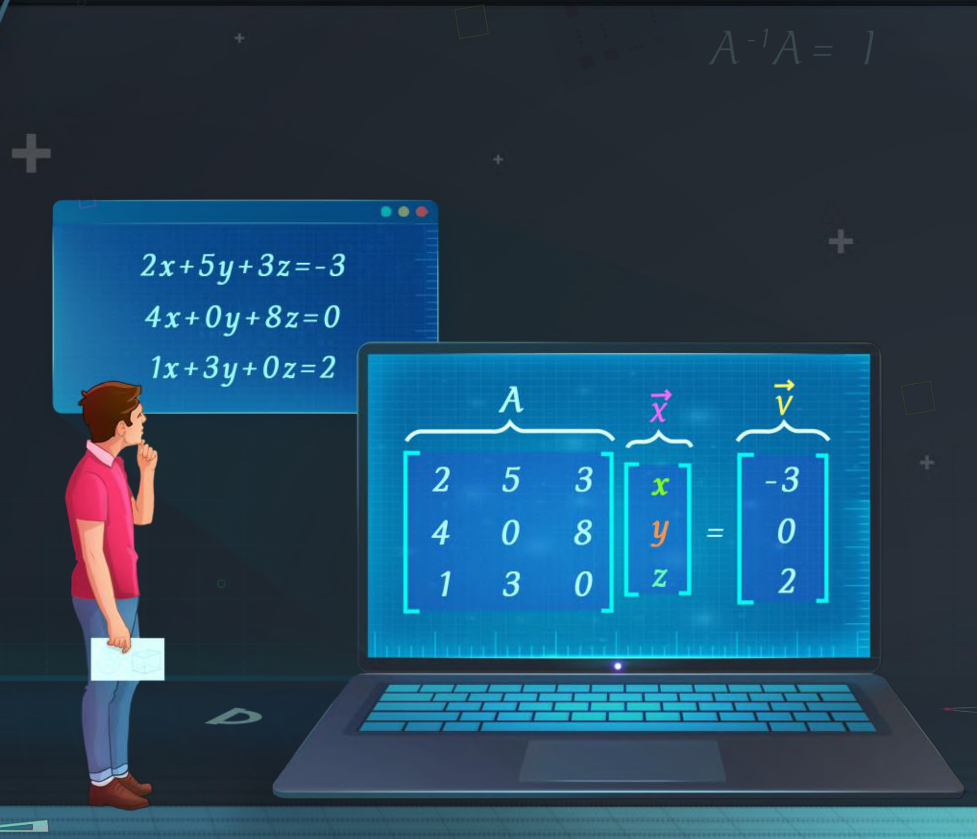
Welcome to

# Aakash + BYJU'S LIVE

## Matrices & Determinants



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# Session 01

## Introduction to Matrices

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## Key Takeaways

- A rectangular arrangement of $m \cdot n$ numbers (real or complex) or expressions (real or complex valued), having $m$ rows and $n$ columns is called a matrix. $(m, n \in N)$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \vdots & \vdots & \vdots \quad\;\; \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mn} \end{bmatrix}$$

Rows

Columns

- An element of a matrix is denoted by

  $a_{ij}$: Element of $i^{th}$ row & $j^{th}$ column.

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# Key Takeaways

- A rectangular arrangement of $m \cdot n$ numbers (real or complex) or expressions (real or complex valued), having $m$ rows and $n$ columns is called a matrix. $(m, n \in N)$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \vdots & \vdots & \vdots \quad\; \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mn} \end{bmatrix} \left.\begin{matrix} \leftarrow \\ \leftarrow \\ \\ \leftarrow \end{matrix}\right\} \text{Rows}$$

Columns

- Number of elements in a matrix

$= \text{Number of rows} \times \text{Number of columns}$

$= m \times n$

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Write $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ for the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 3 & -8 \end{bmatrix}$$

Solution :

$$a_{11} = 1$$

$$a_{12} = 0$$

$$a_{13} = 5$$

$$a_{21} = -2$$

$$a_{22} = 3$$

$$a_{23} = -8$$

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Find the value $a_{23}$ in the following matrix

$$A = \begin{pmatrix} 3 & -4 & 0 \\ -2 & 7 & 10 \\ 5 & -6 & 9 \end{pmatrix}$$

A $-6$

B $0$

C $10$

D $5$

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## Order of a matrix

Order or dimension of a matrix denotes the arrangement of elements as number of rows and number of columns.

- Order = Number of rows × Number of columns $= m \times n$

Name of a matrix

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mn} \end{bmatrix} \Bigg\} \text{Rows}$$

Order of a matrix

Columns

- Thus, a matrix can also be represented as $A = [a_{ij}]_{m\times n}$ or $(a_{ij})_{m\times n}$

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## Types of Matrix:

- Row Matrix (row vector) : A matrix having a single row is called a row matrix.

  $$A = [a_{ij}]_{1\times n} = [a_{11} \quad a_{12} \quad a_{13} \cdots a_{1n}]_{1\times n}$$

  Example: $B = [a \quad b \quad c]_{1\times 3}$

- Column Matrix (column vector) : A matrix having a single column is called a column matrix.

  Example: $B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{4\times 1}$

  $$A = [a_{ij}]_{m\times 1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}$$

- Matrices consisting of one row or one column are called vectors.

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## Types of Matrix:

- Zero Matrix (null matrix) : If all the elements of a matrix are zero, then it is called zero or null matrix

$A = [a_{ij}]_{m\times n}$ is called a zero matrix, if $a_{ij} = 0, \forall\ i \ \& \ j$.

Examples:

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

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## Vertical Matrix

A matrix of order $m \times n$ is known as vertical matrix if $m > n$, where $m$ is equal to the number of rows and $n$ is equal to the number of columns.

Example: $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

- In the matrix example given the number of rows $(m) = 4$, whereas the number of columns $(n) = 2$.

  Therefore, this makes the matrix a vertical matrix.

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## Horizontal Matrix

A matrix of order $m \times n$ is known as vertical matrix if $n > m$, where $m$ is equal to the number of rows and $n$ is equal to the number of columns.

Example: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$

- In the matrix example given the number of rows $(m) = 2$, whereas the number of columns $(n) = 4$.

  Therefore, this makes the matrix a horizontal  matrix.

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If a matrix has 12 elements, then what are the possible orders it can have?

Solution :

Number of elements = Number of rows × Number of columns

$12 = m \times n \ \ (m, n \in N)$

Possible Order $= 1 \times 12\ , 2 \times 6\ ,3 \times 4\ ,4 \times 3, 6 \times 2,,12 \times 1$

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Construct a $2 \times 3$ matrix, whose elements are given by $a_{ij} = \frac{(i+2j)}{3}$.

Solution :

$$a_{ij} = \frac{(i+2j)}{3}$$

$$a_{11} = 1 \qquad a_{12} = \frac{5}{3} \qquad a_{13} = \frac{7}{3}$$

$$a_{21} = \frac{4}{3} \qquad a_{22} = 2 \qquad a_{23} = \frac{8}{3}$$

$$A = \begin{pmatrix} 1 & \frac{5}{3} & \frac{7}{3} \\ \frac{4}{3} & 2 & \frac{8}{3} \end{pmatrix}$$

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# Key Takeaways

- Principal Diagonal of a Matrix : Diagonal containing the elements $a_{ij}$, where $i = j$ is called principal diagonal of a matrix

Examples:

$$A = \begin{bmatrix} 2 & -6 & 10 \\ 5 & 0 & 7 \\ 19 & -3 & -8 \end{bmatrix}_{3\times 3} \qquad B = \begin{pmatrix} 2 & 3 & 4 & -5 \\ 1 & 4 & 0 & 6 \\ -3 & 7 & 8 & 9 \end{pmatrix}_{3\times 4}$$

## Types of Matrix:

- Square Matrix : A matrix where number of rows = number of columns is called square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ \vdots & \vdots & \vdots \quad\;\; \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{bmatrix}_{n\times n}$$

Example:

$$A = \begin{bmatrix} -4 & 5 & 0 \\ 8 & -1 & 3 \\ 9 & 7 & 2 \end{bmatrix}_{3\times 3}$$

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# Key Takeaways

- Trace of a Matrix : Sum of all elements in the principal diagonal of a matrix is called trace of a matrix.

$$Tr\ (A) = \sum_{i=1}^{n} a_{ii}$$

Example:

$$A = \begin{bmatrix} 2 & -6 & 1 \\ 15 & 9 & 0 \\ -7 & 3 & -8 \end{bmatrix}_{3\times3} \quad \Rightarrow Tr\ (A) = 2 + 9 - 8 = 3$$

$$B = \begin{pmatrix} 0 & -3 & 5 & 1 \\ -2 & 3 & 6 & -9 \\ 11 & -8 & -5 & 10 \end{pmatrix}_{3\times4} \quad \Rightarrow Tr\ (B) = 0 + 3 - 5 = -2$$

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If $A = [a_{ij}]_{3\times3}$ where $a_{ij} = i^2 + j^2$. Then the trace of matrix $A$ is

Solution :

Trace is sum of elements in principle diagonal

$$\therefore Tr(A) = a_{11} + a_{22} + a_{33}$$

$$= (1^2 + 1^2) + (2^2 + 2^2) + (3^2 + 3^2)$$

$$= 28$$

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## Key Takeaways

### Types of Matrix:

- Diagonal Matrix : A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if

$$a_{ij} = 0 \,, \forall\, i \neq j.$$

  ➢ A diagonal matrix is represented as : $A = diag.(a_{11}, a_{22}, .., a_{nn})$

Example:

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -10 \end{bmatrix}_{3\times 3}$$

$$A = diag.(-3, 2, -10)$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}_{4\times 4}$$

$$B = diag.(1, 2, 0, -4)$$

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## Types of Matrix:

- Scalar Matrix : A diagonal matrix whose all diagonal elements are equal is called scalar matrix

  $A = [a_{ij}]_n$ is a scalar matrix if

  $a_{ij} = 0 , \forall\ i \neq j \qquad a_{ij} = k , \forall\ i = j$

Example:

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

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## Types of Matrix:

- Unit Matrix (identity matrix) : A diagonal matrix whose all diagonal elements are equal to 1 is called identity matrix

- Unit matrix of order $n$ is denoted by $I_n$ ( $I$ ).

$I_n = [a_{ij}]_n$ such that

$a_{ij} = 0, \forall\, i \neq j$

$a_{ij} = 1, \forall i = j$

Example:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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## Types of Matrix:

- Triangular Matrix :

  $(i)$ Upper Triangular Matrix

  A matrix in which all the elements below the principal diagonal are zero is called an upper triangular matrix.

  $P = [a_{ij}]_n$ such that $a_{ij} = 0 , \forall\, i > j$

  Example:

  $$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -4 & 9 \\ 0 & 0 & -5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -3 & 5 & 1 \\ 0 & 3 & 6 & -9 \\ 0 & 0 & -5 & 10 \end{pmatrix}$$

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## Types of Matrix:

- Triangular Matrix :

  $(ii)$ Lower Triangular Matrix

  A matrix in which all the elements above the principal diagonal are zero is called a lower triangular matrix

  $P = [a_{ij}]_n$ such that $a_{ij} = 0, \forall\, i < j$

  Example:

  $$A = \begin{pmatrix} -7 & 0 & 0 \\ 3 & 4 & 0 \\ -2 & 10 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ -3 & 8 & 6 & 0 \end{pmatrix}$$

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# Key Takeaways

## Comparable Matrix:

Two matrices $A$ & $B$ are said to be comparable if,

order of matrix $A$ = order of matrix $B$

Example: If matrices $A_{3\times5}$ & $B_{m\times n}$ are comparable , then $(m,n) \equiv (3,5)$

## Equal Matrix:

Two matrices are said to be equal if,

$(i)$ They are comparable.

$(ii)$ corresponding elements of them are equal.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$

Then $A = B$ , if $m = p \,;\, n = q$ & $a_{ij} = b_{ij} \,, \forall\, i \,\&\, j$

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Let $A = \begin{bmatrix} \sin\theta & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \cos\theta \\ \cos\theta & \tan\theta \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \sin\theta \\ \cos\theta & \cos\theta \\ \cos\theta & -1 \end{bmatrix}$. Find $\theta$ so that $A = B$.

A $\frac{\pi}{4}$

B $\frac{3\pi}{4}$

C $\frac{5\pi}{4}$

D $\frac{7\pi}{4}$

Solution : Order is same .

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$$

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If $\begin{bmatrix} x-y & 1 & z \\ 2x-y & 0 & w \end{bmatrix} = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, then $x+y+z+w$ is

Solution :

$$2x - y = 0$$

$$\Rightarrow x = 1 \,, y = 2$$

$$z = 4 \,, w = 5$$

Thus , $x + y + z + w = 12$

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## Algebra of Matrix:

### Multiplication of Matrix by a scalar

- Let $k$ be a scaler (real or complex) and $A = [a_{ij}]_{m\times n}$ thus $kA = [b_{ij}]_{m\times n}$, where $b_{ij} = k\,a_{ij}\ \forall\, i\ \&\ j$

Example: If A= $\begin{pmatrix} -1 & 2 & -6 \\ 3 & -4 & 7 \end{pmatrix}$ , then $-A$ is :

Solution: $-A = (-1)A = -1 \times \begin{pmatrix} -1 & 2 & -6 \\ 3 & -4 & 7 \end{pmatrix}$

$= \begin{pmatrix} 1 & -2 & 6 \\ -3 & 4 & -7 \end{pmatrix}$

$-A$ is the negative of matrix $A$

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# Session 02

## Algebra of Matrices
## and
## Multiplication of Matrices

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# Key Takeaways

## Algebra of Matrix:

### Addition/Subtraction of Matrices :

- Let $A$ & $B$ are two comparable matrices , then

$$A \pm B = [a_{ij}]_{m\times n} \pm [b_{ij}]_{m\times n} = [c_{ij}]_{m\times n}\ ,$$ where $c_{ij} = a_{ij} \pm b_{ij}\ \forall\ i\ \&\ j$.

**Example:** If $A = \begin{pmatrix} 2 & -3 & 4 \\ 0 & 1 & 5 \end{pmatrix}$ , $B = \begin{pmatrix} -6 & 0 & -2 \\ 1 & 7 & -8 \end{pmatrix}$ , find $A + B$ , $A - B$.

$$A + B = \begin{pmatrix} -4 & -3 & 2 \\ 1 & 8 & -3 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 8 & -3 & 6 \\ -1 & -6 & 13 \end{pmatrix}$$

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# Algebra of Matrix:

## Properties of Addition/Subtraction of Matrices :

- Let $A$ & $B$ are two comparable matrices having order $m \times n$ , then

$$A + B = B + A \text{ ( commutative )}$$

$$A - B \neq B - A$$

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## Algebra of Matrix:

Example: Let $A = \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 5 & -6 \end{pmatrix}, B = \begin{pmatrix} 1 & 7 \\ -2 & 5 \\ -8 & 10 \end{pmatrix}$

$$A + B = \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 5 & -6 \end{pmatrix} + \begin{pmatrix} 1 & 7 \\ -2 & 5 \\ -8 & 10 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -3 & 9 \\ -3 & 4 \end{pmatrix}$$

$$B + A = \begin{pmatrix} 1 & 7 \\ -2 & 5 \\ -8 & 10 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 5 & -6 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -3 & 9 \\ -3 & 4 \end{pmatrix}$$

- ❑ $A + B = B + A$ ( commutative )

$$A - B = \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 5 & -6 \end{pmatrix} - \begin{pmatrix} 1 & 7 \\ -2 & 5 \\ -8 & 10 \end{pmatrix} = \begin{pmatrix} 2 & -7 \\ 1 & -1 \\ 13 & -16 \end{pmatrix}$$

$$B - A = \begin{pmatrix} 1 & 7 \\ -2 & 5 \\ -8 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 5 & -6 \end{pmatrix} = \begin{pmatrix} -2 & 7 \\ -1 & 1 \\ -13 & 16 \end{pmatrix}$$

- ❑ $A - B \neq B - A$

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If $\begin{pmatrix} x^2+x & x \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -x+1 & x \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 5 & 1 \end{pmatrix}$ then, $x$ is equal to :

A $-1$

B $0$

C $1$

D $-2$

Solution :

$$\begin{pmatrix} x^2+x & x \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -x+1 & x \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x^2+x & x-1 \\ -x+4 & 2+x \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 5 & 1 \end{pmatrix}$$

$$x^2+x=0 \qquad \Rightarrow x = 0, -1$$

$$x-1=-2 \qquad \Rightarrow x=-1$$

$$-x+4=5 \qquad \Rightarrow x=-1$$

$$2+x=1 \qquad \Rightarrow x=-1$$

$$\therefore\ x=-1$$

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## Algebra of Matrix:

### Properties of Addition/Subtraction of Matrices :

- Let $A$ , $B$ & $C$ are two comparable matrices having order $m \times n$ , then

  $A + (B + C) = (A + B) + C$ ( associative )

- Let $A$ is a matrix of order $m \times n$ , then

  $A + O = O + A = A$ $\quad$ ($O = O_{m\times n}$ is the additive identity )

  $A + (-A) = O = (-A) + A$ $\quad$ ( $(-A)$ is the additive inverse of $A$ )

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# Key Takeaways

## Algebra of Matrix:

### Properties of Scalar Multiplication :

- Let $A$ & $B$ are two comparable matrices having order $m \times n$ , then
  - $kA = Ak$ , $k$ is a scalar
  - $k(A \pm B) = kA \pm kB$ , $k$ is a scalar
  - $(k_1 \pm k_2)A = k_1A \pm k_2A$ ; $k_1, k_2$ are scalars
  - $k(\alpha A) = (k\alpha)A = \alpha(kA)$; $k$ , $\alpha$ are scalars

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## Multiplication of Matrix:

### Matrix Multiplication :

- Product of two matrices $A$ & $B$ will exist only when number of columns of $A$ is same as number of rows of $B$ .

i.e. let $A = [a_{ij}]_{m \times p}$ and $B = [b_{ij}]_{p \times n}$

$A_{m \times p} \,.\, B_{p \times n} = C_{m \times n} = [c_{ij}]_{m \times n}$ , where $c_{ij} = \sum_{k=1}^{p} a_{mk} b_{kn}$

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# Key Takeaways

## Multiplication of Matrix:

- $A_{m\times p} . B_{p\times n} = C_{m\times n} = [c_{ij}]_{m\times n}$, where $c_{ij} = \sum_{k=1}^{p} a_{mk} b_{kn}$

**Example:** $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ -7 & -2 \end{bmatrix}$

$$C = AB = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -4 & 6 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ -7 & -2 \end{bmatrix}_{3\times 2}$$

$c_{ij}$ =Dot product of $i^{th}$ row vector of $A$ with $j^{th}$ column vector of $B$

$$= \begin{bmatrix} 2\cdot 1 + 0 + (-1)\cdot(-7) & 2(-3) + 0 + (-1)\cdot(-2) \\ 3\cdot 1 + 0 + 6(-7) & 3(-3) - 4\cdot 5 + 6(-2) \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 9 & -4 \\ -39 & -41 \end{bmatrix}_{2\times 2}$$

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If $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ -7 & -2 \end{bmatrix}$. Find the matrix $BA$.

Solution :

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ -7 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -3 \\ 0 & 5 \\ -7 & -2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 2 & 0 & -1 \\ 3 & -4 & 6 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 2-9 & 0+12 & -1-18 \\ 0+15 & 0-20 & 0+30 \\ -14-6 & 0+8 & 7-12 \end{bmatrix} = \begin{bmatrix} -7 & 12 & -19 \\ 15 & -20 & 30 \\ -20 & 8 & -5 \end{bmatrix}_{3\times 3}$$

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# Key Takeaways

## Properties of Multiplication

- In general, $AB \neq BA$

  If $AB = BA$, then $A$ & $B$ are said to be commute.

  If $AB = -BA$, then $A$ & $B$ are said to be anti – commute.

- $AO = OA = O$, whenever defined .

- Let $A = [a_{ij}]_{m\times n}$. Then $AI_n = A$ & $I_m A = A$,

  where $I_m$ & $I_n$ are identity matrices of order $m$ & $n$ respectively.

- If $k$ is a scalar and product of matrices $A$ & $B$ is defined, then

  $(kA)B = A(kB) = k(AB)$.

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# Key Takeaways

## Properties of Multiplication

- $A(BC) = (AB)C$, whenever defined. (associative)
- $A(B \pm C) = AB \pm AC$, whenever defined. (left distributive)
- $(B \pm C)A = BA \pm CA$, whenever defined. (right distributive)
- $(A + B)^2 = (A + B)(A + \mathrm{B}) = A^2 + AB + BA + B^2$
- $(A + B)(A - B) = A^2 - AB + BA - B^2$

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If $A$ & $B$ be two matrices such that $AB = B$ & $BA = A$, then $A^2 + B^2$ is:

Solution : $AB = B$ (given)

Pre-multiply $B$ on both sides.

$\Rightarrow BAB = B^2$

$\Rightarrow AB = B^2$

$\Rightarrow B = B^2 \ldots (i) \qquad (\because AB = B)$

$BA = A$ (given)

Pre-multiply $A$ on both sides.

$ABA = A^2$

$\Rightarrow BA = A^2$

$\Rightarrow A = A^2 \ldots (ii) \qquad (\because BA = A)$

$A^2 + B^2 = A + B$

A. $2AB$

B. $2BA$

C. $A + B$

D. $AB$

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If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ then a value of α for which $A^2 = B$ is:

Solution :

$$A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix}$$

$$A^2 = B$$

$$\Rightarrow \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow \alpha^2 = 1 \ \& \ \alpha + 1 = 5$$

No real values

A $1$

B $-1$

C $4$

D No real values

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# Power of a Square Matrix

If $A$ is a square matrix of order $n$ ,

- $AI_n = I_nA = A,\ I_n$ is called the multiplicative identity.
- $A^2 = A.A$
- $A^n = A \cdot A \cdots A$ ( up to $n$ times ) , $n \in\ N$
- $A^nA^m = A^{m+n}$ , $m$ , $n \in\ N$

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# Power of a Square Matrix

- If $A = \text{diag}\,.\,(a_1, a_2, \cdots, a_n)$ , then $A^k = \text{diag}\,.\left({a_1}^k,\ {a_2}^k, \cdots,\ {a_n}^k\right)$

Proof: Let $A = \text{diag}\,.\,(a_1, a_2, a_3) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$

$$A^2 = \begin{pmatrix} a_1 & 0\ \ldots & ..0 \\ 0 & a_2\ \ldots & ..0 \\ \vdots & \vdots & \vdots \\ 0 & 0\ \ldots & ..a_n \end{pmatrix} \begin{pmatrix} a_1 & 0\ \ldots & ..0 \\ 0 & a_2\ \ldots & ..0 \\ \vdots & \vdots & \vdots \\ 0 & 0\ \ldots & ..a_n \end{pmatrix} = \begin{pmatrix} a_1^2 & 0\ \ldots & ..0 \\ 0 & {a_2}^2\ \ldots & ..0 \\ \vdots & \vdots & \vdots \\ 0 & 0\ \ldots & ..{a_n}^2 \end{pmatrix}$$

$$\Rightarrow A^k = \begin{pmatrix} a_1^k & 0\ \ldots & ..0 \\ 0 & {a_2}^k\ \ldots & ..0 \\ \vdots & \vdots & \vdots \\ 0 & 0\ \ldots & ..{a_n}^k \end{pmatrix}$$

- $I^k = I$, where $I$ is identity matrix of order $n$ .

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If $A, B, C$ are given square matrices of same order such that $AB = O$ & $BC = I$. Then $(A+B)^2(A+C)^2$ is equal to:

Solution : $BC = I$ , pre multiplying by $A$

$ABC = AI \qquad (\because AB = O)$

$\Rightarrow O = A$

$$(A+B)^2(A+C)^2 = (B)^2(C)^2$$

$$= BBCC$$

$$= BIC$$

$$= BC$$

$$\Rightarrow (A+B)^2\,(A+C)^2 = I$$

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# Session 03

## Transpose of Matrix
## and
## Introduction of Determinants

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# Key Takeaways

## Polynomial Equation in Matrix

A matrix polynomial equation is an equality between two matrix polynomials, which holds for specific matrices.

- If $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ , then

  $f(A) = a_0A^n + a_1A^{n-1} + \cdots + a_nI,$ where $A$ is a square matrix.

- If $f(A) = O$, then $A$ is called zero divisor of the polynomial .

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If $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ & $f(x) = x^2 - 4x + 7$ , then the $f(A)$ is :

A $A$

B $7I$

C $O$

D $A - I$

Solution :

$$f(x) = x^2 - 4x + 7$$

$$f(A) = A^2 - 4A + 7I$$

$$A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$f(A) = O$$

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# Key Takeaways

## Transpose of a Matrix:

The matrix obtained by interchanging rows and columns of a matrix $A$ is called Transpose of matrix $A$.

Let $A = [a_{ij}]_{m\times n}$ , then its transpose is denoted by $A'$ or $A^T = [b_{ij}]_{n\times m}$ , where $b_{ij} = a_{ji}$ , $\forall\ i\ \&\ j$

Example:

$$A = \begin{pmatrix} z & a & x \\ c & e & f \end{pmatrix}_{2\times 3}$$

Its transpose is : $A' = \begin{pmatrix} z & c \\ a & e \\ x & f \end{pmatrix}_{3\times 2}$

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If $A = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ and $A + A^T = I$ where $I$ is $2 \times 2$ unit matrix and $A^T$ is the transpose of $A$, then the value of $\theta$ is equal to

A $\frac{\pi}{6}$

B $\frac{\pi}{2}$

C $\frac{\pi}{3}$

D $\frac{3\pi}{2}$

Solution :

We have $A = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

$\Rightarrow A^T = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

$\Rightarrow A + A^T = \begin{bmatrix} 2\cos 2\theta & 0 \\ 0 & 2\cos 2\theta \end{bmatrix} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow 2\cos 2\theta = 1$

$\Rightarrow \cos 2\theta = \frac{1}{2} = \cos \frac{\pi}{3}$

$\Rightarrow 2\theta = 2n\pi + \frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{6}$

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## Properties of transpose of a matrix:

- For a matrix $A = [a_{ij}]_{m\times n}$, $(A')' = A$
- Let $k$ is a scalar and $A$ is a matrix. Then $(kA)' = kA'$
- $(A_1 \pm A_2 \pm \cdots \pm A_n)' = A_1' \pm A_2' \pm \cdots \pm A_n'$, for comparable matrices $A_i$
- Let $A = [a_{ij}]_{m\times p}$ & $B = [b_{ij}]_{p\times n}$, then $(AB)' = B'A'$

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## Properties of transpose of a matrix:

Let $A = [a_{ij}]_{m\times p}$ & $B = [b_{ij}]_{p\times n}$ then $(AB)' = B'A'$

Example:

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & -3 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 1 \\ 0 & -6 \\ 3 & -1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 0 & -1 \\ 4 & -3 & 5 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 0 & -6 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -7 & 3 \\ 7 & 17 \end{pmatrix}$$

$$(AB)' = \begin{pmatrix} -7 & 7 \\ 3 & 17 \end{pmatrix}$$

$$A' = \begin{pmatrix} 2 & 4 \\ 0 & -3 \\ -1 & 5 \end{pmatrix} \quad B' = \begin{pmatrix} -2 & 0 & 3 \\ 1 & -6 & -1 \end{pmatrix}$$

$$B'A' = \begin{pmatrix} -2 & 0 & 3 \\ 1 & -6 & -1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & -3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} -7 & 7 \\ 3 & 17 \end{pmatrix} \quad \therefore (AB)' = B'A'$$

- ❑ $(A_1A_2 \dots A_n)' = A_n{}'A_{n-1}{}' \dots A_2{}'\, A_1{}'$ , whenever product is defined .

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# Key Takeaways

## Symmetric and skew symmetric Matrix:

A square matrix $A$ is said to be symmetric if, $A' = A$

Let $A = [a_{ij}]_n$ , then $a_{ij} = a_{ji}$ , $\forall\, i$ & $j$

Example:

$$A = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 4 & 5 \\ 2 & 5 & 7 \end{pmatrix} \longrightarrow \begin{array}{l} a_{12} = a_{21} \\ a_{13} = a_{31} \\ a_{23} = a_{32} \end{array}$$

$$A' = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 4 & 5 \\ 2 & 5 & 7 \end{pmatrix} = A$$

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If $A$ and $B$ are symmetric matrices of the same order and $X = AB + BA$ and $Y = AB - BA$, then $XY^T$ is equal to

A. $XY$

B. $YX$

C. $-XY$

D. None of these

Solution : Given : $A$ and $B$ are symmetric.

Then, $A^T = A$ and $B^T = B$

$$XY^T = (AB + BA)(AB - BA)^T$$

$$= (AB + BA)\left((AB)^T - (BA)^T\right)$$

$$= (AB + BA)(B^T A^T - A^T B^T)$$

$$= (AB + BA)(BA - AB)$$

$$= -(AB + BA)(AB - BA)$$

$$= -XY$$

$$\therefore XY^T = -XY$$

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If $A = \begin{bmatrix} 3 & x \\ y & 0 \end{bmatrix}$ and $A = A^T$, then which of the following is correct

Solution :

Given : $A = \begin{bmatrix} 3 & x \\ y & 0 \end{bmatrix}$ and $A = A^T$

It is symmetric

$\therefore x = y$

A $x = 0, y = 3$

B $x + y = 3$

C $x = y$

D $x = -y$

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# Symmetric and skew symmetric Matrix:

A square matrix $A$ is said to be skew symmetric if, $A' = -A$

Let $A = [a_{ij}]_n$ , then $a_{ij} = -a_{ji}$ , $\forall$ $i$ & $j$

Example:

$$A = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -6 \\ -2 & 6 & 0 \end{pmatrix}$$

$$A' = \begin{pmatrix} 0 & 3 & -2 \\ -3 & 0 & 6 \\ 2 & -6 & 0 \end{pmatrix} = -A$$

In skew – symmetric matrix, all diagonal elements are zero .

$$a_{ij} = -a_{ji} \Rightarrow a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$$

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If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew-symmetric, then

A $a = -2$

B $a = 2$

C $b = 3$

D $b = -3$

Solution :

Given : $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew-symmetric

$$A^T = \begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$$

We know that $A$ is skew symmetric if $A = -A^T$

$$\therefore \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -b \\ -a & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

$a = -2$

$\Rightarrow -3 = -b$

$\therefore b = 3$

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# Key Takeaways

## Symmetric and skew symmetric Matrix:

All positive integral power of a symmetric matrix is a symmetric matrix.

Proof:

$A = A^T$

Let $B = A^n, n \in N$

$B^T = (A^n)^T$

$B^T = A^T A^T \dots A^T$ (up to $n$ times)

$B^T = AA \dots A$ (up to $n$ times) $= A^n$

$B^T = B \Rightarrow (A^n)^T = A^n \Rightarrow$ symmetric matrix

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# Key Takeaways

## Symmetric and skew symmetric Matrix:

All odd positive integral power of a skew – symmetric matrix is a skew – symmetric matrix.

All even positive integral power of a skew – symmetric matrix is a symmetric matrix.

Proof:

$A = -A^T$

Let $C = A^n, n \in N$

$C^T = (A^n)^T = A^T A^T \ldots A^T$ (up to $n$ times)

$C^T = (-A)(-A) \ldots (-A)$ (up to $n$ times) $= (-1)^n A^n$

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# Key Takeaways

## Proof:

$C^T = (-A)(-A) \dots (-A)$ (up to $n$ times) $= (-1)^n A^n$

Let $C = A^n, n \in N$ $\quad C^T = (-1)^n A^n \begin{cases} A^n, n \text{ is even} \\ -A^n, n \text{ is odd} \end{cases}$

$$C^T = \begin{cases} C, n \text{ is even} \rightarrow \text{symmetric matrix} \\ -C, n \text{ is odd} \rightarrow \text{skew} - \text{symmetric matrix} \end{cases}$$

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# Key Takeaways

## Symmetric and skew symmetric Matrix:

Every square matrix can be written as sum of a symmetric and a Skew - symmetric matrix .

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{Symmetric matrix}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{Skew – symmetric matrix}}$$

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# Key Takeaways

## Properties of Trace of a Matrix

Let $A = [a_{ij}]_n$ , and $B = [b_{ij}]_n$

- $Tr.(A) = Tr.(A')$
- $Tr.(kA) = k\,Tr.(A)$ , $k$ is scalar
- $Tr.(A \pm B) = Tr.(A) \pm Tr.(B)$
- $Tr.(AB) = Tr.(BA)$

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Number of possible ordered sets of two $n \times n$ matrices $A$ and $B$ for which $AB - BA = I$:

A. Infinite

B. $n^2$

C. $n!$

D. zero

Solution :

$$Tr.(AB - BA) = Tr.(I)$$

$$Tr.(AB) - Tr.(BA) = n$$

$$n = 0$$

$$Tr.(A \pm B) = Tr.(A) \pm Tr.(B)$$

$$Tr.(AB) = Tr.(BA)$$

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## Determinants

- A determinant is a scalar value that is a function( real or complex valued ) of entries of a square matrix .

Let a matrix be : $A = \left[a_{ij}\right]_n$ , then its determinant is denoted as $\det(A) = |A|$

If $A = [a]_{1X1}$ , $|A| = a$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example: $A = \begin{bmatrix} 5 & -1 \\ 4 & 3 \end{bmatrix}$ , its determinant is

$|A| = 15 - (-4) = 19$

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## Minor of an Element

- Let Δ be a determinant

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Minor, of an element $a_{ij}$, denoted by $M_{ij}$ is defined as determinant of a sub – matrix obtained by deleting the $i^{th}$ row and $j^{th}$ column, in which the element is present, of Δ.

$$M_{11} = d \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$M_{12} = c$$

$$M_{21} = b$$

$$M_{22} = a$$

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# Session 04

## Properties of Determinants

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## Co-factor of an Element

- Let Δ be a determinant

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Co – factor, of an element $a_{ij}$ , denoted by $C_{ij}$ is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = M_{11} = d$$

$$C_{12} = -M_{12} = -c$$

$$C_{21} = -M_{21} = -b$$

$$C_{22} = M_{22} = a$$

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Find the minor and co – factors of elements $a_{11}$ , $a_{12}$ , $a_{23}$ , $a_{33}$ of the determinant.

$$\Delta = \begin{vmatrix} -1 & 2 & 4 \\ 0 & -5 & 3 \\ 6 & -7 & -9 \end{vmatrix}$$

Solution :

$$M_{11} = \begin{vmatrix} -5 & 3 \\ -7 & -9 \end{vmatrix} = 66 \qquad C_{11} = 66$$

$$M_{12} = \begin{vmatrix} 0 & 3 \\ 6 & -9 \end{vmatrix} = -18 \qquad C_{12} = 18$$

$$M_{23} = \begin{vmatrix} -1 & 2 \\ 6 & -7 \end{vmatrix} = -5 \qquad C_{23} = 5$$

$$M_{33} = \begin{vmatrix} -1 & 2 \\ 0 & -5 \end{vmatrix} = 5 \qquad C_{33} = 5$$

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## Value of 3 × 3 order determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expansion of determinant can be done by any row or column.

By $1^{st}$ row :

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

By $2^{nd}$ row :

$$= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{23}(a_{11}a_{32} - a_{31}a_{12})$$

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Evaluate value of the determinants

$(i)\ \Delta = \begin{vmatrix} \log_3 8 & \log_3 512 \\ \log_2 \sqrt{3} & \log_4 9 \end{vmatrix}$ $(ii)\ \Delta = \begin{vmatrix} 1 & -3 & 5 \\ 2 & -1 & 0 \\ -7 & 6 & 8 \end{vmatrix}$

$$(i)\ \Delta = \begin{vmatrix} \log_3 8 & \log_3 512 \\ \log_2 \sqrt{3} & \log_4 9 \end{vmatrix}$$

$$\log_a x^k = k \log_a x$$

$$\log_{a^k} x = \frac{1}{k} \log_a x$$

$$= \log_3 8 \log_4 9 - \log_2 \sqrt{3} \log_3 512$$

$$= 3 \log_3 2 \log_2 3 - \frac{1}{2} \log_2 3 \cdot \log_3 2^9$$

$$= 3 - \frac{9}{2} = -\frac{3}{2}$$

$$(ii)\ \Delta = \begin{vmatrix} 1 & -3 & 5 \\ 2 & -1 & 0 \\ -7 & 6 & 8 \end{vmatrix}$$

$$= 1(-8) - (-3)(16) + 5(12 - 7)$$

$$= 65$$

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## Value of determinant in terms of minor and co-factor

By $1^{st}$ row :

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

In terms of minor

$$\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

In terms of co – factor

$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

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# Key Takeaways

## Value of determinant in terms of minor and co-factor

By $2^{nd}$ row :

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{23}(a_{11}a_{32} - a_{31}a_{12})$$

In terms of minor

$$\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$$

In terms of co – factor

$$\Delta = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

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## Value of determinant in terms of minor and co-factor

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Sum of product of elements of a row (column) and corresponding co – factors of elements of the same row (column) gives value of determinant .

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \Delta$$

- Sum of product of elements of a row (column) and corresponding co – factors of elements of any other row (column) is zero.

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$$

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## Value of determinant in terms of minor and co-factor

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$$

Proof: $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$

$$= a_{11}(a_{32}a_{13} \;\; - a_{12}a_{33}) + a_{12}(a_{11}a_{33} \;\; - a_{31}a_{13}) + a_{13}(a_{12}a_{31} \;\; - a_{11}a_{32})$$

$$= 0$$

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If $\Delta = \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix}$ then :

A) $x\,M_{21} - y\,M_{22} + z\,M_{23} = \Delta$

B) $a\,C_{11} + b\,C_{12} + c\,C_{13} = 0$

C) $x\,C_{21} - y\,C_{22} + z\,C_{23} = \Delta$

D) $p\,M_{11} - q\,M_{12} + r\,M_{13} = \Delta$

Solution :

By property,

$a\,C_{11} + b\,C_{12} + c\,C_{13} = 0$ $\quad$ $\boxed{a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0}$

$p\,M_{11} - q\,M_{12} + r\,M_{13} = \Delta$ $\quad$ $\boxed{a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \Delta}$

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## Properties of determinant

- Determinant of upper or lower triangular square matrix is equal to product of its diagonal elements .

Example:

$$A = \begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix} \Rightarrow \begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{vmatrix} = a\begin{vmatrix} b & f \\ 0 & c \end{vmatrix} + 0 + 0$$

$$\Rightarrow |A| = abc$$

The determinant of the transpose of a square matrix is equal to the determinant of the matrix.

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## Properties of determinant

- The determinant of the transpose of a square matrix is equal to the determinant of the matrix.

Example:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \Delta' = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

By $1^{st}$ row, $\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

By $1^{st}$ column, $\Delta' = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

$\Delta = \Delta'$

Value of determinant doesn't change
by interchanging rows with column

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If $\Delta_1 = \begin{vmatrix} -1 & 2 & 4 \\ 5 & -3 & 9 \\ 6 & 7 & -8 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} -1 & 5 & 6 \\ 2 & -3 & 7 \\ 4 & 9 & -8 \end{vmatrix}$; then

A $\Delta_1 + \Delta_2 = 0$

B $\frac{\Delta_1}{\Delta_2} = 2$

C $\frac{\Delta_1}{\Delta_2} = 1$

D $\frac{\Delta_1}{\Delta_2} = -2$

Solution :

Value of determinant and its transpose is same.

$\Delta_1 = \Delta_2$

$\Rightarrow \frac{\Delta_1}{\Delta_2} = 1$

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## Properties of determinant

- If corresponding elements of any two rows (or columns) are identical (or proportional), then value of determinant is zero .

Example:

$$\Delta = \begin{vmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow \Delta = a_{11}(a_{21}a_{33} - a_{23}a_{31}) - a_{11}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{31} - a_{31}a_{21})$$

$$\Rightarrow \Delta = 0$$

$$\Delta = \begin{vmatrix} \sqrt{3} & \sqrt{5} & \sqrt{7} \\ 1 & 2 & 3 \\ \sqrt{3} & \sqrt{5} & \sqrt{7} \end{vmatrix} = 0$$

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## Properties of determinant

- If all the elements of a row or column are zero , then the value of determinant is zero .

Example:

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Rightarrow \Delta = a_{11}(a_{22}a_{33} \ - a_{23}a_{32}) - a_{12}(a_{21}a_{33} \ - a_{31}a_{23}) \\ +a_{13}(a_{21}a_{32} \ - a_{31}a_{22})$$

$$\Rightarrow \Delta = 0 \cdot M_{11} - 0 \cdot M_{12} + 0 \cdot M_{13}$$

$$\Rightarrow \Delta = 0$$

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If $A = \begin{vmatrix} \omega^{501} & \omega^{502} & \omega^{503} \\ \omega^{1101} & \omega^{1102} & \omega^{1102} \\ \omega^{1501} & \omega^{1502} & \omega^{1503} \end{vmatrix}$, where $\omega$ is cube root of unity, then the value of $A$ is:

A $1$

B $0$

C $-1$

D $\omega^2$

$$\omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2, \omega^3 = 1$$

$$A = \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix}$$

$\because$ Two rows are same

$\therefore$ Determinant is zero

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## Properties of determinant

- If any of two rows ( or columns ) of a determinant are interchanged, then its value gets multiplied by (–1).

$$\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \Delta' = \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \qquad \Rightarrow \Delta' = -\Delta$$

$$R_1 \leftrightarrow R_2$$

Proof:

$$\Delta_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Rightarrow \Delta_1 = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

With respect to second row

$$\Rightarrow \Delta_2 = -a_{11}M_{11} + a_{12}M_{12} - a_{13}M_{13}$$

$$\Rightarrow \Delta_2 = -\Delta_1$$

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# Session 05

## Some Special Determinants

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## Properties of determinant

- If elements of a row ( or column ) are multiplied by a constant, then value of determinant also gets multiplied by the same constant .

Proof:

$$\Delta_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_2 = k\,\Delta_1$$

$$\Delta_2 = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_1 = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}$$

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# Key Takeaways

Proof: $\Delta_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix}$

$\Delta_2 = k\,\Delta_1$

$$\Delta_2 = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_1 = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}$$

With $1^{st}$ column

$$\Delta_2 = ka_{11}M_{11} - ka_{21}M_{21} + ka_{31}M_{31}$$

$$\Delta_2 = k(a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}) = k\Delta_1$$

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Let $a, b, c$ be such that $b(c+a) \neq 0$. If

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^{n}c \end{vmatrix} = 0$$

. Then the value of $n$ is :

A Zero

B Any even integer

C Any odd integer

D Any integer

Solution:

$$\underbrace{\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}}_{\Delta_1} + \underbrace{\begin{vmatrix} (-1)^{n+2}a & a+1 & a-1 \\ (-1)^{n+1}b & b+1 & b-1 \\ (-1)^{n}c & c-1 & c+1 \end{vmatrix}}_{\Delta_2} = 0$$

$$\Delta_2 = \begin{vmatrix} (-1)^{n+2}a & (-1)^{n+1}b & (-1)^{n}c \\ a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \end{vmatrix} = \begin{vmatrix} (-1)^{n+2}a & a+1 & a-1 \\ (-1)^{n+1}b & b+1 & b-1 \\ (-1)^{n}c & c-1 & c+1 \end{vmatrix}$$

$$\Delta_1 + \Delta_2 = 0$$

$n$ is odd integer

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$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ & $\Delta_2 = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$, then $\Delta_2 - \Delta_1$ is :

A $(a+b+c)\,\Delta_1$

B $\Delta_1$

C $\mathbf{0}$

D $abc\,\Delta_1$

Solution:

$$\Delta_2 = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Multiply $1^{st}$ column by $a$ and divide $\Delta_2$ by $a$.

Multiply $2^{nd}$ column by $b$ and divide $\Delta_2$ by $b$.

Multiply $3^{rd}$ column by $c$ and divide $\Delta_2$ by $c$.

$$\Delta_2 = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \qquad \Delta_2 = \Delta_1$$

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$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ & $\Delta_2 = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$, then $\Delta_2 - \Delta_1$ is :

A $(a+b+c)\,\Delta_1$

B $\Delta_1$

C $\mathbf{0}$

D $abc\,\Delta_1$

Solution:

$$\Delta_2 = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Multiply $1^{st}$ column by $a$ and divide $\Delta_2$ by $a$.

Multiply $2^{nd}$ column by $b$ and divide $\Delta_2$ by $b$.

Multiply $3^{rd}$ column by $c$ and divide $\Delta_2$ by $c$.

$$\Delta_2 = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Delta_2 = \Delta_1$$

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# Key Takeaways

## Properties of Determinants

- If each element of any row ( or column ) can be expressed as sum of two terms , then the determinant can also be expressed as sum of two determinants .

$$\Delta = \begin{vmatrix} a+x & b+y & c+z \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} x & y & z \\ d & e & f \\ g & h & i \end{vmatrix}$$

Proof:

$$\Delta = (a+x)M_{11} - (b+y)M_{12} + (c+z)M_{13}$$

$$= aM_{11} - bM_{12} + cM_{13} + xM_{11} - yM_{12} + zM_{13}$$

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Find value of the determinant $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$ :

A $\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$

B $\sqrt{5}\sqrt{13}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$

C $0$

D $5\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$

Solution:

$$= \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$$

$$= \begin{vmatrix} \sqrt{13} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{26} & 5 & \sqrt{10} \\ \sqrt{65} & \sqrt{15} & 5 \end{vmatrix} + \begin{vmatrix} \sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15} & 5 & \sqrt{10} \\ 3 & \sqrt{15} & 5 \end{vmatrix}$$

$$= \sqrt{5}\sqrt{13}\begin{vmatrix} 1 & 2\sqrt{5} & 1 \\ \sqrt{2} & 5 & \sqrt{2} \\ \sqrt{5} & \sqrt{15} & \sqrt{5} \end{vmatrix} + 5\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$$= \sqrt{5}\sqrt{13}\times 0 + 5\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix} = 5\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

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# Key Takeaways

## Properties of Determinants

- If $A = [a_{ij}]_n$ , then $\boxed{|kA| = k^n|A|}$ where $k$ is a scalar.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

$$\Rightarrow |kA| = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow |kA| = k^3|A|$$

- If $A = [a_{ij}]_n$, $B = [b_{ij}]_n$ , then $|AB| = |A||B|$

$|A^k| = |A|^k$

$$\Rightarrow \underbrace{|A \cdot A \cdot A \cdots \; A|}_{k \text{ times}} = \underbrace{|A| \cdot |A| \cdot |A| \cdots \; |A|}_{k \text{ times}} = |A|^k$$

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If $A$ & $B$ are square matrices of order $n$, such that $|A| = 3$, $|B| = 5$, then the value of $||2A|B|$ is :

Solution:

$$||2A|B| = |2A|^n|B| \quad \boxed{\text{Since } |\mathrm{k}\mathrm{A}| = \mathrm{k}^{\mathrm{n}}|\mathrm{A}|}$$

$$= (2^n|A|)^n|B|$$

$$= 2^{n^2}|A|^n \cdot |B|$$

$$\boxed{= 2^{n^2} \cdot 3^n \cdot 5}$$

A $5.6^n$

B $2^{n^2} \cdot 15^n$

C $15.2^n$

D $5 \cdot 2^{n^2} \cdot 3^n$

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# Key Takeaways

## Properties of Determinants

- Determinant of an odd order skew – symmetric matrix is zero .

Proof:

$|A| = |-A^T|$ $\boxed{A = -A^T}$

$= (-1)^n \, |A^T|$

If $n$ is odd,

$|A| = -|A| \Rightarrow |A| = 0$

Example: Value of determinant $\begin{vmatrix} 0 & p-q & q-r \\ q-p & 0 & r-p \\ r-q & p-r & 0 \end{vmatrix}$ is $0$.

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Statement 1 : Determinant of a skew- symmetric matrix of odd order is zero.
Statement 2 : For any matrix $A$ , $\det(A^T) = \det(A)$ & $\det(-A) = -\det(A)$ .
where $\det(B)$ denotes determinant of matrix $B$ . Then

A. Both statement are true

B. Both statement are false

C. Statement 1 is true, statement 2 is false

D. Statement 2 is true, statement 1 is false

**Solution:**

Let $A$ is a skew-symmetric matrix $\Rightarrow A^T = -A \cdots (i)$

Taking determinant of $(i)$, we get

$$|A^T| = |-A| \Rightarrow |A| = (-1)|A| \quad (\because |A| = |A^T|)$$

$\Rightarrow |A| = (-1)^n|A|$ where n is order of matrix

Since $n = 3$ is odd

$$\Rightarrow |A| = -|A| \Rightarrow 2|A| = 0$$

Therefore, statement 1 is true.

Hence, option 'C' is correct.

Statement 2 is incorrect $\det(A) = -(\det(A))$ for odd order matrix only

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# Key Takeaways

## Properties of Determinants

- The value of determinant is not altered by adding to the elements of any row ( or column ) a constant multiple of corresponding elements of any other row ( or column ) .

$$\Delta_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$R_1 \to R_1 + pR_2$ , where $p$ is a scalar.

$$\Delta_2 = \begin{vmatrix} a_{11} + pa_{21} & a_{12} + pa_{22} & a_{13} + pa_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\boxed{\Delta_1 = \Delta_2}$$

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# Key Takeaways

## Properties of Determinants

- The value of determinant is not altered by adding to the elements of any row ( or column ) a constant multiple of corresponding elements of any other row ( or column ) .

Proof:

$$\Delta_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} pa_{21} & pa_{22} & pa_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + p\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + p \times 0$$

$$\Delta_2 = \Delta_1$$

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If $a, b, c$ are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then the value of $abc(ab+bc+ca)$ is equal to:

A $a-b-c$

B $a-b+c$

C $a+b+c$

D $0$

Solution:

$$\Rightarrow \begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} + \begin{vmatrix} a & a^3 & -1 \\ b & b^3 & -1 \\ c & c^3 & -1 \end{vmatrix} = 0$$

Taking $a, b, c$ from the first determinant and apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ in both determinants

$$\Rightarrow abc \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix} - \begin{vmatrix} a & a^3 & 1 \\ b-a & b^3-a^3 & 0 \\ c-a & c^3-a^3 & 0 \end{vmatrix} = 0$$

As $a, b, c$ are all distinct and cancelling out $b-a$ and $c-a$

$$\Rightarrow abc \begin{vmatrix} b+a & b^2+a^2+ab \\ c+a & c^2+a^2+ac \end{vmatrix} = \begin{vmatrix} 1 & b^2+a^2+ab \\ 1 & c^2+a^2+ac \end{vmatrix}$$

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If $a, b, c$ are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then the value of $abc(ab+bc+ca)$ is equal to:

Solution:

$$\Rightarrow abc\begin{vmatrix} b+a & b^2+a^2+ab \\ c+a & c^2+a^2+ac \end{vmatrix} = \begin{vmatrix} 1 & b^2+a^2+ab \\ 1 & c^2+a^2+ac \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and then cancelling $c-b$ on both sides, we get

$$\Rightarrow abc\begin{vmatrix} b+a & b^2+a^2+ab \\ 1 & a+b+c \end{vmatrix} = \begin{vmatrix} 1 & b^2+a^2+ab \\ 0 & a+b+c \end{vmatrix}$$

$$\therefore abc(ab+b^2+bc+a^2+ab+ac-b^2-c^2-ab) = a+b+c$$

$$\Rightarrow abc(ab+bc+ca) = a+b+c$$

A $a-b-c$

B $a-b+c$

C $a+b+c$

D $0$

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# Some Important Formula

- $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Proof:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \qquad \begin{array}{l} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{array}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix} \quad \boxed{= (a-b)(b-c)(c-a)}$$

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# Session 06

## Application of Determinants

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## Some Important Determinants

- $$\underbrace{\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}}_{\text{Degree} = 4} = \underbrace{(a-b)(b-c)(c-a)}_{\text{Degree} = 3}\underbrace{(a+b+c)}_{\text{Linear term}}$$

Proof:

$\Delta = 1\,(b^1c^3 - b^3c)$

Put $a = b \Rightarrow \Delta = 0 \;\Rightarrow (a-b)$ is a factor of $\Delta$

$b = c \Rightarrow \Delta = 0 \;\Rightarrow (b-c)$ is a factor of $\Delta$

$c = a \Rightarrow \Delta = 0 \;\Rightarrow (c-a)$ is a factor of $\Delta$

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## Some Important Determinants

- $$\underbrace{\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}}_{\text{Degree} = 5} = \underbrace{(a-b)(b-c)(c-a)}_{\text{Degree} = 3}\underbrace{(ab+bc+ca)}_{2^{nd}\text{ degree terms}}$$

Put $a = b$ or $b = c$ or $c = a$

$\Rightarrow \Delta = 0$

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## Some Important Determinants

- $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3 = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Proof:

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \qquad C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$= (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

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Let $a, b, c \in R$ be all non – zero and satisfy $a^3 + b^3 + c^3 = 2$. If the matrix $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$ satisfies $A^T A = I$, then a value of $abc$ can be :

A. $\frac{2}{3}$

B. $-\frac{1}{3}$

C. $3$

D. $\frac{1}{3}$

Solution: $A^T A = I$

$\Rightarrow |A^T A| = |I|$

$\Rightarrow |A^T||A| = 1$

$\Rightarrow |A|^2 = 1$

$\Rightarrow |A| = \pm 1$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

$\Rightarrow 3abc - a^3 - b^3 - c^3 = \pm 1$

$\Rightarrow 3abc = 1, 3 \quad \Rightarrow abc = \frac{1}{3}, 1$

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## Product of Two Determinants

- Let the two determinants of 2X2 order be :

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad \text{and} \quad \Delta_2 = \begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix}$$

then their product Δ will be :

$$\Delta = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + a_2 m_1 & a_1 l_2 + a_2 m_2 \\ b_1 l_1 + b_2 m_1 & b_1 l_2 + b_2 m_2 \end{vmatrix}$$

Note: Multiplication of same order determinants can be done in four ways –

$$R \times R, R \times C, C \times C, C \times R$$

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Evaluate $\begin{vmatrix} 1 & -2 & 4 \\ 5 & 0 & -6 \\ -3 & 7 & 1 \end{vmatrix} \times \begin{vmatrix} 6 & -1 & 3 \\ -4 & 2 & 8 \\ 0 & -9 & 5 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 1 & -2 & 4 \\ 5 & 0 & -6 \\ -3 & 7 & 1 \end{vmatrix} \times \begin{vmatrix} 6 & -1 & 3 \\ -4 & 2 & 8 \\ 0 & -9 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 6+8+0 & -1-4-36 & 3-16+20 \\ 30+0+0 & -5+0+54 & 15+0-30 \\ -18-28+0 & 3+14-9 & -9+56+5 \end{vmatrix}$$

$$= \begin{vmatrix} 14 & -41 & 7 \\ 30 & 49 & -15 \\ -46 & 8 & 52 \end{vmatrix}$$

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Evaluate $\begin{vmatrix} 1 & \cos(B-A) & \cos(C-A) \\ \cos(A-B) & 1 & \cos(C-B) \\ \cos(A-C) & \cos(B-C) & 1 \end{vmatrix}$

A. $\cos A \cos B \cos C$

B. 1

C. 0

D. $\cos A+\cos B+\cos C$

$$\begin{vmatrix} 1 & \cos(B-A) & \cos(C-A) \\ \cos(A-B) & 1 & \cos(C-B) \\ \cos(A-C) & \cos(B-C) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos(A-A) & \cos(B-A) & \cos(C-A) \\ \cos(A-B) & \cos(B-B) & \cos(C-B) \\ \cos(A-C) & \cos(B-C) & \cos(C-C) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A\cos A+\sin A\sin A & \cos B\cos A+\sin B\sin A & \cos C\cos A+\sin C\sin A \\ \cos A\cos B+\sin A\sin B & \cos B\cos B+\sin B\sin B & \cos C\cos B+\sin C\sin B \\ \cos A\cos C+\sin A\sin C & \cos B\cos C+\sin B\sin C & \cos C\cos C+\sin C\sin C \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 1 \\ \cos B & \sin B & 1 \\ \cos C & \sin C & 1 \end{vmatrix} \times \begin{vmatrix} \cos A & \cos B & \cos C \\ \sin A & \sin B & \sin C \\ 0 & 0 & 0 \end{vmatrix} = 0$$

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If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2,$$ then $k$ is equal to :

A $1$

B $-1$

C $\alpha\beta$

D $\dfrac{1}{\alpha\beta}$

Solution:

$$\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} \qquad \because \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$= \left((1-\alpha)(\alpha-\beta)(\beta-1)\right)^2 \Rightarrow k = 1$$

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# Key Takeaways

## Application of determinants:

- Area of triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is:

$$\Delta = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

**Note:** If $\Delta = 0$ , then points are collinear .

- Equation of straight line passing through points $(x_1, y_1)$ & $(x_2, y_2)$ is :

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

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# Key Takeaways

## Application of determinants:

- The lines:

  $a_1x + b_1y + c_1 = 0$
  $a_2x + b_2y + c_2 = 0$ are concurrent if,
  $a_3x + b_3y + c_3 = 0$

  $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

  **Note:** The converse is not true

- The general 2 – degree equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, represents a pair of straight lines if,

  $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \text{ or } abc + 2hgf - af^2 - bg^2 - ch^2 = 0$$

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Consider the lines given by

$L_1 : x + 3y - 5 = 0$
$L_2 : 3x - ky - 1 = 0$
$L_3 : 5x + 2y - 12 = 0$

| COLUMN $I$ | COLUMN $II$ |
|---|---|
| $(A)$ $L_1, L_2, L_3$ are concurrent, if | $(p)$ $k = -9$ |
| $(B)$ One of $L_1, L_2, L_3$ is parallel to at least one of the other two, if | $(q)$ $k = -\frac{6}{5}$ |
| $(C)$ $L_1, L_2, L_3$ form a triangle, if | $(r)$ $k = \frac{5}{6}$ |
| $(D)$ $L_1, L_2, L_3$ do not form a triangle, if | $(s)$ $k = 5$ |

Solution:

$(A)$ $L_1, L_2, L_3$ are concurrent , if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$L_1 : x + 3y - 5 = 0$
$L_2 : 3x - ky - 1 = 0$
$L_3 : 5x + 2y - 12 = 0$

$$\begin{vmatrix} 1 & 3 & -5 \\ 3 & -k & -1 \\ 5 & 2 & -12 \end{vmatrix} = 0$$

$$\Rightarrow (12k + 2) - 3(-36 + 5) - 5(6 + 5k) = 0$$

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Consider the lines given by

$L_1: x + 3y - 5 = 0$

$L_2: 3x - ky - 1 = 0$

$L_3: 5x + 2y - 12 = 0$

| COLUMN $I$ | COLUMN $II$ |
|---|---|
| $(A)$ $L_1, L_2, L_3$ are concurrent, if | $(p)$ $k = -9$ |
| $(B)$ One of $L_1, L_2, L_3$ is parallel to at least one of the other two, if | $(q)$ $k = -\frac{6}{5}$ |
| $(C)$ $L_1, L_2, L_3$ form a triangle, if | $(r)$ $k = \frac{5}{6}$ |
| $(D)$ $L_1, L_2, L_3$ do not form a triangle, if | $(s)$ $k = 5$ |

Solution:

$\Rightarrow 12k + 2 + 93 - 30 - 25k = 0$

$\Rightarrow 65 - 13k = 0$

$\Rightarrow k = 5$

$(A) \rightarrow (s)$

$(B)$One of $L_1, L_2, L_3$ is parallel to at least one of the other two, if

$\frac{3}{k} = -\frac{1}{3}$ or $-\frac{5}{2}$

$k = -9$ or $-\frac{6}{5}$

$(B) \rightarrow (p), (q)$

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Consider the lines given by

$L_1: x + 3y - 5 = 0$
$L_2: 3x - ky - 1 = 0$
$L_3: 5x + 2y - 12 = 0$

| COLUMN $I$ | COLUMN $II$ |
|---|---|
| $(A)$ $L_1, L_2, L_3$ are concurrent, if | $(p)$ $k = -9$ |
| $(B)$ One of $L_1, L_2, L_3$ is parallel to at least one of the other two, if | $(q)$ $k = -\frac{6}{5}$ |
| $(C)$ $L_1, L_2, L_3$ form a triangle, if | $(r)$ $k = \frac{5}{6}$ |
| $(D)$ $L_1, L_2, L_3$ do not form a triangle, if | $(s)$ $k = 5$ |

Solution:

$(C)$ $L_1, L_2, L_3$ form triangle, if neither they are concurrent nor parallel

$\Rightarrow k \neq 5, -9, -\frac{6}{5}$ $(C) \rightarrow (r)$

$(D)$ $L_1, L_2, L_3$ do not form a triangle, if they are parallel or concurrent

$\Rightarrow k = 5$ or $-9$ or $-\frac{6}{5}$ $(D) \rightarrow (p), (q), (s)$

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# Key Takeaways

## Differentiation of determinant

- If $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

(differentiation can also be done column - wise)

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If $y(x) = \begin{vmatrix} \sin x & \cos x & \sin x + \cos x + 1 \\ 23 & 17 & 13 \\ 1 & 1 & 1 \end{vmatrix}$, $x \in \mathbb{R}$, then $\frac{d^2y}{dx^2} + y$ is equal to :

A $6$

B $4$

C $-10$

D $0$

Solution:

$$y(x) = \begin{vmatrix} \sin x & \cos x & \sin x + \cos x + 1 \\ 23 & 17 & 13 \\ 1 & 1 & 1 \end{vmatrix}$$ , differentiate w.r.t $x$

$$y'(x) = \begin{vmatrix} \cos x & -\sin x & -\sin x + \cos x \\ 23 & 17 & 13 \\ 1 & 1 & 1 \end{vmatrix}$$

$$y'(x) = \begin{vmatrix} \cos x & -\sin x & -\sin x + \cos x \\ 23 & 17 & 13 \\ 1 & 1 & 1 \end{vmatrix}$$ , differentiate w.r.t $x$

$$y''(x) = \begin{vmatrix} -\sin x & -\cos x & -\cos x - \sin x \\ 23 & 17 & 13 \\ 1 & 1 & 1 \end{vmatrix}$$

$$y(x) + y''(x) = \begin{vmatrix} 0 & 0 & 1 \\ 23 & 17 & 13 \\ 1 & 1 & 1 \end{vmatrix} = 6$$

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# Key Takeaways

## Integration/Summation of determinant

- If $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ a & b & c \\ d & e & f \end{vmatrix}$

$$\sum \Delta(x) = \begin{vmatrix} \sum f_1(x) & \sum g_1(x) & \sum h_1(x) \\ a & b & c \\ d & e & f \end{vmatrix}$$

**Note:** If variable is present in more than one row (or column), then first expand the determinant and then apply summation or integration .

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$$\Delta(r) = \begin{vmatrix} 1 & x & n+1 \\ r & y & \frac{n(n+1)}{2} \\ 2r-1 & z & n^2-1 \end{vmatrix}$$, then $\sum_{r=0}^{n} \Delta(r)$ is equal to:

Solution:

$$\sum_{r=0}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=0}^{n} 1 & x & n+1 \\ \sum_{r=0}^{n} (r) & y & \frac{n(n+1)}{2} \\ \sum_{r=0}^{n} (2r-1) & z & n^2-1 \end{vmatrix}$$

$$= \begin{vmatrix} n+1 & x & n+1 \\ \frac{n(n+1)}{2} & y & \frac{n(n+1)}{2} \\ n^2-1 & z & n^2-1 \end{vmatrix}$$

$$= 0$$

A $\frac{n^2(n+1)}{2}$

B $n^3$

C $\frac{n(2n+1)(3n+1)}{2}$

D $0$

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If $\Delta(r) = \begin{vmatrix} 2^{r-1} & 2.3^{r-1} & 4.5^{r-1} \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$, then the value of $\sum_{r=1}^{n} \Delta(r)$

A 0

B $\alpha\beta\gamma$

C $\alpha + \beta + \gamma$

D $\alpha 2^n + \beta 3^n + \gamma 4^n$

Solution:

$$\sum_{r=1}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & \sum_{r=1}^{n} 2.3^{r-1} & \sum_{r=1}^{n} 4.5^{r-1} \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$= \begin{vmatrix} 2^n - 1 & 3^n - 1 & 5^n - 1 \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$$

$$= 0$$

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Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$, Prove that : $\int_0^{\pi/2} f(x)\, dx = -\left(\frac{\pi}{4} + \frac{8}{15}\right)$

Solution:

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Operate $R_1 \to R_1 - \sec x\ R_3$

$$f(x) = \begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

$$= (\sec^2 x + \cot x \operatorname{cosec} x)\,(\cos^4 x - \cos^2 x)$$

$$f(x) = \left(1 + \frac{\cos^3 x}{\sin^2 x} - \cos^3 x\right)(\cos^2 x - 1) = -\sin^2 x \frac{\sin^2 x + \cos^3 x - \cos^3 x \sin^2 x}{\sin^2 x}$$

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Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$, Prove that : $\int_0^{\pi/2} f(x)\, dx = -\left(\frac{\pi}{4} + \frac{8}{15}\right)$

Solution:

$$f(x) = \left(1 + \frac{\cos^3 x}{\sin^2 x} - \cos^3 x\right)(\cos^2 x - 1)$$

$$= -\sin^2 x \frac{\sin^2 x + \cos^3 x - \cos^3 x \sin^2 x}{\sin^2 x}$$

$$f(x) = -\ (\sin^2 x + \cos^5 x)$$

$$\int_0^{\pi/2} f(x)\, dx = \int_0^{\pi/2} (\sin^2 x + \cos^5 x)\, dx$$

$$= -\left(\frac{1}{2} \cdot \frac{\pi}{2} + \frac{4\cdot 2}{5\cdot 3}\right) = -\left(\frac{\pi}{4} + \frac{8}{15}\right)$$

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# Session 07

## Adjoint of Matrix
## and
## Inverse of a Matrix

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# Key Takeaways

## Singular/Non-singular Matrices

- A square matrix $A$ is said to be singular or non – singular according as $|A| = 0$ or $|A| \neq 0$ respectively.

## Co-factor matrix and Adjoint (Adjugate) matrix

- Let $A = [a_{ij}]_n$ be a square matrix
  - The matrix obtained by replacing each element of $A$ by corresponding co factor is called a co factor matrix .

    $C = [c_{ij}]_n$ , where $c_{ij}$ is co factor of $a_{ij}$ , $\forall\ i \,\&\, j$
  - Transpose of co factor matrix of $A$ is called adjoint of matrix $A$ , and is denoted by $adj\ (A)$.

    $adj\ (A) = [d_{ij}]_n$ , where $d_{ij} = c_{ji}$ , $\forall\ i \,\&\, j$

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## Co-factor matrix and Adjoint (Adjugate) matrix

- Let $A = [a_{ij}]_n$ be a square matrix

  $C = [c_{ij}]_n$ , where $c_{ij}$ is co factor of $a_{ij}$ , $\forall\ i\ \&\ j$

  $adj\ (A) = [d_{ij}]_n$ , where $d_{ij} = c_{ji}$ , $\forall\ i\ \&\ j$

  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ , $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

  $adj\ (A) = C^T = \begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

  Note:

  For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad adj\ (A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

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## Example:

Find adjoint of matrix $A = \begin{pmatrix} 2 & 5 & 6 \\ 1 & 3 & 1 \\ 2 & 2 & 3 \end{pmatrix}$.

$$A = \begin{pmatrix} 2 & 5 & 6 \\ 1 & 3 & 1 \\ 2 & 2 & 3 \end{pmatrix} \quad \left\{ \begin{array}{l} C_{11} = 7;\ \ C_{12} = -1;\ \ C_{13} = -4\,;\ C_{21} = -3\,;\ \ C_{22} = -6\,;\ C_{23} = 6\,; \\ \\ C_{31} = -13\,;\ C_{32} = 4;\ \ C_{33} = 1 \end{array} \right.$$

$$\Rightarrow C = \begin{pmatrix} 7 & -1 & -4 \\ -3 & -6 & 6 \\ -13 & 4 & 1 \end{pmatrix}$$

$$\Rightarrow adj(A) = C^T = \begin{pmatrix} 7 & -3 & -13 \\ -1 & -6 & 4 \\ -4 & 6 & 1 \end{pmatrix}$$

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If $A = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$, then $adj\,(3A^2 + 12A)$ is equal to :

A $\begin{pmatrix} 72 & -84 \\ -63 & 51 \end{pmatrix}$

B $\begin{pmatrix} 51 & 63 \\ 84 & 72 \end{pmatrix}$

C $\begin{pmatrix} 51 & 84 \\ 63 & 72 \end{pmatrix}$

D $\begin{pmatrix} 72 & -63 \\ -84 & 51 \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$$

$$\Rightarrow 3A^2 = 3\begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix} = 3\begin{pmatrix} 16 & -9 \\ -12 & 13 \end{pmatrix} = \begin{pmatrix} 48 & -27 \\ -36 & 39 \end{pmatrix}$$

$$12\,A = 12\begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 24 & -36 \\ -48 & 12 \end{pmatrix}$$

$$3A^2 + 12\,A = \begin{pmatrix} 72 & -63 \\ -84 & 51 \end{pmatrix}$$

$$adj\,(3A^2 + 12\,A) = \begin{pmatrix} 51 & 63 \\ 84 & 72 \end{pmatrix}$$

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# Key Takeaways

## Properties of adjoint matrix

- Let $A = [a_{ij}]_n$ be a square matrix .

  $$adj\ (A^T) = (adj\ A)^T$$

  Proof:

  L.H.S = $adj\ (A^T) = (C^T)^T = C$

  R.H.S = $(adj\ A)^T = ((C)^T)^T = C$

  $adj\ (A^T) = (adj\ A)^T$

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# Key Takeaways

## Properties of adjoint matrix

- Let $A = \left[a_{ij}\right]_n$ be a square matrix .

$$A \; adj \; (A) = |A| I_n = adj \; (A) \; A$$

Proof:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$adj \; (A) = C^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

$$A \; adj \; (A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

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## Properties of adjoint matrix

- Let $A = [a_{ij}]_n$ be a square matrix .

$$A\ adj\ (A) = |A|I_n = adj\ (A)\ A$$

Proof:

$$A\ adj\ (A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \Delta$$

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$$

$$A\ adj\ (A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$A\ adj\ (A) = |A|I_n$$

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If $A = [a_{ij}]_{3X3}$ is a scalar matrix with $a_{11} = a_{22} = a_{33} = 2$ and $A\,adj(A) = kI_3$ , then $k$ is equal to :

A 7

B 8

C 2

D $-1$

Solution:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad A\,adj\,(A) = |A|I_n$$

$$\left.\begin{aligned} A\,adj(A) &= 8I_3 \\ A\,adj(A) &= kI_3 \end{aligned}\right\} \quad k = 8$$

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# Key Takeaways

## Properties of adjoint matrix

- Let $A = [a_{ij}]_n$ be a square matrix .

  $|adj\ (A)| = |A|^{n-1}$

  Proof:

  We know, $A\ adj\ (A) = |A| I_n$

  $\Rightarrow |A\ adj\ (A)| = ||A| I_n|$

  $\Rightarrow |A||adj\ (A)| = |A|^n$

  $\Rightarrow |adj\ (A)| = |A|^{n-1}$

  Note:

  $|C| = |adj\ (A)| = |A|^{n-1}$

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If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is adjoint of a $3 \times 3$ matrix $A$ and $|A| = 4$, then $\alpha$ is equal to :

A. 4

B. 11

C. 5

D. 0

$$P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{vmatrix} \qquad R_2 \to R_2 - R_1$$

$$\begin{vmatrix} 1 & \alpha & 3 \\ 0 & 3-\alpha & 0 \\ 2 & 4 & 4 \end{vmatrix} = (3-\alpha)(4-6) = 2\alpha - 6$$

$\because$ $P$ is the adjoint of the matrix $A$

$$\Rightarrow |P| = |A|^2 = 16 \qquad |adj\ (A)| = |A|^{n-1}$$

$$\Rightarrow 2\alpha - 6 = 16 \ \Rightarrow \alpha = 11$$

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If $A$ is a square matrix of order $n$, then $\left|adj\left(adj\ (A)\right)\right|$ is :

$$adj\left(adj(A)\right) = |A|^{n-2}A$$

$$\Rightarrow \left|adj\left(adj(A)\right)\right| = \left||A|^{n-2}A\right|$$

$$\Rightarrow \left|adj\left(adj(A)\right)\right| = |A|^{(n-2)n}|A|$$

$$\Rightarrow \left|adj\left(adj\ (A)\right)\right| = |A|^{(n-1)^2}$$

A $|A|^{n-2}$

B $|A|^{n^2-2n}$

C $|A|^{n^2-n}$

D $|A|^{(n-1)^2}$

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## Key Takeaways

### Properties of adjoint matrix

- Let $A = [a_{ij}]_n$ be a square matrix .

$$adj(adj(A)) = |A|^{n-2}A$$

Proof:

$$A\,adj(A) = |A|I$$

$$\Rightarrow adj(A)adj(adj(A)) = |adj(A)|I$$

$$\Rightarrow A\,adj(A)adj(adj(A))$$

$$\Rightarrow |A|adj(adj(A)) = A|A|^{n-1}$$

$$\Rightarrow adj(adj(A)) = |A|^{n-2}A$$

$$A \rightarrow adj\,(A)$$

$$|adj\,(A)| = |A|^{n-1}$$

$$A\,adj\,(A) = |A|I_n = adj(A)\,A$$

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If the matrices $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$, $B = adj\,(A)$ and $C = 3A$, then $\frac{|adj\,(B)|}{|C|}$ is equal to :

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A 8

B 2

C 16

D 72

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{vmatrix}$$

$R_3 \to R_3 - R_1$

$R_2 \to R_2 - R_1$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -2 & 1 \end{vmatrix} = 6$$

$$\frac{|adj\,(B)|}{|C|} = \frac{|adj\,(adj\,(A))|}{|3A|} = \frac{|A|^{(3-1)^2}}{3^3|A|} = \frac{6^3}{3^3}$$

$$\Rightarrow \frac{|adj\,(B)|}{|C|} = 8$$

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## Properties of adjoint matrix

- If $A$ is a symmetric matrix , then $adj\ (A)$ is also a symmetric matrix.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow adj\ (A) = \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$

- If $A$ is a singular matrix , then $adj\ (A)$ is also a singular matrix.

$$|A| = 0 \quad \Rightarrow |adj\ (A)| = 0 \qquad |adj\ (A)| = |A|^{n-1}$$

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## Key Takeaways

### Inverse of a matrix (Reciprocal matrix)

- If $A, B$ are square matrices of order $n$ and $|A| \neq 0,$

  $AB = I_n = BA,$ then $B$ is multiplicative inverse of $A$ i.e. $B = A^{-1}$

  $\Rightarrow AA^{-1} = I = A^{-1}A$

  To find inverse of a matrix :

  We know , $A\ adj\ (A) = |A|I_n = adj\ A \cdot A$

  $\Rightarrow A \cdot \left(\frac{adj\ A}{|A|}\right) = I_n = \left(\frac{adj\ A}{|A|}\right) \cdot A$

  $\Rightarrow A \cdot A^{-1} = I_n = A^{-1} \cdot A \Rightarrow A^{-1} = \frac{adj\ (A)}{|A|}$

  Note: For a matrix to be invertible, it must be non – singular .

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Find the inverse of matrix $A \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$:

$$|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} \quad R_3 \rightarrow R_3 - R_1$$

$$|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow |A| = 1$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{cases} C_{11} = 7;\ C_{12} = -1;\ C_{13} = -1\,;\ C_{21} = -3\,;\ C_{22} = 1\,;\ C_{23} = 0\,; \\ C_{31} = -3;\ C_{32} = 0;\ C_{33} = 1 \end{cases}$$

$$C = \begin{pmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix};\ adj(A) = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

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Find the inverse of matrix $A \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$adj(A) = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{adj\ (A)}{|A|}$$

$$A^{-1} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$|A| = 1$$

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If $|A| = \begin{vmatrix} e^{-t} & e^{-t}\cos t & e^{-t}\sin t \\ e^{-t} & -e^{-t}\cos t - e^{-t}\sin t & e^{-t}\cos t - e^{-t}\sin t \\ e^{-t} & 2e^{-t}\sin t & -2e^{-t}\cos t \end{vmatrix}$, then $A$ is

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A Non-invertible for any $t \in \mathbb{R}$

B Invertible only if $t = \frac{\pi}{2}$

C Invertible only if $t = \pi$

D Invertible for all $t \in \mathbb{R}$

$$|A| = \begin{vmatrix} e^{-t} & e^{-t}\cos t & e^{-t}\sin t \\ e^{-t} & -e^{-t}\cos t - e^{-t}\sin t & e^{-t}\cos t - e^{-t}\sin t \\ e^{-t} & 2e^{-t}\sin t & -2e^{-t}\cos t \end{vmatrix}$$

$$\Rightarrow |A| = e^{-3t}\begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\cos t - \sin t & \cos t - \sin t \\ 1 & 2\sin t & -2\cos t \end{vmatrix}$$

$$R_1 = R_1 + R_2 + \frac{1}{2}R_3$$

$$\Rightarrow |A| = e^{-3t}\begin{vmatrix} \frac{5}{2} & 0 & 0 \\ 1 & -\cos t - \sin t & \cos t - \sin t \\ 1 & 2\sin t & -2\cos t \end{vmatrix}$$

$$\Rightarrow |A| = e^{-3t}\cdot\frac{5}{2}(2\cos^2 t + 2\sin t\cos t - 2\sin t\cos t + 2\sin^2 t)$$

$\Rightarrow |A| = e^{-3t}(5\,) \neq 0$ $\quad\therefore$ $A$ is invertible for all $t \in \mathbb{R}$

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## Matrix Properties :

- $adj(AB) = adj\ (B)\ Adj\ (A)$

Proof :

$$(AB)^{-1} = \frac{adj(AB)}{\det(AB)}$$

Or $adj(AB) - (AB)^{-1} \cdot \det(AB) \cdots (1)$

It is also known $= (AB)^{-1} \cdot \det(AB)$

And $\det(AB) = \det(A) \cdot \det(B) \cdots (2)$

Also,$A^{-1} = \frac{adj(A)}{\det(A)}$ $B^{-1} = \frac{adj(B)}{\det(B)}$

Or $adj\ (B) \cdot adj\ (A) = \det A \cdot \det B \cdot B^{-1} \cdot A^{-1} \cdots (3)$

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## Matrix Properties :

Proof :

$adj(AB) - (AB)^{-1} \cdot \det(AB) \cdots (1)$

$\det(AB) = \det(A) \cdot \det(B) \cdots (2)$

$adj\ (B) \cdot adj\ (A) = \det A \cdot \det B \cdot B^{-1} \cdot A^{-1} \cdots (3)$

Putting (2) in equation (1)

$adj\ (AB) = \det(A) \cdot \det(B) \cdot B^{-1} \cdot A^{-1} \cdots (4)$

From (3) and (4)

$adj\ (AB) = adj\ (B) \cdot adj\ (A)$

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## Matrix Properties :

- $adj(O) = O$

Proof :

As we know that $|O| = 0$

Also, cofactors of $a_{ij} = 0$ for all $i$ and $j$.

So, $adj\ (O) = O$

- $adj(I) = I$

Proof : As we know that $[I] = 1$

Also, cofactors of $a_{ij} = 1$ when $i = j$ and $0$ when $i \neq j$.

So, $adj\ (I) = [a_{ij}]' = I' = I$

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# Session 08

## Properties of Inverse Matrix

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If $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\dots\begin{bmatrix}1 & n-1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 78\\0 & 1\end{bmatrix}$, then the inverse of $\begin{bmatrix}1 & n\\0 & 1\end{bmatrix}$ is :

JEE MAIN APRIL 2019

A $\begin{bmatrix}1 & 0\\12 & 1\end{bmatrix}$

B $\begin{bmatrix}1 & 2\\13 & 1\end{bmatrix}$

C $\begin{bmatrix}1 & -12\\0 & 1\end{bmatrix}$

D $\begin{bmatrix}1 & -13\\0 & 1\end{bmatrix}$

Solution:

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\dots\begin{bmatrix}1 & n-1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1+2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\dots\begin{bmatrix}1 & n-1\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1+2+3\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}\dots\begin{bmatrix}1 & n-1\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1+2+\dots+(n-1)\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & \frac{n(n-1)}{2}\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 78\\0 & 1\end{bmatrix}$$

$$\Rightarrow \frac{n(n-1)}{2} = 78 \;\Rightarrow n = 13$$

Inverse of $\begin{bmatrix}1 & n\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 13\\0 & 1\end{bmatrix}$ $\boxed{= \begin{bmatrix}1 & -13\\0 & 1\end{bmatrix}}$

$$\Rightarrow |B| = 1 \;\Rightarrow B^{-1} = Adj\, B$$

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# Key Takeaways

## Properties of Inverse of a matrix

If $A$ is a non – singular matrix ,

- $|A^{-1}| \neq 0$

$$AA^{-1} = I$$

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

$$\det(I) = 1$$

$$\Rightarrow \det(A \cdot A^{-1}) = \det(I)$$

$$\Rightarrow |A|\,|A^{-1}| = 1$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|} \quad (\because |A| \neq 0)$$

$$\Rightarrow \quad |A^{-1}| = \frac{1}{|A|}$$ → non singular

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Let $A$ & $B$ be two invertible matrices of order $3 \times 3$. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then $\det(BA^{-1}B^T)$ is equal to :

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A. $16$

B. $1$

C. $\frac{1}{16}$

D. $\frac{1}{4}$

Solution:

$$|ABA^T| = 8$$

$$\Rightarrow |A|\,|B||A^T| = 8$$

$$\Rightarrow |A|^2|B| = 8$$

$$\det(AB^{-1}) = 8$$

$$|AB^{-1}| = 8$$

$$\Rightarrow |A||B^{-1}| = 8$$

$$\Rightarrow \frac{|A|}{|B|} = 8$$

$$|A|^3 = 64$$

$$\Rightarrow |A| = 4 \quad \& \ |B| = \frac{1}{2}$$

$$\det(BA^{-1}B^T)$$

$$= |B| \cdot \frac{1}{|A|} \cdot |B|$$

$$= \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2}$$

$$\boxed{= \frac{1}{16}}$$

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# Key Takeaways

## Properties of Inverse of a matrix

If $A$ is a non – singular matrix , $\Rightarrow A^{-1}$ is also non singular

- $(A^{-1})^{-1} = A$ Let $B = A^{-1}$

$BB^{-1} = I \Rightarrow A^{-1}(A^{-1})^{-1} = I$ (Pre multiply by $A$ on both sides)

$AA^{-1}(A^{-1})^{-1} = A\,I$

$\Rightarrow (A^{-1})^{-1} = A$

- If $A =$ diag $(a_1, a_2, \dots, a_n)$ , then $A^{-1} =$ diag $( {a_1}^{-1}, {a_2}^{-1},\dots, {a_n}^{-1})$

$$A = \begin{bmatrix} a_1 & \cdots & \cdots \\ \cdots & a_2 & \cdots \\ \cdots & \cdots & a_3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{a_1} & \cdots & \cdots \\ \cdots & \frac{1}{a_2} & \cdots \\ \cdots & \cdots & \frac{1}{a_3} \end{bmatrix}$$

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# Key Takeaways

## Properties of Inverse of a matrix

If $A$ is a non – singular matrix ,

- $(A^{-1})^{-1} = A$
- If $A = \text{diag}\,(a_1, a_2, \dots, a_n)$ , then $A^{-1} = \text{diag}\,(a_1{}^{-1}, a_2{}^{-1}, \dots, a_n{}^{-1})$

Proof : $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ $\quad |A| = a_1 \cdot a_2 \cdot a_3$ , $|A| \neq 0 \quad \Rightarrow A^{-1} = \dfrac{adj\,(A)}{|A|}$

$$\Rightarrow adj\,(A) = \begin{pmatrix} a_2 a_3 & 0 & 0 \\ 0 & a_1 a_3 & 0 \\ 0 & 0 & a_2 a_1 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{a_1 a_2 a_3} \begin{pmatrix} a_2 a_3 & 0 & 0 \\ 0 & a_1 a_3 & 0 \\ 0 & 0 & a_2 a_1 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \\ 0 & 0 & \frac{1}{a_3} \end{pmatrix}$$

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If $A = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$ & $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $X$ be a matrix such that $A = BX$, then $X$ is equal to :

A $\frac{1}{2}\begin{pmatrix} 2 & 4 \\ 3 & -5 \end{pmatrix}$

B $\frac{1}{2}\begin{pmatrix} -2 & 4 \\ 3 & 5 \end{pmatrix}$

C $\begin{pmatrix} 2 & 4 \\ 3 & -5 \end{pmatrix}$

D $\begin{pmatrix} -2 & 4 \\ 3 & 5 \end{pmatrix}$

Solution:

$$A = BX$$

$$X = B^{-1}A$$

$$X = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$$

$$\Rightarrow X = \frac{1}{2}\begin{pmatrix} 2 & 4 \\ 3 & -5 \end{pmatrix}$$

Since, $|B| \neq 0$

$$B^{-1} = \frac{adj(B)}{|B|}$$

$$adj(B) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

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# Key Takeaways

## Properties of Inverse of a matrix

If matrix $A$ is invertible , then

- $A^{-k} = (A^{-1})^k, k \in \mathbb{N}$

$$A^{-2} = (A^{-1})^2 = A^{-1} \cdot A^{-1}$$

$$A^{-3} = (A^{-1})^3 = A^{-1} \cdot A^{-1} \cdot A^{-1}$$

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If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then the matrix $A^{-50}$ when $\theta = \frac{\pi}{12}$, is equal to :

**JEE MAIN JAN 2019**

A $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

B $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

C $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

D $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Solution: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\qquad A^{-1} = \dfrac{adj(A)}{|A|}$

$$|A| = \cos^2\theta + \sin^2\theta = 1$$

$$adj\,(A) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow A^{-1} = adj\ A$$

$$A^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$A^{-2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$$

$$A^{-3} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{pmatrix}$$

Similarly , $A^{-50} = \begin{pmatrix} \cos 50\theta & \sin 50\theta \\ -\sin 50\theta & \cos 50\theta \end{pmatrix}$

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If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then the matrix $A^{-50}$ when $\theta = \frac{\pi}{12}$, is equal to :

**JEE MAIN JAN 2019**

Solution: $A^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Similarly , $A^{-50} = \begin{pmatrix} \cos 50\theta & \sin 50\theta \\ -\sin 50\theta & \cos 50\theta \end{pmatrix}$

$$A^{-50}{}_{\theta=\frac{\pi}{12}} = \begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}$$

$$A^{-50}{}_{\theta=\frac{\pi}{12}} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

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## Properties of Inverse of a matrix

If matrix $A$ is invertible , then

- $(A^{-1})^T = (A^T)^{-1}$

Proof :

$$A^{-1} = \frac{adj\ (A)}{|A|}$$

$$(A^{-1})^T = \frac{\left(adj(A)\right)^T}{|A|} \qquad \left(adj(A)\right)^T = adj(A^T)$$

$$= \frac{adj(A^T)}{|A^T|}$$

$$= (A^T)^{-1}$$

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If $A$ is $3 \times 3$ non singular matrix such that $AA^T = A^T A$ and $B = A^{-1}A^T$, then $BB^T$ equals :

**JEE MAIN 2014**

A $B^{-1}$

B $(B^{-1})^T$

C $I + B$

D $I$

Solution:

$$BB^t = A^{-1}A^T(A^{-1}A^T)^T$$

$$= A^{-1}A^T A(A^{-1})^T$$

$$= A^{-1}AA^T(A^{-1})^T$$

$$= IA^T(A^{-1})^T$$

$$= I$$

$$A^{-1}A = I = AA^{-1}$$

$$(A^{-1})^T = (A^T)^{-1}$$

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# Key Takeaways

## Properties of Inverse of a matrix

If matrix $A$ is invertible , then

- $(kA)^{-1} = \frac{1}{k} A^{-1}$, Where $k$ is a scalar

Proof : $(kA)(kA)^{-1} = I$

$AA^{-1} = I$

$\Rightarrow A \cdot (kA)^{-1} = \frac{1}{k} \cdot I \ (\because |A| \neq 0)$

Premultiply by $A^{-1}$

$\Rightarrow A^{-1} \cdot A \cdot (kA)^{-1} = \frac{1}{k} \cdot (A^{-1} \cdot I)$

$$\boxed{(kA)^{-1} = \frac{1}{k} A^{-1}}$$

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If $|B| = \frac{1}{3}$, then $(3A)^{-1} A\, adj(B)$ is equal to :

A $3B^{-1}$

B $B^{-1}$

C $\frac{1}{9} B^{-1}$

D $I$

Solution:

$$(3A)^{-1} A\, adj(B) = \frac{1}{3} (A)^{-1} A \cdot adj\ B$$

$$= \frac{1}{3} \cdot I \cdot B^{-1} \cdot |B|$$

$$= \frac{1}{3} \cdot I \cdot B^{-1} \cdot \frac{1}{3}$$

$$\boxed{= \frac{1}{9} B^{-1}}$$

$$(kA)^{-1} = \frac{1}{k} A^{-1}$$

$$adj\ (A) = |A| A^{-1}$$

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# Key Takeaways

## Properties of Inverse of a matrix

If matrix $A$ is invertible , then

- $adj(kA) = k^{n-1} adj\ (A)$ , where $k$ is scalar & $n$ is the order of matrix

Proof : $adj\ (kA) = |kA|(kA)^{-1}$

$$= k^n |A| \frac{1}{k} A^{-1}$$

$$= k^{n-1} |A| A^{-1}$$

$$\boxed{adj\ (kA) = k^{n-1} adj\ (A)}$$

$adj\ (A) = |A| A^{-1}$

$|kA| = k^n |A|$

$(kA)^{-1} = \frac{1}{k} A^{-1}$

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If $A$ is a square matrix of order 4 and $|A| = 2$, then $\frac{1}{2}\, adj\,(5A)$ equals :

Solution:

$$\frac{1}{2}\, adj\,(5A) = \frac{1}{|A|} 5^3 adj(A) \qquad \frac{1}{|A|} adj\,(A) = A^{-1}$$

$$= 5^3 A^{-1}$$

$$= 125A^{-1}$$

A. $A^{-1}$

B. $125A^{-1}$

C. $50\, I$

D. $\frac{5}{2} A^{-1}$

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## Properties of Inverse of a matrix

If matrix $A$ is invertible , then

- $(AB)^{-1} = B^{-1}A^{-1}$

Proof : $(AB)(AB)^{-1} = I$

$AA^{-1} = I$

$A^{-1}(AB)(AB)^{-1} = A^{-1}I$

$B(AB)^{-1} = A^{-1}I$

$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

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Let $M$ & $N$ be two $2n \times 2n$ non singular , skew symmetric matrices such that $MN = NM$ . If $P^T$ denotes the transpose of $P$, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to :

IIT JEE 2011

A $M^2$

B $-N^2$

C $-M^2$

D $MN$

Solution: $M^T = -M$ $\quad N^T = -N$ $\quad MN = NM$

$$M^2N^2(M^TN)^{-1}(MN^{-1})^T$$

$(AB)^{-1} = B^{-1}A^{-1}$

$$= M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^TM^T$$

$(AB)^T = B^TA^T$

$(A^{-1})^T = (A^T)^{-1}$

$$= -M^2N^2N^{-1}M^{-1}N^{-1}M$$

$$= -M^2NM^{-1}N^{-1}M$$

$$= -MNN^{-1}M$$

$$= -M^2$$

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## Properties of Inverse of a matrix

If $|A|, |B| \neq 0$ , then

- $adj(AB) = (adj\ B)(adj\ A)$

Proof : $(AB)^{-1} = B^{-1}A^{-1}$

$$\frac{adj(AB)}{|AB|} = \frac{adj(B)}{|B|}\frac{adj(A)}{|A|}$$

$$adj(AB) = (adj\ B)(adj\ A)$$

Note : $adj(A_1 \cdot A_2 \cdots A_n) = (adj\ A_n) \cdots (adj\ A_2)(adj\ A_1)$

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## Properties of Inverse of a matrix

Generally, $AB = O \nRightarrow A = 0$ or $B = O$

$$AB = O \begin{cases} \text{both are singular matrices} \\ \text{if one is non singular, other will be a null matrix.} \end{cases}$$

Proof : $AB = O$

$\Rightarrow |AB| = 0 \;\Rightarrow |A| \cdot |B| = 0$

If $A$ is non singular $|A| \neq 0 \;\Rightarrow A^{-1}$ exists

$A \cdot B = 0$ ( Premultiply by $A^{-1}$ )

$A^{-1}AB = O \;\Rightarrow B = O$

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## Properties of Inverse of a matrix

If $A$ is a non-singular matrix, then

$AB = AC \Rightarrow B = C$

**Proof :** $AB = AC$

$\Rightarrow AB - AC = O \Rightarrow A(B - C) = O$

Since $A$ is non singular

$\Rightarrow (B - C) = 0$ (has to be null)

$\Rightarrow B = C$

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If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A\ adj\ (A) = AA^T$, then $5a + b$ is equal to :

**JEE MAIN 2016**

$$A\ adj\ (A) = AA^T$$

$$adj(A) = A^T$$

$$\Rightarrow \begin{bmatrix} 2 & b \\ -3 & 5a \end{bmatrix} = \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$$

$$5a = 2\ ;\ b = 3$$

$$\boxed{\Rightarrow 5a + b = 5}$$

$$AB = AC$$

$$\Rightarrow B = C$$

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# Session 09

## System of Linear Equations

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## Key Takeaways

### Inverse of a matrix by elementary transformations :

- Elementary row/column transformation include the following operations :

$(i)$ Interchanging two rows ( columns ).

$(ii)$ Multiplication of all elements of a row (column) by a non – zero scalar.

$(iii)$ Addition of a constant multiple of a row (column) to another row(column).

Note:

Two matrices are said to be equivalent if one is obtained from other using elementary transformation $A \approx B$.

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# Key Takeaways

## Inverse of a matrix by elementary transformations :

Example: By using elementary row transformation, find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$A = IA$$

$$\begin{array}{cc} A & = IA \\ \downarrow & \downarrow \\ I & A^{-1} \end{array}$$

By applying transformation, reduce to convert $'A'$ matrix into $'I'$ matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

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## Inverse of a matrix by elementary transformations :

Example: By using elementary row transformation, find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

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# Key Takeaways

## Inverse of a matrix by elementary transformations :

Example: By using elementary row transformation, find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad R_3 \to R_3 - 5R_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \qquad R_3 \to \frac{1}{2} R_3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

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# Key Takeaways

## Inverse of a matrix by elementary transformations :

Example: By using elementary row transformation, find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A \qquad R_1 = R_1 + R_3$$

$$R_1 = R_2 - 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{7}{2} & \frac{1}{2} \end{bmatrix} A$$

$$= A^{-1}$$

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# Key Takeaways

## Inverse of a matrix by elementary transformations :

Example: By using elementary row transformation, find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_3 \rightarrow R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$R_3 \rightarrow \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

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# Key Takeaways

## Inverse of a matrix by elementary transformations :

Example: By using elementary row transformation, find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

$$R_1 = R_1 + R_3$$

$$R_2 = R_2 - 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

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The inverse of $\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ is

A $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 1 \end{bmatrix}$

B $\begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & 1 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

C $\begin{bmatrix} \frac{2}{5} & \frac{3}{5} & -1 \\ -\frac{1}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

D $\begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & 0 \\ -\frac{3}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$R_1 \rightarrow R_1 + R_2$

$$\begin{bmatrix} 5 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$R_1 \rightarrow \frac{R_1}{5}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

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The inverse of $\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ is

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$R_2 \to -\frac{1}{2} R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & -\frac{3}{10} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

A $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 1 \end{bmatrix}$

B $\begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & 1 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

C $\begin{bmatrix} \frac{2}{5} & \frac{3}{5} & -1 \\ -\frac{1}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

D $\begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & 0 \\ -\frac{3}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

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The inverse of $\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ is

A $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 1 \end{bmatrix}$

B $\begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & 1 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

C $\begin{bmatrix} \frac{2}{5} & \frac{3}{5} & -1 \\ -\frac{1}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

D $\begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & 0 \\ -\frac{3}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & -\frac{3}{10} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix} \cdot A$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix} \cdot A$$

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The inverse of $\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ is

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix} \cdot A$$

This is of the form $I = A^{-1} \cdot A$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$$

A $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 1 \end{bmatrix}$

B $\begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & 1 \\ -\frac{1}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

C $\begin{bmatrix} \frac{2}{5} & \frac{3}{5} & -1 \\ -\frac{1}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

D $\begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & 0 \\ -\frac{3}{5} & \frac{3}{10} & 1 \\ \frac{1}{5} & -\frac{3}{10} & 0 \end{bmatrix}$

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# Key Takeaways

## System of linear equations (Cramer's rule):

Two variables :

Consider system of equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

$$\Delta_1(\Delta_x) = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1$$

$$\Delta_2(\Delta_y) = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$

Solution : $x = \frac{\Delta_x}{\Delta}$ ; $y = \frac{\Delta_y}{\Delta}$

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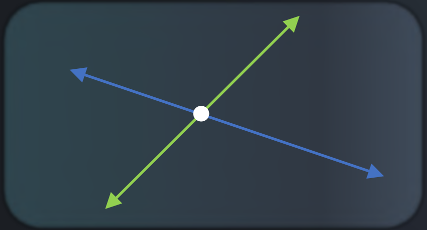
## Two variables :

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

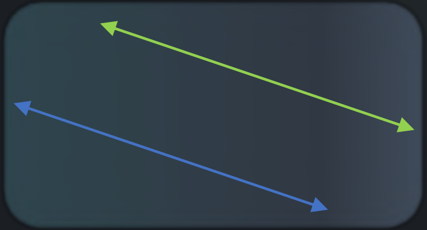
$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

## Consistent System:

$(i)$ If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then system of equations has unique solution.



$(ii)$ If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then system of equations has infinite solution.



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## Two variables :

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

## Inconsistent System:

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then system of equations has no solution.

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The number of values of $k$ , for which the system of equations :
$(k+1)x+8y=4k$ ; $kx+(k+3)y=3k-1$, has no solution, is :

A Infinite

B 1

C 2

D 3

Solution:

$$\left.\begin{aligned}(k+1)x+8y&=4k\\ kx+(k+3)y&=3k-1\end{aligned}\right\}\text{ no solution}$$

For no solution : $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

$$\frac{k+1}{k}=\frac{8}{k+3}\neq\frac{4k}{3k-1}$$

$$\frac{k+1}{k}=\frac{8}{k+3}\Rightarrow k=1,3$$

For $k=1$ $\frac{8}{1+3}=\frac{4\times1}{3\times1-1}$ (not possible)

For $k=3$ $\frac{8}{3+3}\neq\frac{4\times3}{3\times3-1}$ (possible)

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# Key Takeaways

## System of linear equations (Cramer's rule):

Two variables :

Consider system of equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$
$$a_3x + b_3y = c_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

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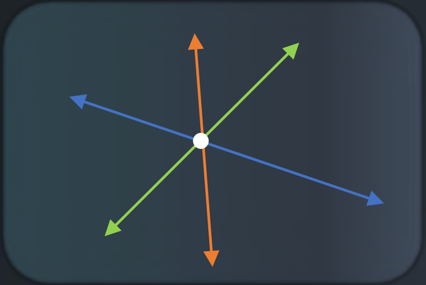
# Key Takeaways

Two variables :

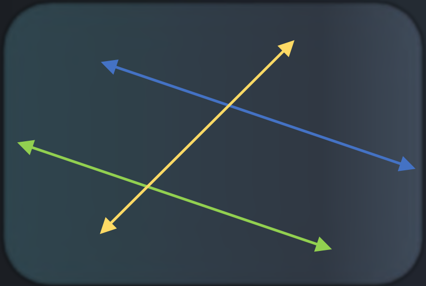
$i$) For consistent system , $\Delta = 0$ ( concurrent lines )

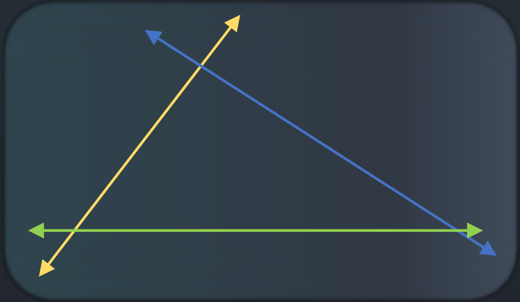


$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$
$$a_3x + b_3y = c_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$ii$) For inconsistent system , $\Delta \neq 0$





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If the system of equations : $2x + y = 1$ ; $kx + 3y + 5 = 0$ ; $x - 2y = 3$ is consistent , then the value of $k$ is :

A $5$

B $-2$

C $3$

D $-7$

Solution:

For consistent system : $\Delta = 0$

$$\begin{vmatrix} 2 & 1 & 1 \\ k & 3 & -5 \\ 1 & -2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow -5k + 30 - 40 = 0$$

$$\Rightarrow k = -2$$

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# Key Takeaways

## System of linear equations (Cramer's rule):

Three variables : Consider system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Solution : $x = \frac{\Delta_x}{\Delta}$ ; $y = \frac{\Delta_y}{\Delta}$ ; $z = \frac{\Delta_z}{\Delta}$

$\Delta \neq 0$

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# Key Takeaways

## System of linear equations (Cramer's rule):

Three variables : Consider system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

For $(0,0,0)$

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = d_3$$

If $d_1, d_2, d_3$ all are zero simultaneously, then we have HOMOGENEOUS SYSTEM.

Note: $(x, y, z) = (0,0,0)$ is always a solution of this equation and it's called Trivial solution.

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## System of linear equations (Cramer's rule):

Three variables : HOMOGENEOUS SYSTEM

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$(i)$ If $\Delta \neq 0$ , then system has trivial solution.

$(ii)$ If $\Delta = 0$ , then system has non - trivial solution (infinitely many solutions).

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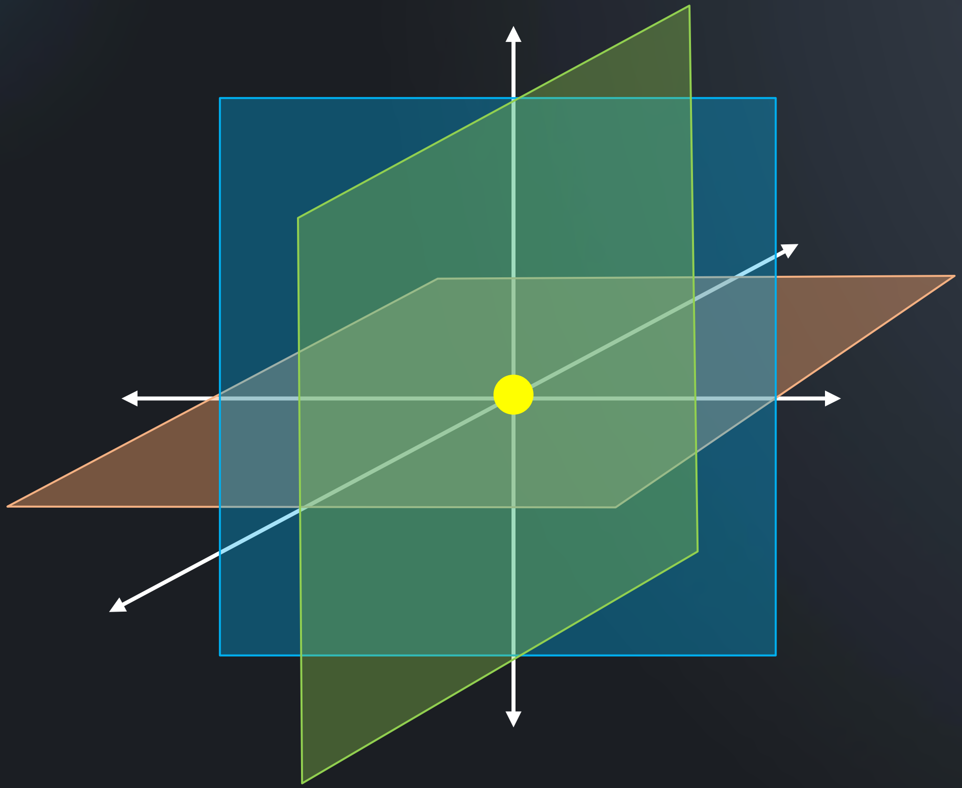
$(i)$ If $\Delta \neq 0$ , then system has trivial solution.



$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

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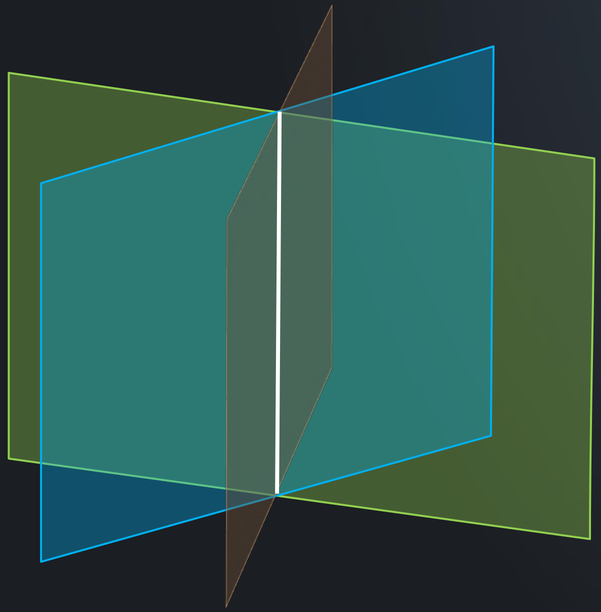
$(ii)$ If $\Delta = 0$ , then system has non - trivial solution (infinitely many solutions).



$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

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The number of values of $\theta \in (0,\pi)$ for which the system of linear equations $x + 3y + 7z = 0$ ; $\sin 3\theta\, x + \cos 2\theta\, y + 2z = 0$ ; $-x + 4y + 7z = 0$, has a non – trivial solution, is :

A Four

B Three

C Two

D One

Solution: $\theta \in (0,\pi)$

$$\left.\begin{aligned} &x + 3y + 7z = 0 \\ &\sin 3\theta\, x + \cos 2\theta\, y + 2z = 0 \\ &-x + 4y + 7z = 0 \end{aligned}\right\} \text{non – trivial solution}$$

For non – trivial solution : $\Delta = 0$

$$\begin{vmatrix} 1 & 3 & 7 \\ -1 & 4 & 7 \\ \sin 3\theta & \cos 2\theta & 2 \end{vmatrix} = 0 \qquad R_1 \to R_1 + R_2$$

$$\begin{vmatrix} 0 & 7 & 14 \\ -1 & 4 & 7 \\ \sin 3\theta & \cos 2\theta & 2 \end{vmatrix} = 0 \qquad C_3 \to C_3 - 2C_2$$

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 4 & -1 \\ \sin 3\theta & \cos 2\theta & 2 - 2\cos 2\theta \end{vmatrix} = 0$$

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The number of values of $\theta \in (0, \pi)$ for which the system of linear equations $x + 3y + 7z = 0$ ; $\sin 3\theta \, x + \cos 2\theta \, y + 2z = 0$ ; $-x + 4y + 7z = 0$, has a non – trivial solution, is :

A Four

B Three

C Two

D One

Solution: $\theta \in (0, \pi)$

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 4 & -1 \\ \sin 3\theta & \cos 2\theta & 2 - 2\cos 2\theta \end{vmatrix} = 0$$

$$\Rightarrow -1(2 - 2\cos 2\theta) + \sin 3\theta = 0$$

$$\Rightarrow \sin 3\theta + 2\cos 2\theta = 2$$

$$\Rightarrow \sin 3\theta = 4\sin^2\theta$$

$$\Rightarrow 3\sin\theta - 4\sin^3\theta - 4\sin^2\theta = 0$$

$$\Rightarrow -\sin\theta\,(4\sin^2\theta + 4\sin\theta - 3) = 0$$

$$\Rightarrow \sin\theta = 0, \frac{1}{2}, -\frac{3}{2}$$

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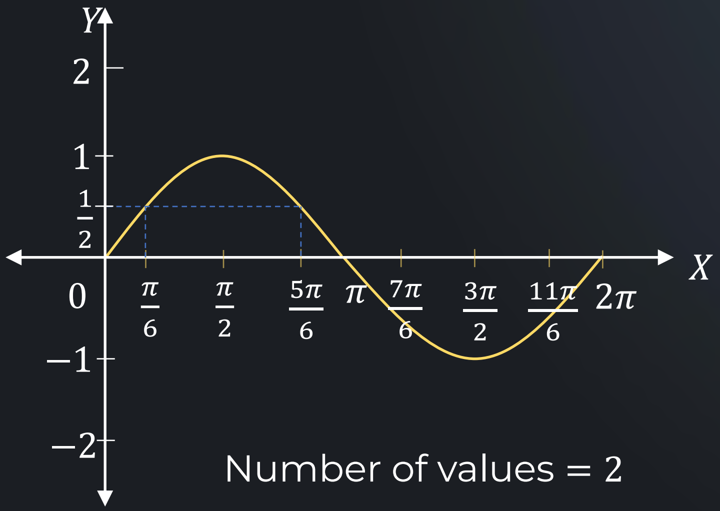
The number of values of $\theta \in (0, \pi)$ for which the system of linear equations $x + 3y + 7z = 0\,;\ \sin 3\theta\, x + \cos 2\theta\, y + 2z = 0\,;\ -x + 4y + 7z = 0$, has a non – trivial solution, is :

$\theta \in (0, \pi)$

$$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 4 & -1 \\ \sin 3\theta & \cos 2\theta & 2 - 2\cos 2\theta \end{vmatrix} = 0 \qquad \Rightarrow \sin\theta = 0, \boxed{\frac{1}{2}}, -\frac{3}{2}$$




Number of values = 2

A Four

B Three

C Two

D One

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# Session 10

## System of Linear Equations (Matrix Inversion) and Homogeneous System of Equations

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If the system of linear equations $2x+3y-z=0$ ; $x+ky-2z=0$ & $2x-y+z=0$, has a non – trivial solution $(x,y,z)$, then $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+k$ is equal to :

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A $\frac{1}{2}$

B $\frac{3}{4}$

C $-\frac{1}{4}$

D $-4$

Solution:

$$\left.\begin{aligned} 2x+3y-z&=0\\ x+ky-2z&=0\\ 2x-y+z&=0 \end{aligned}\right\}\ \text{non – trivial solution}\quad \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+k=?$$

For non – trivial solution : $\Delta = 0$

$$\begin{vmatrix} 2 & 3 & -1\\ 1 & k & -2\\ 2 & -1 & 1 \end{vmatrix}=0 \qquad \begin{array}{l} R_1\to R_1-2R_2\\ R_3\to R_3-2R_2 \end{array} \qquad \Rightarrow \begin{vmatrix} 0 & 3-2k & 3\\ 1 & k & -2\\ 0 & -1-2k & 5 \end{vmatrix}=0$$

$$\Rightarrow -1(15-2k+3+6k)=0 \Rightarrow 18-4k=0$$

$$\Rightarrow\ k=\frac{9}{2}$$

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If the system of linear equations $2x + 3y - z = 0$ ; $x + ky - 2z = 0$ &

$2x - y + z = 0$, has a non – trivial solution $(x, y, z)$, then $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + k$ is equal to :

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Solution:

So, the equations will be :

$$2x + 3y - z = 0 \cdots (i)$$

$$x + \frac{9}{2}y - 2z = 0 \cdots (ii)$$

$$2x - y + z = 0 \cdots (iii)$$

$$(i) - (iii) : 4y = 2z \qquad \Rightarrow \frac{y}{z} = \frac{1}{2}$$

$$(i) + (iii) : 4x + 2y = 0 \qquad \Rightarrow \frac{x}{y} = -\frac{1}{2}$$

$$(i) + 3(iii) : 8x + 2z = 0 \qquad \Rightarrow \frac{z}{x} = -4$$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + k = \frac{1}{2}$$

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Let $S$ be the set of all integer solutions $(x, y, z)$, of the system of equations $x - 2y + 5z = 0$ ; $-2x + 4y + z = 0$ ; $-7x + 14y + 9z = 0$, such that $15 \leq x^2 + y^2 + z^2 \leq 150$. Then, the number of elements in the set $S$ is ___

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Solution: $x - 2y + 5z = 0 \cdots (i)$

$-2x + 4y + z = 0 \cdots (ii)$ $\qquad 15 \leq x^2 + y^2 + z^2 \leq 150$

$-7x + 14y + 9z = 0 \cdots (iii)$

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ -7 & 14 & 9 \end{vmatrix} = 0$$

Let $x = k$, in $(i)$ & $(ii)$

$k - 2y + 5z = 0 \qquad \Rightarrow 2y - 5z = k$

$\Rightarrow -2k + 4y + z = 0 \qquad \Rightarrow 4y + z = 2k$

$\Rightarrow z = 0, y = \frac{k}{2}$ Since $x$, $y$, $z$ are integers, $k =$ even integer

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Let $S$ be the set of all integer solutions $(x, y, z)$, of the system of equations $x - 2y + 5z = 0\,; -2x + 4y + z = 0\,;\ -7x + 14y + 9z = 0,$ such that $15 \le x^2 + y^2 + z^2 \le 150$. Then, the number of elements in the set $S$ is ___

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Solution:

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ -7 & 14 & 9 \end{vmatrix} \qquad \Delta = 0 \qquad x = k,$$

$$z = 0, y = \frac{k}{2}$$

Since $x$, $y$, $z$ are integers , $k$ = even integer

$$15 \le \frac{5k^2}{4} \le 150$$

$$\Rightarrow 12 \le k^2 \le 120 \quad \Rightarrow k^2 \in [12\,, 120]$$

$$k \in \{\pm 4, \pm 6, \pm 8, \pm 10\}$$

$N$ number of elements in the set $S$ is = 8.

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# Key Takeaways

## System of linear equations (Cramer's rule):

- Three variables: NON-HOMOGENEOUS SYSTEM (If $d_1, d_2, d_3$ are not all simultaneously zero)

  Consider system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Solution: $x = \frac{\Delta_x}{\Delta}; y = \frac{\Delta_y}{\Delta}; z = \frac{\Delta_z}{\Delta}$

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# Key Takeaways

## System of linear equations (Cramer's rule):

$(i)$ If $\Delta \neq 0$, system of equation is consistent and has unique solution

- If at least one of $\Delta_x$ , $\Delta_y$ , $\Delta_z \neq 0$
  Unique non-trivial solution.
- If all $\Delta_x$ , $\Delta_y$ , $\Delta_z = 0$
  Unique trivial solution.

$(ii)$ If $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$ , system of equation has infinite solution.

**Example:**

$$x + 2y + z = 1$$
$$2x + 4y + 2z = 2$$
$$4x + 8y + 4z = 4$$

Infinite solution

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# Key Takeaways

## System of linear equations (Cramer's rule):

$(iii)$ If $\Delta = 0$, but at least one of $\Delta_x$, $\Delta_y$, $\Delta_z \neq 0$,
system of equations is inconsistent and has no solution.

| | | | |
|---|---|---|---|
| $\Delta \neq \mathbf{0}$ | | Consistent system | Unique solution |
| $\Delta = \mathbf{0}$ | $\Delta_x = \Delta_y = \Delta_z = 0$ | Consistent system | Infinite solution |
| | at least one of $\Delta_x$, $\Delta_y$, $\Delta_z \neq 0$ | Inconsistent system | No solution |

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The system of linear equations $x+y+z=2\,;2x+3y+2z=5\,;$ $2x+3y+(a^2-1)z=a+1$

A. Has a unique solution for $|a|=\sqrt{3}$

B. Is inconsistent for $|a|=\sqrt{3}$

C. Has infinitely many solutions for $a=4$

D. Is inconsistent for $a=4$

Solution: $x+y+z=2$

$2x+3y+2z=5$

$2x+3y+(a^2-1)z=a+1$

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ 2 & 3 & a^2-1\end{vmatrix} \quad R_3\to R_3-R_2 \Rightarrow \begin{vmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ 0 & 0 & a^2-3\end{vmatrix}=0$$

$$\Rightarrow |a|=\sqrt{3}$$

For $|a|=\sqrt{3}$, Equations become:

$x+y+z=2$

$2x+3y+2z=5$

$2x+3y+2z=\pm\sqrt{3}+1$

Inconsistent system

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Let $S$ be the set of all $\lambda \in \mathbb{R}$ for which the system of linear equations $2x - y + 2z = 2$; $x - 2y + \lambda z = -4$ ; $x + \lambda y + z = 4$, has no solution. Then the set $S$

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A Contains more than two elements

B Contains exactly two elements

C Is a singleton

D Is an empty set

Solution: $S$ be the set of all $\lambda \in \mathbb{R}$

$$\left.\begin{aligned} 2x - y + 2z &= 2 \\ x - 2y + \lambda z &= -4 \\ x + \lambda y + z &= 4 \end{aligned}\right\} \text{No solution}$$

$$\Delta = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & \lambda \\ 1 & \lambda & 1 \end{vmatrix} = 0$$

$C_1 \rightarrow C_1 - C_3$

$$\Delta = \begin{vmatrix} 0 & -1 & 2 \\ 1-\lambda & -2 & \lambda \\ 0 & \lambda & 1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 1)(-1 - 2\lambda) = 0$$

$$\Rightarrow \lambda = 1, -\frac{1}{2}$$

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Let $S$ be the set of all $\lambda \in \mathbb{R}$ for which the system of linear equations $2x - y + 2z = 2$; $x - 2y + \lambda z = -4$ ; $x + \lambda y + z = 4$, has no solution. Then the set $S$

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Solution: $S$ be the set of all $\lambda \in \mathbb{R}$

$$\left.\begin{aligned} 2x - y + 2z &= 2 \\ x - 2y + \lambda z &= -4 \\ x + \lambda y + z &= 4 \end{aligned}\right\} \text{No solution}$$

If $\Delta = 0$ , but at least one of $\Delta_x$ , $\Delta_y$ , $\Delta_z \neq 0$ ,system of equations is inconsistent and has no solution.

For $\lambda = 1$

$$\Delta_x = \begin{vmatrix} 2 & -1 & 2 \\ -4 & -2 & 1 \\ 4 & 1 & 1 \end{vmatrix} \neq 0$$

$$\Delta_x = -6$$

For $\lambda = \frac{1}{2}$

$$\Delta_x = \begin{vmatrix} 2 & -1 & 2 \\ -4 & -2 & -\frac{1}{2} \\ 4 & -\frac{1}{2} & 1 \end{vmatrix} \neq 0$$

$$\Delta_x = \frac{27}{2}$$

Then the set $S$ contains two values

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If the system of linear equations $x + y + z = 5$ ; $x + 2y + 2z = 6$ & $x + 3y + \lambda z = \mu$, $(\lambda, \mu \in \mathbb{R})$ has infinitely many solutions, then the value of $\lambda + \mu$ is:

Solution:

$$\left.\begin{aligned} x + 3y + \lambda z &= \mu \\ x + y + z &= 5 \\ x + 2y + 2z &= 6 \end{aligned}\right\} \text{infinitely many solutions}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & \lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda = 3 \qquad \Delta = \Delta_x = \Delta_y = \Delta_z = 0$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 6 \\ 1 & 3 & \mu \end{vmatrix} = 0$$

$$\Rightarrow 2\mu - 18 - (\mu - 6) + 5(3 - 2) = 7$$

$$\Rightarrow \mu - 7 = 0 \quad \Rightarrow \mu = 7$$

Putting $\lambda = 3$ and $\mu = 7$

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If the system of linear equations $x+y+z=5$ ; $x+2y+2z=6$ & $x+3y+\lambda z=\mu$, $(\lambda,\mu \in \mathbb{R})$ has infinitely many solutions, then the value of $\lambda+\mu$ is:

Solution:

$\Rightarrow \mu - 7 = 0 \quad \Rightarrow \mu = 7$

Putting $\lambda = 3$ and $\mu = 7$

$$\Delta_x = \begin{vmatrix} 5 & 1 & 1 \\ 6 & 2 & 2 \\ 7 & 3 & 3 \end{vmatrix} = 0$$

$$\Delta_y = \begin{vmatrix} 1 & 5 & 1 \\ 1 & 6 & 2 \\ 1 & 7 & 3 \end{vmatrix} = 0$$

$\lambda + \mu = 10$

A 10

B 9

C 12

D 7

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## System of linear equations (Matrix inversion):

- Consider system of equations ( If $d_1, d_2, d_3$ are not all simultaneously zero )

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Thus, we have, in matrix form $AX = B$

where $A$ is a square matrix .

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# Key Takeaways

## System of linear equations (Matrix inversion):

Thus, we have, in matrix form $AX = B$ where $A$ is a square matrix .

$AX = B$

- $|A| \neq 0$ → Unique solution: $X = A^{-1}B$
- $|A| = 0$
  - $(adj\ A)B = O$ → Infinite solution
  - $(adj\ A)B \neq O$ → No solution

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Solve the system of equations :

$x + y + z = 6\,; x - y + z = 2; 2x + y - z = 1$, using matrix inverse.

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$C_3 \to C_2 + C_3 \quad |A| = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \quad |A| = 6$$

$|A| = 2(1 + 2) = 6 \neq 0$ (Unique solution)

$$\therefore X = A^{-1}B$$

$$\text{Adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} \quad A^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} \Rightarrow x = 1\,,\ y = 2\,,\ z = 3$$

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## Homogenous system of equations (Matrix inversion):

- Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

  Thus, we have, in matrix form $AX = B$

  where $A$ is a square matrix.

Consider system of equations

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

- If $|A| \neq 0$, then system has trivial solution $(x, y, z) = (0, 0, 0)$

  $A^{-1}AX = A^{-1} \cdot 0 \Rightarrow X = 0$

- If $|A| = 0$, then system has non–trivial (infinite) solution.

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The set of all values of $\lambda$ for which the system of equations $x - 2y - 2z = \lambda x; x + 2y + z = \lambda y; -x - y = \lambda z$ has a non –trivial solution

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Solution:

$$|A| = \begin{vmatrix} 1-\lambda & -2 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda(2-\lambda)-1) + 2(\lambda-1) - 2(1+\lambda-2) = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$\Rightarrow (\lambda - 1)^3 = 0$$

$$\Rightarrow \lambda = 1$$

A. Is a singleton

B. Contains exactly two elements

C. Is an empty set

D. Contains more than two elements

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# Session 11

## Cayley – Hamilton Theorem
## &
## Special Types of Matrices

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# Key Takeaways

## Characteristic polynomial and characteristic equation:

Let $A$ be a square matrix.

The polynomial $|A - \lambda \mathrm{I}|$ is called characteristic polynomial of $A$ and equation $|A - \lambda \mathrm{I}| = 0$ is called characteristic equation of $A$.

(here $\lambda$ is called eigen value of $A$)

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A - \lambda \mathrm{I}| = \begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{vmatrix} = 0$$ will be the characteristic equation .

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# Key Takeaways

- Cayley – Hamilton Theorem

Every square matrix $A$ satisfies its characteristic equation $|A-\lambda \mathrm{I}| = 0.$

If $a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$ is the characteristic equation of $A$

$\therefore a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI = O$

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If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ is a root of the polynomial $x^3 - 6x^2 + 7x + k = 0$, then the value of $k$ is:

A 2

B 4

C $-2$

D 1

Solution: $x^3 - 6x^2 + 7x + k = 0$ $\quad A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$

$A^3 - 6A^2 + 7A + kI = 0 \quad \cdots (i)$

In order to get characteristics equation $|A - \lambda I| = 0$

$$|A - \lambda| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\big((2-\lambda)(3-\lambda) - 0\big) + 2\big(0 - 2(2-\lambda)\big) = 0$

$\Rightarrow (2-\lambda)\big((1-\lambda)(3-\lambda) - 4\big) = 0 \Rightarrow (2-\lambda)(\lambda^2 - 4\lambda - 1) = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$ → characteristic equation

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If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ is a root of the polynomial $x^3 - 6x^2 + 7x + k = 0$, then the value of $k$ is:

Solution: $A^3 - 6A^2 + 7A + kI = 0 \quad \cdots (i)$ $\quad A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$

$\Rightarrow \lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$ → characteristic equation

∴ By Cayley – Hamilton Theorem ,

$A^3 - 6A^2 + 7A + 2I = 0 \cdots (ii)$

By $(i)$ & $(ii)$, $k = 2$

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If $A = \begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $10A^{-1}$ is equal to :

A $A - 6I$

B $4I - A$

C $6I - A$

D $A - 4I$

Solution: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 9 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(4-\lambda) - 18 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 8 - 18 = 0$$ → characteristic equation

By Cayley – Hamilton theorem,

$$A^2 - 6A - 10I = 0$$

$$\Rightarrow A^{-1}A^2 - 6A^{-1}A - 10A^{-1}I = 0$$

$$\Rightarrow A - 6I - 10A^{-1} = 0$$

$$\Rightarrow 10A^{-1} = A - 6I$$

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If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ & $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^{-1} = \frac{1}{6}(A^2 + cA + dI)$, then the ordered pair $(c, d)$ is:

A $(-6, -11)$

B $(6, -11)$

C $(-6, 11)$

D $(6, 11)$

Solution: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ $A^{-1} = \frac{1}{6}(A^2 + cA + dI)$ $(c, d) = ?$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\big((1-\lambda)(4-\lambda) + 2\big) = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ $\rightarrow$ characteristic equation

By Cayley – Hamilton theorem,

$A^3 - 6A^2 + 11A - 6I = 0$

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If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ & $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^{-1} = \frac{1}{6}(A^2 + cA + dI)$, then the ordered pair $(c, d)$ is:

A $(-6, -11)$

B $(6, -11)$

C $(-6,11)$

D $(6,11)$

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$$

$$A^3 - 6A^2 + 11A - 6I = 0$$

$$A^{-1}A^3 - 6A^{-1}A^2 + 11A^{-1}A - 6A^{-1}I = 0$$

$$6A^{-1} = A^2 - 6A + 11I$$

$$\Rightarrow A^{-1} = \frac{1}{6}(A^2 - 6A + 11I)$$

$$(c, d) \equiv (-6, 11)$$

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# Key Takeaways

## Special types of Matrices

- Orthogonal Matrix

A square matrix $A$ is said to be orthogonal if $AA^T = I = A^TA$

For orthogonal matrix $A$, $A^T = A^{-1}\ (|A| = \pm 1)$

Example: If $A$ is orthogonal and $ABA = B^T$ , then show that $BA$ is symmetric .

$ABA = B^T$

Pre multiply $A^T$ on both sides $A^TABA = A^TB^T$ $AA^T = I$

$\Rightarrow I \cdot BA = A^TB^T \Rightarrow BA = A^T \cdot B^T$

$BA = (BA)^T \Rightarrow BA$ is symmetric .

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# Key Takeaways

## Special types of Matrices

- Involutory matrix :

A square matrix $A$ is said to be involutory if $A^2 = I$.

$\Rightarrow A \cdot A = I \;\; \Rightarrow A = A^{-1}$

$\Rightarrow A^3 = A^2 \cdot A = I \cdot A$

$\Rightarrow A^3 = A$

Note:

If $A$ is involutory , then $A = A^{-1}$

$A^3 = A \,; \;\; A^4 = I$

$A^{2k} = I \;\; ; \; A^{2k+1} = A \;\; , k \in$ Integer

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If $P$ is an orthogonal matrix and $Q = PAP^T$ and $B = P^TQ^{1000}P$, then $B^{-1}$ is ( where $A$ is involutory matrix )

A. $A$

B. $A^{1000}$

C. $I$

D. None of these

Solution:

$$B = P^TQ^{1000}P$$

$$= P^T(PAP^T)^{1000}P$$

$$= P^TPAP^T \cdot PAP^T \cdots PAP^TP \quad P^TP = I$$

$$= A^{1000} \quad = I \quad A^{2k} = I$$

$$B = I$$

$$B^{-1} = I$$

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### Special types of Matrices

#### Idempotent matrix

A square matrix $A$ is said to be idempotent if $A^2 = A$.

Note:

If $A$ is idempotent, then $A^n = A, \forall\, n \geq 2, n \in \mathbb{N}$

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If $A$ is idempotent and $(I+A)^{10} = I + kA$, then $k$ is:

A $1023$

B $2047$

C $1024$

D $2048$

Solution: $(I+A)^{10} = {}^{10}C_0 I + {}^{10}C_1 I \cdot A + {}^{10}C_2 I \cdot A^2 + \cdots + {}^{10}C_{10} A^{10}$ $A^n = A$

$= I + {}^{10}C_1 A + {}^{10}C_2 A + \cdots + {}^{10}C_{10} A$ $A^n = A$

$= I + ({}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_{10})A$

$= I + (2^{10} - 1)A$

$= I + (1024 - 1)A$

$\therefore k = 1023$

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## Special types of Matrices

- Nilpotent Matrix

A square matrix $A$ is said to be nilpotent matrix of order $p$

If $A^p = O$ and $A^{p-1} \neq O$

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Show that the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$ is nilpotent of order 3.

Solution:

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ $A$ is a nilpotent matrix of order 3.

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Let $\omega \neq 1$, be a cube root of unity and $S$ be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ where each of $a, b, \& c$ is either $\omega$ or $\omega^2$. Then number of distinct matrices in set $S$ is:

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A 2

B 6

C 4

D 8

Solution:

$$\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$$

$$1 - a\omega - c\omega + ac\omega^2 \neq 0$$

$$\Rightarrow (1 - a\omega)(1 - c\omega) \neq 0 \ \Rightarrow a \neq \frac{1}{\omega} \ \& \ c \neq \frac{1}{\omega}$$

So, $a = c = \omega$ , while $b$ can take $\omega$ or $\omega^2$

Number of matrices = 2

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If $P$ is a $3 \times 3$ matrix such that $P^T = 2P + I$, where $P^T$ is transpose of $P$ and $I$ is the $3 \times 3$ identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that

Solution: $P^T = 2P + I$

$P = 2P^T + I$

$= 4P + 3I$

$\Rightarrow P = -I$

$PX = -X$

A $PX = X$

B $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

C $PX = -X$

D $PX = 2X$

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Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and $I$ is an identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31}+q_{32}}{q_{21}}$ equals :

A 52

B 103

C 201

D 205

Solution: $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ $\quad Q = [q_{ij}]$ $\quad P^{50} - Q = I$ $\quad \frac{q_{31}+q_{32}}{q_{21}} = ?$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 48 & 8 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 48 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 96 & 12 & 1 \end{bmatrix}$$

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Solution: $P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 48 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 96 & 12 & 1 \end{bmatrix}$

Similarly,

$$P^n = \begin{bmatrix} 1 & 0 & 0 \\ 4n & 1 & 0 \\ 16\frac{n(n+1)}{2} & 4n & 1 \end{bmatrix}$$

$$\therefore P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 8 \cdot 50 \cdot 51 & 200 & 1 \end{bmatrix}$$

$$P^{50} - I = \begin{bmatrix} 0 & 0 & 0 \\ 200 & 0 & 0 \\ 8 \cdot 50 \cdot 51 & 200 & 0 \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} 0 & 0 & 0 \\ 200 & 0 & 0 \\ 8 \cdot 50 \cdot 51 & 200 & 0 \end{bmatrix} \quad \therefore \frac{q_{31}+q_{32}}{q_{21}} = \frac{400 \cdot 51 + 200}{200} = 103$$

A 52

B 103

C 201

D 205

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THANK YOU

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