Regional Mathematical Olympiad (RMO) – 2023

Instructions:

1. The RMO 2023 question paper consists of six questions.
2. The response to each question requires writing detailed mathematical arguments.
3. Duration: 3 hrs

1. Let N be the set of all +ve integer and $S = \{ (a,b,c,d) \in N^4 : a^2 + b^2 + c^2 = d^2\}$. Find the largest +ve integer m such that m divides $a\ b\ c\ d$ for all $(a,b,c,d) \in S$.

Ans: Let smallest pair $\{1,2,2,3\}$

$a^2 + b^2 + c^2 = d^2\ Mod\ 3$

$d^2 \equiv 0,1\ Mod\ 3$

If $d^2 \equiv 0\ Mod\ 3$ then (a^2, b^2, c^2) in $Mod\ 3$ following possibility

$(0,0,0)(1,1,1)$

In each case we have a multiple of 3.

If $d^2 \equiv 0,1\ Mod\ 3$ then

$(a^2, b^2, c^2)\ \ Mod\ 3$ can we $(0,0,1)$

So, we will get a multiple of 3

Note take $Mod\ 4$

If $d^2 \equiv 0\ Mod\ 4$

It can we $(0,0,0)$ give as a multiple of $mod\ 4$.

If $d^2 \equiv 1\ Mod\ 4$

$(a^2, b^2, c^2) \equiv (0,1,0)\ Mod\ 4$

Here 2 of a, b, c must we even

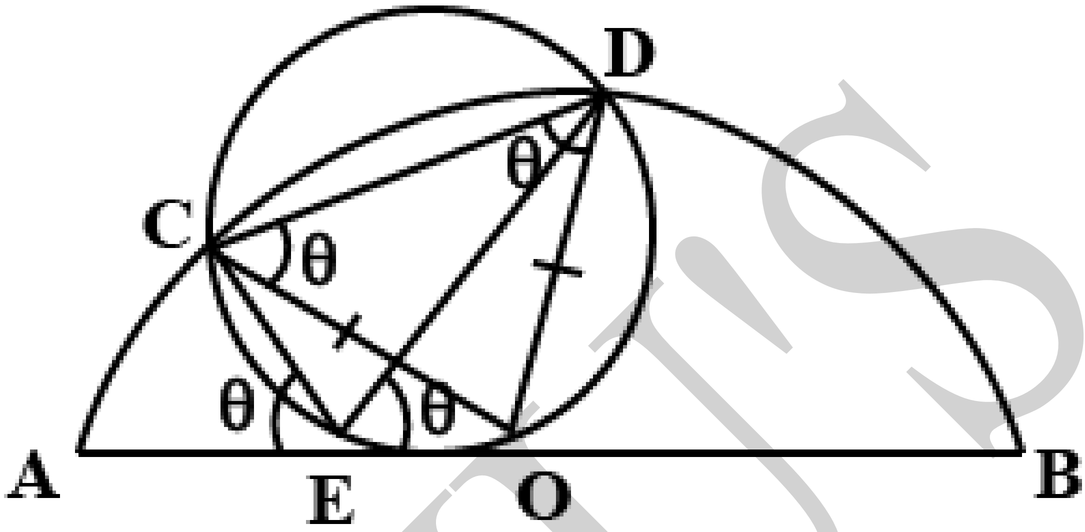
Hence, $m = 12$

2. Let ω be a semicircle with AB as the bounding diameter and let CD be a variable chord of the semicircle of constant length such that C, D lie in the interior of the arc AB. Let E be a point on the diameter AB such that CE and DE are equally inclined to the line AB. Prove that

(a) The measure of $\angle CED$ is a constant ;

(b) The circumcircle of triangle CED passes through a fixed point.

Ans:



Draw a circumcircle for ΔCED which is passes through O. Centre of semicircle.
$AO = OB$
And join C to O and D to O
Now $OC = OD$ =Radius
Which means Δ OCD is Isoceles Triangle
$\angle ODC = \angle OCD = \theta$
$\Rightarrow \angle COD = 180 - 2\theta$ ….(1)
Now take OD as a chord then
$\angle OCD = \angle OED = \theta$
$\angle CEA = \angle CDO = \theta$
Then, $\angle CED = 180 - 2\theta$ which is constant.

3. For any natural number n, expressed in base 10, let $s(n)$ denote the sum of all its digits. Find all natural numbers $m\ and\ n$ such that $m < n$ and $(s(n))^2 = m\ and\ s(m))^2 = n$.

Ans: Give $(\mathrm{s(n)})^2 = m\ \ and\ m < n$

$(\mathrm{s(m)})^2 = n$

No of digit	S(n)	Max $(\mathrm{s(n)})^2$
1	9	81
2	18	824
3	27	729
4	36	1296
5	45	2025
6	54	2916

Now, n can +ve 5 digit no.

If $n\ is\ 4$ –digit no then m will be 4 –digit no.

$=\ m \leq 1296$

= max value of $\mathrm{s(m)}$ is when $\mathrm{m} = 999$

= max $(\mathrm{s(m)})^2 = 27$

= max $(\mathrm{s(m)})^2 = 729$ and

$n\ \leq 729\ as\ n =\ (\mathrm{s(m)})^2$

Now,

If n is a 3 digit no then $m \leq 729$

$n\ \leq 729$

$\Rightarrow m \leq (9 + 9 + 9)^2 = 729$

$\Rightarrow n \leq (6 + 9 + 9)^2 = 576$

$\Rightarrow m \leq (4 + 9 + 9)^2 = 484$

$\Rightarrow n \leq (3 + 9 + 9)^2 = 441$

So

$m < n < 441$

$1 \rightarrow 1$	$121 \rightarrow 16$
$4 - 16$	$144 - 81$
$9 - 81$	$169 - 256$
$16 - 49$	$196 - 256$
$25 - 49$	$225 - 81$

Continue with the pattern.

$1 - 1$

$81 - 81$

$169 - 256$

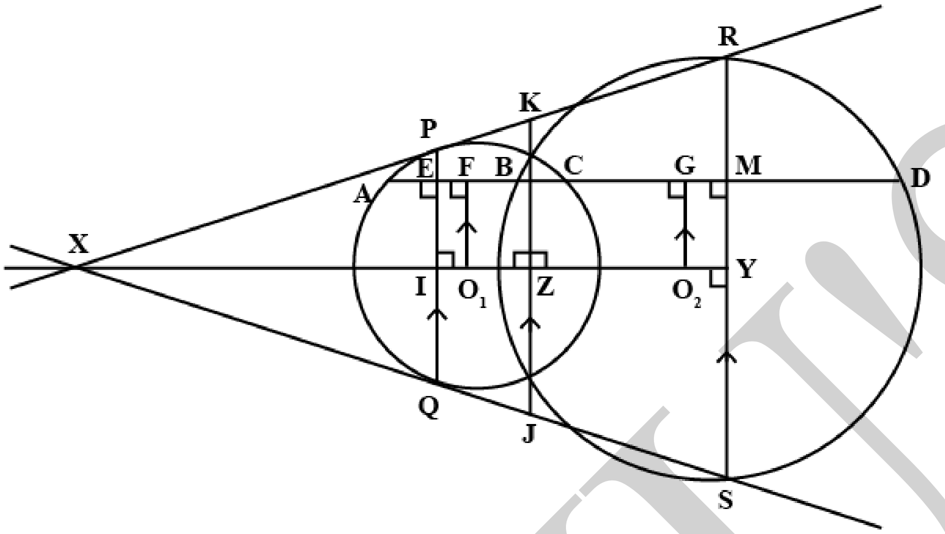
Are the solution.

Since $m < n$

$m = 169,\ \ n = 256$

4. Let Ω_1, Ω_2 be two intersecting circles with centres O_1, O_2 respectively. Let l be a line that intersects - Ω_1 at points A, C and Ω_2 at points B, D such that A, B, C, D are collinear in that order. Let the perpendicular bisector of segment A B intersect Ω_1 at points P, Q; and the perpendicular bisector of segment C D intersect Ω_2 at points R, S such that P, R are on the same side of l. Prove that the midpoints of P R, Q S and $O_1 O_2$ are collinear.

Ans:



Join P to R and Q to S and extend, which is meet at X.
Now,
$PQ \perp AB \Rightarrow AE = EB$
$RS \perp CD \Rightarrow CH = HD$
Now O_1, O_2 is center of circle.
and extend the line which intersect at x.
Now PQ II RS
PQSQ is Trapezium
Now, $\Delta XRS \sim \Delta XPQ$
and $\frac{XP}{PR} = \frac{XQ}{QS}$
Now, K is midpoint of PR.
$\frac{XP}{PK} = \frac{XQ}{QJ}$
Now PQIIKJIIRS
Let $AB = a,\ BC = b\ \ CD = c$
$AF = \frac{a+b}{2}$ $\quad GD = \frac{b+c}{2}$
$AE = \frac{a}{2}$ $\quad HD = \frac{c}{2}$
$EF = \frac{b}{2}$ $\quad GH = \frac{b}{2}$
Now using intercept theorem.
We can say
$PK = KR$
$QJ = JS$
$IZ - IO_1 = ZY - O_2Y$
$O_1Z = O_2Z$
So Z is midpoint of O_1O_2.

5. Let $n > k > 1$ be positive integers. Determine all positive real numbers $a_1, a_2, \ldots, a_n$ which satisfy $\sum_{i=1}^{n} \sqrt{\frac{ka_1^k}{(k-1)a_i^k+1}} = \sum_{i=1}^{n} a_1 = n$.

Ans:

$A.M. \geq G.M.$

$\frac{\underbrace{a_i^k+a_i^k+a_i^k+\cdots a_i^k}_{k-1\,time}++1}{k} \geq \sqrt[k]{a_i^{k(k-1)}}$

$\frac{(k-1)a_i^k+1}{k} \geq a_i^{k-1}$

$\frac{(k-1)a_i^k+1}{k} \geq a_i^{k-1}$

$(k-1)a_i^k + 1 \geq ka_i^{k-1}$

$\frac{1}{(k-1)\ a\_i\hat{}k+1} \leq \frac{1}{ka_i^{k-1}}$

$\frac{ka_i^k}{(k-1)a_i^k+1} \leq \frac{ka_i^k}{ka_i^{k-1}} = a_i$

$\sqrt{\frac{ka_i^k}{(k-1)a_i^k+1}} \leq \sqrt{a_i}$

$\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k+1}} \leq \sum_{i=1}^{n} \sqrt{a_1}$

Now A.M. ≥ RMS

$\frac{\sqrt{a_1}+\sqrt{a_2}+\cdots\sqrt{a_n}}{n} \leq \sqrt{\frac{a_1+a_2+\cdots a_n}{n}} \leq \sqrt{\frac{n}{n}} \leq 1$

$\sqrt{\frac{ka_i^k}{(k-1)a_i^k+1}} \leq n$

Here this to be true only

$a_1 = a_2 \ldots a_n = 1$

$\sum_{i=1}^{n} \sqrt{a_1} = n$

6. Consider a set of 16 points arranged in a 4×4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

Ans:

If there is a $n \times n$ grid then If we take (2n-1) vertices then there will be at least one right isosceles triangle.
4×4
$2n - 1 = 7$
$A_1 \quad A_2 \quad A_3 \quad A_4$
$A_5 \quad A_6 \quad A_7 \quad A_8$
$A_9 \quad A_{10} \quad A_{11} \quad A_{12}$
$A_{13} \quad A_{14} \quad A_{15} \quad A_{16}$
There are 3 types of lattice points
Corners → A_1, A_{13}, A_4, A_{16}
Edges → A_2, A_3, A_5 , $A_9, A_{14}, A_{15}, A_8, A_{12}$
Centres → A_6, A_7, A_{10}, A_{11}
Let us assume a 7 colouring exists without blue isosceles right Δ.
I: There can't be more than 4 edges coloured blue.
Assume 2 subgroups within edge points
$\{A_2, A_8, A_{15}, A_9\}$ $\{A_3, A_{12}, A_{14}, A_5\}$
Choosing any 3 points from either group will form a Right isosceles Δ . Thus, by PHP if 5 edges are coloured, we will get a contradiction. Now same way we can choose max 2 corners & 2 centres.
now possible colouring of the points will be as follows
1. 2 corner 2 centre 3 edge
2. 1 corner 2 centre 4 edge
3. 2 corner 1 centre 4 edge
Thus 1 centre must be selected.
Let the centre point A_6
& consider the following groups
$\{A_1, A_2, A_5\}$, $\{A_{12}, A_{13}\}$, $\{A_{10}, A_{11}, A_7\}$
$\{A_4, A_{15}\}$, $\{A_{14}, A_{16}, A_8\}$ Remaining
A_3 & A_3 .
Off these groups if 2 or more points are coloured along with A_6 will form Isosceles Right Δ. There we can select max 1 point to be coloured in each group' since there are 5 groups, we have to colour either A_3 or A_1
Since the points are symmetric let's colour A_9 since A_6 and A_3 are coloured blue A_{10} A_{11} and A_1, A_5 con not be colowed blue.
⇒ A_2, A_1 must be coloured Blue, but A_2, A_7, A_8 will form a Right isosceles Δ.
Thus, a colouring does not exist your 16 points in a 4×4 grid.