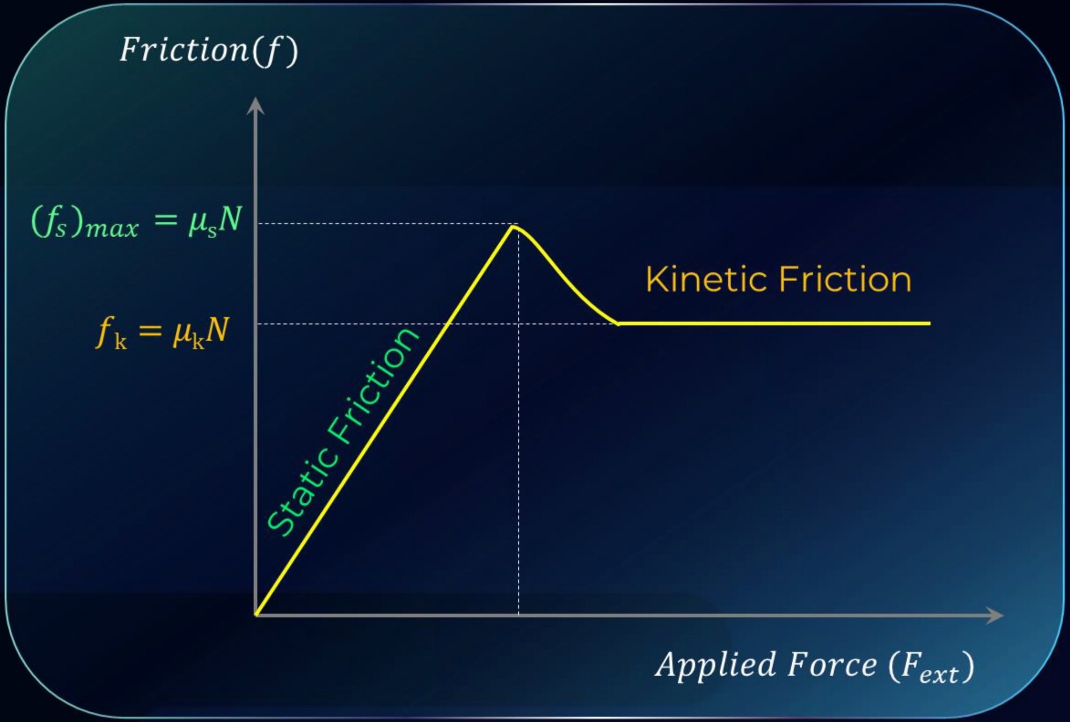
c. $\vec{f} = (\mu Mg\cos\theta)\hat{\jmath}$ $\vec{a} = (-g\sin\theta - \mu g\cos\theta)\hat{\imath}$

d. $\vec{f} = -(\mu Mg\cos\theta)\hat{\jmath}$ $\vec{a} = (-g\sin\theta + \mu g\cos\theta)\hat{\imath}$

Friction Graph (Theoretical Way)

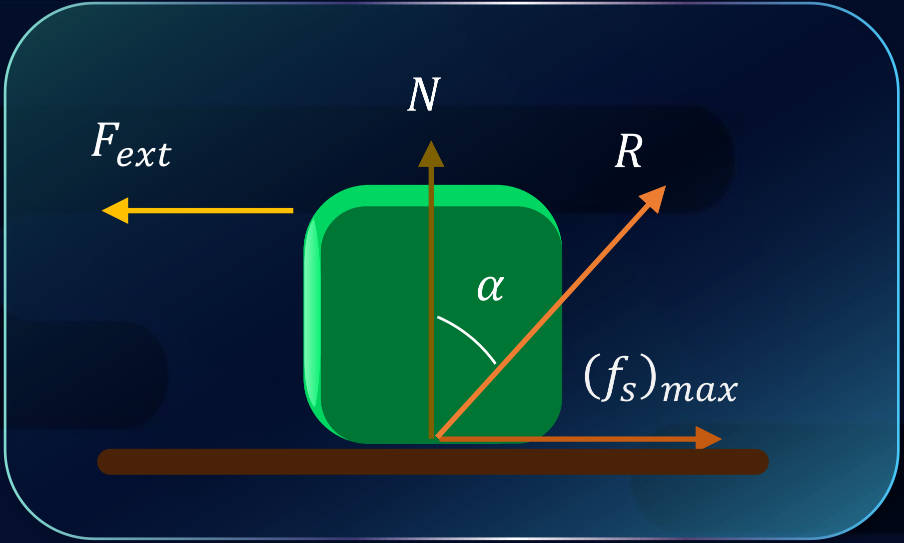


Friction Graph (Practical Way)



Angle of Friction (α)

The angle made by the resultant of the normal reaction and limiting friction with the normal reaction.



$$R = \sqrt{N^2 + f_{s_{max}}^2}$$

$$(f_s)_{max} = \mu N$$

$$\tan\alpha = \frac{(f_s)_{max}}{N}$$

$$\alpha = \tan^{-1}\left(\frac{\mu N}{N}\right)$$

$$\alpha = \tan^{-1}(\mu)$$

Consider a block kept on horizontal surface as shown in the figure. For the given values of F, find the acceleration of the block, force of friction and the nature of friction in each case. Given $\mu_s = 0.5$, $\mu_k = 0.2$ and $g = 10\ m/s^2$.

Solution:

$$f_{s_{max}} = \mu_s N = \mu_s mg$$

$$f_{s_{max}} = \mu_s mg = 0.5 \times 4 \times 10 = 20\ N$$

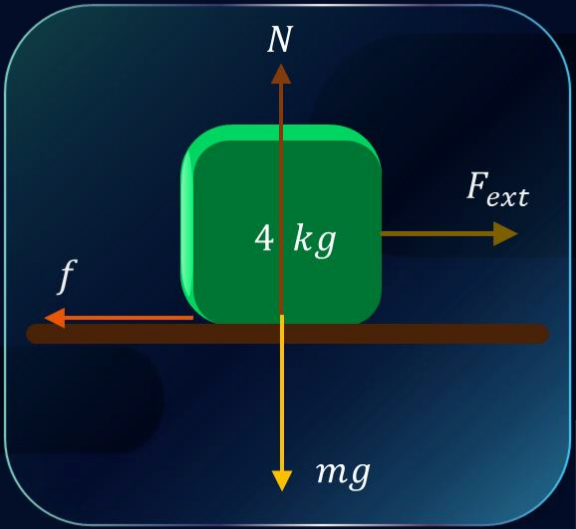
$$f_k = \mu_k N = \mu_k mg$$

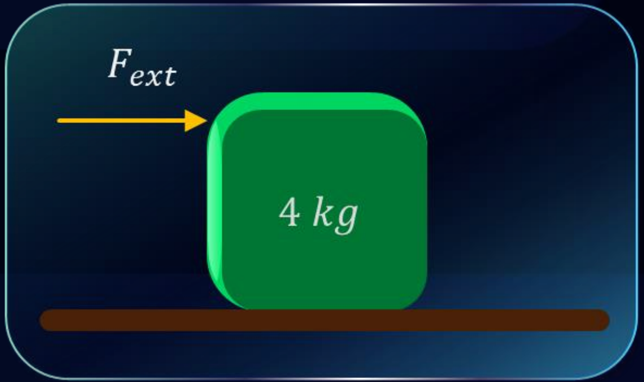
$$f_k = \mu_k mg = 0.2 \times 4 \times 10 = 8\ N$$

$$24 - f_k = ma$$

$$24 - 8 = 4 \times a$$

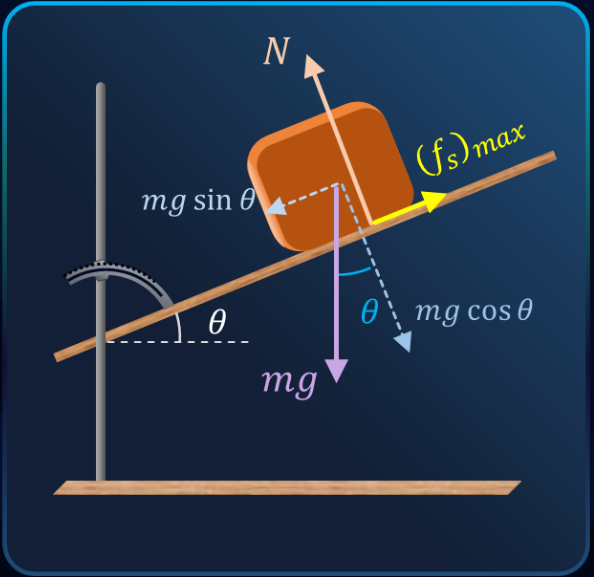
$$a = 4\ m/s^2$$





F	$f(N)$	$a(m/s^2)$	Nature
$10\ N$	10	0	Static
$15\ N$	15	0	Static
$20\ N$	20	0	Static
$24\ N$	8	4	Kinetic

Angle of Repose



- It is the minimum angle (θ) made by an inclined plane with the horizontal such that an object on the inclined surface just begins to slide.

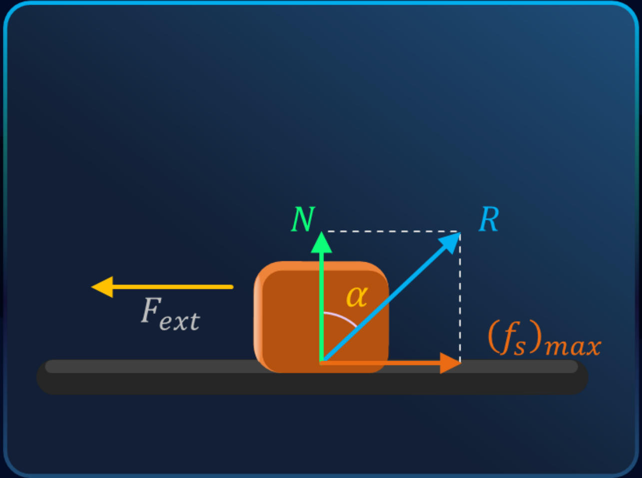
$(f_s)_{max} = mg\sin\theta$ (Along the inclined plane)

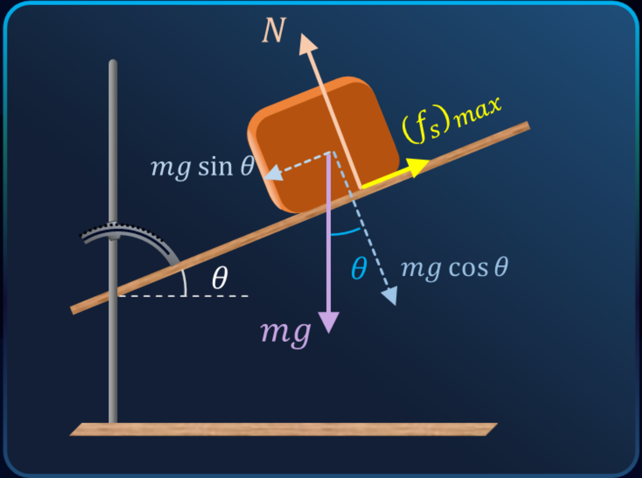
$N = mg\cos\theta$ (Perpendicular to the inclined plane)

$\mu mg\cos\theta = mg\sin\theta$ $(\because f_{s_{max}} = \mu N)$

$$\boxed{\theta = \tan^{-1}\mu}$$

Angle of Friction vs Angle of Repose





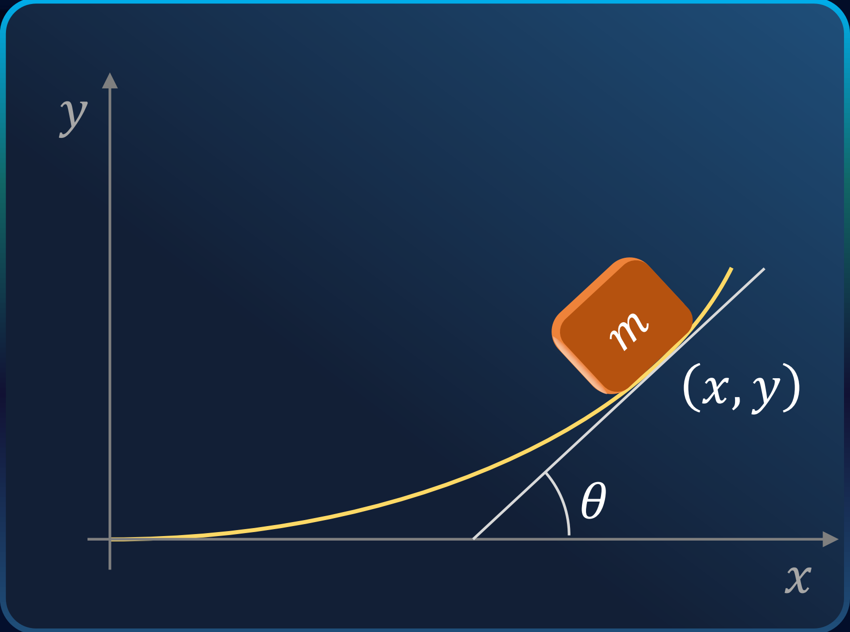
The angle of repose (θ) and angle of friction (α) are in magnitude the same, given by the inverse tangent of the coefficient of friction.

$$\boxed{\alpha = \tan^{-1}\mu}$$

$$\boxed{\theta = \tan^{-1}\mu}$$

A block of mass m is placed on a surface with a vertical cross-section given by $y = \frac{x^3}{6}$. If the coefficient of friction is 0.5, the maximum height above the ground at which the block can be placed without slipping is

Solution:



At limiting equilibrium,

$$\tan\theta = \mu$$

$$\frac{d}{dx}\left(\frac{x^3}{6}\right) = 0.5$$

$$\frac{x^2}{2} = 0.5$$

$$x = \pm 1$$

Maximum height above the ground at which the block can be placed is,

$$y(x=1) = \frac{(1)^3}{6}$$

$$y_{max} = \frac{1}{6}\,m$$

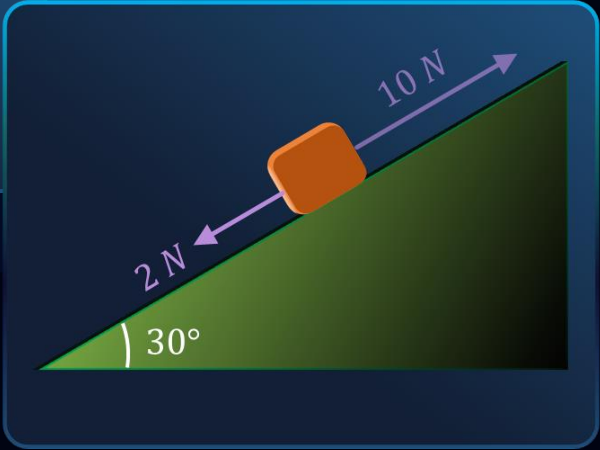
a. $\frac{1}{6}\,m$

b. $\frac{2}{3}\,m$

c. $\frac{1}{3}\,m$

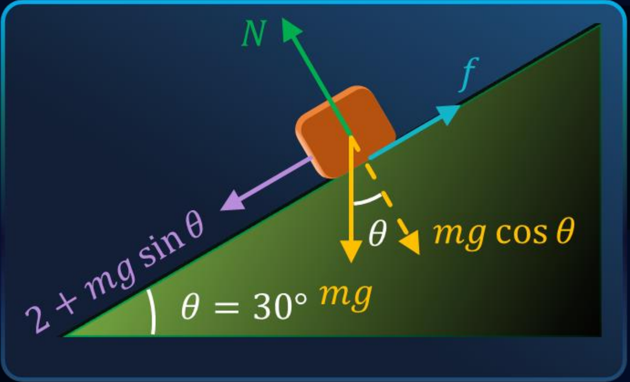
d. $\frac{1}{2}\,m$

A block kept on a rough inclined plane, as shown in the figure, remains at rest upto a maximum force $2\ N$ down the inclined plane. The maximum external force up the inclined plane that does not move the block is $10\ N$. The coefficient of static friction between the block and the plane is (JEE 2014)



Solution:

Block does not move upto a maximum applied force of $2\ N$ down the inclined plane.



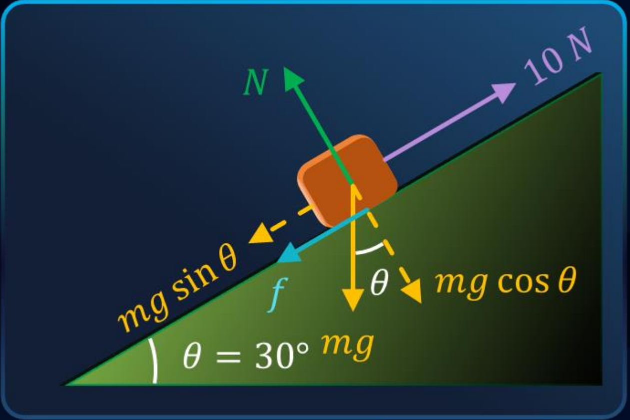
$$N = mg\cos\theta$$

$$2 + mg\sin\theta = f$$

$$2 + mg\sin\theta = \mu mg\cos\theta \quad .....(1)$$

$$(\because f = (f_s)_{max} = \mu N)$$

Similarly, block also does not move upto a maximum applied force of $10\ N$ up the plane.



$$N = mg\cos\theta$$

$$mg\sin\theta + f = 10$$

$$mg\sin\theta + \mu mg\cos\theta = 10 \quad ....(2) \quad (\because f = (f_s)_{max} = \mu N)$$

$$2 + mg\sin\theta = \mu mg\cos\theta \quad .....(1)$$

$$mg\sin\theta + \mu mg\cos\theta = 10 \quad ....(2)$$

Solving both equations , we get,

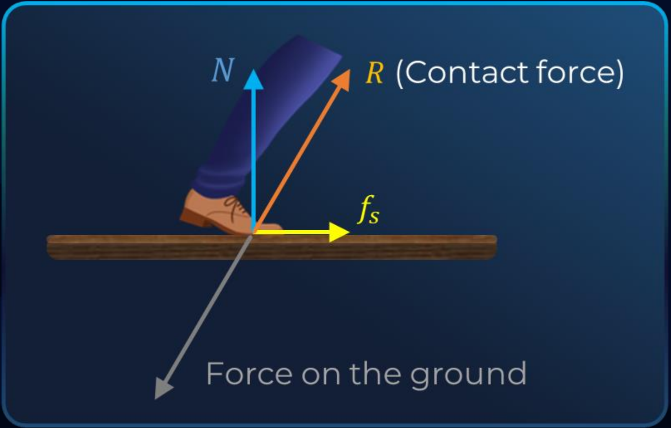
$$mg\sin\theta = 4, \qquad \mu mg\cos\theta = 6$$

Dividing these equations,

$$\mu\cot\theta = \frac{3}{2} \qquad \mu\cot 30^\circ = \frac{3}{2}$$

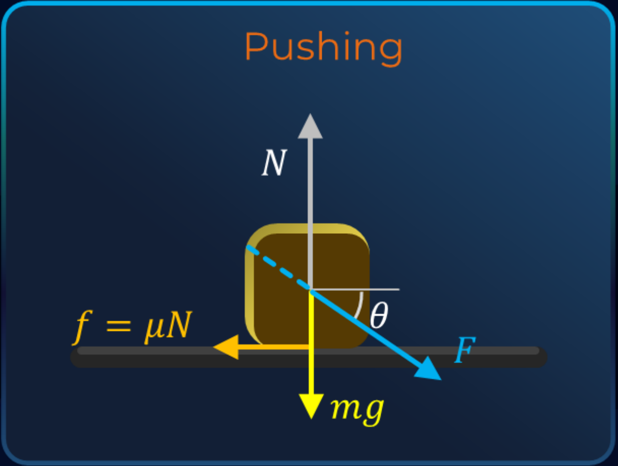
$$\boxed{\mu = \frac{\sqrt{3}}{2}}$$

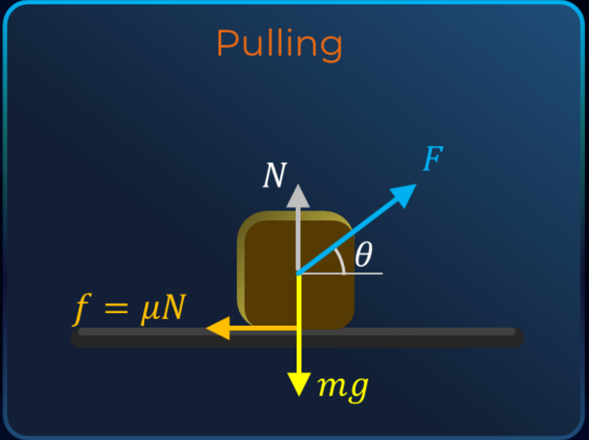
Role of Friction in Walking



- While walking, the foot presses the ground at an angle in the backward direction. The ground, in turn, exerts a reaction force R on the foot.
- The frictional force prevents us from slipping.

Pulling is easier than Pushing

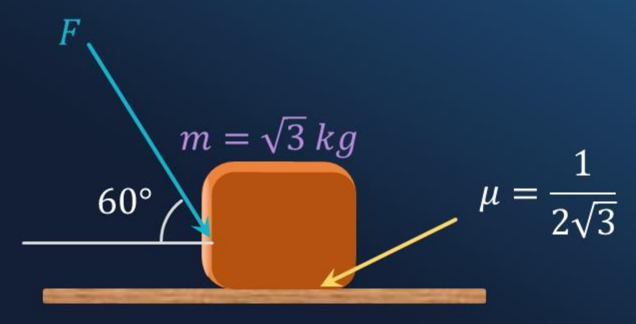




Pushing	Pulling
$N = mg + F\sin\theta$	$N = mg - F\sin\theta$
$\mu N = \mu mg + \mu F\sin\theta$	$\mu N = \mu mg - \mu F\sin\theta$
$F\cos\theta = \mu mg + \mu F\sin\theta$	$F\cos\theta = \mu mg - \mu F\sin\theta$

$$\boxed{F = \frac{\mu mg}{\cos\theta - \mu\sin\theta}} > \boxed{F = \frac{\mu mg}{\cos\theta + \mu\sin\theta}}$$

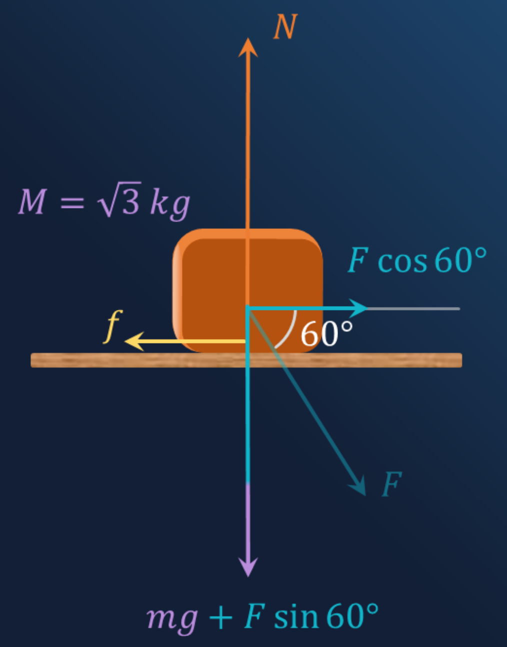
What is the maximum value of the force F such that the block shown in the arrangement does not move?



Solution: For vertical equilibrium of the block,

$$N = mg + F \sin 60°$$

$$N = \sqrt{3}g + \frac{\sqrt{3}}{2}F$$



For block to remain stationary,

$$f_{max} \geq F \cos 60°$$

$$\mu N \geq \frac{F}{2}$$

$$\frac{1}{2\sqrt{3}}\left(\sqrt{3}g + \frac{\sqrt{3}}{2}F\right) \geq \frac{F}{2}$$

$$F \leq 20\ N$$

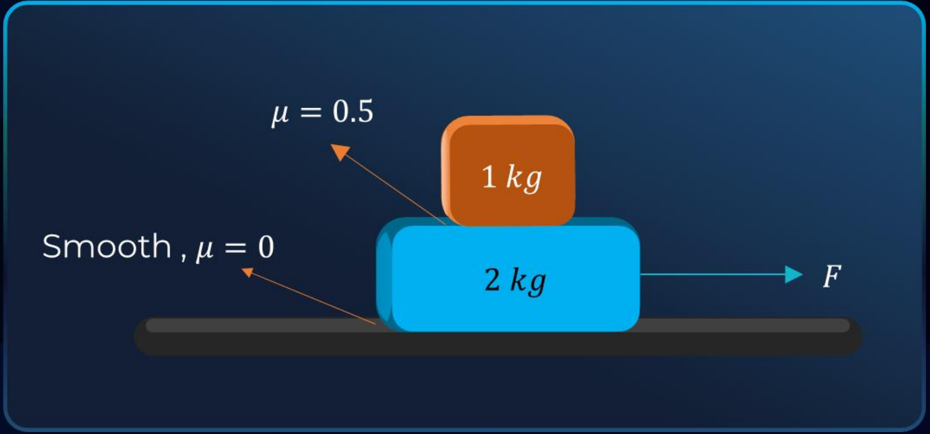
a. $20\ N$

b. $10\ N$

c. $12\ N$

d. $15\ N$

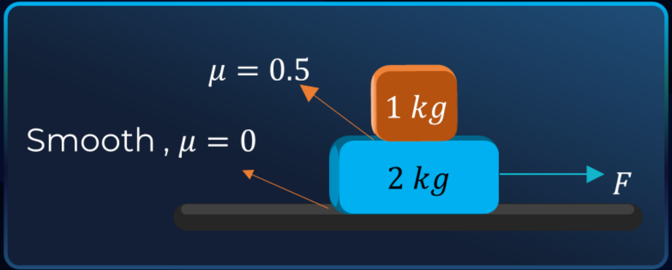
The coefficient of static friction between two blocks is 0.5 and the ground surface is smooth. The maximum horizontal force that can be applied to move the blocks together is (take g = $10\ m/s^2$)



Solution: $F = (m_1 + m_2)a$

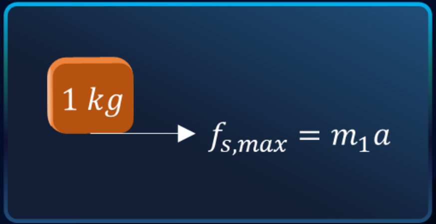
$F = (1+2)a$

$F = 3a$



$f_{s,max} = m_1 a$

$\mu \times N = 1 \times a$

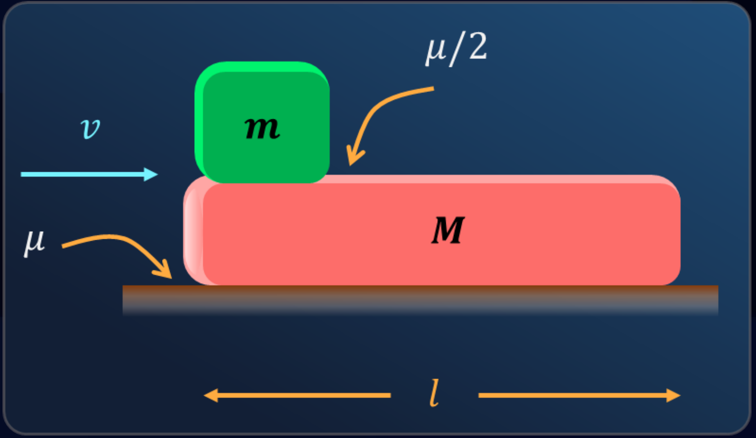


$0.5 \times 1 \times 10 = 1 \times a$

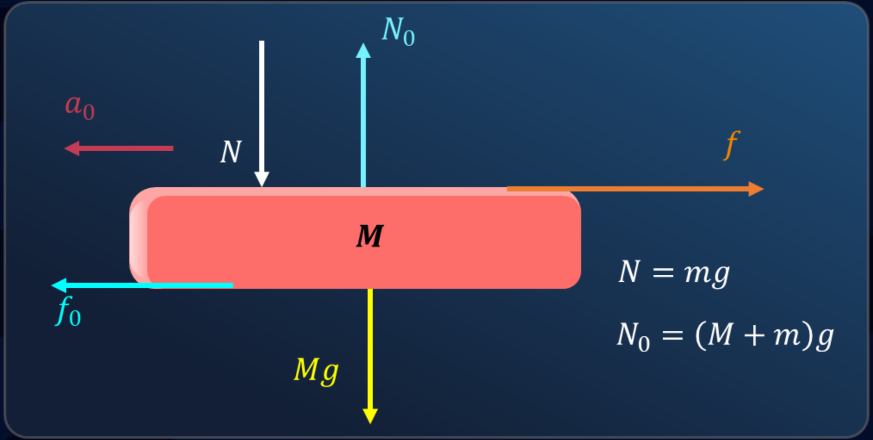
$a = 5\ m/s^2$

So, $\boxed{F = 3a = 3 \times 5 = 15\ N}$

A block of mass m is placed on another block of mass M and length l as shown in the figure. If the system starts moving with velocity v towards the right, then find time elapsed before the smaller block separates from the bigger block.



FBD of M (Ground frame)



$N = mg$

$N_0 = (M+m)g$

$$f_0 - f = Ma_0$$

$$\mu(M+m)g - \frac{\mu}{2}mg = Ma_0$$

$$\mu\left(M + \frac{m}{2}\right)g = Ma_0$$

$$a_0 = \frac{\mu(2M+m)g}{2M}$$

$$a_0 = \frac{\mu(2M+m)g}{2M}$$

$$ma_0 - f = ma$$

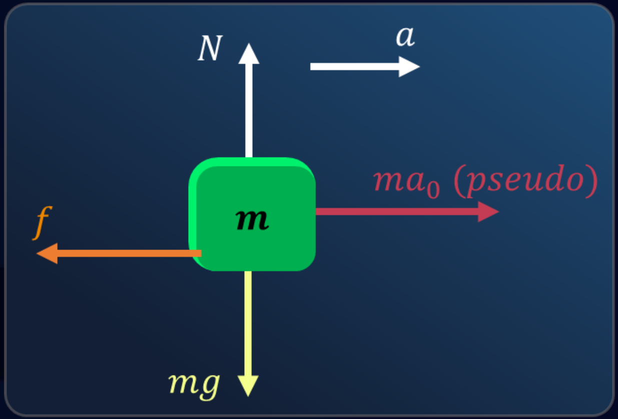
$$\frac{\mu m(2M+m)g}{2M} - \frac{\mu}{2}mg = ma$$

$$a = \frac{\mu(M+m)g}{2M}$$

$$x = ut + \frac{1}{2}at^2$$

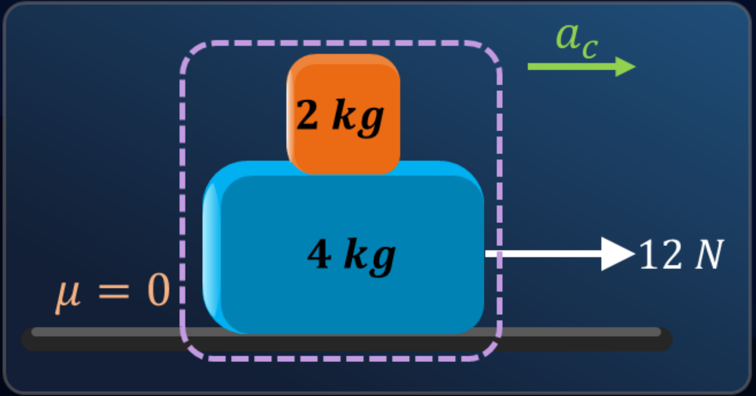
$$l = \frac{1}{2}\frac{\mu(M+m)g}{2M}T^2$$

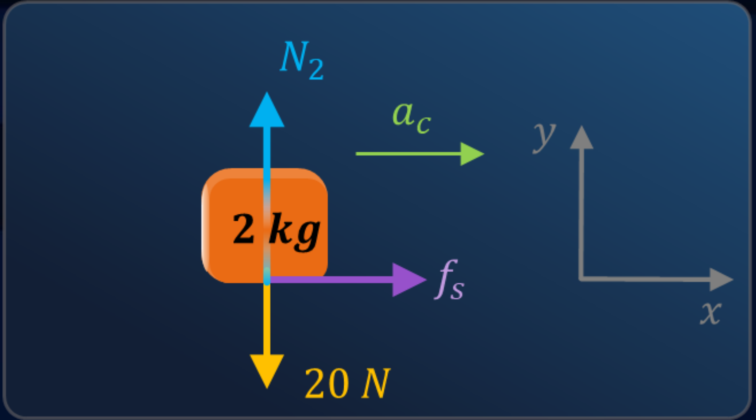
$$\boxed{T = \sqrt{\frac{4Ml}{\mu(M+m)g}}}$$



FBD of m (M frame)(non-inertial)

A $2\ kg$ block is placed over a $4\ kg$ block and both are placed on a smooth horizontal surface. The coefficient of friction between the blocks is 0.2. Find the acceleration of the two blocks if a horizontal force of $12\ N$ is applied on the lower block.
(Take $g = 10\ m/s^2$)



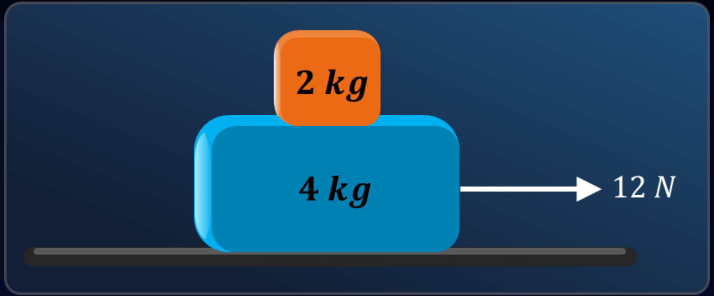


FBD of $2\ kg$ block

Assuming motion with common acceleration,

$(2+4)a_c = F = 12\ N$ (Along the x-axis)

$$\Rightarrow a_c = \frac{12}{6} = 2\ m/s^2$$



$N_2 = 2g = 20\ N$ (Along y axis)

$f_s = 2a_c = 4\ N$ (Along x axis)

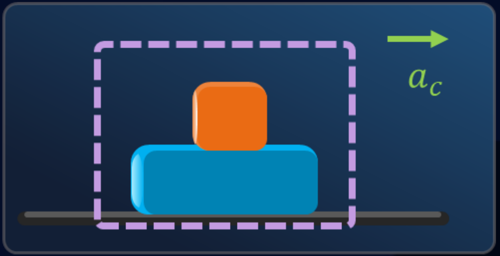
But, $f_s \le (f_s)_{max}$, where $(f_s)_{max} = \mu N_2 = 4\ N$

Hence, the blocks move with a common acceleration,

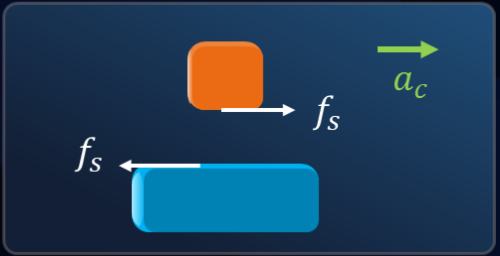
$$\boxed{a_c = 2\ m/s^2}$$

Steps- Block on Block

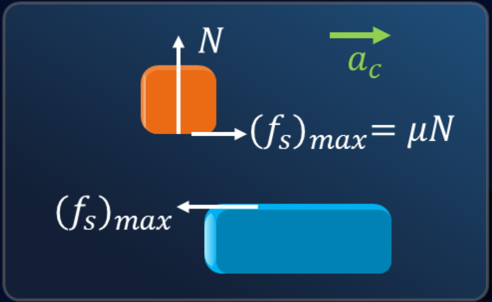
Step 1: Assume that all blocks move together with a common acceleration a_c



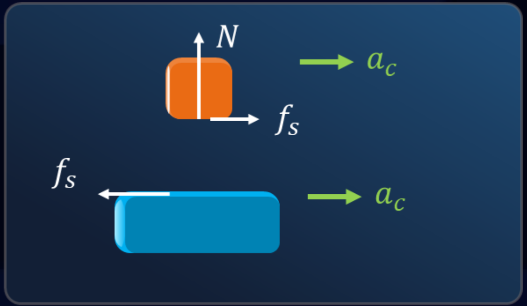
Step 2: Find out the required static frictional force (f_s),i.e., driving force on the other block such that the blocks move with a common acceleration



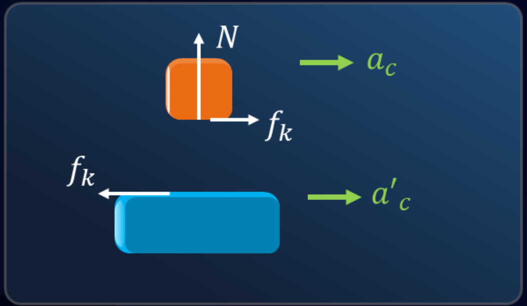
Step 3: Find out the limiting friction $(f_s)_{max}$



Step 4: For any two blocks stacked up, if $(f_s) \leq (f_s)_{max}$, then both will move together with common acceleration



Step 5: If it is seen that $(f_s) > (f_s)_{max}$, then there will be a relative motion between them. The assumption is wrong and they will move with different accelerations. Find out the individual accelerations from their respective FBDs



Two blocks A and B of masses $m_A = 1\ kg$ and $m_B = 3\ kg$ are kept on the table as shown. The coefficient of friction between A and B is 0.2 and between B and the surface of the table is also 0.2. The maximum force F that can be applied on B horizontally, so that the block A does not slide over the block B is

Solution:

The system of blocks accelerates with acceleration a under the influence of F and f_{BT}

$$a = \frac{F - f_{BT}}{m_A + m_B}$$

$$= \frac{F - \mu_{BT}(m_A + m_B)g}{m_A + m_B} = \frac{F - 0.2(1+3)10}{1+3}$$

$$= \frac{F - 8}{4}$$

Considering block A as system,

Maximum value of f_{AB} for which the block A doesn't slip over B,

$$f_{AB} = m_A a$$

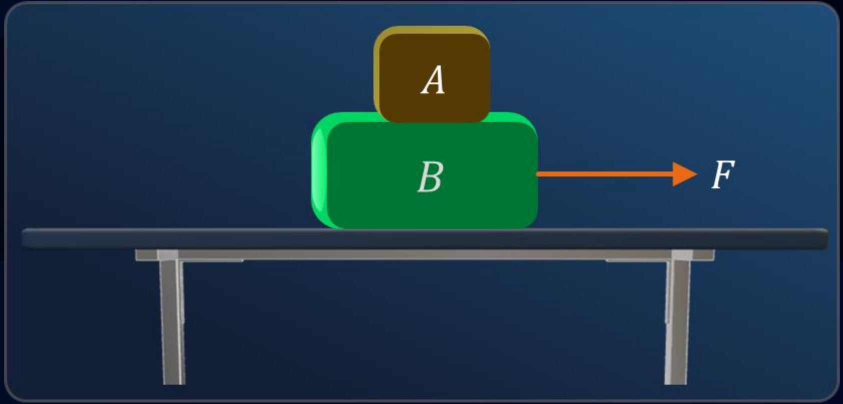
$$\mu_{AB} m_A g = m_A a$$

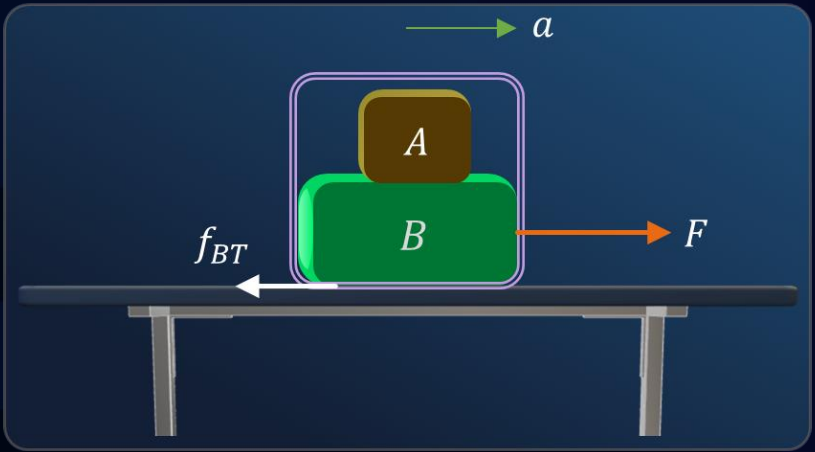
$$a = 2\ m/s^2$$

$$a = \frac{F - 8}{4}$$

$$2 = \frac{F_{max} - 8}{4}$$

$$\boxed{F_{max} = 16\ N}$$

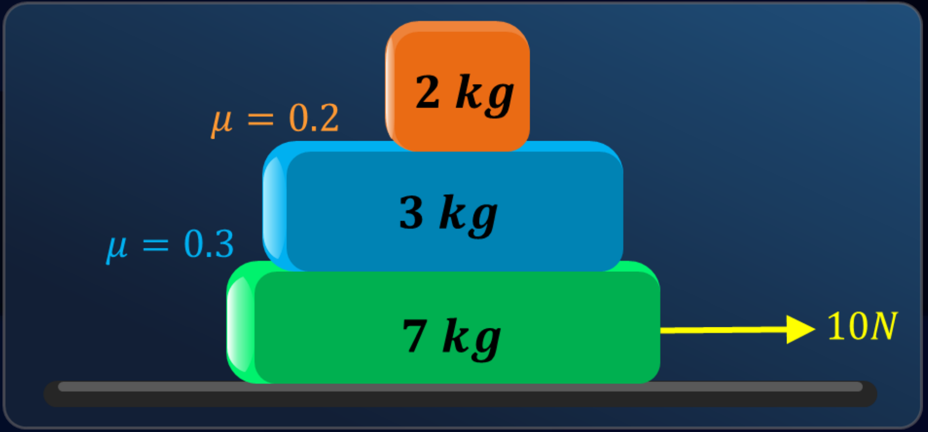




Find the accelerations $a_2,\ a_3,\ a_7$ of the three blocks shown in the figure placed on a smooth horizontal surface, if a horizontal force of $10\ N$ is applied on the $7\ kg$ block. (Take $g\ = 10\ m/s^2$)

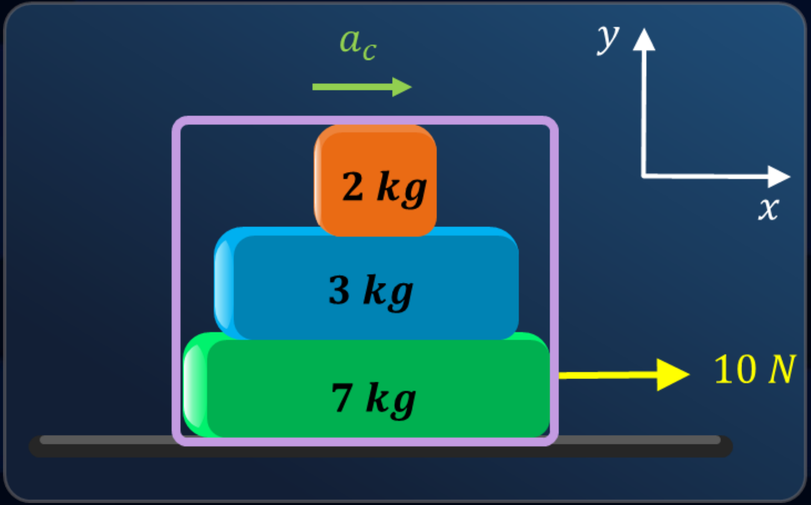
Solution:

Assumption: Motion with common acceleration, a_c



Along x axis, $(2+3+7)a_c = 10\ N$

Thus, $a_c = \dfrac{10}{12} = \dfrac{5}{6}\ m/s^2$



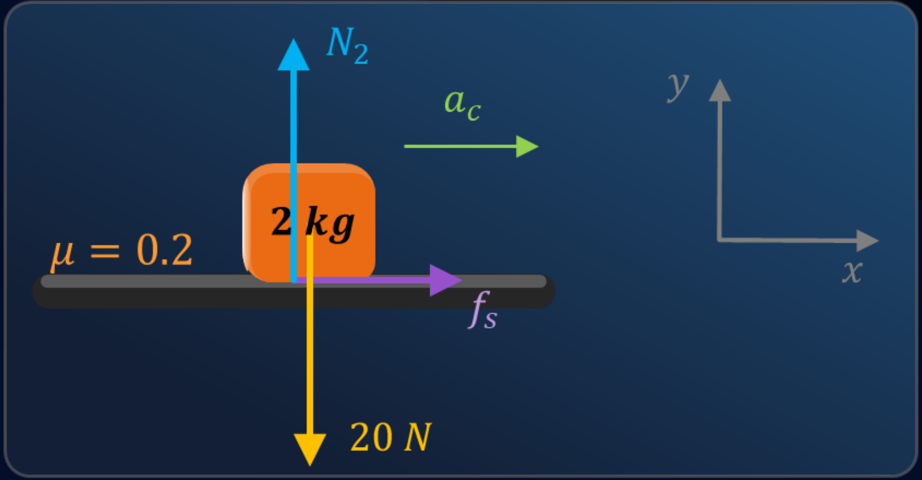
$N_2 = 2g = 20\ N$ (Along y axis)

$f_s = 2a_c = \dfrac{5}{3}\ N$ (Along x axis)

$(f_s)_{max} = \mu N_2 = (0.2)20 = 4\ N$

Since $f_s \le (f_s)_{max}$

Blocks of $2\ kg$ and $3\ kg$ move together



FBD of $2\ kg$ block

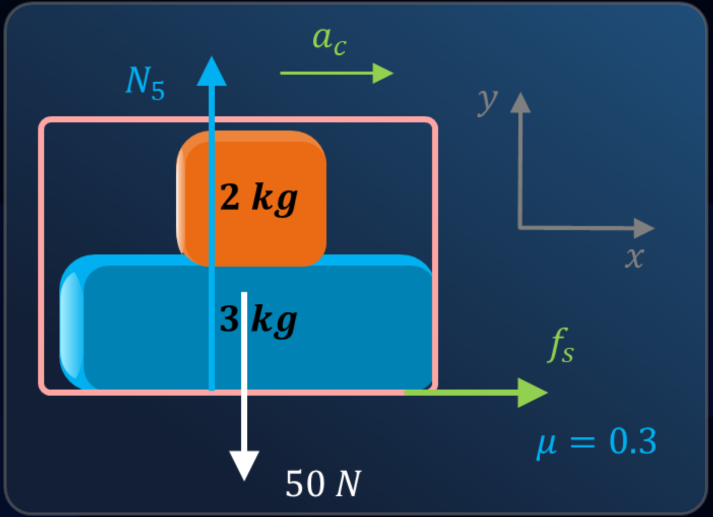
$N_2 = 2g = 20\ N$ (Along y axis)

$f_s = 2a_c = \frac{5}{3}\ N$ (Along x axis)

$(f_s)_{max} = \mu N_2 = (0.2)20 = 4\ N$

Since $f_s \leq (f_s)_{max}$

Blocks of $2\ kg$ and $3\ kg$ move together



FBD of $(2 + 3)\ kg$ block

Now consider the top two blocks to be a system

$N_5 = 5g = 50\ N$ (Along y axis)

$f_s = 5a_c = \frac{25}{6}\ N$ (Along x axis)

$(f_s)_{max} = \mu N_5 = (0.3)50 = 15\ N$

Since, $f_s \leq (f_s)_{max}$

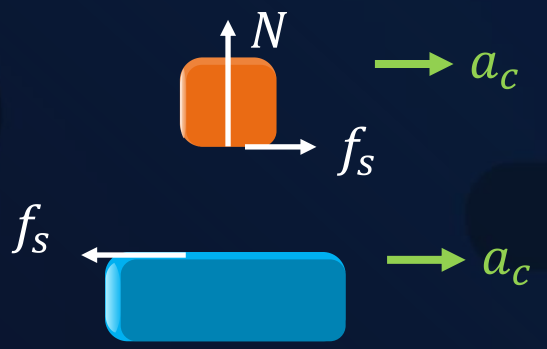
All blocks move together

$$a_2 = a_3 = a_7 = a_c = \frac{5}{6}\ m/s^2$$

IF $(f_s) \leq (f_s)_{max}$

YES

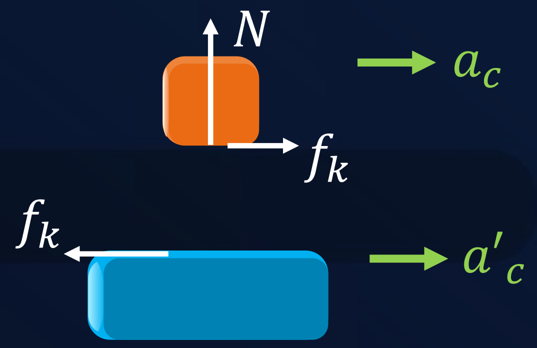
Blocks will move together with common acceleration



NO

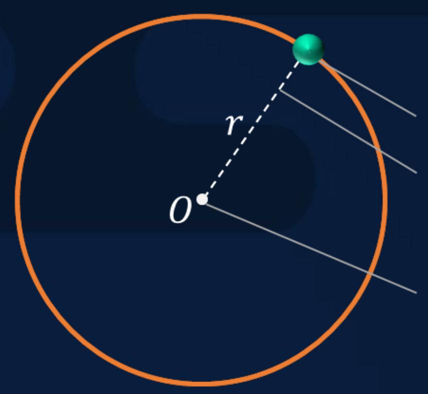
There is relative motion between blocks (Assumption is wrong)

$\Rightarrow$ Find acceleration of blocks from their respective FBDs

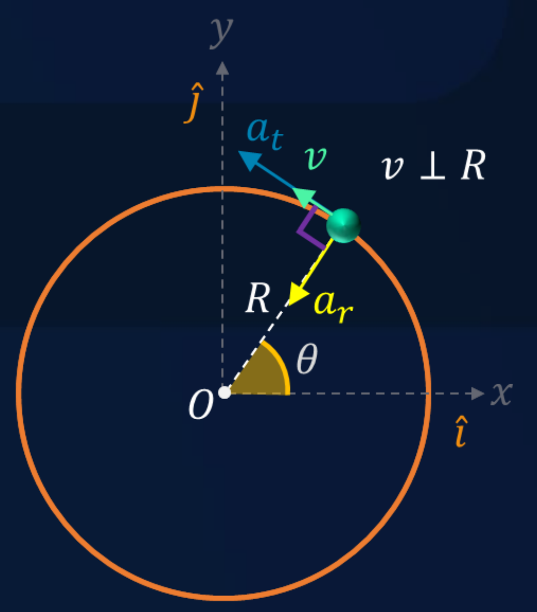


Circular Motion

- The trajectory/locus of a particle is called a Circle,
  - When a particle moves in a plane and,
  - Distance from a reference point remains constant



Position and Velocity



Position Vector:

$$\vec{r} = R\cos\omega t\,\hat{\imath} + R\sin\omega t\,\hat{\jmath}$$

$$|\vec{r}| = R$$

Velocity Vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = -R\omega\sin\omega t\,\hat{\imath} + R\omega\cos\omega t\,\hat{\jmath}$$

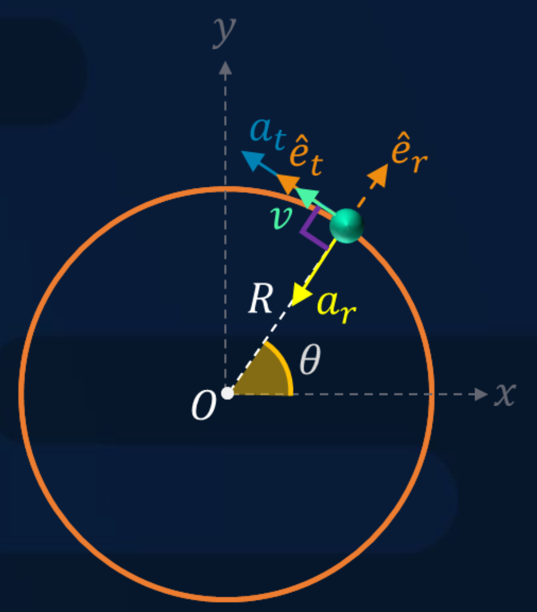
$$|\vec{v}| = R\omega$$

$$\boxed{\vec{v} = R\omega(-\sin\omega t\,\hat{\imath} + R\cos\omega t\,\hat{\jmath})}$$

Tangential Acceleration

- Rate of change of linear speed (Only magnitude) w.r.t. t
- Acts along tangent
- Should be zero for UCM
- It is a variable vector quantity
- If v is along a_t: Speed Increases
- If v is opposite to a_t: Speed decreases

$$a_t = \frac{d|\vec{v}|}{dt} = R\frac{d\omega}{dt} = R\alpha$$



Centripetal/Radial Acceleration

- Responsible for change in direction
- Necessary to move the body in circular motion
- Acts towards centre
- Always perpendicular to velocity
- Although it might be having constant magnitude but its direction is always variable and so its a variable vector quantity

$$a_r = R\omega^2 = \frac{v^2}{R}$$

Net Acceleration

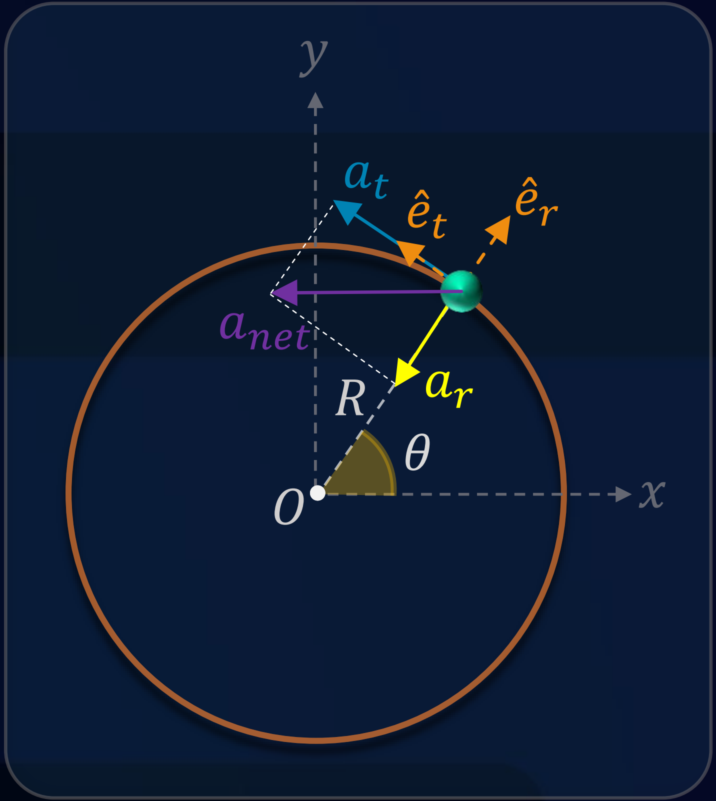
In vector form:

$$\vec{a} = \frac{d\vec{v}}{dt} = a_t\hat{e}_t - a_r\hat{e}_r$$

$$\boxed{\vec{a} = R\frac{d\omega}{dt}\hat{e}_t - R\omega^2\hat{e}_r}$$

Net acceleration:

$$\boxed{a_{net} = \sqrt{a_r^2 + a_t^2}}$$



- If angle between v and a_{net} is zero: Circular motion is not possible
- If angle between v and a_{net} is greater than zero and less than 90° : Speed increasing

Summary

Parameter	Formula
Position Vector	$\vec{r} = R\cos\omega t\,\hat{\imath} + R\sin\omega t\,\hat{\jmath}$
Velocity Vector	$\vec{v} = R\omega(-\sin\omega t\,\hat{\imath} + R\cos\omega t\,\hat{\jmath})$
Tangential Acceleration	$a_t = \dfrac{d\lvert\vec{v}\rvert}{dt} = R\dfrac{d\omega}{dt} = R\alpha$
Centripetal Acceleration	$a_r = R\omega^2 = \dfrac{v^2}{R}$
Net acceleration	$a_{net} = \sqrt{a_r^2 + a_t^2}$

Force in Circular Motion

Centripetal Force:

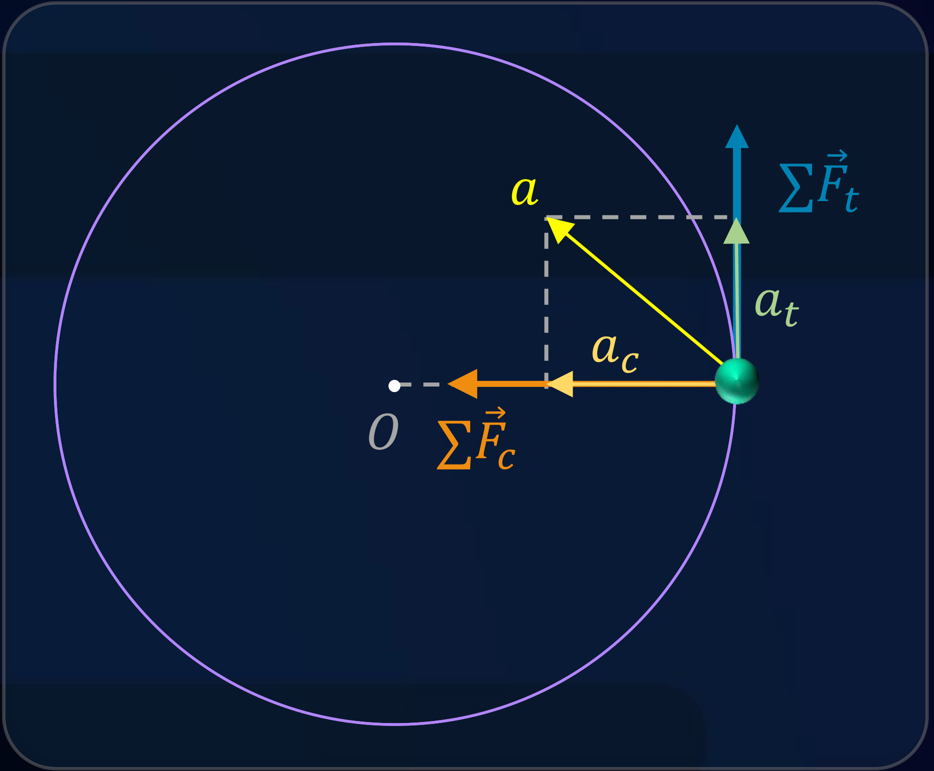
$$\Sigma\vec{F}_c = m\vec{a}_c$$

$$\Sigma|\vec{F}_c| = m|\vec{a}_c|$$

$$\boxed{\Sigma F_c = \frac{mv^2}{R} = m\omega^2 R}$$

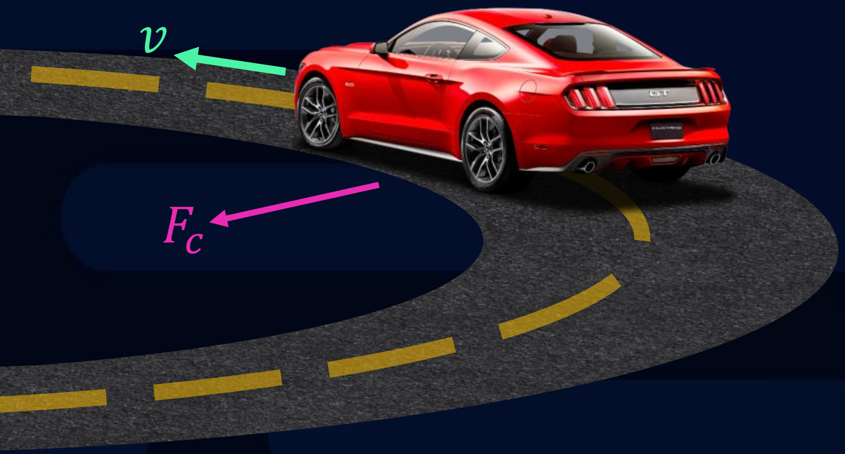
Tangential Force:

$$\Sigma\vec{F}_t = m\vec{a}_t$$



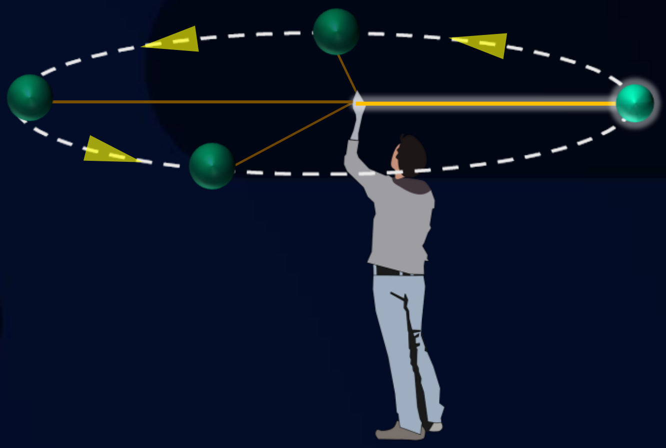
Examples of Centripetal Force

Friction Force:



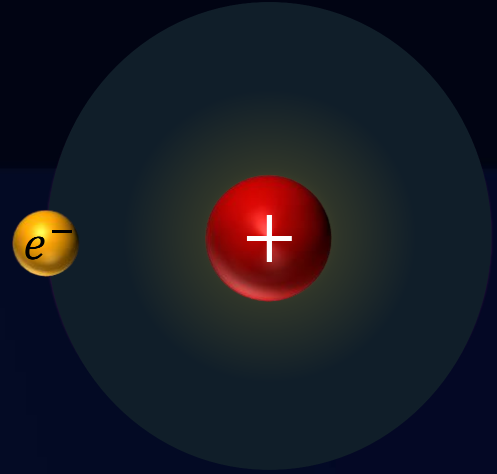
$$F_c = \frac{mv^2}{R} = m\omega^2 R$$

Tension in string:



$$T = \frac{mv^2}{R} = m\omega^2 R$$

Electrostatic Force:



$$F_e = \frac{mv^2}{R} = m\omega^2 R$$

Linear vs Circular Motion

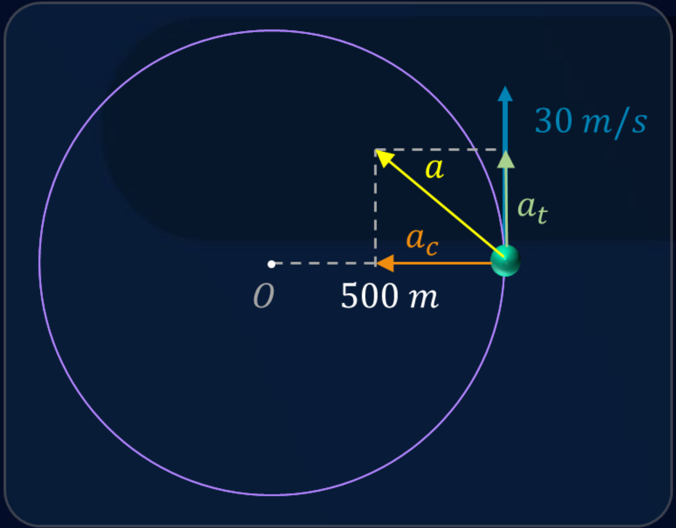
Parameter	Linear Motion	Circular Motion	Vector Form
Position/Angle	x	θ	$\vec{x} = \vec{\theta} \times \vec{r}$
Velocity/Angular Velocity	$v = \dfrac{dx}{dt}$	$\omega = \dfrac{d\theta}{dt}$	$\vec{v} = \vec{\omega} \times \vec{r}$
Acceleration/Angular Acceleration	$a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$	$\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$	$\vec{a} = \vec{\alpha} \times \vec{r}$

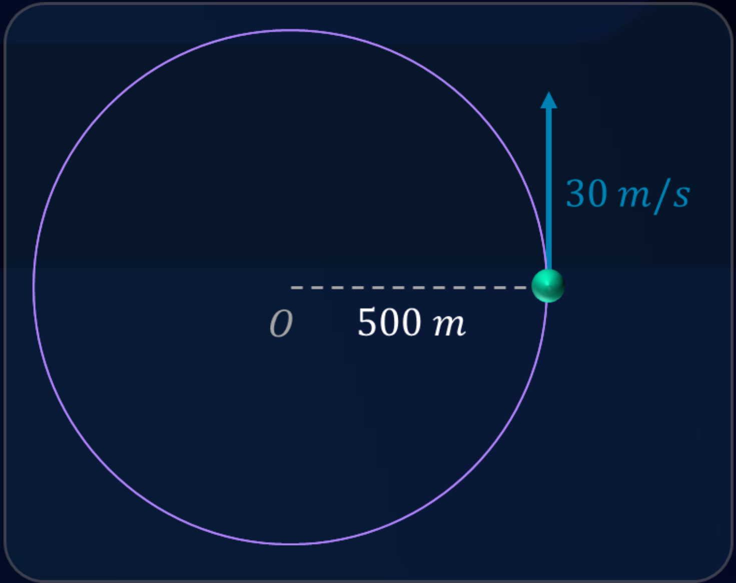
Equations of Motion

Linear Motion	Circular Motion
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$
$s_n = u + \frac{a}{2}(2n - 1)$	$\theta_n = \omega_0 + \frac{\alpha}{2}(2n - 1)$

A particle is revolving in a circular path of radius $500\ m$. It is increasing its speed at the rate of $2\ m/s^2$. What is its net acceleration when its speed is $30\ m/s$?

Solution:





Centripetal Acceleration:

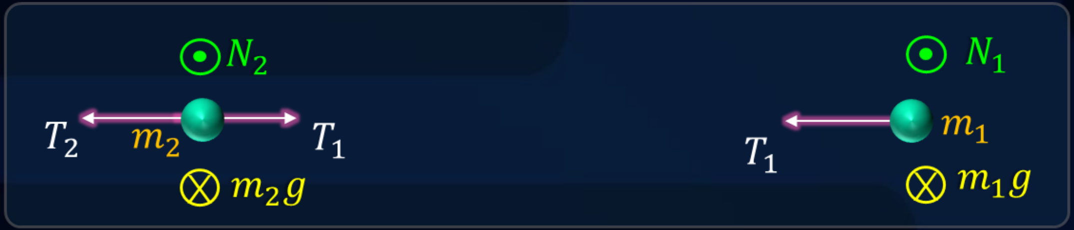
$$a_c = \frac{u^2}{r} = \frac{30^2}{500} = \frac{9}{5}\ m/s^2$$

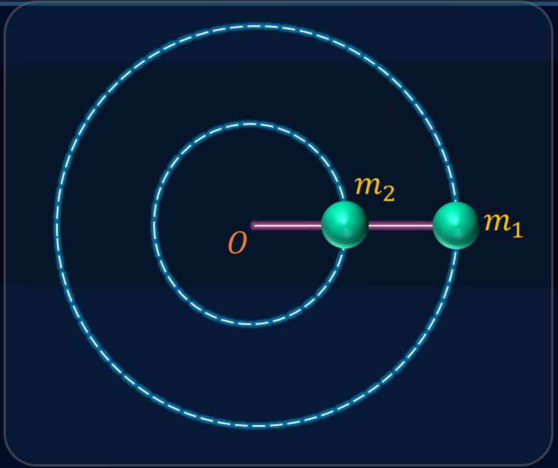
Total acceleration:

$$a = \sqrt{a_c{}^2 + a_t{}^2} = \sqrt{2^2 + \left(\frac{9}{5}\right)^2} = \boxed{\frac{\sqrt{181}}{5}\ m/s^2}$$

A particle of mass m_1 is fastened to one end of a string and one of m_2 to the middle point, the other end of the string being fastened to a fixed point on a smooth horizontal table. The particles are then projected, so that the two portions of the string are always in the same straight line and describes horizontal circles. Find the ratio of tensions in the two parts of the string.

Solution:





Along the radial direction:

$$\Sigma \vec{F}_c = m\vec{a}_c$$

$$T_2 - T_1 = m_2 \omega^2 l \quad \ldots(1)$$

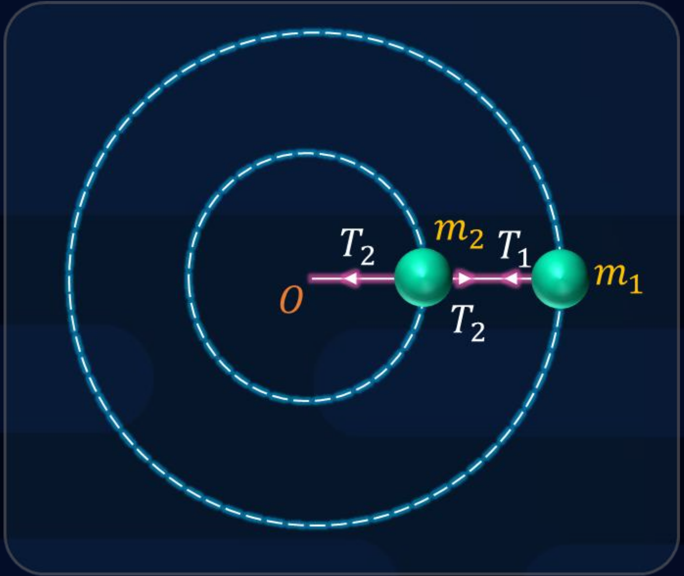
Along the radial direction:

$$\Sigma \vec{F}_c = m\vec{a}_c$$

$$T_1 = m_1 \omega^2 (2l) \quad \ldots(2)$$

$$\frac{T_2 - T_1}{T_1} = \frac{m_2 \omega^2 l}{2m_1 \omega^2 l}$$

$$\Rightarrow \boxed{\frac{T_2}{T_1} = \frac{2m_1 + m_2}{2m_1}}$$



a. $\dfrac{m_1}{m_1 + m_2}$

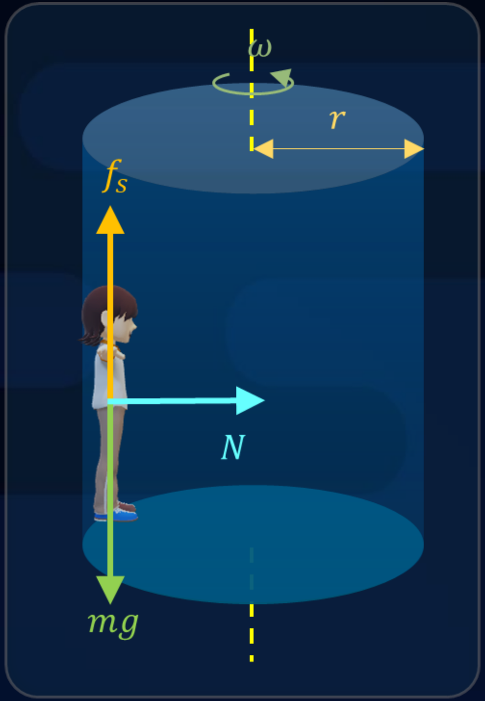
b. $\dfrac{m_1 + m_2}{m_1}$

c. $\dfrac{2m_1 + m_2}{2m_1}$

d. $\dfrac{2m_1}{m_1 + m_2}$

A rotor, a hollow vertical cylindrical structure rotates about its axis and a person rests against the inner wall. At a particular speed of the rotor, the floor below the person is removed and the person hangs resting against the wall without any floor. If the radius of the rotor is $2\ m$ and the coefficient of static friction between the wall and the person is 0.2, find the minimum speed at which the floor may be removed (Take $g = 10\ m/s^2$)

Solution:



Along vertical direction:

$$f_s = mg \quad \dots(1)$$

Along radial direction:

$$\Sigma \vec{F}_c = M\vec{a}_c$$

$$N = \frac{mv^2}{r} \quad \dots(2)$$

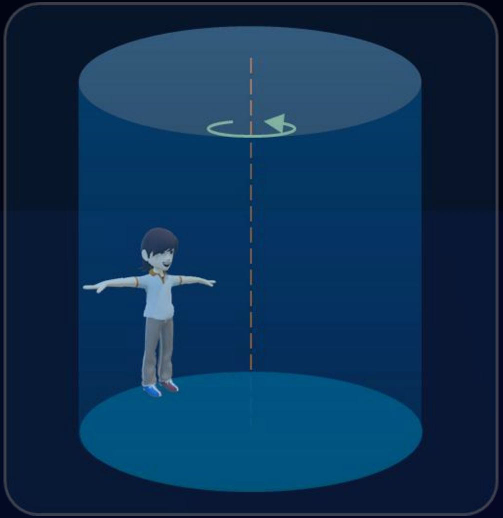
For person to hang without floor,

$$f_s \leq (f_s)_{max}$$

$$mg \leq \mu N$$

$$mg \leq \mu \frac{mv^2}{r}$$

$$\boxed{v \geq \sqrt{\frac{rg}{\mu}}}$$



$$v_{min} = \sqrt{\frac{10 \times 2}{0.2}}$$

$$\boxed{v_{min} = 10\ m/s}$$

Conical Pendulum

Tension in the string:

For mass m along vertical direction,

$$\Sigma \vec{F}_v = 0$$

$$\Rightarrow T\cos\theta = mg$$

$$\Rightarrow T\sqrt{\frac{L^2 - r^2}{L^2}} = mg \qquad \left[\because \cos\theta = \sqrt{\frac{L^2 - r^2}{L^2}}\right]$$

$$\Rightarrow \boxed{T = \frac{mgL}{\sqrt{L^2 - r^2}}}$$

Angle with vertical:

$$T\cos\theta = mg \quad \ldots(1)$$

For mass m along radial direction,

$$\Sigma \vec{F}_c = m\vec{a}_c$$

$$\Rightarrow T\sin\theta = \frac{mv^2}{r} \quad \ldots(2)$$

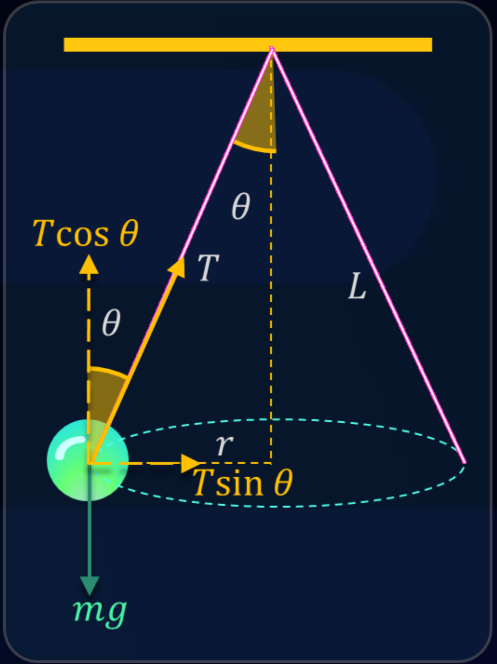
$$\Rightarrow \boxed{\tan\theta = \frac{v^2}{gr}}$$

Velocity of mass:

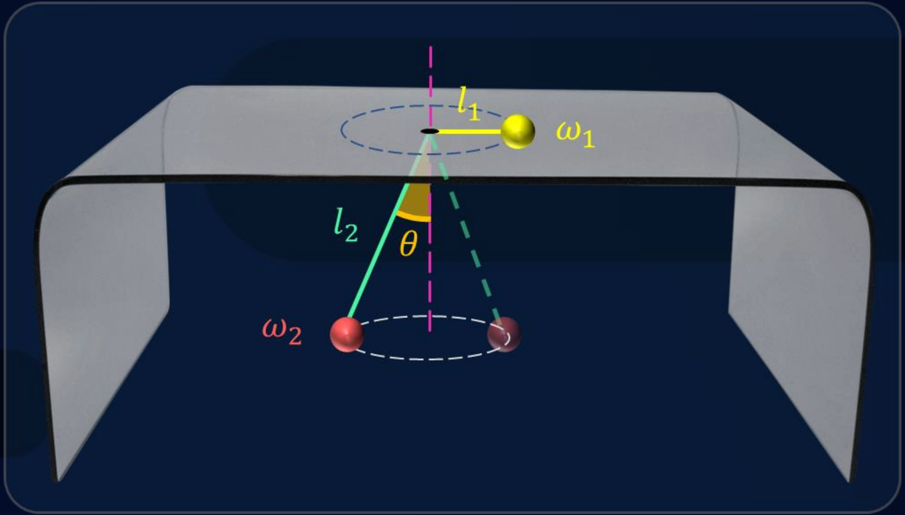
$$\tan\theta = \frac{v^2}{gr} \quad \ldots(1)$$

$$\Rightarrow v = \sqrt{rg\tan\theta}$$

$$\boxed{v = \frac{r\sqrt{g}}{(L^2 - r^2)^{\frac{1}{4}}}} \qquad \left[\because \tan\theta = \frac{r}{\sqrt{L^2 - r^2}}\right]$$



Two particles each of mass m are attached to the two ends of a light string of length l which passes through a hole at the center of a table. One particle describes a circle on the table with angular velocity ω_1 and the other describes a circle as a conical pendulum with angular velocity ω_2 below the table as shown in figure. If l_1 and l_2 are the lengths of the portion of the string above and below the table, then



a. $\frac{l_1}{l_2} = \frac{\omega_2^2}{\omega_1^2}$

b. $\frac{l_1}{l_2} = \frac{\omega_1^2}{\omega_2^2}$

c. $\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} = \frac{l}{g}$

d. $\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} > \frac{l}{g}$

Solution:

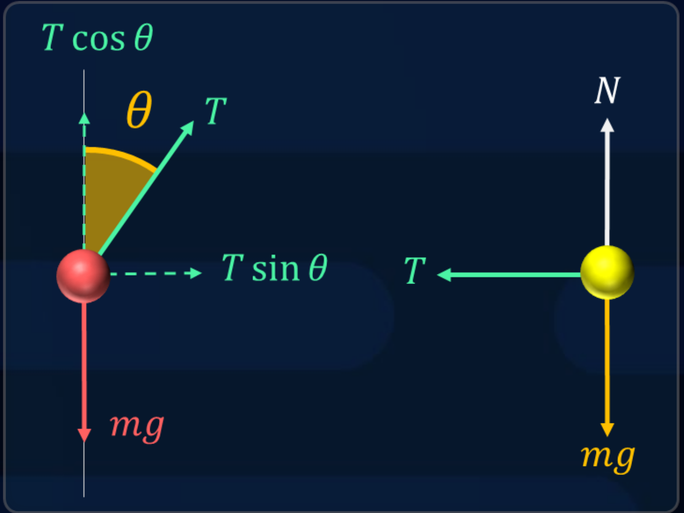
For particle of conical pendulum:

$$T \sin\theta = m\omega_2^2 r$$

$$T = m\omega_2^2 l_2 \quad ...(1) \qquad [\because r = l_2 \sin\theta]$$

For particle describing circle on table:

$$T = m\omega_1^2 l_1 \quad ...(2)$$



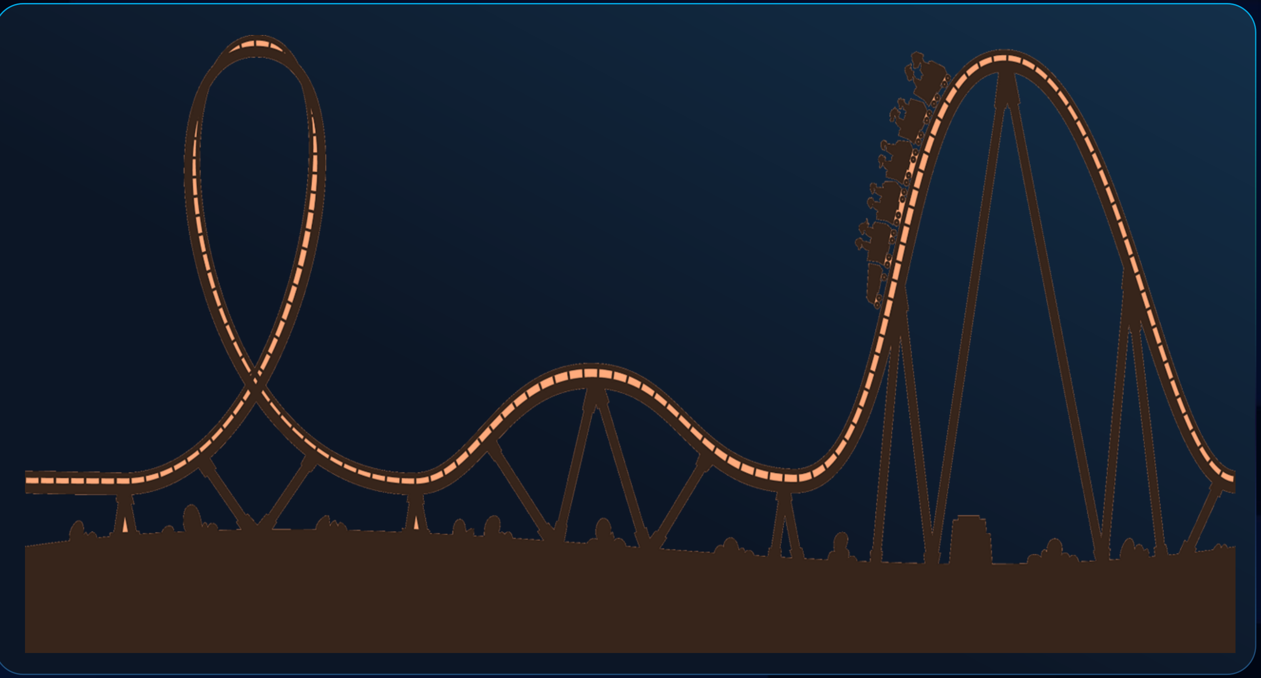
From equation (1) and (2),

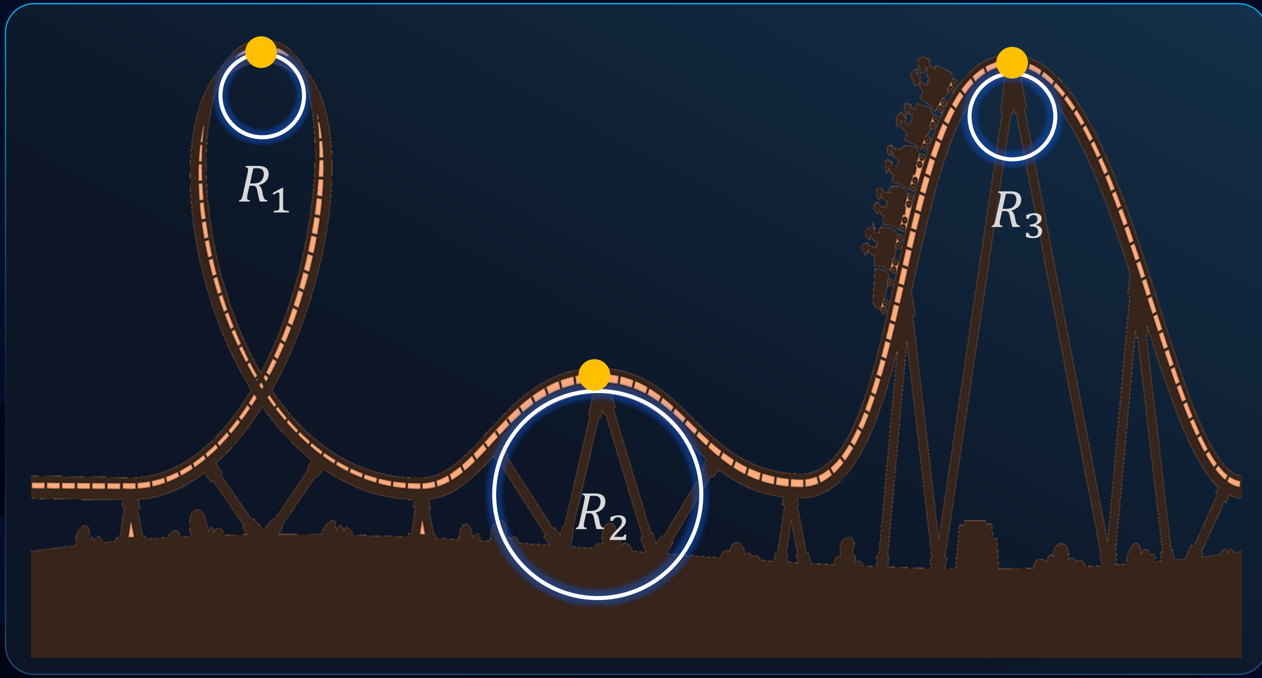
$$m\omega_1^2 l_1 = m\omega_2^2 l_2$$

$$\boxed{\frac{l_1}{l_2} = \frac{\omega_2^2}{\omega_1^2}}$$

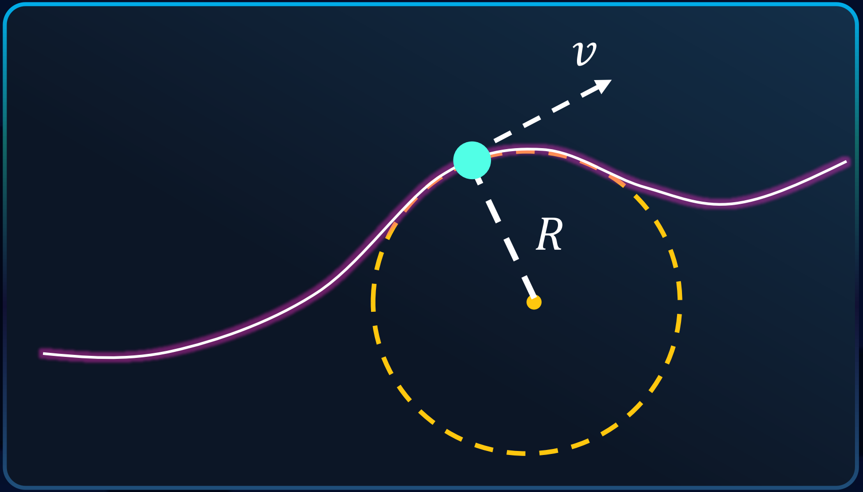
Radius of Curvature

- Any curved path can be assumed to be made of infinitesimal circular arcs.
- Radius of curvature at a point is the radius of the circular arc which fits the curve at that point.





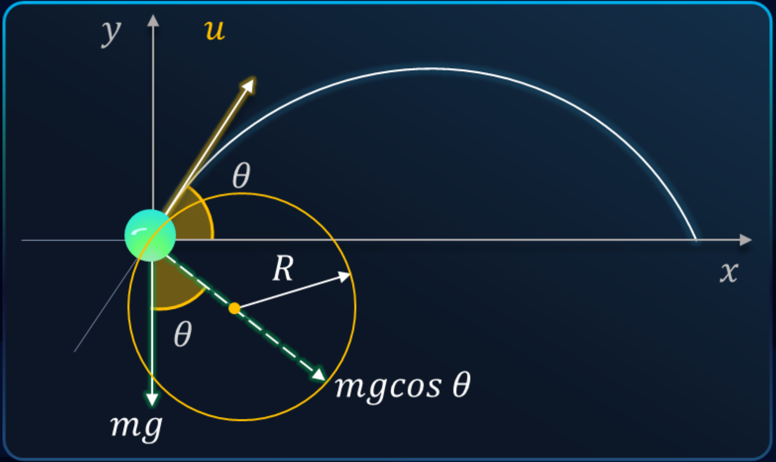
Finding Radius of Curvature

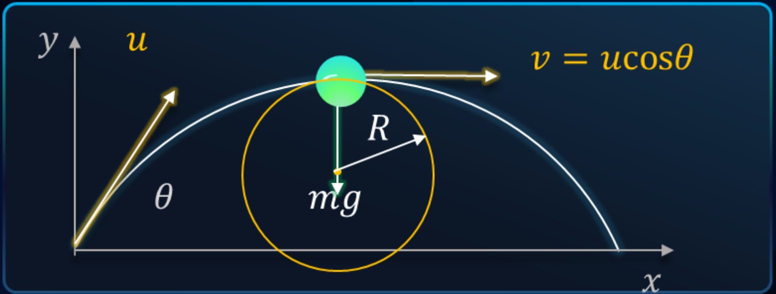


- Draw the velocity vector tangent to the chosen point and complete an approximated circle with an imagined center.
- Draw the F.B.D.
- Avail the component of the force along the center.
- Solve for $\sum F_c = ma_c = \frac{mv^2}{R}$

A particle of mass m is projected with speed u at an angle θ with the horizontal. Find the radius of curvature of the path traced out by the particle at the point of projection and also at the highest point of trajectory.

Solution:





$$\sum F_C = \frac{mv^2}{R}$$

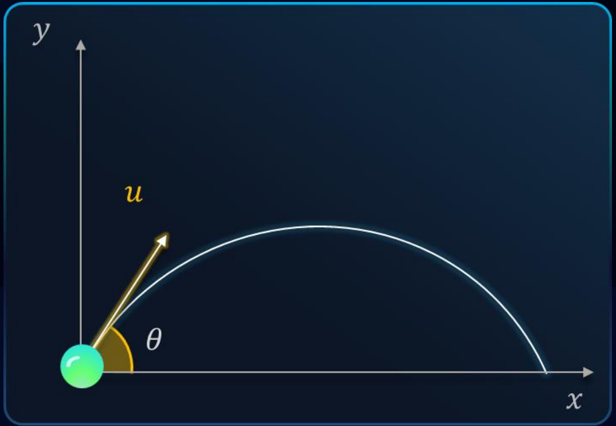
$$mg\cos\theta = \frac{mu^2}{R}$$

$$R = \frac{u^2}{g\cos\theta}$$

$$\sum F_C = \frac{mv^2}{R}$$

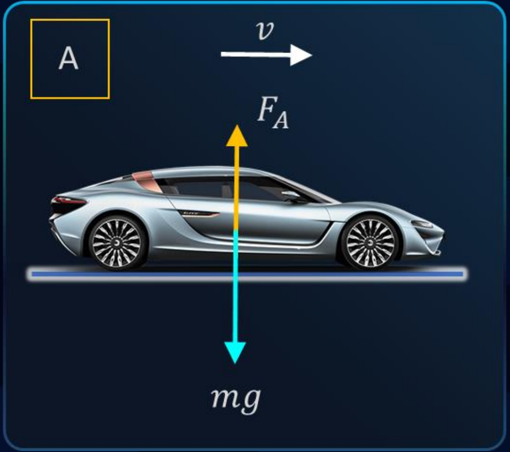
$$mg = \frac{m(u\cos\theta)^2}{R}$$

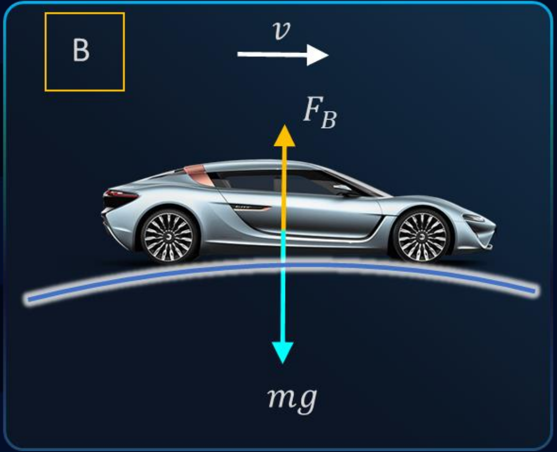
$$R = \frac{u^2\cos^2\theta}{g}$$

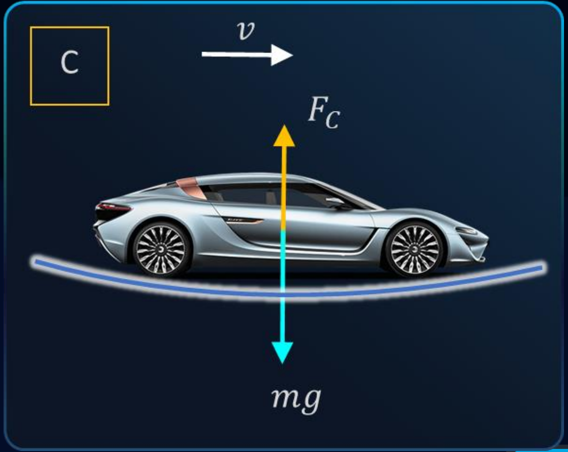


Three identical cars A, B and C are moving at the same speed on three bridges. The car A goes on a plane bridge, B on a bridge concave downward and C goes on a bridge concave upward. Let F_A, F_B and F_C be the normal forces exerted by the cars on the bridges when they are at the middle of bridges.

Solution:







$$F_A = mg$$

$$mg - F_B = \frac{mv^2}{r}$$

$$F_B = mg - \frac{mv^2}{r}$$

$$F_C - mg = \frac{mv^2}{r}$$

$$F_C = mg + \frac{mv^2}{r}$$

a. F_A is the maximum of the three forces.

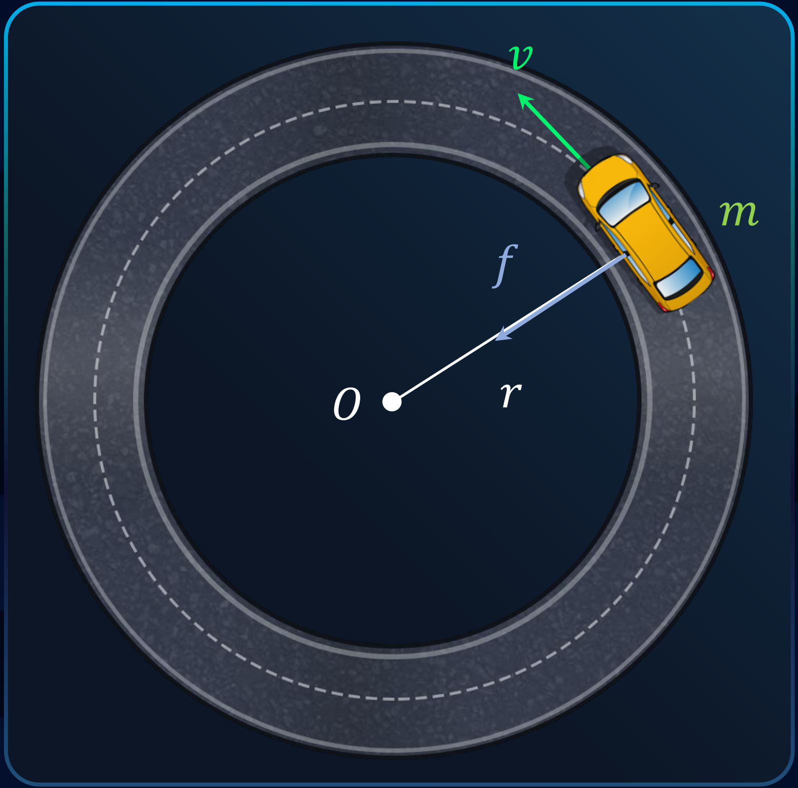
b. F_B is the maximum of the three forces.

c. F_C is the maximum of the three forces.

d. $F_A = F_B = F_C$

$F_A < F_B < F_C$ $\Rightarrow$ F_C is the maximum force.

Car in a Circular Turn



Friction force (f) provides the centripetal force, $f = \frac{mv^2}{r}$

Limiting Frictional force, $f_{max} = \mu N' = \mu mg$

For a safe turn without sliding,

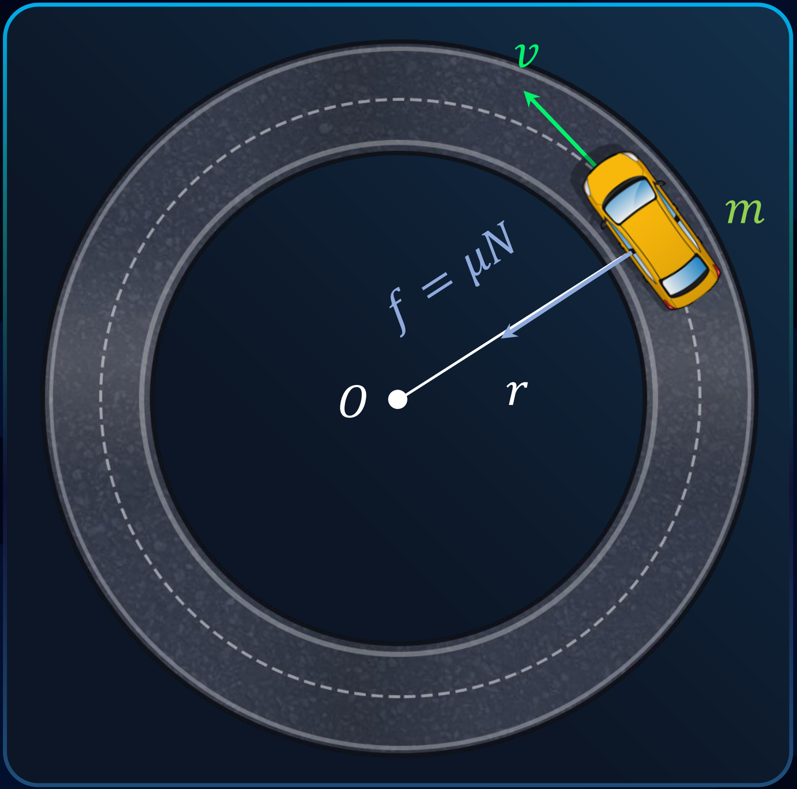
$$f \leq f_{max} \Rightarrow \frac{m(v)^2}{r} \leq \mu mg$$

$$\Rightarrow v \leq \sqrt{\mu r g}$$

If v is beyond v_{max}, the car skids outwards.

If v below v_{max}, friction adjusts itself to prevent skidding.

Consider a normal friction situation,



$$\mu = 1$$

$$r = 6.4\, m$$

$$g = 10\, m/s^2$$

$$v_{max} = \sqrt{\mu r g}$$

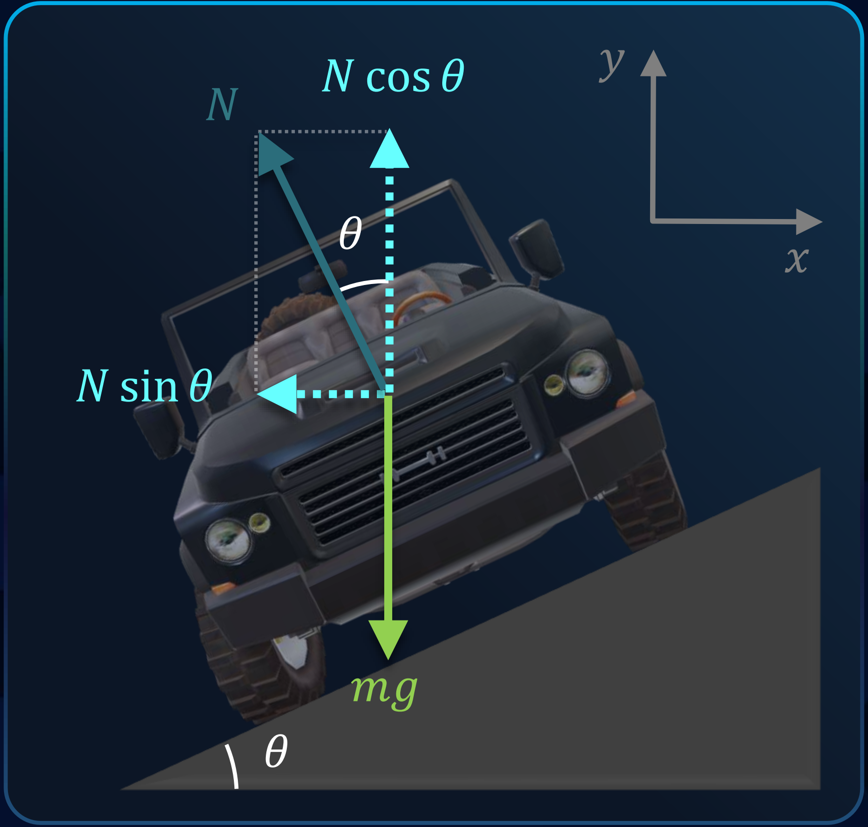
$$v_{max} = \sqrt{1 \times 6.4 \times 10}$$

$$v_{max} = 8\, m/s = 8 \times \frac{18}{5} \approx 29\, km/hr$$

$29\, km/hr$ is very less speed, So we need another alternative to provide the centripetal force.

That is where Banking of roads comes, this helps to provide more centripetal force and v_{max} increases.

Banking of roads without Friction



$N\cos\theta = mg$ (Along the y – axis)

$N\sin\theta = \dfrac{mv^2}{r}$ (Along the x – axis)

$\Rightarrow \tan\theta = \dfrac{v^2}{rg}$

$v = \sqrt{rg\tan\theta}$ (Safe Speed / Correct Speed)

$\theta = \tan^{-1}\dfrac{v^2}{rg}$ (Angle of banking)

(Safe Speed / Correct Speed) $v = \sqrt{rg\tan\theta}$

- If the speed is more than v, it tends to skid outwards
- If the speed is less than v, it tends to skid inwards

What should be the angle of banking of a circular track of radius $600\ m$ which is designed for cars with an average speed of $180\ km/hr$?

Solution :

$$v_{avg} = 180\ km/hr = 180 \times \frac{5}{18}\ m/s = 50\ m/s$$

$$r = 600\ m$$

$$\tan\theta = \frac{v^2}{rg}$$

$$\tan\theta = \frac{50 \times 50}{600 \times 10}$$

$$\tan\theta = \frac{5}{12}$$

$$\theta = \tan^{-1}\left(\frac{5}{12}\right)$$

a. $\theta = \tan^{-1}\left(\frac{5}{12}\right)$

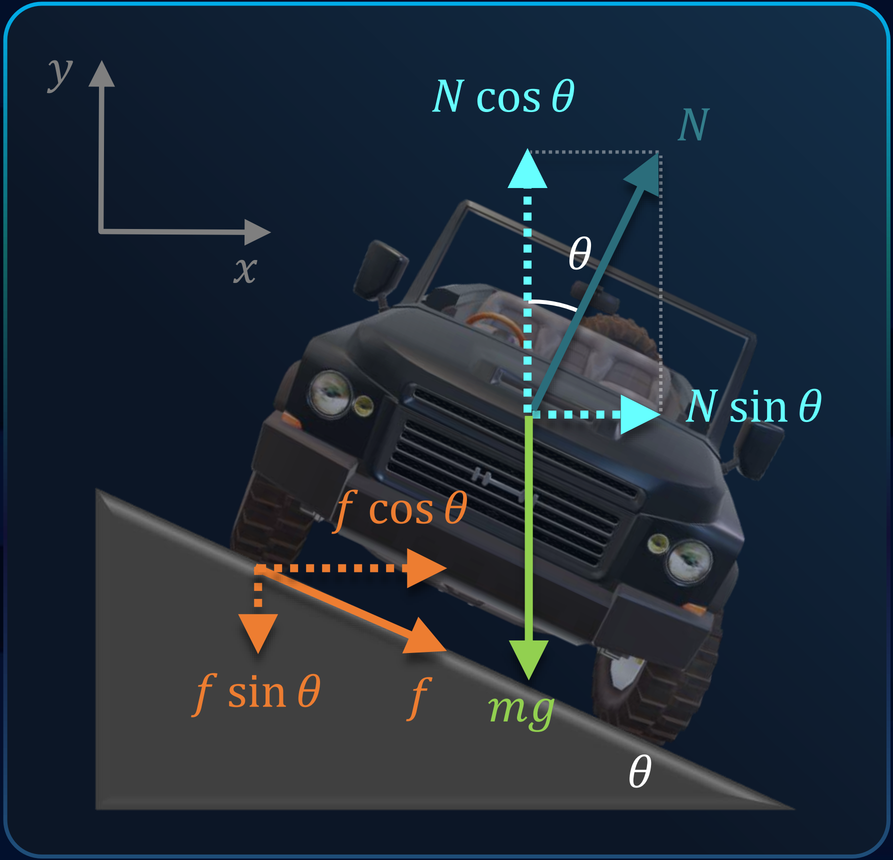
b. $\theta = \tan^{-1}\left(\frac{15}{12}\right)$

c. $\theta = \tan^{-1}\left(\frac{1}{12}\right)$

d. $\theta = \tan^{-1}\left(\frac{10}{12}\right)$

Banking with Friction

Case 1: with Friction speed $v > \sqrt{rg\tan\theta}$



At v_{max} : $f = f_{max} = \mu N$

$$N\sin\theta + f_{max}\cos\theta = \frac{mv^2}{r} \qquad \text{.........(1)}$$

$$N\cos\theta - f_{max}\sin\theta = mg \qquad \text{.........(2)}$$

From (1)/(2)

$$\frac{N\sin\theta + \mu N\cos\theta}{N\cos\theta - \mu N\sin\theta} = \frac{v^2}{rg}$$

$$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

$$v_{max} = \sqrt{\frac{rg(\tan\theta + \mu)}{(1 - \mu\tan\theta)}}$$

Banking with Friction

Case 2: with Friction speed $v < \sqrt{rg\tan\theta}$

At v_{min} $\Rightarrow$ $f = f_{max} = \mu N$

$$N\sin\theta - \mu N\cos\theta = \frac{m(v_{min})^2}{r}$$ (Along the x – axis)

$$N\cos\theta + \mu N\sin\theta = mg$$ (Along the y – axis)

$$\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} = \frac{(v_{min})^2}{rg}$$

$$v_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu}{1 + \mu\tan\theta}\right)}$$

Banking with Friction

$$v_{min} \leq v_{safe} \leq v_{max}$$

$$\sqrt{\frac{rg\,(\tan\theta - \mu)}{(1 + \mu\tan\theta)}} \leq v_{safe} \leq \sqrt{\frac{rg\,(\tan\theta + \mu)}{(1 - \mu\tan\theta)}}$$

- At speeds higher than v_{max}, the car will skid outwards
- At speeds lower than v_{min}, the car will skid inwards

Key Takeaways

$v_{required}$

Only Friction

$\mu \neq 0 \quad \theta = 0$

$$v = \sqrt{\mu r g}$$

Only Banking

$\mu = 0 \quad \theta \neq 0$

$$v = \sqrt{rg\tan\theta}$$

Banking with friction

$\mu \neq 0 \quad \theta \neq 0$

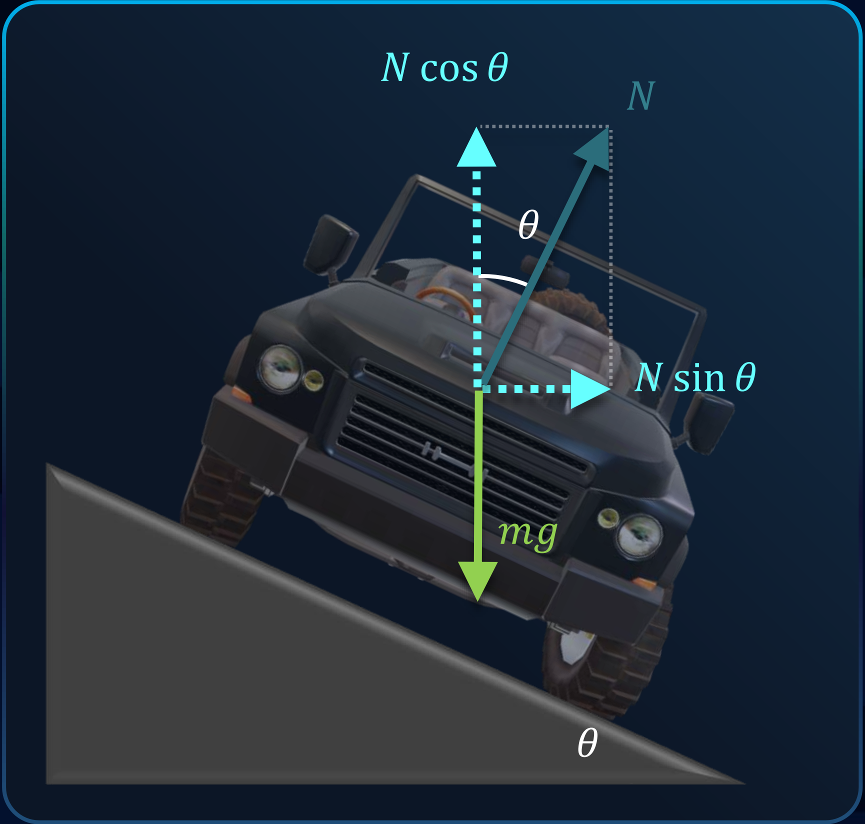
$$\sqrt{\frac{rg(\tan\theta - \mu)}{(1+\mu\tan\theta)}} < v < \sqrt{\frac{rg(\tan\theta + \mu)}{(1-\mu\tan\theta)}}$$

μ is coefficient of friction.

θ is the Angle of banking.

A circular road of radius r is banked for a speed $v = 40\ km/hr$. A car of mass m attempts to go on the circular road. The friction coefficient between the tire and the road is negligible.

Solution :



$$N\cos\theta = mg$$

$$N = \frac{mg}{\cos\theta}$$

$$N\sin\theta = \frac{mv^2}{r}$$

$$N = \frac{\left(\frac{mv^2}{r}\right)}{\sin\theta}$$

As $\cos\theta$ and $\sin\theta$ are < 1

$N > mg$

$N > \frac{mv^2}{r}$

a. The car cannot make a turn without skidding.

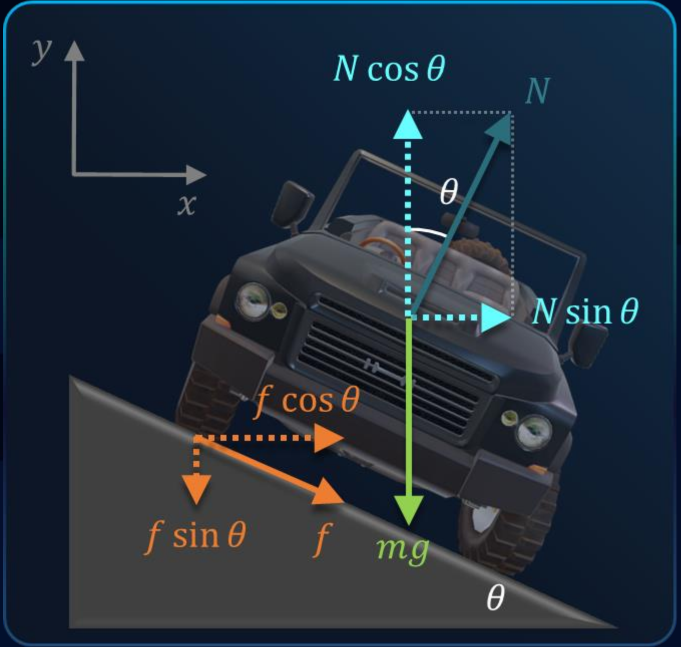
b. If the car turns at a speed less than $40\ km/hr$, it will slip down.

c. If the car turns at the correct speed of $40\ km/hr$, the force by the road on the car is equal to $\frac{mv^2}{r}$.

d. If the car turns at the correct speed of $40\ km/hr$, the force by the road on the car is greater than mg as well as greater than $\frac{mv^2}{r}$.

A circular road of radius $1\ km$ has a banking angle of $45°$. What will be the maximum safe speed (in m/s) of a car whose mass is $2{,}000\ kg$ and the coefficient of friction between the tire and the road is 0.5 ?

Solution :



$$\sqrt{\frac{rg\,(\tan\theta-\mu)}{(1+\mu\tan\theta)}} \le v \le \sqrt{\frac{rg\,(\tan\theta+\mu)}{(1-\mu\tan\theta)}}$$

$$v_{max} = \sqrt{\frac{rg\,(\tan\theta+\mu)}{(1-\mu\tan\theta)}}$$

$$v_{max} = \sqrt{\frac{1000\times 10(1+0.5)}{(1-0.5\times 1)}}$$

$$v_{max} = 100\sqrt{3}$$

$$v_{max} = 173.2\ m/s \approx 172\ m/s$$

a. $172\ m/s$

b. $124\ m/s$

c. $99\ m/s$

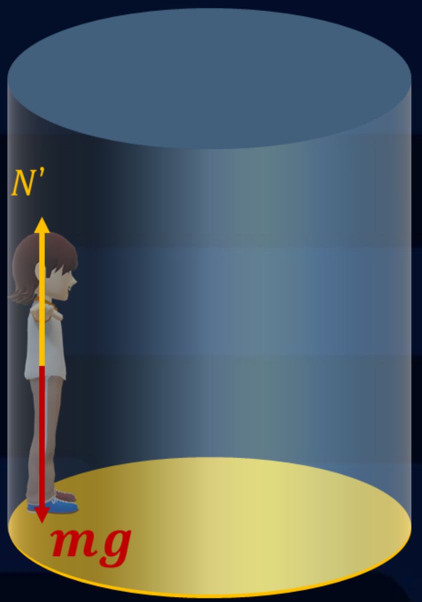
d. $86\ m/s$

The Devil's Hole

The Rotor - an amusement park ride designed by German engineer Ernst Hoffmeister in the late 1940s.

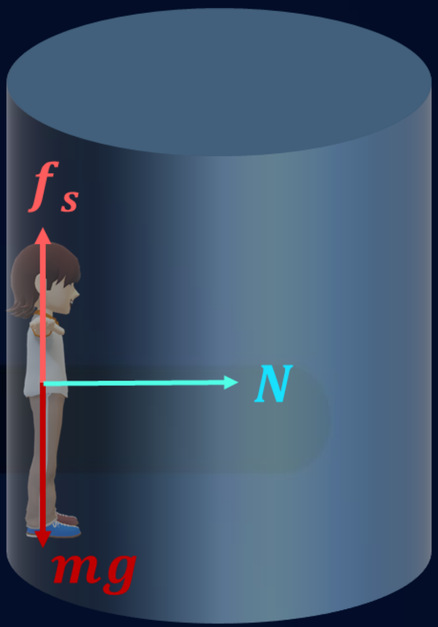
The Devil's Hole

Ground frame :



$$N' = mg$$

After floor is removed

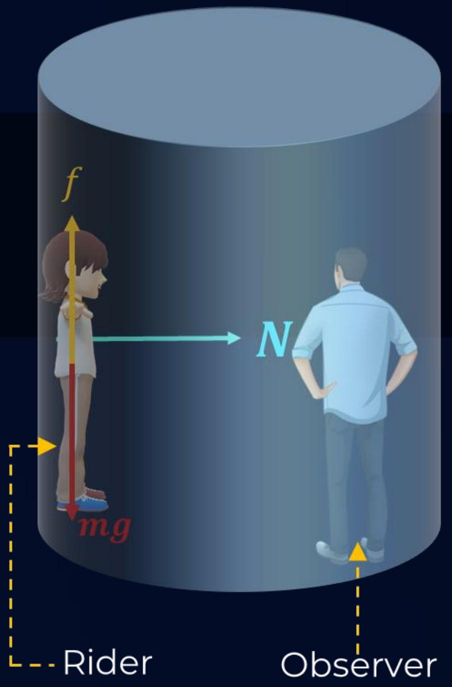


$$f_s = mg$$

$$N = m\omega^2 r$$

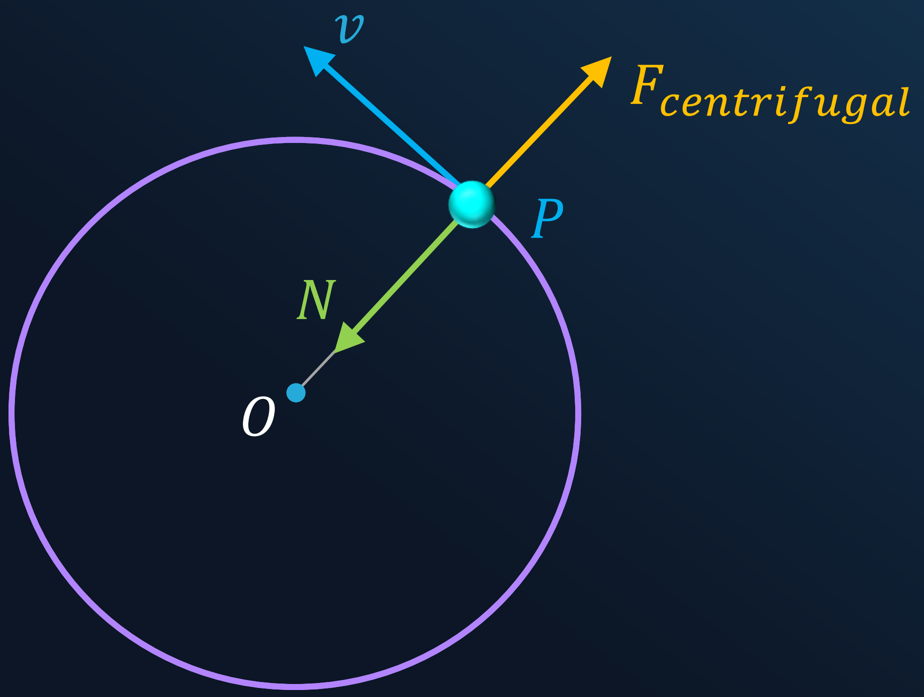
$$f_s \leq (f_s)_{max} = \mu N$$

Observer's frame :



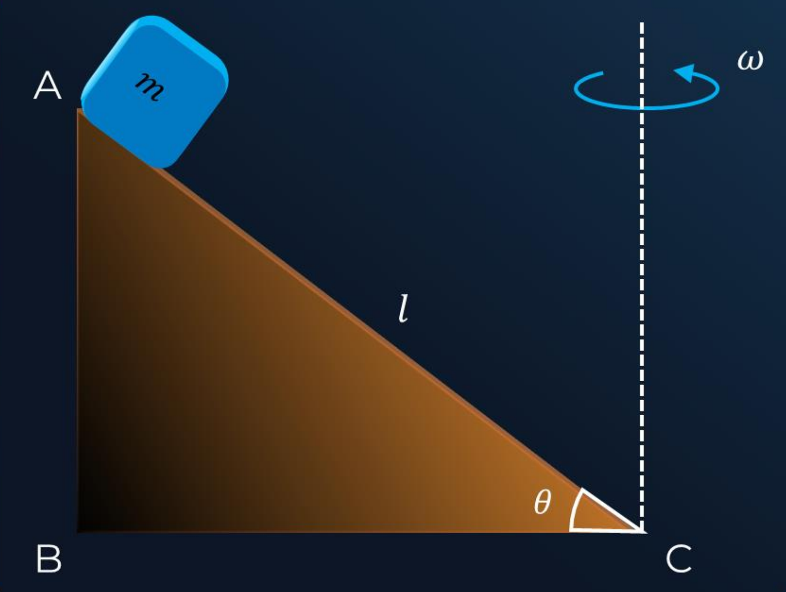
- As the observer is rotating, he is a non-inertial frame.
- Rider is at rest w.r.t. to the observer.
- To counter the non-inertility, we have to add a pseudo force in this frame.
- In this case pseudo force is applied due to the accelerated frame in circular motion and it acts away from the center, hence the name centrifugal force.

Centrifugal force



- It is a fictitious or pseudo force.
- Applied on an object when observation is made from a rotating frame of reference.
- $\left|\vec{F}_{centrifugal}\right| = \dfrac{m_{system}v^2}{r} = m_{system}\omega^2 r$
- Centrifugal force is a sufficient pseudo force, only if we are analyzing the particles at rest in a uniformly rotating frame.
- If we analyze the motion of a particle that moves in the rotating frame, we may have to assume other pseudo forces, together with the centrifugal force. Such forces are called the Coriolis forces.

A block of mass m is placed at the top of a smooth wedge ABC. The wedge is rotated about an axis passing through C as shown in the figure. The value of angular speed ω such that the block does not slip on the wedge is



Wedge frame :

Effective radius= $r = l\cos\theta$

Wedge is at rest w.r.t. to wedge frame.

$N\sin\theta = m\omega^2 r$ (Along $x - axis$)

$N\sin\theta = m\omega^2 l\cos\theta$ .....(1)

$N\cos\theta = mg$ .....(2) (Along $y - axis$)

$$\frac{\sin\theta}{\cos\theta} = \frac{\omega^2 l\cos\theta}{g} \quad \left(\because \frac{(1)}{(2)}\right)$$

$$\omega^2 = \frac{g\sin\theta}{l\cos^2\theta}$$

$$\Rightarrow \omega = \sqrt{\frac{g\sin\theta}{l}}\sec\theta$$